Transitive Relations, Topologies and Partial Orders

Steven Finch

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Let S be a set with n elements. A subset R of $S \times S$ is a **binary relation** (or **relation**) on S. The number of relations on S is 2^{n^2}. Equivalently, there are 2^{n^2} labeled bipartite graphs on $2n$ vertices, assuming the bipartition is fixed and equitable.

A relation R on S is **reflexive** if for all $x \in S$, we have $(x, x) \in R$. The number of reflexive relations on S is $2^{n(n-1)}$.

A relation R on S is **antisymmetric** if for all $x, y \in S$, the conditions $(x, y) \in R$ and $(y, x) \in R$ imply that $x = y$. The number of antisymmetric relations on S is $2^n \cdot 3^{n(n-1)/2}$.

A relation R on S is **transitive** if for all $x, y, z \in S$, the conditions $(x, y) \in R$ and $(y, z) \in R$ imply that $(x, z) \in R$. There is no known general formula for the number T_n of transitive relations on S. It is surprising that such a simply-stated counting problem remains unsolved [1, 2, 3, 4, 5, 6].

A **topology** on S is a collection Σ of subsets of S that satisfy the following axioms:

- $\emptyset \in \Sigma$ and $S \in \Sigma$
- the union of any two sets in Σ is in Σ
- the intersection of any two sets in Σ is in Σ.

Note that since S is finite, our phrasing of the second axiom is correct. No one knows a general formula for the number U_n of topologies on S. Also, a topology on S is a **T0 topology** if it additionally satisfies a (weak) separation axiom:

- for any pair of distinct points in S, there is a set in Σ containing one point but not the other.

Again, no one knows a general formula for the number V_n of T0 topologies [7].

A **quasi-order** on S is a relation that is both reflexive and transitive. Let Q_n denote the number of such relations. Other uses of the phrase "quasi-order" exist and so care must be taken when reviewing the literature. There is a one-to-one

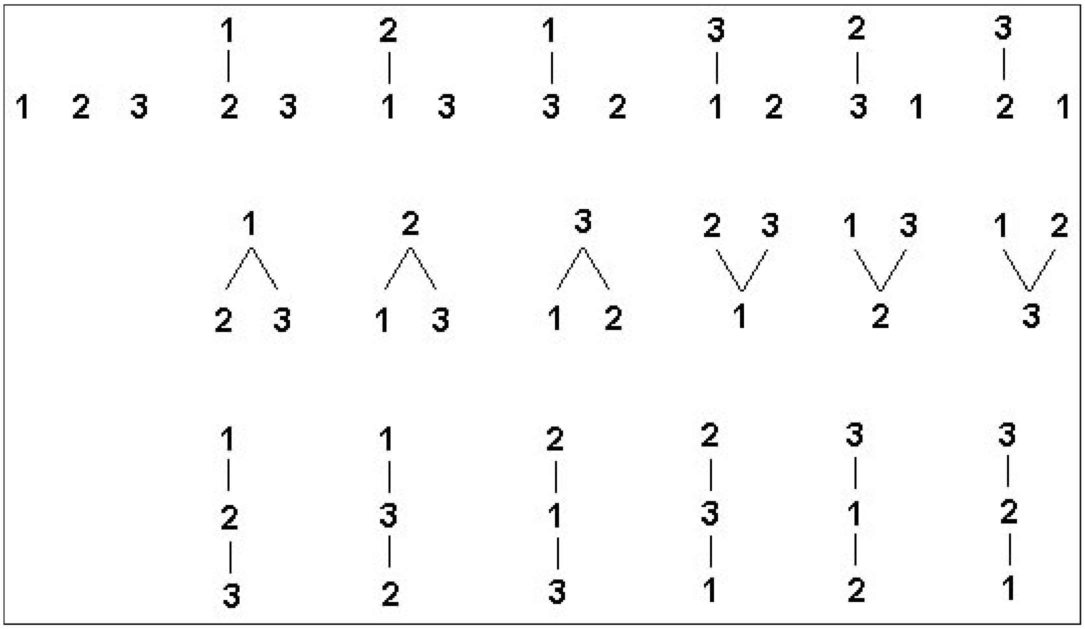

Figure 1: There are 19 labeled posets with 3 elements, that is, $P_3 = 19$.

correspondence between the topologies on S and the quasi-orders on S; hence $Q_n = U_n$.

A **partial order** on S is a quasi-order that is antisymmetric as well. Let P_n denote the number of such relations. We usually write $x \leq y$ if $(x, y) \in R$ and, moreover, $x < y$ if $x \neq y$. There is a one-to-one correspondence between the T0 topologies on S and the partial orders on S; hence $P_n = V_n$.

Further connections between P_n and Q_n, and between P_n and T_n, can be expressed in terms of Stirling numbers of the second kind [1, 8]:

$$Q_n = \sum_{k=1}^{n} S_{n,k} P_k, \qquad T_n = \sum_{k=1}^{n} \left(\sum_{j=0}^{k} \binom{n}{j} S_{n-j,k-j} \right) P_k$$

and hence [9, 10]

$$Q_n \sim P_n, \qquad T_n \sim 2^n P_n$$

as $n \to \infty$. It is therefore sufficient to focus on just one of these sequences; we choose $\{P_n\}$, which enumerates labeled posets (see Figure 1) as opposed to $\{p_n\}$, which enumerates unlabeled posets (see Figure 2). The existence of an edge (x, y) in any of the graphs pictured here indicates that $x < y$ and y is drawn above x.

Even though a closed-form expression for P_n is unknown, progress has been made

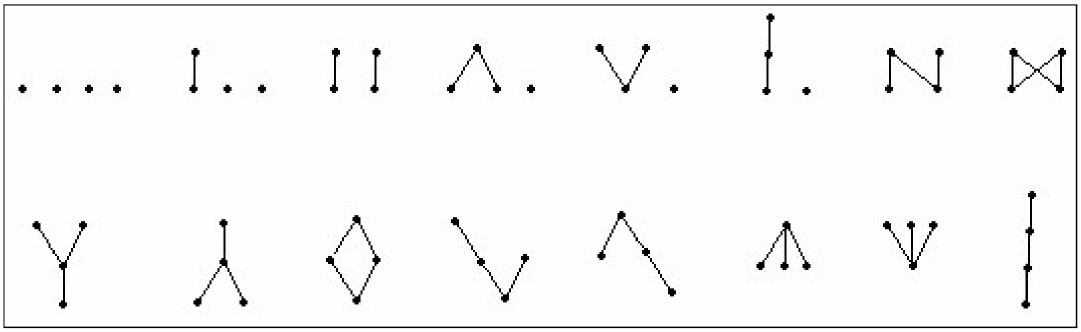

Figure 2: There are 16 unlabeled posets with 4 elements, that is, $p_4 = 16$.

in understanding the asymptotics of

$$\{P_n\}_{n=1}^{\infty} = \{1, 3, 19, 219, 4231, 130023, 6129859, 431723379, \ldots\}.$$

Kleitman & Rothschild [11] deduced that

$$\frac{\ln(P_n)}{\ln(2)} = \frac{n^2}{4} + O\left(n^{\frac{3}{2}} \ln(n)\right)$$

and later sharpened this to [12]

$$\frac{\ln(P_n)}{\ln(2)} = \frac{n^2}{4} + \frac{3n}{2} + O\left(\ln(n)\right).$$

Building on their work, several authors [10, 13, 14, 15, 16] gave the following improvement:

$$P_n \sim C_a \cdot \sqrt{\frac{2}{\pi}} \cdot 2^{\frac{n^2}{4}+\frac{3n}{2}+\frac{1}{4}} \cdot n^{-\frac{1}{2}}$$

where $n \equiv a \bmod 2$ and $a \in \{0, 1\}$, and where

$$C_1 = \sum_{k=-\infty}^{\infty} 2^{-k^2} = 2.1289368272... = \pi \cdot (0.8058800428...) \cdot 2^{-\frac{1}{4}},$$

$$C_0 = \sum_{k=-\infty}^{\infty} 2^{-(k-\frac{1}{2})^2} = 2.1289312505... = \pi \cdot (0.8058779318...) \cdot 2^{-\frac{1}{4}}.$$

It is interesting that the constant depends on the parity of n.

The asymptotics of the unlabeled case [17, 18]:

$$\{p_n\}_{n=1}^{\infty} = \{1, 2, 5, 16, 63, 318, 2045, 16999, \ldots\}$$

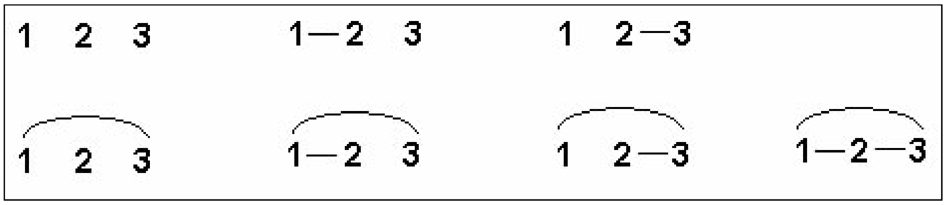

Figure 3: There are 7 natural partial orders on $\{1, 2, 3\}$, that is, $\sigma_3 = 7$.

turn out to satisfy

$$p_n \sim \frac{P_n}{n!} \sim C_a \cdot \frac{1}{\pi} \cdot 2^{\frac{n^2}{4}+\frac{3n}{2}+\frac{1}{4}} \cdot e^n \cdot n^{-n-1}$$

thanks to a general result due to Prömel [19].

See [20, 21] for more appearances of the constants C_0 and C_1. It's believed that, for any asymptotic enumeration problem where a typical member is based on a bipartite graph, these constants are likely to occur. Alternative representations include [16, 22]:

$$C_1 = \sqrt{\frac{\pi}{\ln(2)}} \sum_{k=-\infty}^{\infty} \exp\left(\frac{-\pi^2}{\ln(2)}k^2\right), \qquad C_0 = \sqrt{\frac{\pi}{\ln(2)}} \sum_{k=-\infty}^{\infty} (-1)^k \exp\left(\frac{-\pi^2}{\ln(2)}k^2\right)$$

from which the strict inequality $C_0 < C_1$ becomes obvious.

0.1. Natural Partial Orders. Consider the set $S = \{1, 2, \ldots n\}$ equipped with the usual total ordering $\leq$. A **natural partial order** $\preceq$ on S is a partial ordering that is compatible with $\leq$ (meaning that if $x \preceq y$, then $x \leq y$). This is equivalent to saying that $(S, \leq)$ is a **linear extension** of $(S, \preceq)$. Define σ_n to be the number of natural partial orders on S, then [23, 24, 25]

$$\{\sigma_n\}_{n=1}^{\infty} = \{1, 2, 7, 40, 357, 4824, 96428, 2800472, \ldots\}$$

(see Figure 3).

Brightwell, Prömel & Steger [16] proved that

$$\sigma_n \sim \begin{cases} \frac{1}{2}\eta^2 \cdot C_1 \cdot 2^{\frac{n^2}{4}} \cdot n = (12.7636300229...) \cdot 2^{\frac{n^2}{4}} \cdot n & \text{if } n \text{ is even,} \\ \frac{1}{2}\eta^2 \cdot C_0 \cdot 2^{\frac{n^2}{4}} \cdot n = (12.7635965889...) \cdot 2^{\frac{n^2}{4}} \cdot n & \text{if } n \text{ is odd} \end{cases}$$

where

$$\eta = \prod_{j=1}^{\infty} \left(1 - 2^{-j}\right)^{-1} = 3.4627466194...$$

is a digital search tree constant [26]. These constants also arise when determining the average number λ_n of linear extensions of S, where S is a *random* poset on n points [16, 27]:

$$\lambda_n \sim \begin{cases} \dfrac{\eta^2 C_1}{2^{5/4} C_0} \cdot (\frac{n}{2})!^2 \cdot n \cdot 2^{-n/2} = (5.0414454338...) \cdot (\frac{n}{2})!^2 \cdot n \cdot 2^{-n/2}, \\ \dfrac{\eta^2 C_0}{2^{5/4} C_1} \cdot (\frac{n-1}{2})! \cdot (\frac{n+1}{2})! \cdot n \cdot 2^{-n/2} = (5.0414190220...) \cdot (\frac{n-1}{2})! \cdot (\frac{n+1}{2})! \cdot n \cdot 2^{-n/2} \end{cases}$$

when n is even, respectively, n is odd.

Consider instead the set S of all 2^n subsets of $\{1, 2, \ldots, n\}$ equipped with the usual partial ordering $\subseteq$. Define τ_n in a manner analogous to σ_n. We observe that $\lambda_n \cdot P_n \sim n! \cdot \sigma_n$ and wonder what the corresponding asymptotics for τ_n might be.

0.2. Evolving Posets. An interesting variation is as follows. What is the number N_ρ of partial orders on S with the property that a specified fraction ρ of the $n(n-1)/2$ pairs of distinct points are comparable? (If necessary, $\rho n(n-1)/2$ is rounded to the nearest integer.) Dhar [28, 29] investigated this question in the limit as $n \to \infty$ and proposed a lattice gas model (with infinitely many phase transitions) based on the evolution of N_ρ as ρ increases. Prömel, Steger, & Taraz [30, 31, 32] recently completed a highly intricate analysis of Dhar's model, and we hope to report on this later.

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