Formalizing Alexander duality through BDDs ⋆

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Abstract. In this work we explore the possibility of representing in a proof assistant simplicial complexes as Boolean functions, and of trying to take advantage of this relationship to prove properties of the former based on such a representation. These relationship and properties are to be proven in a proof assistant (Isabelle/HOL).
Simplicial complexes have been for long an object of study in Algebraic Topology. They can be briefly depicted as "sets of sets of vertices". Saks et al. [2] proposed that simplicial complexes were isomorphic to monotone Boolean functions in as many variables as vertices in the simplicial complex. From there on, Saks et al. and others (see, for instance, Scoville [7]) studied the relationship between concepts related to simplicial complexes (such as, for instance, Betti numbers or the Euler Characteristic) and the possibility of defining them over Boolean functions. One of these properties is *evasiveness*. For a Boolean function, being *evasive* means that we need to know the value of every single variable in order to get its output. If the Boolean function representing a simplicial complex is evasive, for the simplicial complex this has implications in its topological properties (in particular, it means that it is not *collapsible*).
A well-known and interesting way to encode a Boolean function is by means of a Binary Decision Diagram (BDD, for short) [4], since it identifies (and removes) non-relevant variables for the function output, and it minimizes the number of evaluations that have to be done to obtain the function's output. Since simplicial complexes *are* monotone Boolean functions, it seems sensible to study the BDDs generated by simplicial complexes. For instance, in this setting, an *evasive* Boolean function (and then a non-collapsible simplicial complex) will satisfy that the height of a corresponding BDD will be equal to its number of vertices.
In the previous vein, the *Alexander dual* of a simplicial complex is defined as its *complementary* (in the sense that for each *simplex* in the original simplicial complex, its Alexander dual *will not contain* its complementary set of vertices). It can be proven that the previous construction gives place to a *simplicial complex*, but it can be also analysed the BDDs corresponding to the simplicial complex and to its *Alexander dual*. Following Knuth's notion of *bead* (see [3]), it is easy to check that given the BDD corresponding to a simplicial complex K we can obtain a (isomorphic, in some sense) BDD for the Alexander dual of K.

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Being the formalisation of Mathematics an interesting topic by itself, in this work we plan to formalise the previous results in a proof assistant (namely, Isabelle/HOL [6]). In fact, following the presentation by Scoville [7, Chs. 2, 6], we have already formalised the definitions of simplicial complexes and monotone Boolean functions, as well as the result stating that they are equivalent. We have also formalised theorems stating what certain binary operations over Boolean functions (such as union, intersection or product) represent in terms of simplicial complexes. Finally, we have also connected our representation of Boolean functions (where the number of inputs of a Boolean function is crucial) with a previous representation of Boolean functions in the Isabelle library. Taking advantage of this connection and of a development in the Isabelle Archive of Formal Proofs (AFP) about ROBDDs (Reduced Ordered Binary Decision Diagrams) [5], we have been able to compute the BDD representation of a given simplicial complex, and also to produce *executable* definitions of the *height* of a BDD and therefore of obtaining information about the *evasiveness* of a simplicial complex. The previous work has been published in the Isabelle AFP [1].
Our next goal will be to provide a definition in Isabelle of the Alexander dual of a simplicial complex, and then to prove the relation between the BDDs associated to the simplicial complex and its Alexander dual. Some other topics that could be studied would include the computation of some simplicial complexes properties (such as Betti's numbers and Euler's characteristic) in terms of their associated BDDs, as well as the exploration of some Discrete Morse Theory objects (such as gradient vector fields) in terms of BDDs.

Keywords: Simplicial Complexes · Boolean functions · Formalization of mathematics.

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