PARTITIONS INTO PRIMES

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1. Let $P(n)$ denote the number of distinct ways in which a positive integer n can be expressed as a sum of the prime numbers 2, 3, 5, 7, 11, . . . repeated any number of times.

Thus $\quad 5 = 5,\ 2+3;\quad$ and $\quad 8 = 3+5,\ 2+3+3,\ 2+2+2+2;$

so that $\quad P(5) = 2 \quad$ and $\quad P(8) = 3.$

We take $\quad P(0) = 1.$

In this paper, we give a table of values of $P(n)$ for $n \leq 300$. The calculations were made by us independently and discrepancies corrected. After the check, which Dr. H. Gupta has applied, there can be no doubt about the correctness of our results.

2. $P(n)$ may be evaluated in two different ways:

First method.

Denote * by $P(n, p)$ the number of those of the $P(n)$ partitions of n into primes, which contain the prime p as the largest summand.
Then

$$P(n) = \sum_{p} P(n, p) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

where p runs through all primes less than or equal to n.
It may be noticed that

$$P(p, p) = 1.$$

Denoting the rth prime by p_r, with $p_1 = 2$, $p_2 = 3$ and so on, we have

$$P(n, p_r) = \sum_{i=1}^{r} P(n-p_r, p_i) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

because if from a partition of n into primes with p_r as the largest summand, we take away p_r, we are left with a partition of $n-p_r$ with p_r or a smaller prime as the largest summand. From (2), we readily obtain the recurrence formula

$$\begin{aligned} P(n, p_r) &= P(n-p_r, p_r) + \sum_{i=1}^{r-1} P(n-p_r, p_i) \\ &= P(n-p_r, p_r) + P(n-p_r+p_{r-1}, p_{r-1}). \qquad \ldots \quad \ldots \quad (3) \end{aligned}$$

Evidently, for $p_r \geqslant \frac{n}{2}$,

$$P(n, p_r) = P(n-p_r) \qquad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (4)$$

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* $P(n, p_r)$ is the coefficient of x^{n-p_r} in the expansion of $\prod_{i=1}^{r} (1-x^{p_i})^{-1}$.

As an illustration, we write down the table for $P(n, p)$ for values of n up to 9 and calculate the values for $n = 10$.

$n \backslash p$	2	3	5	7	$P(n)$
1	0				0
2	1				1
3	0	1			1
4	1	0			1
5	0	1	1		2
6	1	1	0		2
7	0	1	1	1	3
8	1	1	1	0	3
9	0	2	1	1	4
10	1	1	2	1	5

$$P(10, 2) = P(8, 2) = 1;$$
$$P(10, 3) = P(7, 3)+P(9, 2) = 1+0 = 1;$$
$$P(10, 5) = P(5, 5)+P(8, 3) = 1+1 = 2;$$
and
$$P(10, 7) = P(3, 7)+P(8, 5) = 0+1 = 1.$$

Hence
$$P(10) = P(10, 2)+P(10, 3)+P(10, 5)+P(10, 7) = 5.$$

Second method.

$P(n)$ is the coefficient of x^n in the expansion of
$$f(x) \equiv \prod_{r=1}^{\infty} (1-x^{p_r})^{-1}.$$

Now* $\{f(x)\}^{-1} = \prod_{r=1}^{\infty} (1-x^{p_r}) = 1-x^2-x^3+x^8+x^9-x^{11}+x^{16}-x^{19}-x^{23}+\cdots$

Hence
$$1 = (1-x^2-x^3+x^8+x^9-x^{11}+\cdots) \sum_{n=0}^{\infty} P(n)x^n$$
so that
$$P(n) = P(n-2)+P(n-3)-P(n-8)-P(n-9)+P(n-11)-\cdots.$$

Thus
$$P(10) = P(8)+P(7)-P(2)-P(1) = 3+3-1 = 5.$$

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* See H. Gupta, 'Partitions into distinct primes'.

n	$p(n)$	n	$p(n)$	n	$p(n)$
1	0	51	899	101	43709
2	1	52	987	102	46696
3	1	53	1083	103	49871
4	1	54	1186	104	53243
5	2	55	1298	105	56826
6	2	56	1420	106	60631
7	3	57	1552	107	64671
8	3	58	1695	108	68957
9	4	59	1850	109	73506
10	5	60	2018	110	78331
11	6	61	2198	111	83447
12	7	62	2394	112	88874
13	9	63	2605	113	94625
14	10	64	2833	114	1 00719
15	12	65	3079	115	1 07175
16	14	66	3344	116	1 14014
17	17	67	3630	117	1 21255
18	19	68	3936	118	1 28923
19	23	69	4268	119	1 37038
20	26	70	4624	120	1 45627
21	30	71	5007	121	1 54709
22	35	72	5419	122	1 64320
23	40	73	5861	123	1 74482
24	46	74	6336	124	1 85225
25	52	75	6845	125	1 96583
26	60	76	7393	126	2 08585
27	67	77	7979	127	2 21265
28	77	78	8608	128	2 34658
29	87	79	9282	129	2 48807
30	98	80	10003	130	2 63745
31	111	81	10776	131	2 79516
32	124	82	11603	132	2 96161
33	140	83	12488	133	3 13727
34	157	84	13435	134	3 32258
35	175	85	14445	135	3 51808
36	197	86	15527	136	3 72427
37	219	87	16681	137	3 94170
38	244	88	17914	138	4 17088
39	272	89	19232	139	4 41250
40	302	90	20636	140	4 66711
41	336	91	22134	141	4 93538
42	372	92	23732	142	5 21804
43	413	93	25436	143	5 51573
44	456	94	27251	144	5 82925
45	504	95	29186	145	6 15933
46	557	96	31246	146	6 50686
47	614	97	33439	147	6 87262
48	677	98	35772	148	7 25757
49	744	99	38257	149	7 66262
50	819	100	40899	150	8 08872

n	$p(n)$	n	$p(n)$	n	$p(n)$
151	8 53692	201	103 12927	251	914 77898
152	9 00827	202	108 01607	252	953 24698
153	9 50393	203	113 12080	253	993 24684
154	10 02502	204	118 45265	254	1034 83632
155	10 57278	205	124 02104	255	1078 07529
156	11 14849	206	129 83601	256	1123 02573
157	11 75344	207	135 90769	257	1169 75172
158	12 38904	208	142 24686	258	1218 31963
159	13 05679	209	148 86458	259	1268 79839
160	13 75815	210	155 77234	260	1321 25912
161	14 49471	211	162 98212	261	1375 77558
162	15 26812	212	170 50639	262	1432 42423
163	16 08014	213	178 35813	263	1491 28391
164	16 93247	214	186 55065	264	1552 43647
165	17 82712	215	195 09801	265	1615 96652
166	18 76598	216	204 01462	266	1681 96183
167	19 75108	217	213 31548	267	1750 51297
168	20 78460	218	223 01629	268	1821 71385
169	21 86867	219	233 13328	269	1895 66161
170	23 00576	220	243 68324	270	1972 45661
171	24 19812	221	254 68365	271	2052 20288
172	25 44843	222	266 15264	272	2135 00804
173	26 75925	223	278 10910	273	2220 98343
174	28 13326	224	290 57246	274	2310 24409
175	29 57342	225	303 56317	275	2402 90920
176	31 08265	226	317 10220	276	2499 10190
177	32 66409	227	331 21140	277	2598 94960
178	34 32097	228	345 91339	278	2702 58409
179	36 05666	229	361 23177	279	2810 14186
180	37 87467	230	377 19090	280	2921 76374
181	39 77861	231	393 81607	281	3037 59545
182	41 77239	232	411 13365	282	3157 78780
183	43 85994	233	429 17077	283	3282 49663
184	46 04537	234	447 95565	284	3411 88297
185	48 33306	235	467 51763	285	3546 11368
186	50 72740	236	487 88713	286	3685 36090
187	53 23313	237	509 09557	287	3829 80254
188	55 85505	238	531 17571	288	3979 62282
189	58 59833	239	554 16136	289	4135 01182
190	61 46816	240	578 08759	290	4296 16642
191	64 47003	241	602 99078	291	4463 28963
192	67 60967	242	628 90875	292	4636 59174
193	70 89299	243	655 88056	293	4816 28967
194	74 32618	244	683 94674	294	5002 60763
195	77 91567	245	713 14927	295	5195 77753
196	81 66824	246	743 53172	296	5396 03890
197	85 59069	247	775 13908	297	5603 63918
198	89 69035	248	808 01816	298	5818 83402
199	93 97474	249	842 21750	299	6041 88762
200	98 45164	250	877 78708	300	6273 07270

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