A TRIPTYCH OF DISCRETE DISTRIBUTIONS RELATED TO THE STABLE LAW

Luc Devroye
School of Computer Science
McGill University

Abstract. We derive useful distributional representations for three discrete laws: the discrete stable distribution of Steutel and Van Harn, the discrete Linnik distribution introduced by Pakes, and a distribution of Sibuya. These representations may be used to obtain simple uniformly fast random variate generators.

Keywords and phrases. Discrete stable distribution. Random variate generation. Distribution theory. Linnik's distribution.

1991 Mathematics Subject Classifications: 65C10, 68J05.

Authors' address: School of Computer Science, McGill University, 3480 University Street, Montreal, Canada H3A 2K6. The authors' research was sponsored by NSERC Grant A3456 and FCAR Grant 90-ER-0291.

1. The discrete stable family.

Steutel and Van Harn (1979) introduced the discrete stable family via the generating function

$$g(s) = \mathbf{E}s^X = \exp(-\lambda(1-s)^\gamma), |s| \le 1,$$

where $\lambda > 0$ and $\gamma \in (0,1]$ are two parameters. X (or $X(\lambda,\gamma)$) is called a discrete stable random variable. For $\gamma = 1$, we obtain the Poisson distribution with parameter λ. This distribution is infinitely divisible, and has several other stability properties. In this note, we are motivated by the need to efficiently generate i.i.d. random variates from this family. Since the discrete stable distribution is a compound Poisson distribution, random variates can be obtained as

$$\sum_{1 \le i \le Y} Z_i,$$

where Y is Poisson (λ), and $Z_1, Z_2, \ldots$ are i.i.d. discrete random variables with generating function

$$h(s) = 1 - (1-s)^\gamma.$$

This is best seen by noting that

$$g(s) = \mathbf{E}s^X = \mathbf{E}\prod_{i \le Y} s^{Z_i} = \mathbf{E}\left(\mathbf{E}s^{Z_1}\right)^Y = \mathbf{E}(h(s))^Y = e^{\lambda(h(s)-1)} = e^{-\lambda(1-s)^\gamma}.$$

A random variable Z with generating function $h(s)$ will be called a Sibuya (γ) random variable, after Sibuya (1979). If we can generate the Z_i's at unit expected time cost, then the expected time for this method is proportional to $\mathbf{E}Y = \lambda$:

```
(S,X) ← (E,0), where E is exponential
while S < λ do
  generate E exponential, Z Sibuya (γ)
  (S,X) ← (S+E, X+Z)
return X
```

The above method does not have uniformly bounded time in both parameters. It is nevertheless attractive for small values of λ. A Sibuya generator is derived in section 3. The main point of this note is to present a generator that is uniformly fast in both parameters. This would follow from the following simple theorem:

THEOREM. *A discrete stable (λ, γ) random variable is distributed as a Poisson random variable with parameter $\lambda^{1/\gamma} S_{\gamma,1}$, where $S_{\gamma,1}$ is a positive stable random variate with parameter γ:*

$$X(\lambda, \gamma) \stackrel{\mathcal{L}}{=} \text{Poisson}(\lambda^{1/\gamma} S_{\gamma,1}) \ .$$

PROOF. We recall that $S_{\gamma,1}$ has characteristic function

$$\varphi(t) = e^{-t^\gamma e^{-i\pi\gamma/2}} \ .$$

For $\gamma = 1$, we obtain the degenerate function with atom one at $x = 1$. Also, the Laplace transform is given by

$$\mathbf{E} e^{-sS_{\gamma,1}} = e^{-s^\gamma} \ , \ \Re(s) > 0$$

(Zolotarev,1986, p. 114). The characteristic function of X is now obtained as

$$\mathbf{E} e^{itX} = \mathbf{E} e^{\lambda^{1/\gamma} S_{\gamma,1}(e^{it}-1)} = e^{-\lambda(1-e^{it})^\gamma} \ ,$$

in which we recognize the characteristic function of the discrete stable distribution. □

The above result leads directly to a uniformly fast generator. It suffices to note that there is a plentiful supply of uniformly fast Poisson generators (see Devroye, 1986), and that $S_{\gamma,1}$ can be obtained very easily by a variety of methods. For example, Kanter's method (1975) uses the distributional identity

$$S_{\gamma,1} \stackrel{\mathcal{L}}{=} \left(\frac{\sin(1-\gamma)\pi U}{E \sin \gamma\pi U} \right)^{(1-\gamma)/\gamma} \left(\frac{\sin \gamma\pi U}{\sin \pi U} \right)^{1/\gamma} \ ,$$

where E is exponential and U is uniform $[0, 1]$ and independent of E.

2. The discrete Linnik distribution.

Pakes (1993) studies the discrete distribution with generating function

$$g(s) = \frac{1}{(1 + (1-s)^\gamma)^\beta} \ ,$$

where $\beta > 0$ and $\gamma \in (0, 1]$. We call this the discrete Linnik distribution and denote a typical random variable by $L(\beta, \gamma)$. A continuous distribution with an analogous characteristic function $1/(1 + |t|^\delta)^\beta$ was introduced by Linnik (1962). From the Theorem given above we deduce the following simple distributional identity:

$$L(\beta, \gamma) \stackrel{\mathcal{L}}{=} \text{Poisson}(G_\beta^{1/\gamma} S_{\gamma,1}) \ ,$$

where G_β is a gamma (β) random variable, independent of $S_{\gamma,1}$. This is easily proved by computing the generating function:

$$\mathbf{E}s^L = \mathbf{E}e^{-G_\beta(1-s)^\gamma} = \frac{1}{(1+(1-s)^\gamma)^\beta} \ .$$

As gamma variates are easily available in uniformly bounded time (Devroye, 1986, has a list of references), discrete Linnik variates can be generated in uniformly bounded expected time as well. The representation given above may also be used to derive various properties of the discrete Linnik distribution.

3. Sibuya's distribution.

The last of our triptych of discrete distributions was introduced by Sibuya in 1979. Described by one parameter $\gamma \in (0,1]$, it has generating function

$$h(s) = 1 - (1-s)^\gamma.$$

If $S(\gamma)$ denotes a Sibuya (γ) random variate, then we obtain via the binomial expansion,

$$\mathbf{P}\{S(\gamma) = n\} = \begin{cases} \frac{\gamma(1-\gamma)\cdots(n-1-\gamma)}{n!} & (n > 1) \\ \gamma & (n = 1) \end{cases} \ .$$

For $\gamma = 1$, the distribution is monoatomic with an atom at 1. Uniformly fast generators are described in Devroye (1992). The generalized hypergeometric distribution of type B3 (or: GHgB3) with parameters $a, b, c > 0$ is a discrete distribution on the nonnegative integers defined by

$$p_n = \frac{\Gamma(a+c)\Gamma(b+c)}{\Gamma(a+b+c)\Gamma(c)} \frac{(a)_n(b)_n}{n!(a+b+c)_n} \ (n \geq 0),$$

where $(a)_n$ is Pochhammer's symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)\ldots(a+n-1) & (n \geq 1) \\ 1 & (n = 0). \end{cases}$$

Other names for the distribution include the inverse Pólya-Eggenberger distribution (Johnson and Kotz, 1982), the generalized Waring distribution (Irwin, 1975) and the negative binomial beta distribution. When $a = 1$ or $b = 1$ the distribution is called the Waring distribution. From Sibuya (1979), Shimizu (1968), Sibuya and Shimizu (1981) and Devroye (1992), we recall that a GHgB3 random variate may be generated as

$$\text{Poisson}\left(\frac{G_a G_b}{G_c}\right) \ ,$$

where G_a, G_b and G_c are independent gamma random variates with parameters a, b and c. Good Poisson generators are given in Schmeiser and Kachitvichyanukul (1981), Ahrens and Dieter (1982) and Stadlober (1990). Efficient gamma generators are given in

Marsaglia (1977), Best (1978), Cheng and Feast (1979, 1980), Ahrens and Dieter (1982), Ahrens, Kohrt and Dieter (1983) and Le Minh (1988). Sibuya (1979) points out that $S(\gamma)$ is distributed as one plus a GHgB3 $(1, 1-\gamma, \gamma)$ random variable, In our case, we thus have

$$S(\gamma) \stackrel{\mathcal{L}}{=} 1 + \text{Poisson}\left(\frac{EG_{1-\gamma}}{G_\gamma}\right) ,$$

where E is an exponential random variable, independent of the gamma random variables.

4. References.

J. H. Ahrens and U. Dieter, "Sampling from binomial and Poisson distributions: a method with bounded computation times," *Computing*, vol. 25, pp. 193–208, 1980.

J. H. Ahrens and U. Dieter, "Computer generation of Poisson deviates from modified normal distributions," *ACM Transactions on Mathematical Software*, vol. 8, pp. 163–179, 1982.

J. H. Ahrens and U. Dieter, "Generating gamma variates by a modified rejection technique," *Communications of the ACM*, vol. 25, pp. 47–54, 1982.

J. H. Ahrens, K. D. Kohrt, and U. Dieter, "Algorithm 599. Sampling from gamma and Poisson distributions," *ACM Transactions on Mathematical Software*, vol. 9, pp. 255–257, 1983.

D. J. Best, "Letter to the editor," *Applied Statistics*, vol. 27, p. 181, 1978.

R. C. H. Cheng and G. M. Feast, "Some simple gamma variate generators," *Applied Statistics*, vol. 28, pp. 290–295, 1979.

R. C. H. Cheng and G. M. Feast, "Gamma variate generators with increased shape parameter range," *Communications of the ACM*, vol. 23, pp. 389–393, 1980.

L. Devroye, *Non-Uniform Random Variate Generation*, Springer-Verlag, New York, 1986.

L. Devroye, "Random variate generation for the digamma and trigamma distributions," *Journal of Statistical Computation and Simulation*, vol. 43, pp. 197–216, 1992.

J. O. Irwin, "The generalized Waring distribution, part I," *Journal of the Royal Statistical Society*, vol. A138, pp. 18–31, 1975.

N. L. Johnson and S. Kotz, "Developments in discrete distributions, 1969-1980," *International Statistical Review*, vol. 50, pp. 71–101, 1982.

M. Kanter, "Stable densities under change of scale and total variation inequalities," *Annals of Probability*, vol. 3, pp. 697–707, 1975.

Yu. V. Linnik, "Linear forms and statistical criteria: I, II," *Selected Translations in Mathematical Statistics and Probability*, vol. 3, pp. 1–40, 41-90, 1962.

G. Marsaglia, "The squeeze method for generating gamma variates," *Computers and Mathematics with Applications*, vol. 3, pp. 321–325, 1977.

D. Le Minh, "Generating gamma variates," *ACM Transactions on Mathematical Software*, vol. 14, pp. 261–266, 1988.

A. Pakes, Personal communication, 1993.

B. W. Schmeiser and V. Kachitvichyanukul, "Poisson random variate generation," Research Memorandum 81-4, School of Industrial Engineering, Purdue University, West Lafayette, Indiana, 1981.

R. Shimizu, "Generalized hypergeometric distributions," *Proceedings of the Institute of Statistical Mathematics*, vol. 16, pp. 147–165, 1968.

M. Sibuya, "Generalized hypergeometric, digamma and trigamma distributions," *Annals of the Institute of Statistical Mathematics*, vol. 31, pp. 373–390, 1979.

M. Sibuya and R. Shimizu, "The generalized hypergeometric family of distributions," *Annals of the Institute of Statistical Mathematics*, vol. 33, pp. 177–190, 1981.

E. Stadlober, "The ratio of uniforms approach for generating discrete random variates," *Journal of Computational and Applied Mathematics*, vol. 31, pp. 181–189, 1990.

F. W. Steutel and K. Van Harn, "Discrete analogues of self-decomposability and stability," *Annals of Probability*, vol. 7, pp. 893–899, 1979.

V. M. Zolotarev, *One-Dimensional Stable Distributions*, American Mathematical Society, Providence, R.I., 1986.