

## Text S1

We prove Theorem 2 which states:

**Theorem 2.** Let  $C_\alpha$  be a  $d$ -copula,  $I := \{1, \dots, d\}$ ,  $S \subseteq I$ ,  $P_U(\bigcap_{i \in S} \{U_i \leq u_i\}) := C_\alpha(\mathbf{u})$  a measure, and  $C_{\alpha,S}^F(\mathbf{u}) := P_U(\left(\bigcap_{i \in S} \{U_i > 1 - u_i\}\right) \cap \left(\bigcap_{i \in \bar{S}} \{U_i \leq u_i\}\right))$ . Then  $C_{\alpha,S}^F$  is a  $d$ -copula and can be expressed as

$$C_{\alpha,S}^F(\mathbf{u}) = \sum_{A \subseteq S} (-1)^{|A|} C_\alpha(\kappa_{S,A}(1, \mathbf{u}), \dots, \kappa_{S,A}(d, \mathbf{u})), \quad (1)$$

$$\text{where } \bar{S} = I \setminus S \text{ and } \kappa_{S,A}(i, \mathbf{u}) = \begin{cases} 1 - u_i & \text{if } i \in A, \\ 1 & \text{if } i \in S \setminus A, \\ u_i & \text{if } i \in \bar{S}. \end{cases}$$

We need a short lemma for the proof:

**Lemma 3.** Let  $(\Omega, \mathcal{F}, P)$  be a probability space with  $A \in \mathcal{F}$ . Then  $P^A := P(\bullet \cap A)$  is a measure on  $\mathcal{F}$ .

*Proof.* Verification of the measure properties:

1.  $P^A(\emptyset) = P(\emptyset \cap A) = P(\emptyset) = 0$ .
2. For  $E_1, E_2, \dots \in \mathcal{F}$  pairwise disjoint:

$$P^A\left(\bigcup_{i=1}^{\infty} E_i\right) = P\left(\left(\bigcup_{i=1}^{\infty} E_i\right) \cap A\right) = P\left(\bigcup_{i=1}^{\infty} (E_i \cap A)\right) = \sum_{i=1}^{\infty} P(E_i \cap A) = \sum_{i=1}^{\infty} P^A(E_i).$$

□

*Proof of the Main Theorem.* We start with Eqn 1. By definition  $P_U$  is a probability mass function. Let  $B := \left(\bigcap_{i \in \bar{S}} \{U_i \leq u_i\}\right)$ . We can apply the generalized inclusion exclusion principle for arbitrary measures [1]:

$$\begin{aligned} C_{\alpha,S}^F(\mathbf{u}) &= P_U^B\left(\bigcap_{i \in S} \{U_i > 1 - u_i\}\right) = P_U^B(B) - P_U^B\left(\overline{\bigcap_{i \in S} \{U_i > 1 - u_i\}}\right) \\ &= P_U^B(B) - P_U^B\left(\bigcup_{i \in S} \{U_i \leq 1 - u_i\}\right) \\ &\stackrel{\text{Lemma 3}}{\equiv} P_U^B(B) - \sum_{k=1}^{|S|} (-1)^{k-1} \sum_{\substack{A \subseteq S \\ |A|=k}} P_U^B\left(\bigcap_{i \in A} \{U_i \leq 1 - u_i\}\right) \\ &= \sum_{k=0}^{|S|} (-1)^k \sum_{\substack{A \subseteq S \\ |A|=k}} P_U^B\left(\bigcap_{i \in A} \{U_i \leq 1 - u_i\}\right) \\ &= \sum_{A \subseteq S} (-1)^{|A|} P_U^B\left(\bigcap_{i \in A} \{U_i \leq 1 - u_i\}\right) \\ &= \sum_{A \subseteq S} (-1)^{|A|} P_U\left(\left(\bigcap_{i \in A} \{U_i \leq 1 - u_i\}\right) \cap \left(\bigcap_{i \in \bar{S}} \{U_i \leq u_i\}\right)\right) \\ &= \sum_{A \subseteq S} (-1)^{|A|} C_\alpha(\kappa_{S,A}(1, \mathbf{u}), \dots, \kappa_{S,A}(d, \mathbf{u})). \end{aligned}$$

Now we will show that  $C_{\alpha,S}^F$  is a copula. We verify the copula properties:

1. Let  $\mathbf{u} \in [0, 1]^d$  with  $\exists a \in I : u_a = 0$ . We distinguish two cases:

- For  $a \in \bar{S}$ :

$$\begin{aligned} C_{\alpha, S}^F(\mathbf{u}) &= \sum_{A \subseteq S} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(a-1, \mathbf{u}), 0, \kappa_{S, A}(a+1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &= \sum_{A \subseteq S} (-1)^{|A|} 0 = 0. \end{aligned}$$

- For  $a \in S$ :

$$\begin{aligned} C_{\alpha, S}^F(\mathbf{u}) &= \sum_{A \subseteq S \setminus \{a\}} (-1)^{|A \cup \{a\}|} C_{\alpha}(\kappa_{S, A \cup \{a\}}(1, \mathbf{u}), \dots, \kappa_{S, A \cup \{a\}}(a-1, \mathbf{u}), 1, \kappa_{S, A \cup \{a\}}(a+1, \mathbf{u}), \dots, \\ &\quad \kappa_{S, A \cup \{a\}}(d, \mathbf{u})) + \sum_{A \subseteq S \setminus \{a\}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(a-1, \mathbf{u}), 1, \kappa_{S, A}(a+1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &= \sum_{A \subseteq S \setminus \{a\}} (-1)^{|A|+1} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(a-1, \mathbf{u}), 1, \kappa_{S, A}(a+1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &\quad + \sum_{A \subseteq S \setminus \{a\}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(a-1, \mathbf{u}), 1, \kappa_{S, A}(a+1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &= 0. \end{aligned}$$

2. Let  $\mathbf{u} \in [0, 1]^d$  with  $u_i = 1$  if  $i \neq a$ . We distinguish the same cases:

- For  $a \in \bar{S}$ :

$$\begin{aligned} C_{\alpha, S}^F(\mathbf{u}) &= (-1)^{|\emptyset|} C_{\alpha}(\kappa_{S, \emptyset}(1, \mathbf{u}), \dots, \kappa_{S, \emptyset}(d, \mathbf{u})) + \sum_{\substack{A \subseteq S \\ A \neq \emptyset}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &\stackrel{(1)}{=} (-1)^0 C_{\alpha}(\kappa_{S, \emptyset}(1, \mathbf{u}), \dots, \kappa_{S, \emptyset}(a-1, \mathbf{u}), \kappa_{S, \emptyset}(a, \mathbf{u}), \kappa_{S, \emptyset}(a+1, \mathbf{u}), \dots, \kappa_{S, \emptyset}(d, \mathbf{u})) \\ &\stackrel{(2)}{=} C_{\alpha}(\underbrace{1, \dots, 1}_{a-1 \text{ times}}, u_a, \underbrace{1, \dots, 1}_{d-a \text{ times}}) = u_a. \end{aligned}$$

(1)  $\forall A \subseteq S, A \neq \emptyset : a \in \bar{S} \Rightarrow a \notin A \Rightarrow \exists i \in A : i \neq a$ , thus  $\kappa_{S, A}(i, \mathbf{u}) = 1 - u_i \stackrel{i \neq a}{=} 0$ . Therefore, we have:

$$\begin{aligned} &\sum_{\substack{A \subseteq S \\ A \neq \emptyset}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &= \sum_{\substack{A \subseteq S \\ A \neq \emptyset}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(i-1, \mathbf{u}), 0, \kappa_{S, A}(i+1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &= \sum_{\substack{A \subseteq S \\ A \neq \emptyset}} (-1)^{|A|} 0 = 0. \end{aligned}$$

(2)  $\kappa_{S, \emptyset}(a, \mathbf{u}) \stackrel{a \in \bar{S}}{=} u_a$ ;  $\forall i \in S : \kappa_{S, \emptyset}(i, \mathbf{u}) = 1$ ;  $\forall i \in \bar{S} \setminus \{a\} : \kappa_{S, \emptyset}(i, \mathbf{u}) = 1$ .

- For  $a \in S$ :

$$\begin{aligned} C_{\alpha, S}^F(\mathbf{u}) &= (-1)^{|\emptyset|} C_{\alpha}(\kappa_{S, \emptyset}(1, \mathbf{u}), \dots, \kappa_{S, \emptyset}(d, \mathbf{u})) \\ &\quad + (-1)^{|\{a\}|} C_{\alpha}(\kappa_{S, \{a\}}(1, \mathbf{u}), \dots, \kappa_{S, \{a\}}(d, \mathbf{u})) \\ &\quad + \sum_{\substack{A \subseteq S \\ A \neq \emptyset, A \neq \{a\}}} (-1)^{|A|} C_{\alpha}(\kappa_{S, A}(1, \mathbf{u}), \dots, \kappa_{S, A}(d, \mathbf{u})) \\ &\stackrel{(3)}{=} C_{\alpha}(\kappa_{S, \emptyset}(1, \mathbf{u}), \dots, \kappa_{S, \emptyset}(d, \mathbf{u})) - C_{\alpha}(\kappa_{S, \{a\}}(1, \mathbf{u}), \dots, \kappa_{S, \{a\}}(d, \mathbf{u})) \\ &\stackrel{(4)}{=} C_{\alpha}(\underbrace{1, \dots, 1}_{d \text{ times}}) - C_{\alpha}(\kappa_{S, \{a\}}(1, \mathbf{u}), \dots, \kappa_{S, \{a\}}(d, \mathbf{u})) \end{aligned}$$

$$\stackrel{(5)}{=} 1 - C_\alpha(\underbrace{1, \dots, 1}_{a-1 \text{ times}}, \underbrace{1 - u_a, 1, \dots, 1}_{d-a \text{ times}})$$

$$= 1 - (1 - u_a) = u_a.$$

(3)  $\forall A \subseteq S, A \neq \emptyset, A \neq \{a\} : \exists i \in A : i \neq a$ , remainder as in (1).

(4)  $\kappa_{S, \emptyset}(a, \mathbf{u}) = 1; \quad \forall i \in S \setminus \{a\} : \kappa_{S, \emptyset}(i, \mathbf{u}) = 1; \quad \forall i \in \bar{S} : \kappa_{S, \emptyset}(i, \mathbf{u}) = 1.$

(5)  $\kappa_{S, \{a\}}(a, \mathbf{u}) = 1 - u_a; \quad \forall i \in S \setminus \{a\} : \kappa_{S, \{a\}}(i, \mathbf{u}) = 1; \quad \forall i \in \bar{S} : \kappa_{S, \{a\}}(i, \mathbf{u}) = u_i = 1.$

3. We show that  $C_{\alpha, S}^F$  is  $d$ -increasing, i.e.  $\forall B = \times_{i=1}^d [u_i, v_i] : V_{C_{\alpha, S}^F}(B) \geq 0$ ,

where  $V_{C_{\alpha, S}^F}(B) = \sum_{i_1=1}^2 \cdots \sum_{i_d=1}^2 (-1)^{i_1 + \cdots + i_d} C_{\alpha, S}^F(g_{1, i_1}, \dots, g_{d, i_d}), g_{i, 1} = u_i, g_{i, 2} = v_i$ . In the definition of  $V_{C_{\alpha, S}^F}$  we assume that  $u_i \leq v_i$  for all  $i \in I$ . More generally, we can define  $V_{C_{\alpha, S}^F}(B) = \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_{\alpha, S}^F(\mathbf{z})$  where  $N_I(\mathbf{z}, \mathbf{u}, \mathbf{v}) := |\{k \in I | z_k = \min\{u_k, v_k\}\}|$ . Hence, there is no need to swap  $u_i$  and  $v_i$  if  $u_i > v_i$  for any  $i$ . Let  $B = \times_{i=1}^d [u_i, v_i]$ . We define  $\psi_S(z_k) := \chi_S(k)(1 - z_k) + \chi_{\bar{S}}(k)(z_k)$ , where  $\chi$  is the indicator function:

$$\chi_A(i) := \begin{cases} 1 & \text{if } i \in A, \\ 0 & \text{if } i \notin A. \end{cases}$$

It holds:

$$\begin{aligned} V_{C_{\alpha, S}^F}(B) &= \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} \sum_{A \subseteq S} (-1)^{|A|} C_\alpha(\kappa_{S, A}(1, \mathbf{z}), \dots, \kappa_{S, A}(d, \mathbf{z})) \\ &= \sum_{A \subseteq S} (-1)^{|A|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\kappa_{S, A}(1, \mathbf{z}), \dots, \kappa_{S, A}(d, \mathbf{z})) \\ &= (-1)^{|S|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\kappa_{S, S}(1, \mathbf{z}), \dots, \kappa_{S, S}(d, \mathbf{z})) \\ &\quad + \underbrace{\sum_{\substack{A \subseteq S \\ A \neq S}} (-1)^{|A|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\kappa_{S, A}(1, \mathbf{z}), \dots, \kappa_{S, A}(d, \mathbf{z}))}_{=0, \text{ see (6)}} \\ &= (-1)^{|S|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\chi_S(1)(1 - z_1) + \chi_{\bar{S}}(1)(z_1), \dots, \chi_S(d)(1 - z_d) + \chi_{\bar{S}}(d)(z_d)) \\ &\stackrel{(7)}{=} (-1)^{|S|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\psi_S(\mathbf{z}), \psi_S(\mathbf{u}), \psi_S(\mathbf{v}))} (-1)^{|S|} C_\alpha(\psi_S(\mathbf{z})) \\ &\stackrel{(8)}{=} \sum_{\mathbf{z} \in \times_{i=1}^d \{\psi_S(u_i), \psi_S(v_i)\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\mathbf{z}) = V_{C^\alpha}(\psi_S(B)). \end{aligned}$$

(6) For  $S = \emptyset$  we have  $C_{\alpha, S}^F = C_\alpha$ . Therefore, suppose  $S \neq \emptyset$ . For  $A \subseteq S, A \neq S$  there is a  $s_A \in S \setminus A$  with  $\kappa_{S, A}(s_A, z) = 1$ . Hence:

$$\begin{aligned} &\sum_{\substack{A \subseteq S \\ A \neq S}} (-1)^{|A|} \sum_{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\kappa_{S, A}(1, \mathbf{z}), \dots, \kappa_{S, A}(d, \mathbf{z})) \\ &= \sum_{\substack{A \subseteq S \\ A \neq S}} (-1)^{|A|} \left( \sum_{\substack{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\} \\ z_{s_A} = u_{s_A}}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\dots) + \sum_{\substack{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\} \\ z_{s_A} = v_{s_A}}} (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\dots) \right) \\ &= \sum_{\substack{A \subseteq S \\ A \neq S}} (-1)^{|A|} \left( \sum_{\substack{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\} \\ z_{s_A} = u_{s_A}}} \left( (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} C_\alpha(\dots) + (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v}) - 1} C_\alpha(\dots) \right) \right) \end{aligned}$$

$$= \sum_{\substack{A \subset S \\ A \neq S}} (-1)^{|A|} \sum_{\substack{\mathbf{z} \in \times_{i=1}^d \{u_i, v_i\} \\ z_S A = u_S A}} 0 = 0.$$

(7) We show  $(-1)^{N_I(\psi_S(\mathbf{z}), \psi_S(\mathbf{u}), \psi_S(\mathbf{v}))} = (-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} (-1)^{|S|}$ . Let  $k := |\{z(i) = u(i) | i \in I \setminus S\}|$  and  $l := |\{z(i) = v(i) | i \in S\}|$ . Then  $N_I(\mathbf{z}, \mathbf{u}, \mathbf{v}) = k + |S| - l$  and  $N_I(\psi_S(\mathbf{z}), \psi_S(\mathbf{u}), \psi_S(\mathbf{v})) = k + l$ .

$$(-1)^{N_I(\mathbf{z}, \mathbf{u}, \mathbf{v})} (-1)^{|S|} = (-1)^{k+2|S|-l} = (-1)^{k-l} = (-1)^{k+l} = (-1)^{N_I(\psi_S(\mathbf{z}), \psi_S(\mathbf{u}), \psi_S(\mathbf{v}))},$$

where we used that  $(-1)^{2|S|} = (-1)^{|S|}(-1)^{|S|} = (-1)^{|S|}/(-1)^{|S|} = 1$  and  $(-1)^{-l} = (-1)^l$ .

(8)  $\psi_S : [0, 1]^d \rightarrow [0, 1]^d$  is bijective.

$C_\alpha$  is  $d$ -increasing since it is a copula. Thus,  $V_{C_{\alpha, S}^F}(B) = V_{C_\alpha}(\phi_S(B)) \geq 0$ . This completes the proof. □

## References

- [1] Comtet L (1974) Advanced Combinatorics: the Art of Finite and Infinite Expansions. Dordrecht: Reidel, 176–177 pp.