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A LIMIT LAW FOR THE ROOT VALUE OF MINIMAX TREES

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Abstract

We consider minimax trees with independent, identically distributed leaf values that have a continuous distribution function F_V being strictly increasing on the range where $0 < F_V < 1$. It was shown by Pearl that the root value of such trees converges to a deterministic limit in probability without any scaling. We show that after normalization we have convergence in distribution to a nondegenerate limit random variable.

1 Introduction and result

We study trees related to the analysis of game-searching methods for two-person perfect information games like Chess or Go. In these games two players A and B start with an initial

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position and take alternate turns, choosing each time among $d \geq 2$ possible moves. A terminal position is reached after $2k$, $k \geq 0$, moves. The terminal position does not necessarily terminate the game, but instead it terminates the horizon of a player or machine searching for best possible moves. One would like to assign a value to each position that indicates the chances of each player winning the game when starting from that position. Although, assuming best possible moves of both players, it is deterministic how the game terminates, the horizon $2k$ of players or machines is usually limited so that they cannot plan their moves up to the very end of the game. To overcome this problem one assigns values V to terminal positions, where large values of V indicate that the position favors player A, small values favor player B. Given the values of all $n = d^{2k}$ terminal nodes one can search for best possible moves for the starting position and calculate its value.
The possible moves and its terminal positions can be represented in a rooted tree with fixed branching degree $d \geq 2$ and height $2k$, $k \geq 0$. Leaves are assigned random values $V_1, \ldots, V_n$, $n = d^{2k}$, and all other nodes are labeled with $\wedge$ on even levels and with $\vee$ on odd levels, cf. Figure 1 for the case $d = 2$ and $k = 2$.

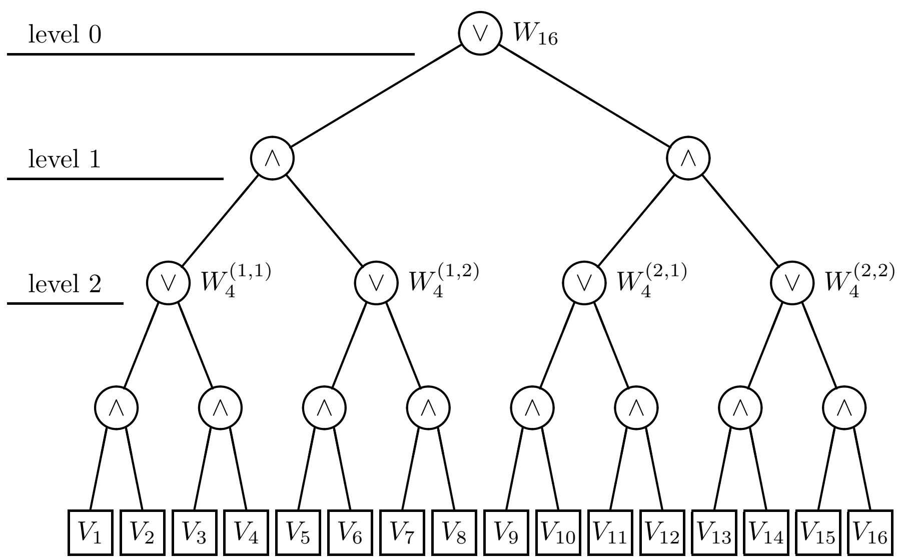

Figure 1: A minimax tree with branching degree 2 and height 4.

These trees are called minimax trees. The value of a node is given as the value of the operator labeled at that node applied to the values of its children. This corresponds to player A always choosing the move with maximal value, player B always choosing a minimal value move. Thus from $V_1, \ldots, V_n$ one could first calculate the values of all nodes on level $2k-1$ and successively determine the values on higher levels leading finally the root's value. The most popular algorithm for finding the leaf whose value trickles up to the root (which corresponds to the best moves when two intelligent players are playing) is alpha-beta pruning. Analysis of this algorithm for various models is given in Knuth and Moore [4], see also Zhang [11].
Minimax trees where $V_1, \ldots, V_n$ only take the values 0 and 1 are also known as AND/OR trees or boolean decision trees. A randomized algorithm for finding the root's value of AND/OR

trees was given in Snir [10]. For probabilistic analysis of Snir's algorithm see Saks and Wigderson [9], Karp and Zhang [3], and Ali Khan and Neininger [1].
In this paper we are not concerned with the complexity of algorithms to determine the root's value of minimax trees. We study the value W_n of the root itself asymptotically as $k \to \infty$.
We consider Pearl's model where the leaves' values $V_1, \dots, V_n$ are independent and identically distributed random variables with a distribution $\mathcal{L}(V)$ having a distribution function $F_V(x) = \mathbb{P}(V \le x)$ that is continuous and strictly increasing on the range, where $0 < F_V < 1$, see Pearl [8]. For an alternative incremental model, see Nau [5, 6, 7] and the analysis in Devroye and Kamoun [2].
Pearl [8] showed for his model that W_n converges, without any scaling, in probability to a deterministic limit q_V as $n = d^{2k}$ tends to infinity,

$$W_n \xrightarrow{\mathbb{P}} q_V, \quad (k \to \infty),$$

and characterized the value of q_V.
In this paper we derive a limit law for W_n after appropriate rescaling. We denote the distribution function of W_n by F_n. Note that this is defined for all $n = d^{2k}$ with $k \in \mathbb{N}_0$ and that we have $F_1 = F_V$. Moreover, for $k \ge 1$, we have $F_n = f \circ F_{n/d^2}$ with

$$f(x) = \left(1 - (1-x)^d\right)^d, \quad x \in [0,1]. \tag{1}$$

This is implied by the recursive structure of the tree: The values of the d^2 nodes on level 2 are independent and identically distributed with distribution $\mathcal{L}(W_{n/d^2})$. We denote these values by $W_{n/d^2}^{(i,j)}$ with $i, j = 1, \dots, d$, see Figure 1 for the case $d = 2$. Hence, by independence we have

$$\begin{aligned} F_n(x) &= \mathbb{P}\left(\bigvee_{i=1}^{d} \bigwedge_{j=1}^{d} W_{n/d^2}^{(i,j)} \le x\right) = \left(1 - \left(1 - \mathbb{P}\left(W_{n/d^2}^{(i,j)} \le x\right)\right)^d\right)^d \\ &= f(F_{n/d^2}(x)). \end{aligned}$$

Function f has the fixed points 0 and 1 and there is a unique fixed point in the open unit interval $(0,1)$ that we denote by q, cf. Lemma 2 below. It was shown by Pearl [8] that we have

$$q_V = F_V^{-1}(q).$$

We denote the slope of f in q by $\xi = f'(q)$. Then the following limit law holds.

Theorem 1 *With F_V, q and ξ as above and $d \ge 2$ we have the following convergence in distribution for the value W_n of the minimax tree in Pearl's model. With $\alpha = \log(\xi)/\log(d^2) \in (0,1)$,*

$$n^\alpha (F_V(W_n) - q) \xrightarrow{\mathcal{L}} W, \quad k \to \infty. \tag{2}$$

The random variable W does not depend upon $\mathcal{L}(V)$, has a continuous distribution function F_W with $0 < F_W < 1$, $F_W(0) = q$ and

$$F_W(x) = f\left(F_W(x/\xi)\right), \quad x \in \mathbb{R}, \tag{3}$$

where f is the function defined in (1).

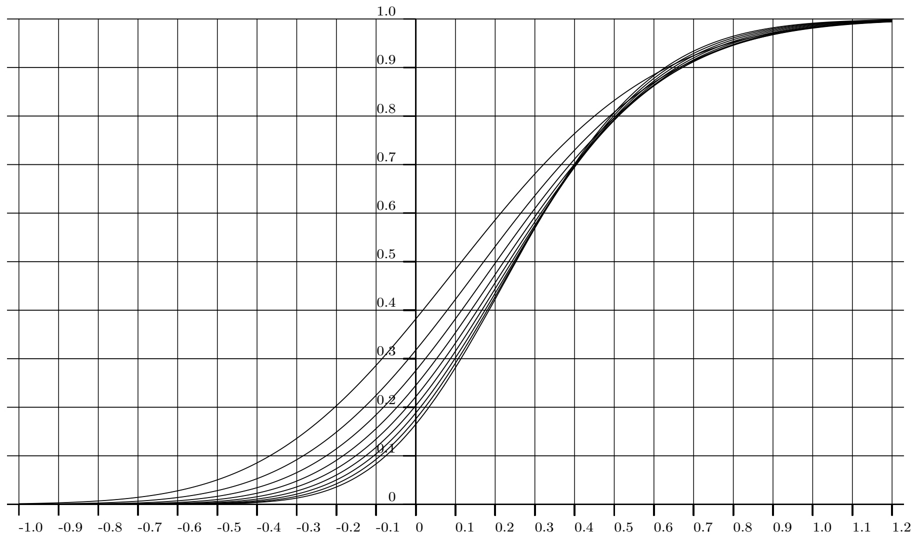

Figure 2: Approximations of the limit distribution function F_W for $d = 2, \ldots, 10$. They can be distinguished by $F_W(0) = q_d$ being decreasing in d. As approximations the functions g_6 defined in (7) are plotted.

An approximation of the limit distribution function F_W is plotted in Figure 2 for the cases $d = 2, \ldots, 10$.

Note that the transformation $F_V(W_n)$ of W_n in (2) allows to rewrite $F_V(W_n)$ as follows: The random variable $F_V(W_n)$ is distributed as the root's value W_n' of a minimax tree with same branching degree and height where the independent, identically distributed leaves now have distribution $\mathcal{L}(V') = \mathcal{L}(F_V(V)) = \mathrm{unif}[0,1]$, the uniform distribution on $[0,1]$. Hence without loss of generality one may assume that $\mathcal{L}(V) = \mathrm{unif}[0,1]$.

The rest of this note contains a proof of Theorem 1. We first collect some properties of f in section 2, since later on the recurrence relation $F_n = f \circ F_{n/d^2}$ is exploited. Section 3 contains the proof of Theorem 1.

2 Technical preliminaries

We collect some properties of the function f defined in (1).

Lemma 2 *There is a unique $q \in (0,1)$ with $f(q) = q$. We have $\xi = f'(q) = d^2q^2/(1-q)^2 \in (1, d^2)$. Furthermore, for $z := 1 - 1/(d+1)^{1/d}$, we have*

$$f''(x) \begin{cases} > 0 & \text{for } 0 < x < z, \\ = 0 & \text{for } x = z, \\ < 0 & \text{for } z < x < 1. \end{cases} \tag{4}$$

We have $q < z$, thus $f''(q) > 0$.

Proof. For $0 < x < 1$ we have

$$f'(x) = d^2(1-x)^{d-1}(1-(1-x)^d)^{d-1}, \tag{5}$$

$$f''(x) = d^2(d-1)(1-x)^{d-2}(1-(1-x)^d)^{d-2}((d+1)(1-x)^d - 1). \tag{6}$$

From this the (in-)equalities (4) follow with $z = z_d = 1 - 1/(d+1)^{1/d}$. For existence and uniqueness of the fixed point q of f in (0,1) we first show:
Claim: $f(z_d) - z_d > 0$ for all $d \geq 2$.
The claim follows for $d = 2, 3$ by explicit calculation. Furthermore we have $f(z_d) = (1-1/(d+1))^d \downarrow 1/e$ as $d \to \infty$, hence $f(z_d) \geq 1/e$ for all $d \geq 4$. It is easily seen that z_d is decreasing in d, thus $z_d \leq z_4$ for all $d \geq 4$. Consequently, for all $d \geq 4$

$$f(z_d) - z_d \geq \frac{1}{e} - z_4 = \frac{1}{e} + 1 - \frac{1}{5^{1/4}} > 0,$$

which implies the claim.
Since $f(0) = f'(0) = 0$, there exists $0 < \varepsilon < z_d$ with $f(x) - x < 0$ for all $0 < x \leq \varepsilon$. Together with the previous claim, continuity and the intermediate value theorem we obtain a fixed point of f in (ε, z_d). We denote by $q = q_d$ the smallest fixed point of f in $(0, z_d)$, which exists by continuity and satisfies $q > \varepsilon > 0$. Then we have $f(x) < x$ for all $x \in (0, q)$. For $x \in (q, z)$ we have $f(x) > x$ by convexity of f on $[0, z]$: Otherwise there was an $x \in (q, z)$ with $f(x) \leq x$. For arbitrary $y \in (0, q)$, and $\lambda \in (0, 1)$ with $q = \lambda y + (1-\lambda)x$ this implied $f(q) \leq \lambda f(y) + (1-\lambda)f(x) < \lambda y + (1-\lambda)x = q$, a contradiction. Similarly, concavity of f on $[z, 1]$ implies $f(x) > x$ for all $x \in (z, 1)$: For all such x there is a $\lambda \in (0, 1)$ with $x = \lambda z + (1-\lambda)1$ thus $f(x) \geq \lambda f(z) + (1-\lambda)f(1) > \lambda z + (1-\lambda)1 = x$. Altogether, q is the unique fixed point of f in $(0, 1)$.
It remains to prove that $\xi = \xi_d = f'(q) = d^2q^2/(1-q)^2 \in (1, d^2)$. For this note that the function $u_d : [0, 1] \to [0, 1]$, $x \mapsto (1-x)^d$, has a unique fixed point in $(0, 1)$. Since $f = u_d \circ u_d$ this fixed point must be $q = q_d$, hence we obtain the relation $q = (1-q)^d$. Using this relation in (5) implies $\xi = f'(q) = d^2q^2/(1-q)^2$. Moreover, since $u_{d'} \leq u_d$ for all $2 \leq d \leq d'$ the sequence $(q_d)_{d\geq 2}$ is decreasing. Thus $q_d \leq q_2 = (3-\sqrt{5})/2 < 1/2$ for all $d \geq 2$, hence $\xi_d < d^2$. Finally, $q = (1-q)^d$, $f''(q) > 0$ and the representation (6) imply $q > 1/(d+1)$, hence $q/(1-q) > 1/d$ and $\xi > d^2/d^2 = 1$. □

In the following, it is convenient to extend function f defined in (1) to the real line by setting $f(x) = 0$ for $x < 0$ and $f(x) = 1$ for $x > 1$. We denote the iterations of f by $f_k = f \circ f_{k-1}$ for $k \geq 1$ and $f_0(x) = x$ for all $x \in \mathbb{R}$. In particular, we have $f_1 = f$. Using $F_n = f \circ F_{n/d^2}$ we obtain for $n = d^{2k}$ that $F_n = f_k \circ F_1 = f_k \circ F_V$.
For the quantities $n^\alpha(F_V(W_n) - q)$ of Theorem 1 we obtain with the relation $n^\alpha = \xi^k$

$$\begin{aligned}\mathbb{P}(n^\alpha(F_V(W_n) - q) \leq x) &= \mathbb{P}\left(W_n \leq F_V^{-1}\left(q + \frac{x}{\xi^k}\right)\right) \\ &= F_n \circ F_V^{-1}\left(q + \frac{x}{\xi^k}\right) \\ &= f_k\left(q + \frac{x}{\xi^k}\right).\end{aligned}$$

Thus, the functions $g_k : \mathbb{R} \to \mathbb{R}$ defined by

$$g_k(x) = f_k\left(q + \frac{x}{\xi^k}\right), \quad x \in \mathbb{R}, \tag{7}$$

are the distribution functions of $n^\alpha(F_V(W_n)-q)$ for $n=d^{2k}$, $k\geq 0$.
Subsequently we will need bounds for g_k valid locally around $x=0$ and uniformly in $k\geq 0$.

Lemma 3 *Denote $h_1(x):=q+x$ and $h_2(x):=q+x+cx^2$ for $x\in\mathbb{R}$ with $c:=1+f''(q)/(2\xi(\xi-1))>1$. Then there exists an $\varepsilon>0$ such that for all $k\geq 0$ and $|x|<\varepsilon$*

$$h_1(x)\leq g_k(x)\leq h_2(x).$$

Proof. We prove the assertion by induction on k. For $k=0$ we have, for all $x\in\mathbb{R}$,

$$h_1(x)=q+x=g_0(x)\leq h_2(x).$$

Assume that the assertion is true for some $k-1\geq 0$ and $\varepsilon>0$. Since f is increasing and $|x|/\xi<\varepsilon$ for all $|x|<\varepsilon$ we obtain

$$g_k(x)=f_k\left(q+\frac{x}{\xi^k}\right)=f\left(f_{k-1}\left(q+\frac{x/\xi}{\xi^{k-1}}\right)\right)=f\left(g_{k-1}\left(\frac{x}{\xi}\right)\right)\geq f\left(h_1\left(\frac{x}{\xi}\right)\right),$$

and analogously

$$g_k(x)\leq f\left(h_2\left(\frac{x}{\xi}\right)\right).$$

Thus, the induction proof is completed by showing that for some $\varepsilon>0$ we have

$$f\left(h_1\left(\frac{x}{\xi}\right)\right)\geq h_1(x),\quad f\left(h_2\left(\frac{x}{\xi}\right)\right)\leq h_2(x), \tag{8}$$

for all $|x|<\varepsilon$.
Taylor expansion of $x\mapsto f(h_i(x/\xi))$ around $x=0$ yields for each $i=1,2$

$$f(h_i(x/\xi))=q+x+\frac{1}{2}\left(\frac{h_i''(0)}{\xi}+\frac{f''(q)}{\xi^2}\right)x^2+O(x^3),$$

for all x in a bounded neighborhood of 0. We have

$$\frac{1}{2}\left(\frac{h_1''(0)}{\xi}+\frac{f''(q)}{\xi^2}\right)=\frac{1}{2}\frac{f''(q)}{\xi^2}>0$$

by Lemma 2. From $h_2''(0)=2c$ and the definition of c it follows

$$\frac{1}{2}\left(\frac{h_2''(0)}{\xi}+\frac{f''(q)}{\xi^2}\right)=\frac{f''(q)}{2\,\xi\,(\xi-1)}+\frac{1}{\xi}<\frac{f''(q)}{2\,\xi\,(\xi-1)}+1=c.$$

Thus, there exists an $\varepsilon>0$ with (8) for all $|x|<\varepsilon$. □

3 Proof of the theorem

We proof the claims of Theorem 1.

Convergence in distribution: We show that $n^\alpha(F_V(W_n)-q)$ converges in distribution by showing that its distribution functions g_k, $n=d^{2k}$, convergence pointwise to a distribution function g.

Fix $x \in \mathbb{R}$. Since $q < z$ and $f'(q) = \xi > 1$ there is $k_0(x)$ such that $0 < q + x/\xi^k < z$, for all $k \geq k_0(x)$. By Lemma 2 the function f is convex on $[0, z]$ and satisfies $f(q) = q$. Hence, for all $k \geq k_0(x)$

$$f\left(q + \frac{x}{\xi^k}\right) \geq f(q) + f'(q)\frac{x}{\xi^k} = q + \frac{x}{\xi^{k-1}}$$

and, since f_{k-1} is monotone increasing,

$$g_k(x) = f_k\left(q + \frac{x}{\xi^k}\right) = f_{k-1}\left(f\left(q + \frac{x}{\xi^k}\right)\right) \geq f_{k-1}\left(q + \frac{x}{\xi^{k-1}}\right) = g_{k-1}(x). \tag{9}$$

Thus, the sequence $(g_k(x))_{k \geq k_0(x)}$ is monotone increasing and upper bounded, hence convergent. We denote its limit by

$$g(x) := \lim_{k\to\infty} g_k(x), \quad x \in \mathbb{R}.$$

Since g_k is nondecreasing for all $k \geq 1$ its limit g is a nondecreasing function. Since $g_k(0) = f_k(q) = q$ for every $k \geq 0$, we have $g(0) = q$. Continuity of f and $g_k(x) = f(g_{k-1}(x/\xi))$ yields, with $k \to \infty$, the functional equation $g(x) = f(g(x/\xi))$.
Monotonicity of g and $0 \leq g \leq 1$ imply that $\lim_{x\to\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$ exist. Continuity of f and $\xi > 0$ yield with the functional equation for g that

$$\lim_{x\to-\infty} g(x) = f\Big(\lim_{x\to-\infty} g(x)\Big), \quad \lim_{x\to\infty} g(x) = f\Big(\lim_{x\to\infty} g(x)\Big).$$

Hence, both limits are fixed points of f. Lemma 3 and convergence of g_k yield, with ε as in Lemma,

$$h_1(x) < g(x) < h_2(x), \quad -\varepsilon < x < \varepsilon.$$

In a left neighborhood of 0 we have $h_2 < q$. Thus, for some $x < 0$ we have $g(x) < q$, and for appropriate $x > 0$ we have $g(x) > h_1(x) > q$. Since f has only the fixed points 0, q and 1 we obtain $\lim_{x\to-\infty} g(x) = 0$ and $\lim_{x\to\infty} g(x) = 1$.
Hence, $\bar{g}(x) = \lim_{y\downarrow x} g(y)$ for $x \in \mathbb{R}$ is a distribution function with $g_k(x) \to \bar{g}(x)$ for all continuity points x of $\bar{g}$. This implies that $n^\alpha(F_V(W_n) - q) \to W$ in distribution with a random variable W with distribution function $F_W = \bar{g}$.
Note that up to now we only know $g(x) = \bar{g}(x)$ for continuity points x of $\bar{g}$. (We will see below that g is continuous, hence $g(x) = \bar{g}(x) = F_W(x)$ for all $x \in \mathbb{R}$.) □

Continuity of g: We show that g is continuous in all $x \in \mathbb{R}$ by distinguishing the three cases $x < 0$, $x = 0$ and $x > 0$. Note that for all $x \in \mathbb{R}$ it is sufficient to show that there exists a $\delta > 0$ with

$$\sup\left\{g_k'(y)\Big||x-y| < \delta, k \geq 0\right\} =: C < \infty. \tag{10}$$

From this we obtain $|g_k(x) - g_k(y)| \leq C|x-y|$ for all $k \in \mathbb{N}$ and $|x-y| < \delta$, hence $|g(x)-g(y)| \leq C|x-y|$, in particular g is continuous in x.
Case $x < 0$: The chain rule and induction imply

$$g_k'(x) = \frac{1}{\xi^k} \prod_{i=0}^{k-1} f'\left(f_i\left(q + \frac{x}{\xi^k}\right)\right). \tag{11}$$

For $x \le 0$ we have $f_i(q + x/\xi^k) \le q$. Since f' is monotone increasing on $(-\infty, q]$ we obtain $g_k'(x) \le (f'(q)/\xi)^k = 1$ for all $x \le 0$ and $k \ge 0$. Hence, for all $x < 0$ we have (10) with $C = 1$.
Case $x = 0$: By Lemma 3 and $\bar{g}(x) = g(x)$ for all $x < 0$ we obtain

$$\begin{aligned}\mathbb{P}(W < 0) &= \lim_{\ell\to\infty} \mathbb{P}\left(W \le -\frac{1}{\ell}\right) = \lim_{\ell\to\infty} \bar{g}\left(-\frac{1}{\ell}\right) = \lim_{\ell\to\infty} g\left(-\frac{1}{\ell}\right) \\ &\ge \lim_{\ell\to\infty} h_1\left(-\frac{1}{\ell}\right) = q. \end{aligned} \tag{12}$$

Since g is a monotone function it has at most countably many discontinuity points. Hence there exists a sequence $(x_\ell)_{\ell\ge1}$ of continuity points of g with $x_\ell \downarrow 0$. Then, with Lemma 3 we obtain

$$\begin{aligned}\mathbb{P}(W > 0) &= 1 - \lim_{\ell\to\infty} \mathbb{P}\left(W \le x_\ell\right) = 1 - \lim_{\ell\to\infty} \bar{g}\left(x_\ell\right) = 1 - \lim_{\ell\to\infty} g\left(x_\ell\right) \\ &\ge 1 - \lim_{\ell\to\infty} h_2\left(x_\ell\right) = 1 - q. \end{aligned} \tag{13}$$

Inequalities (12) and (13) together imply $\mathbb{P}(W = 0) = 0$, hence g is continuous in $x = 0$. Since we have $g(0) = q$ this implies $F_W(0) = g(0) = q$.
Case $x > 0$: We first show the following assertion:
Claim: There exists a $0 < \varepsilon \le z - q$ such that g_k' is a monotone increasing function on $[0, \varepsilon]$ for all $k \ge 0$.
The claim is shown as follows: Since g is continuous in 0 and $g(0) = q < z$ there exists a $0 < \varepsilon < z - q$ with $g(y) \le z$ for all $0 \le y \le \varepsilon$. By monotonicity of the f_i, we have for all $k \ge 0$, $0 \le i \le k$ and $0 < y' < y \le \varepsilon$

$$f_i(q + y'/\xi^k) \le f_i(q + y/\xi^k) \le f_i(q + y/\xi^i) = g_i(y) \le g(y) \le z.$$

For the second last inequality in the latter display note that $(g_i)_{i\ge0}$ is increasing on $(-\infty, z-q)$, cf. (9). Since f' is monotone increasing on $(-\infty, z]$ this yields

$$f'(f_i(q + y'/\xi^k)) \le f'(f_i(q + y/\xi^k)),$$

thus by (11) we obtain $g_k'(y') \le g_k'(y)$ which implies the claim.
Now, assume g is discontinuous in some $x' > 0$. Let ε be as in the previous claim. Note that all the points x'/ξ^k, $k \ge 0$, are discontinuities of g by the functional equation $g(x) = f(g(x/\xi))$ and continuity of f. Hence there exists a discontinuity $0 < x < \varepsilon/2$ of g. By (10), we have for all $0 < \delta < (\varepsilon/2 - x) \wedge x$,

$$\sup\left\{ g_k'(y) \,\middle|\, y : |y - x| < \delta,\, k \ge 0 \right\} = \infty. \tag{14}$$

Fix such a δ. By (14) and the claim we have $g_m'(x + \delta) \ge 4/\varepsilon$ for a sufficiently large m. Now, the claim implies $g_m'(y) \ge 4/\varepsilon$ for all $y \in [\varepsilon/2, \varepsilon]$. Then,

$$g_m(\varepsilon) - g_m(\varepsilon/2) = \int_{\varepsilon/2}^{\varepsilon} g_m'(y)\, dy \ge \int_{\varepsilon/2}^{\varepsilon} \frac{4}{\varepsilon}\, dy = 2.$$

This is a contradiction, since g_m is a distribution function. □

$0 < F_W < 1$: Assume that $F_W(x) = g(x) \in \{0, 1\}$ for some $x \in \mathbb{R}$. Then $g(x/\xi^k) = g(x)$ for all $k \ge 0$. Hence by continuity of g, we obtain $g(0) \in \{0, 1\}$. Since $g(0) = q \in (0, 1)$ this is a contradiction. □

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References

[1] Ali Khan, T. and Neininger, R. Probabilistic analysis for randomized game tree evaluation. *Mathematics and computer science. III*, 163–174, Trends Math., *Birkhäuser, Basel*, 2004.

[2] Devroye, L. and Kamoun, O. Random minimax game trees. *Random discrete structures (Minneapolis, MN, 1993)*, 55–80, IMA Vol. Math. Appl., 76, *Springer, New York*, 1996

[3] Karp, R. M. and Zhang, Y. Bounded branching process and AND/OR tree evaluation. *Random Structures Algorithms* **7** (1995), 97–116.

[4] Knuth, D. E. and Moore, R. W. An analysis of alpha-beta pruning. *Artificial Intelligence* **6** (1975), 293–326.

[5] Nau, D. S. The last player theorem. *Artificial Intelligence* **18** (1982), 53–65.

[6] Nau, D. S. An investigation of the causes of pathology in games. *Artificial Intelligence* **19** (1982), 257–278.

[7] Nau, D. S. Pathology on game trees revisited, and an alternative to minimaxing. (English) *Artificial Intelligence* **21** (1983), 221–244 (1983).

[8] Pearl, J. Asymptotic properties of minimax trees in game-searching procedures. *Aritificial Intelligence* **14** (1980), 113–126.

[9] Saks, M. and Wigderson, A. Probabilistic boolean decision trees and the complexity of evaluating game trees. *Proceedings of the 27th Annual IEEE Symposium on Foundations of Computer Science*, 29–38, Toronto, Ontario, 1986.

[10] Snir, M. Lower bounds on probabilistic linear decision trees. *Theor. Comput. Sci.* **38** (1985), 69-82.

[11] Zhang, Y. The variance of two game tree algorithms. *J. Algorithms* **28** (1998), 21–39.