PARTITIONS OF n INTO DIVISORS OF m

by HANSRAJ GUPTA, F.N.A., 402 *Mumfordganj, Allahabad* 211002

(*Received* 21 *May* 1974)

Let $m > 1$ be a given natural number, t the number of positive divisors of m; and

$$n = mq + r, \; q \geqslant 0, \; 0 \leqslant r < m$$

an arbitrary natural number. Using a new technique, it is here shown that $P_m(n)$ the number of partitions of n into the divisors of m, repetitions allowed, is a polynomial in q of degree exactly $(t - 1)$.

1. INTRODUCTION

Gupta (1975) has described a remarkably simple technique which is of immense help in the study of partition problems of various types, when the summands come from a finite set of natural numbers. Here we apply the technique to the case when the given set consists of the (positive) divisors of a given natural number $m > 1$.

In what follows, $t = t(m)$ denotes the number of divisors of m; $\sigma = \sigma(m)$ is the sum of these divisors; d runs through the divisors of m;

$$n = mq + r, \; q \geqslant 0, \; 0 \leqslant r < m;$$

is an arbitrary natural number, and

$$u = mt - \sigma.$$

We denote by $P(n) = P_m(n)$ the number of partitions of n into the divisors of m, repetitions allowed. We take

$$P(0) = 1.$$

Evidently

$$G = \sum_{n=0}^{\infty} P(n)\, x^n = 1/\Pi \;(1 - x^d) \qquad \ldots(1)$$

is the generating function of $P(n)$.

The technique consists in writing

$$G = (1 - x^m)^{-t} \;\Pi\; (1 + x^d + x^{2d} + \ldots + x^{m-d}) \qquad \ldots(2)$$

On expansion, the product Π on the right of (2), gives a polynomial in x of degree u with positive integral coefficients for all powers of x and 1 as the constant term. Let

$$\Pi\ (1 + x^d + x^{2d} + \ldots + x^{m-d}) = \sum_{a=0}^{u} c(a)\, x^a. \qquad \ldots(3)$$

Then here, as in Gupta (1975) we have

$$c(a) = c(u - a),\ 0 \leqslant a \leqslant u. \qquad \ldots(4)$$

As already stated above, in this range of values of a, $c(a)$'s are all positive integers, and we decide to take

$$c(a) = 0 \text{ for } a > u \text{ as well as for } a < 0. \qquad \ldots(5)$$

Combinatorially, $c(a)$ represents the number of those partitions of a into the divisors of m, in which no divisor d of m occurs more than $(m - d)/d$ times in the partition.

From the foregoing, it immediately follows that

$$P(n) = c(r)\binom{q+t-1}{t-1} + c(r+m)\binom{q+t-2}{t-1} + c(r+2m)\binom{q+t-3}{t-1} + \ldots + c(r+tm-m)\binom{q}{t-1};$$

with $\qquad n = mq + r. \qquad \ldots(6)$

It is noteworthy that in (6), the c's depend only on the residue class to which n belongs modulo m, while the combinatory functions involved depend only on q the quotient obtained when n is divided by m.

Moreover, as already shown in Gupta (1975) we have in the present case

$$c(r) + c(r+m) + c(r+2m) + \ldots + c(r+tm-m) = m^{(t-2)/2} \qquad \ldots(7)$$

The right-hand side in (7) is independent of r and is a positive integer. Hence, we have the following theorem.

Theorem — For each non-negative $r < m$, $P(n)$ is a polynomial in q of degree exactly $(t - 1)$. [For a similar result see Intrator 1968, Gupta 1971].

In the tables in section 3, where the values of c's are given for each composite $m \leqslant 40$, the values of

$$c(r),\ c(r+m),\ c(r+2m),\ \ldots$$

appear in the same column and in this order. For the first column $r = 0$, for the second it is 1, and so on. This arrangement enables one to use formula (6) readily as will be exemplified in section 4. When $m = p$ (a prime), we have

$$c(0) = c(1) = c(2) = \ldots = c(p-1) = 1. \qquad \ldots(8)$$

2. Computation of c's

This involves the expansion of the product Π on the left of (3). The first factor in the product is

$$1 + x + x^2 + \dots + x^{m-1}.$$

Let $1 = d_1 < d_2 < \dots < d_t = m$ be all the divisors of m.

Suppose the expansion of the product has been carried up to the factor involving the divisor d_{j-1}, and at the end of this factor we get the expression

$$h(0) + h(1)\, x + h(2)\, x^2 + \dots + h(v_{j-1})\, x^{v_{j-1}}; \qquad \dots(9)$$

where

$$v_{j-1} = m(j-1) - (d_1 + d_2 + \dots + d_{j-1}).$$

Then, to get the expression at the next step, we have to multiply (9) by the factor

$$1 + x^{d_j} + x^{2d_j} + \dots + x^{m-d_j}\,.$$

The coefficients in the resulting product are most easily found by using the following algorithm.

Let H_j denote the matrix.

$$\begin{bmatrix} h(0) & h(1) & h(2) & \dots & h(d_j - 1) \\ h(d_j) & h(d_j+1) & h(d_j+2) & \dots & h(2d_j - 1) \\ \cdot & \cdot & \cdot & \dots & \cdot \\ h(kd_j) & h(kd_j+1) & h(kd_j+2) & \dots & h(kd_j + d_j - 1) \end{bmatrix}$$

with $k = [v_{j-1}/d_j]$ and $h(s) = 0$ for $s > v_{j-1}$.

This is obtained by writing the h's in (9) in order in rows of d_j and adding some zeros at the end if necessary. Evidently H_j is a matrix of the $(k+1) \times d_j$ type.

Now, let M_j stand for the $\left(\frac{m}{d_j} + k\right) \times (k+1)$ matrix in which b_{ef} the element in the eth row and fth column is given by

$$b_{ef} = 1 \text{ if } e = f,\ f+1,\ f+2, \dots, f + \frac{m}{d_j} - 1;$$

$$= 0 \text{ otherwise.}$$

Note that the 1's in M_j all appear in a belt of depth m/d_j.

Then the algorithm states that the coefficients in the product at the jth step appear in order as the elements of the rows in the product $M_j H_j$.

The algorithm is not difficult to prove.

The following example with $m = 10$; will clarify the procedure.

To start with, we have the expression

$$1 + x + x^2 + \ldots + x^9.$$

This gives for $d_2 = 2$,

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}; \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so that

$$M_2H_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \\ 5 & 5 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 \end{bmatrix}.$$

This means that

$$(1 + x + x^2 + \ldots + x^9)(1 + x^2 + x^4 + \ldots + x^8)$$
$$= 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + \ldots + x^{16} + x^{17}.$$

We next have $d_3 = 5$;

$$H_3 = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 3 & 4 & 4 & 5 & 5 \\ 4 & 4 & 3 & 3 & 2 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}; \qquad M_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and

$$M_3H_3 = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 \\ 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 7 & 8 & 7 \\ 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Finally for $d_4 = 10$, we get

$$H_4 = \begin{bmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 7 & 8 & 7 & 8 & 7 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $M_4H_4 = H_4$.

Thus the c's in the final product have been obtained and recorded in the table. Notice that the column sums in H_4 are all equal, adding up to 10 in each case. The column sums for M_3H_3 are each $= 20$; and those for M_2H_2 each $= 25$.

One other feature is noteworthy. The columns for any r modulo m and $(u - r)$ modulo m, have the same entries but in the opposite order. Thus, for $m = 10$, the columns for $r = 3$ and $u - r = 22 - 3 \equiv 9 \pmod{10}$ are

$$\begin{matrix} 2 \\ 8 \end{matrix} \text{ and } \begin{matrix} 8 \\ 2 \end{matrix}.$$

Since the product Π is independent of the order in which the divisors $d_2, d_3, \ldots, d_{t-1}$ are taken, the algorithm can be applied sometimes with advantage with the divisors in a changed order. In the case of $m = 36$ for example, we took the divisors in the order 2, 4, 3, 6, 12, 9, 18, 36.

3. Tables of Values of c's

1. $m = 4, \quad t = 3, \quad u = 5.$

1	1	2	2
1	1		

2. $m = 6, \quad t = 4, \quad u = 12.$

1	1	2	3	4	5
4	5	4	3	2	1
1					

3. $m = 8, \quad t = 4, \quad u = 17.$

1	1	2	2	4	4	6	6
6	6	6	6	4	4	2	2
1	1						

4. $m = 9, \quad t = 3, \quad u = 14.$

1	1	1	2	2	2	3	3	3
2	2	2	1	1	1			

5. $m = 10, \quad t = 4, \quad u = 22.$

1	1	2	2	3	4	5	6	7	8
7	8	7	8	7	6	5	4	3	2
2	1	1							

6. $m = 12, \quad t = 6, \quad u = 44.$

1	1	2	3	5	6	10	12	17	21	28	33
39	46	54	60	67	75	78	85	87	90	88	90
87	85	78	75	67	60	54	46	39	33	28	21
17	12	10	6	5	3	2	1	1			

7. $m = 14, \quad t = 4, \quad u = 32.$

1	1	2	2	3	3	4	5	6	7	8	9	10	11
10	11	10	11	10	11	10	9	8	7	6	5	4	3
3	2	2	1	1									

8. $m = 15, \quad t = 4, \quad u = 36.$

1	1	1	2	2	3	4	4	5	6	7	8	9	10	11
10	11	12	11	12	11	10	11	10	9	8	7	6	5	4
4	3	2	2	1	1	1								

9. $m = 16, \quad t = 5, \quad u = 49.$

1	1	2	2	4	4	6	6	10	10	14	14	20	20	26	26
31	31	36	36	40	40	44	44	44	44	44	44	40	40	36	36
31	31	26	26	20	20	14	14	10	10	6	6	4	4	2	2
1	1														

10. $m = 18, \quad t = 6, \quad u = 69.$

1	1	2	3	4	5	8	9	12
75	84	93	105	114	126	135	147	156
198	198	195	186	183	174	165	156	147
50	41	34	30	23	19	16	12	9

16	19	23	30	34	41	50	57	66
165	174	183	186	195	198	198	201	201
135	126	114	105	93	84	75	66	57
8	5	4	3	2	1	1		

11. $m = 20, \quad t = 6, \quad u = 78.$

1	1	2	2	4	5	7	8	11	13
86	95	107	118	129	139	152	164	176	187
245	247	240	238	230	227	215	208	195	187
68	57	51	42	37	29	26	20	18	13
18	20	26	29	37	42	51	57	68	76
195	208	215	227	230	238	240	247	245	248
176	164	152	139	129	118	107	95	86	76
11	8	7	5	4	2	2	1	1	

12. $m = 21, \quad t = 4, \quad u = 52.$

1	1	1	2	2	2	3	4	4	5	6
14	15	16	15	16	17	16	15	16	15	14
6	5	4	4	3	2	2	2	1	1	1
6	7	8	9	10	11	12	13	14	15	
15	14	13	12	11	10	9	8	7	6	

13. $m = 22, \quad t = 4, \quad u = 52.$

1	1	2	2	3	3	4	4	5	5	6
16	17	16	17	16	17	16	17	16	17	16
5	4	4	3	3	2	2	1	1		
7	8	9	10	11	12	13	14	15	16	17
15	14	13	12	11	10	9	8	7	6	5

14. $m = 24, \quad t = 8, \quad u = 132.$

1	1	2	3	5	6	10	12	18	22	30	36
450	508	598	670	778	868	994	1102	1249	1375	1546	1693
4530	4778	5012	5254	5474	5708	5908	6128	6300	6500	6636	6808
6916	6808	6636	6500	6300	6128	5908	5708	5474	5254	5012	4778
1877	1693	1546	1375	1249	1102	994	868	778	670	598	508
50	36	30	22	18	12	10	6	5	3	2	1
50	58	76	90	115	133	168	193	239	274	332	378
1877	2048	2256	2440	2664	2872	3104	3328	3568	3800	4048	4288
6916	7052	7112	7220	7238	7298	7280	7298	7238	7220	7112	7052
4530	4288	4048	3800	3568	3328	3104	2872	2664	2440	2256	2048
450	378	332	274	239	193	168	133	115	90	76	58
1											

15. $m = 25, \quad t = 3, \quad u = 44.$

1	1	1	1	1	2	2	2	2	2	3	3	3
4	4	4	4	4	3	3	3	3	3	2	2	2
3	3	4	4	4	4	4	5	5	5	5	5	
2	2	1	1	1	1	1						

16. $m = 26, \quad t = 4, \quad u = 62.$

1	1	2	2	3	3	4	4	5	5	6	6	7
19	20	19	20	19	20	19	20	19	20	19	20	19
6	5	5	4	4	3	3	2	2	1	1		
8	9	10	11	12	13	14	15	16	17	18	19	20
18	17	16	15	14	13	12	11	10	9	8	7	6

17. $m = 27, \quad t = 4, \quad u = 68.$

1	1	1	2	2	2	3	3	3	5	5	5	7	7
19	19	19	20	20	20	21	21	21	20	20	20	19	19
7	7	7	5	5	5	3	3	3	2	2	2	1	1
7	9	9	9	12	12	12	15	15	15	18	18	18	
19	18	18	18	15	15	15	12	12	12	9	9	9	
1													

18. $m = 28, \quad t = 6, \quad u = 112.$

1	1	2	2	4	4	6	7	10	11
150	162	178	191	206	221	238	252	268	284
482	485	478	480	472	470	458	455	442	435
150	136	126	111	102	89	82	70	64	54
1									
14	16	20	22	28	31	38	41	50	54
302	317	334	351	364	380	392	410	418	435
418	410	392	380	364	351	334	317	302	284
50	41	38	31	28	22	20	16	14	11
64	70	82	89	102	111	126	136		
442	455	458	470	472	480	478	485		
268	252	238	221	206	191	178	162		
10	7	6	4	4	2	2	1		

19. $m = 30, \quad t = 8, \quad u = 168.$

1	1	2	3	4	6	9	11	15	19
734	829	932	1045	1168	1300	1442	1599	1764	1943
8280	8662	9039	9414	9788	10156	10515	10866	11208	11538
13761	13590	13406	13198	12963	12712	12448	12158	11857	11538
4097	3813	3533	3267	3018	2778	2546	2335	2131	1943
127	105	88	73	59	48	40	31	25	19
25	31	40	48	59	73	88	105	127	148
2131	2335	2546	2778	3018	3267	3533	3813	4097	4408
11857	12158	12448	12712	12963	13198	13406	13590	13761	13888
11208	10866	10515	10156	9788	9414	9039	8662	8280	7908
1764	1599	1442	1300	1168	1045	932	829	734	648
15	11	9	6	4	3	2	1	1	

(continued)

177	208	243	282	328	378	436	500	571	648
4713	5036	5370	5716	6063	6424	6784	7152	7525	7908
14014	14104	14167	14200	14218	14200	14167	14104	14014	13888
7525	7152	6784	6424	6063	5716	5370	5036	4713	4408
571	500	436	378	328	282	243	208	177	148

20. $m = 32, \quad t = 6, \quad u = 129.$

1	1	2	2	4	4	6	6	10	10	14
196	196	226	226	260	260	294	294	330	330	366
630	630	630	630	620	620	610	610	590	590	570
196	196	166	166	140	140	114	114	94	94	74
1	1									

14	20	20	26	26	36	36	46	46	60	60
366	404	404	442	442	476	476	510	510	540	540
570	540	540	510	510	476	476	442	442	404	404
74	60	60	46	46	36	36	26	26	20	20

74	74	94	94	114	114	140	140	166	166
570	570	590	590	610	610	620	620	630	630
366	366	330	330	294	294	260	260	226	226
14	14	10	10	6	6	4	4	2	2

21. $m = 33, \quad t = 4, \quad u = 84.$

1	1	1	2	2	2	3	3	3	4	4	5	6	6
22	23	24	23	24	25	24	25	26	25	26	25	24	25
10	9	8	8	7	6	6	5	4	4	3	3	3	2

7	8	8	9	10	10	11	12	13	14	15	16	17	18
24	23	24	23	22	23	22	21	20	19	18	17	16	15
2	2	1	1	1									

19	20	21	22	23
14	13	12	11	10

22. $m = 34, \quad t = 4, \quad u = 82.$

1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
25	26	25	26	25	26	25	26	25	26	25	26	25	26	25	26	25
8	7	7	6	6	5	5	4	4	3	3	2	2	1	1		

10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
24	23	22	21	20	19	18	17	16	15	14	13	12	11	10	9	8

23. $m = 35, \quad t = 4, \quad u = 92.$

1	1	1	1	1	2	2	3	3	3	4	4	5	5	6	7	7	8
22	23	24	25	26	25	26	25	26	27	26	27	26	27	26	25	26	25
12	11	10	9	8	8	7	7	6	5	5	4	4	3	3	3	2	2

8	9	10	11	12	12	13	14	15	16	17	18	19	20	21	22	23
26	25	24	23	22	23	22	21	20	19	18	17	16	15	14	13	12
1	1	1	1	1												

24. $m = 36, \quad t = 9, \quad u = 233.$

1	1	2	3	5	6	10	12	17
2225	2483	2794	3123	3493	3870	4328	4776	5293
43463	45850	48265	50757	53292	55923	58520	61256	63973
132495	133741	134793	135717	136471	137115	137565	137887	138039
89620	86692	83849	80964	78122	75195	72415	69531	66752
12004	11060	10145	9300	8496	7779	7063	6445	5840
128	109	88	72	57	48	35	29	22
22	29	35	48	57	72	88	109	128
5840	6445	7063	7779	8496	9300	10145	11060	12004
66752	69531	72415	75195	78122	80964	83849	86692	89620
138039	137887	137565	137115	136471	135717	134793	133741	132495
63973	61256	58520	55923	53292	50757	48265	45850	43463
5293	4776	4328	3870	3493	3123	2794	2483	2225
17	12	10	6	5	3	2	1	1
161	187	226	267	318	366	440	503	589
13055	14127	15289	16509	17807	19155	20600	22096	23677
92375	95261	98017	100773	103441	106155	108653	111237	113647
131203	129677	128087	126345	124515	122493	120493	118255	116015
41188	38940	36785	34692	32677	30723	28855	27056	25328
1954	1744	1532	1350	1178	1044	895	789	680
680	789	895	1044	1178	1350	1532	1744	1954
25328	27056	28855	30723	32677	34692	36785	38940	41188
116015	118255	120493	122493	124515	126345	128087	129677	131203
113647	111237	108653	106155	103441	100773	98017	95261	92375
23677	22096	20600	19155	17807	16509	15289	14127	13055
589	503	440	366	318	267	226	187	161

25. $m = 38, \quad t = 4, \quad u = 92.$

1	1	2	2	3	3	4	4	5	5	6	6	7
28	29	28	29	28	29	28	29	28	29	28	29	28
9	8	8	7	7	6	6	5	5	4	4	3	3
7	8	8	9	9	10	11	12	13	14	15	16	17
29	28	29	28	29	28	27	26	25	24	23	22	21
2	2	1	1									
18	19	20	21	22	23	24	25	26	27	28	29	
20	19	18	17	16	15	14	13	12	11	10	9	

26. $m = 39, \quad t = 4, \quad u = 100.$

1	1	1	2	2	2	3	3	3	4	4	4	5
26	27	28	27	28	29	28	29	30	29	30	31	30
12	11	10	10	9	8	8	7	6	6	5	4	4

(continued)

6	6	7	8	8	9	10	10	11	12	12	13	14
29	30	29	28	29	28	27	28	27	26	27	26	25
4	3	3	3	2	2	2	1	1	1			
15	16	17	18	19	20	21	22	23	24	25	26	27
24	23	22	21	20	19	18	17	16	15	14	13	12

27. $m = 40$, $t = 8$, $u = 230$.

1	1	2	2	4	5	7	8	12	14
1362	1472	1644	1774	1970	2120	2344	2516	2767	2963
17642	18298	18972	19636	20304	20970	21630	22304	22944	23612
33092	32984	32680	32508	32148	31920	31496	31192	30726	30366
11437	10845	10338	9770	9292	8745	8307	7800	7388	6910
466	400	364	310	282	240	216	180	163	135
20	22	30	34	44	50	63	71	88	98
3242	3466	3776	4027	4371	4650	5028	5338	5750	6094
24232	24892	25484	26124	26680	27300	27814	28406	28872	29428
29852	29428	28872	28406	27814	27300	26680	26124	25484	24892
6532	6094	5750	5338	5028	4650	4371	4027	3776	3466
122	98	88	71	63	50	44	34	30	22
122	135	163	180	216	240	282	310	364	400
6532	6910	7388	7800	8307	8745	9292	9770	10338	10845
29852	30366	30726	31192	31496	31920	32148	32508	32680	32984
24232	23612	22944	22304	21630	20970	20304	19636	18972	18298
3242	2963	2767	2516	2344	2120	1970	1774	1644	1472
20	14	12	8	7	5	4	2	2	1
466	508	587	641	734	800	910	989	1119	1212
11437	11984	12600	13168	13808	14400	15056	15672	16336	16976
33092	33320	33358	33530	33492	33600	33492	33530	33358	33320
17642	16976	16336	15672	15056	14400	13808	13168	12600	11984
1362	1212	1119	989	910	800	734	641	587	508
1									

4. Use of the Formula

We conclude by computing for $m = 28$, the value of $P(123)$. Since $123 = 28.4 + 11$, we look for the values of the c's, we need, in column 12 of the table for $m = 28$. This gives

$$c(11) = 16, \quad c(39) = 317, \quad c(67) = 410, \quad c(95) = 41.$$

From formula (6), we now obtain

$$\begin{aligned} P(123) &= 16.\tbinom{9}{5} + 317.\tbinom{8}{5} + 410.\tbinom{7}{5} + 41.\tbinom{6}{5}, \\ &= 16.126 + 317.56 + 410.21 + 41.6, \\ &= 28624. \end{aligned}$$

References

Gupta, H. (1971). On partitions of n into k summands. *Proc. Edinb. math. Soc.*, **17**, 337-39; *MR* **46** (1973), # 7155.

———— (1975) A technique in partitions. *Univ. Beograd. Publ. Elektrotehn Fak. ser. Mat. Fiz.*, # 511, 73-76.

Intrator, J. (1968). Partitions I. *Czech. Math. J.*, **18** (93), 16-24; *MR* **37** (1969), # 181.