Scan

A3441
etc

R.C. Read,
preprint

Ent'd

(f) f91

1004 ~~N339~~
2054 ~~N1607~~
2055 ~~N1982~~
2056 ~~N2115~~
3441
3457
~~N1325~~
1683

On General Dissections of a Polygon

by Ronald C. Read
University of Waterloo

Abstract: This paper solves the problem of finding the number of ways in which a regular polygon can be divided into polygonal regions by non-intersecting chords. The main problem is that for which two dissections are regarded as the same if one can be obtained from the other by rotation or reflection, but the solution when rotation only is allowed is also given. For fixed polygons the result is already known but is proved here as a necessary preliminary.

All results are given in terms of the number of sides of the polygon and the number of regions into which it is dissected. Tables of values are given, and a few points of interest about their preparation are briefly discussed.

MS. Received On [illegible] 26/74
Revised MS. Received On..................
Final MS. Received On..................

Aequationes Math 18 370 1978
378

McMahon is N 363, N1044, N416
[AMM 82 922 75]

On General Dissections of a Polygon

by Ronald C. Read
University of Waterloo

I. INTRODUCTION

A type of problem of recurrent interest relates to the number of ways of dividing up the interior of a convex polygon into a number of smaller polygons by means of nonintersecting diagonals, i.e. line segments joining two nonadjacent vertices of a polygon. The simplest problem of this type is that of determining the number of ways of dissecting a fixed n-gon into $n - 2$ triangles. This problem, for which the answer is the sequence of Catalan numbers, has been extensively studied; an excellent account of it and of its literature is given in [1]. Similar results for the dissection of an n-gon into k-sided polygons $(k > 3)$, were obtained by Motzkin [7]. In these problems the n-gon is regarded as fixed in the plane, so that, for example, the two dissections of a hexagon shown in figures 1(a) and 1(b) are regarded as different.

If we choose to regard two dissections as equivalent if one can be obtained from the other by rotating the n-gon (now assumed to be regular) about its centre, and ask for the number of nonequivalent dissections, we have a different, and more difficult, problem. Under these conditions dissections 1(a) and 1(b) become equivalent, but are distinct from 1(c). If we also allow reflection in our definition of equivalence then we have yet another type of problem, in which, for example, dissections 1(a), 1(b) and 1(c) are all equivalent. This latter problem, for dissections into triangles only, was solved some years ago by Guy [2] (see also [7]).

It was only recently that the solution was given for the problem of enumerating dissections of an n-gon into k-gons ($k \geq 3$) subject to equivalence under rotation, and possibly reflection as well (there being, therefore, two problems for each value of k according as reflection is or is not allowed). The stratagem used in that paper - one that enabled the symmetries of the configurations to be easily perceived - was to convert the problem into a type of cell-growth problem. This, so to speak, turns the problem inside out; instead of starting with a polygon and dividing it up, one starts with a number of small polygons (cells) and sticks them together to make the larger polygon (or a homeomorph of it). Such an assemblage of regular cells will be called a "cluster". Since no vertex of the dissected n-gon can belong to more than 2 cells, it follows that the structure of these clusters is essentially tree-like, as observed in [5]. It is this which makes their enumeration feasible, in contrast to the general cell-growth problem, which appears to be quite intractable. (For general information on the cell-growth problem see [3, 4, 6, 9]).

The precise details of this new way of looking at the dissection problem are given in [5], to which the reader is referred; but the general principle can be gleaned from Figures 2 and 3. Figure 2(a) shows a 27-gon dissected into eight regions. In Figure 2(b) these regions have been made into regular polygonal cells, a procedure which, of course, distorts the boundary of the figure as a whole. Figure 3 not only illustrates the two ways of drawing a particular dissection of a 25-gon into 9 regions, but shows how, by forcing the cells to be regular polygons, we may cause the drawing of the cluster to overlap itself as shown in Figure 3(b). This overlapping is allowed, and is of no consequence; it does not matter which of two overlapping portions is drawn on top of the other.

Little attention appears to have been paid to the problem of dissecting an n-gon into polygons with arbitrary (i.e. not necessarily equal) numbers of sides, although this problem for a fixed n-gon has been solved by Motzkin [7]. In this paper we shall give the solutions to the two problems of this general type for which equivalence excludes and includes reflection as well as rotation. In these problems there is no connection between the number of cells and the number of sides of the polygon being dissected. Thus our dissections will be classified according to these two numbers, and we shall therefore be working with generating functions in two variables. This is the most noticeable difference between this paper and [5]. The general plan of campaign is otherwise much the same, and to avoid needless repetition some theorems and results that were discussed in [5] are merely quoted here.

The edges of a cluster that lie on its boundary, and therefore correspond to the edges of the n-gon being dissected, are called "outer" edges; the others, corresponding to the diagonals, are called "inner" edges. A cluster is said to be "rooted" at a certain edge (the root edge) if that edge is distinguished from the others. Two such rooted clusters will be regarded as equivalent if, and only if, there is a mapping of one onto the other (by rotation or reflection) which makes their root edges correspond. In an analogous way we define a cluster "rooted at a cell".

2. CLUSTERS ROOTED AT AN OUTER EDGE

Following the same general plan of campaign that was adopted in [5] we first consider clusters that are rooted at an outer edge (called, for brevity, "out-rooted clusters"), and we let $V_{r,s}$ be the number of these for which the number of cells is r and the number of outer edges, not counting the root edge, is s. For the time being we shall regard such a cluster and its mirror image as being distinct unless, of course, they are identical. We shall include the "empty out-rooted cluster" consisting of the root edge alone; thus $V_{0,1} = 1$. The counting series for these clusters will be denoted by $V(x, y)$. Thus

$$V(x, y) = \sum_{r,s} V_{r,s} x^r y^s$$

Consider those out-rooted clusters for which the cell containing the root edge is a $(k + 1)$-gon. Such clusters can be constructed by taking a $(k + 1)$-gon, one side of which is to be the root edge, and attaching an out-rooted cluster (possibly empty) to each of the other sides (see Figure 4). It is easily seen that the counting series for such clusters is

$$xV^k(x, y)$$

the factor x being required to accommodate the original $(k + 1)$-gon, which adds 1 to the number of cells.

Now every out-rooted cluster is either empty or is a cluster of the above type for some value of $k \geq 2$. Hence we have

$$V(x, y) = y + x[V^2(x, y) + V^3(x, y) + \dots]$$

or

$$(2.1) \qquad V(x, y) = y + \frac{xV^2(x, y)}{1 - V(x, y)}$$

Applying Lagrange's Theorem to equation (2.1), regarding $V(x, y)$ as a power series in x, we find that the coefficient of x^r in $V(x, y)$ is

$$\frac{1}{r!}\left[\left(\frac{d}{dz}\right)^{r-1}\left\{\frac{z^2}{1-z}\right\}^r\right]_{z=y} = \frac{1}{r!}\left[\left(\frac{d}{dz}\right)^{r-1} \sum_{m=0}^{\infty} \binom{r+m-1}{r-1} z^{2r+m}\right]_{z=y}$$

$$= \frac{1}{r!} \sum_{m=0}^{\infty} \binom{r+m-1}{r-1} \frac{(2r+m)!}{(r+m+1)!} y^{r+m+1}$$

provided $r \geq 1$. From this it follows that

$$(2.2) \quad V_{r,s} = \frac{1}{r!}\binom{s-2}{r-1}\frac{(r+s-1)!}{s!} = \frac{1}{r}\binom{s-2}{r-1}\binom{r+s-1}{s}$$

3. CLUSTERS ROOTED AT A CELL

If, in a cluster, one cell is distinguished from the others, the cluster will be said to be rooted at that cell, and will be called a cell-rooted cluster. Such clusters can be obtained by taking a k-gon as the root cell and attaching to each of its sides an out-rooted cluster. As before, we do not regard two clusters as being the same if they are no more than mirror images of each other, but we do regard two clusters as the same if one can be obtained from the other by a rotation about the centre of the root cell. Thus the problem of counting cell-rooted clusters is a straightforward application of Pólya's Theorem (see [4, 8]) for which the figure counting series is $V(x, y)$, and the group is the cyclic group C_k. From the theorem we have that the counting series for these clusters for a given value of k is $Z(C_k; V(x, y))$. Therefore, if we let $F(x, y)$ be the counting series for all such clusters (any value of k) we have

$$(3.1) \qquad F(x, y) = \sum_{k=3} Z(C_k; V(x, y))$$

This expression for $F(x, y)$ is adequate as a theoretical result, but presents some difficulties when it comes to actual computation. We shall consider later the interesting computational aspects of this problem, and in particular see how a certain amount of manipulation enables the coefficients in $F(x, y)$ to be easily calculated. For the present we continue deriving the necessary theoretical formulae.

4. UNROOTED CLUSTERS

The transition from cell-rooted to unrooted clusters is effected by means of the formula

$$(4.1) \qquad 1 = p^* - q^* + s$$

(see [5] for the necessary details). We shall sum equation (4.1) for all unrooted clusters, thus obtaining on the left-hand side the number of these clusters, which is what we want to find. Summing p^* - the number of equivalence clases of cells under rotations which leave the cluster invariant - gives us the total which counts each cluster as many times as it has equivalence classes of cells, i.e. this total is the number of cell-rooted clusters, enumerated by $F(x, y)$. In a similar way the summation of q^* - the number of equivalence classes of inner edges - gives us the number of clusters rooted at an inner edge.

These latter clusters, which we may call "in-rooted clusters" can be obtained by taking an edge and adding, on each side of it, a non-empty out-rooted cluster. Since rotation is allowed we can find the counting series for these clusters by an application of Pólya's Theorem in which the figure counting series is $U(x, y) = V(x, y) - y$ and the group is S_2, the symmetric group of degree 2. We obtain

$$\frac{1}{2}[U^2(x, y) + U(x^2, y^2)]$$

as the counting series for these clusters.

The summation of s over all clusters simply gives us the number of clusters having a symmetric edge. These can be obtained by taking an edge, choosing any nonempty out-rooted cluster, and then placing this cluster on one side of the edge and the same cluster (rotated through 180°) on the other side (see Figure 5). This gives the counting series $U(x^2, y^2)$.

Collecting these results together we find that the counting series for unrooted clusters is

$$(4.2) \qquad H(x, y) = \sum_{r,s} H_{r,s} x^r y^s = F(x, y) - \frac{1}{2}[U^2(x, y) - U(x^2, y^2)]$$

5. ENUMERATION OF CLUSTERS WHEN MIRROR IMAGES ARE NOT REGARDED AS DISTINCT CELL-ROOTED CLUSTERS

When, as a first step, we come to enumerate cell-rooted clusters under the condition that mirror images are not regarded as different, we run into a difficulty that was remarked in [5]. The root cell, a k-gon, say, can be mapped onto itself by any element of the dihedral group D_k ; but half the elements of this group will not only permute the figures (the out-rooted clusters attached to the sides of the root cell) but will also replace these by other figures, namely their mirror images. Under these circumstances Pólya's Theorem cannot be directly applied. Instead we make use of Burnside's Lemma (see [4], page 38) and first ask which figures are left invariant by a given permutation of D_k.

Those permutations of D_k that do not correspond to reflection, and which are therefore elements of C_k, give no trouble; it is easily verified that the results that we get from them will be exactly what we get by applying Pólya's Theorem for the elements of C_k. We therefore look at the elements of D_k which are reflections.

If k is odd, say $k = 2m + 1$, these elements have one cycle of length 1 and m cycles of length 2, and correspond to terms of the form $s_1 s_2^m$ in the cycle index $Z(D_k)$. To be invariant under such a reflection, the out-rooted clusters attached to corresponding pairs of edges (that are interchanged) must be identical, and the cluster rooted at the edge that maps onto itself must be symmetric, i.e. symmetrical about the perpendicular bisector of its root edge (see Figure 6). From this it follows that the counting series for such cell-rooted clusters is

$$(5.1) \qquad xT(x, y)\, V^m(x^2, y^2)$$

where $T(x, y)$ is the counting series for symmetric clusters - a counting series that we shall derive shortly.

Now the cycle index of the dihedral group D_k for k odd can be written

$$(5.2) \qquad Z(D_k) = \tfrac{1}{2}Z(C_k) + \tfrac{1}{2}s_1 s_2^m$$

In the first term on the right-hand side we make the usual substitution of $V(x^i, y^i)$ for s_i, as for Pólya's Theorem; the second term must be replaced by $xT(x, y)\, V^m(x^2, y^2)$. Thus the counting series which enumerates the cell-rooted clusters for a given odd value of k is

$$(5.3) \qquad f_k(x, y) = \tfrac{1}{2}F_k(x, y) + \tfrac{x}{2}T(x, y)\, V^m(x^2, y^2)$$

If k is even, say $k = 2m + 2$, then the elements of D_k which represent reflections correspond to terms of the form s_2^{m+1} or $s_1^2 s_2^m$. To the first of these there corresponds the counting series $V^{m+1}(x^2, y^2)$. To the second there corresponds $T^2(x, y)\, V^m(x^2, y^2)$, since the two edges that map onto themselves must independently receive a symmetric out-rooted cluster, while the other out-rooted clusters must be allocated in symmetrical pairs. Since, for $k = 2m + 2$, we have

$$Z(D_k) = \tfrac{1}{2}Z(C_k) + \tfrac{1}{4}s_2^{m+1} + \tfrac{1}{4}s_1^2 s_2^m$$

we derive

$$(5.4) \qquad \begin{aligned} f_k(x, y) &= \tfrac{1}{2}F_k(x, y) + \tfrac{x}{4}V^{m+1}(x^2, y^2) \\ &\quad + \tfrac{x}{4}T^2(x, y)\, V^m(x^2, y^2) \end{aligned}$$

To find the counting series, call it $f(x, y)$, for the total number of cell-rooted clusters we must sum equations (5.3) and (5.4) for all relevant values of m. We obtain

$$(5.5) \qquad \begin{aligned} f(x, y) &= \tfrac{1}{2}F(x, y) + \tfrac{x}{2}T(x, y)[V(x^2, y^2) + V^2(x^2, y^2) + \dots] \\ &\quad + \tfrac{x}{4}[V^2(x^2, y^2) + V^3(x^2, y^2) + \dots] \\ &\quad + \tfrac{2}{4}T^2(x, y)[V(x^2, y^2) + V^2(x^2, y^2) + \dots] \\ &= \tfrac{1}{2}F(x, y) + \tfrac{1}{4}[2T(x, y) + V(x^2, y^2) + T^2(x, y)]K(x, y) \end{aligned}$$

where $$R(x, y) = \frac{xV(x^2, y^2)}{1 - V(x^2, y^2)}$$

It now only remains to determine $T(x, y)$, the counting series for symmetric out-rooted clusters. Let the cell containing the root edge of such a cluster have k other edges. Then if k is even, equal to $2m$ say, we can obtain these symmetric out-rooted clusters by disposing m out-rooted clusters in symmetric pairs on the m pairs of edges that change places under reflection. Thus the counting series is $xV^m(x^2, y^2)$. If k is odd, equal to $m + 1$ say, we must, in addition, place a symmetric out-rooted cluster on the edge opposite the root-edge (see Figure 7). The counting series is then $xT(x, y)\, V^m(x^2, y^2)$. Summing these counting series for $m \geq 1$, and remembering that the empty out-rooted cluster (which must be included) is also symmetric, we have

$$\begin{aligned} T(x, y) &= y + x[1 + T(x, y)]\{V(x^2, y^2) + V^2(x^2, y^2) + \ldots\} \\ &= y + \frac{xV(x^2, y^2)}{1 - V(x^2, y^2)}[1 + T(x, y)] \qquad (5.6) \end{aligned}$$

whence

$$T(x, y) = \frac{y + R(x, y)}{1 - R(x, y)} \quad .$$

6. IN-ROOTED CLUSTERS

With the same convention about mirror images we now count the clusters rooted at an internal edge. As in [5] we do this by enumerating those in-rooted clusters that are invariant under the four kinds of transformation of the plane that map the root edge onto itself, namely,

(a) The identity mapping,

(b) Reflection about the root edge,

(c) Rotation about 180°,

(d) Reflection about the perpendicular bisector of the root edge.

Those clusters invariant under (a) are enumerated by $U^2(x, y)$. Those invariant under (b) are enumerated by $U(x^2, y^2)$ since whichever (non-empty) out-rooted cluster goes on the one side of the root-edge must be duplicated on the other. The same is true of the clusters that are invariant under (c). Clusters invariant under (d) are obtained by placing together at the root edge two independently chosen non-empty symmetric out-rooted clusters, and are therefore enumerated by $W^2(x, y)$ where $W(x, y) = T(x, y) - y$.

Hence by Burnside's Lemma the in-rooted clusters are enumerated by

$$\frac{1}{4}\{U^2(x, y) + 2U(x^2, y^2) + W^2(x, y)\}$$

7. UNROOTED CLUSTERS

We now determine the counting series for the unrooted clusters. This will be done using the method given in Section 5, and there is no point in repeating the argument given there. The only necessary counting series that we have not already derived is that which enumerates the clusters having a symmetric edge. Such clusters can be constructed by taking an out-rooted cluster and joining it at the root-edge to a duplicate of itself. For a symmetric out-rooted cluster this can be done in only one way, whereas for an asymmetric cluster this can be done in two distinct ways. Thus we see, as in [5], that there is a one-to-one correspondence between the clusters that we are now enumerating and the out-rooted clusters, which also occur singly when symmetrical and in pairs when asymmetrical. It follows that the counting series for non-empty clusters with a symmetric edge is $U(x^2, y^2)$.

Applying Equation (4.1) we obtain the counting series $h(x, y)$ where

$$h(x, y) = f(x, y) - \frac{1}{4}[U^2(x, y) - 2U(x^2, y^2) + W^2(x, y)].$$

This, in theory, completes the enumeration of the clusters that we are interested in. These formulae however are not in a very suitable form for computation, and we shall now consider how they may be improved.

8. SOME COMPUTATIONAL CONSIDERATIONS

It is common, in papers on enumeration problems, to include some tables of coefficients for the counting series that have been obtained, and we do so here. It is not customary to describe in detail how these numerical results were obtained, but in this respect the present problem is unusual. To obtain numerical results from the counting series that have been derived requires the manipulation on a computer of double power series. This is quite feasible, and for the purposes of this paper routines were written in APL for the 360/75 computer at the University of Waterloo to handle such series. It was found that the multiplication of two double series of any size was a very time-consuming process, and this prompted the investigation of ways and means of modifying the calculations so that multiplication could be avoided wherever possible. These modifications are of some interest and will briefly be described.

The method of computing the coefficients in the function $V(x, y)$ has already been described; they are given by equation (2.2).

The function $F(x, y)$ is given by equation (3.1) but its coefficients are not at all easily computer thereby. Instead we write equation (3.1) as

(8.1)
$$F(x, y) = -V(x, y) - \frac{1}{2}[V^2(x, y) + V(x^2, y^2)] + \sum_{k=1}^{\infty} Z(C_k; V(x, y))$$

and note that

$$\sum_{k=1}^{\infty} Z(C_k; V(x, y)) = \sum_{k=1}^{\infty} \{\frac{1}{k} \sum_{\alpha/k} \phi(\alpha) V^{k/\alpha}(x^\alpha, y^\alpha)\}$$

$$= \sum_{m,\alpha} \{\frac{\phi(\alpha)}{m\alpha} V^m(x^\alpha, y^\alpha)\}$$

(8.2)
$$= \sum_{\alpha} \{\frac{\phi(\alpha)}{\alpha} \sum_{m} \frac{1}{m} V^m(x^\alpha, y^\alpha)\}$$

$$= \sum_{\alpha} \{-\frac{\phi(\alpha)}{\alpha} \log[1 - V(x^\alpha, y^\alpha)]\} .$$

Now by Lagrange's Theorem, the coefficient of x^r in $L(x, y) = -\log(1 - V)(x, y)$, for $r \geq 1$, is

$$\frac{1}{r}[(\frac{d}{dz})^{r-1}(\frac{z^2}{1-z})^r \frac{1}{1-z}]_{z=y}$$

which reduces to

$$\frac{1}{r!} \sum_{i=0} \binom{r+1}{i} \frac{(2r+i)!}{(r+i+1)!} y^{r+i+1}$$

Thus the coefficient of $x^r y^s$ in $L(x, y)$ is, for $r > 0$,

$$\frac{1}{r!}\binom{s-1}{r}\frac{(r+s-1)!}{s!} = \frac{1}{s}\binom{s-1}{r}\binom{r+s-1}{r}$$

which formula also gives the correct value $(1/s)$ when $r = 0$.

Using this result the summation (8.2) can be evaluated without the need for any multiplication of two double series. Even the $V^2(x, y)$ which occurs in equation (8.1) can be evaluated without multiplication by a further simple application of Lagrange's Equation. The coefficient of $x^r y^s$ in $V^2(x, y)$ turns out to be

$$\frac{2}{r}\binom{s-3}{r-1}\binom{r+s-1}{r-1}$$

for $r \geq 1$, while the term independent of x is y^2.

The coefficients in $H(x, y)$ give little trouble. We avoid the multiplication implied in $U^2(x, y)$ by using the fact that

$$U^2(x, y) = V^2(x, y) - 2y\,V(x, y) + y^2$$

and that the coefficients in $V^2(x, y)$ are already known. The multiplication of $V(x, y)$ by y is, of course, accomplished by merely increasing the indices of y in $V(x, y)$.

We must next deal with series $R(x, y)$. From equation (2.1), we have

$$(1 + x)\,V(x, y) = y + \frac{xV^2(x, y)}{1 - V(x, y)} + xV(x, y)$$

$$= y + \frac{xV(x, y)}{1 - V(x, y)} .$$

Replacing x by x^2 and y by y^2 we have

$$(1 + x^2)V(x^2, y^2) = y^2 + \frac{x^2 V(x^2, y^2)}{1 - V(x^2, y^2)}$$

whence

$$xR(x, y) = (1 + x^2)V(x^2, y^2) - y^2$$

from which $R(x, y)$ can be computed without any need for multiplication except of a trivial kind.

The computation of $T(x, y)$ is another matter. There probably exists some elegant method of avoiding multiplication in using equation (5.7), but I have been unable to find one. The need here is not so urgent, however, since $R(x, y)$ is a more sparse series, half the coefficients being zeros. A suitable method, using multiplication, is to define

$$Y_1 = R(x, y)$$

and $$Y_n = (1 + Y_{n-1})R(x, y).$$

Continue until n is the highest power of y that is being retained, then multiply by $1 + y$. This gives the function $W(x, y)$, needed later, and $T(x, y)$ is $W(x, y) + y$. Multiplication is also needed to find $W^2(x, y)$ but $T^2(x, y)$ can then be obtained without it.

One further multiplication is needed to find $f(x, y)$, but $h(x, y)$ can then be found without multiplication from data already available. Tables 1-5 were computed in just this sort of way. Note that the extent of computation was not limited by the size of the numbers but by the work space size. It was occasionally necessary to have two double series in the work space at one and the same time (in addition to the necessary programmes) and the tables represent the most that could be accommodated under these conditions.

9. TOTAL NUMBERS OF DISSECTIONS

It is of some interest to look at the total numbers of ways of dissecting a polygon into other polygons by chords. These numbers will, of course, be the column sums in Tables 1-5, so that this problem has, in a sense, already been solved. But considered on its own merits, this problem is one which can be solved by manipulating counting series in a single variable, and one might reasonably expect that with the same computing facilities one could extend the results much further, since single series take up so much less computer space than double series. This is indeed the case and I briefly outline the method.

The total numbers of out-rooted clusters with s outer edges are enumerated by the series $V(1, y)$. These numbers are the same as the numbers of dissections of a fixed polygon, and their enumeration has been studied by Motzkin [7] who gives what is probably the easiest way of calculating them. If we write

$$v(y) = V(1, y) = \sum_{s=1}^{\infty} v_s y^s$$

then his result is equivalent to

$$v_{n+1} = v_2 v_n + 2(v_3 v_{n-1} + \ldots + v_n v_2)$$

from which the v_s are readily computed.

Since multiplication of single series is no great problem we need not break our necks trying to avoid it, and equations (3.1), (4.2), (5.5), with $x = 1$ can be used in a straightforward manner. Note that we get an unexpected bonus in the computation of $T(1, y)$. It will be left as an exercise for the reader to show that

$$\frac{R(1, y)}{1 - R(1, y)}$$

reduces to $y^{-2}v(y^2) - 1$, thus enabling the computation of $W(1, y)$, and hence of $T(1, y)$ to be greatly simplified.

The numbers in Table 6 represent these calculations taken as far as the integer size limit (16 digits) of APL would allow. Note that the value of h for $s = 8$ does not agree with the Figure of 73 given in Motzkin [7], and hence in

Sloane ([10], sequence number 339). However, Motzkin's result appears to have been obtained by an exhaustive enumeration - a method that is notoriously prone to error. By way of confirmation that the figure of 75 is indeed the correct one, I display 75 dissections of the octagon in figure 8. They appear to be all distinct.

References

1. Brown, W.G., Historical Note on a Recurrent Combinatorial Problem, Amer. Math. Monthly 72(1965), 973-977.
2. Guy, R.K., Dissecting a Polygon into Triangles, Bull. Malayan Math. Soc. 5(1968), 57-60.
3. Harary, F., Graph Theory, Addison-Wesley, Reading, Massachusetts 1969.
4. Harary, F. and Palmer, E.M., Graphical Enumeration, Academic Press, New York 1973.
5. Harary, F., Palmer, E.M. and Read, R.C., On the Cell-Growth Problem for Arbitrary Polygons, Discrete Math., To Appear.
6. Klarner, D., Cell Growth Problems, Canad. J. Math. 19(1967), 851-863.
7. Motzkin, T., Relations Between Hyper-Surface Cross-Ratios and a Combinatorial Formula for Partitions of a Polygon, for Permanent Preponderance and for Non-Associative Products, Bull. Amer. Math. Soc. 54(1948), 352-360.
8. Pólya, G., Kombinatorische Anzahlbestimmungen fur Graphen, Graphen und chemische Verbindungen. Acta Math. 68(1937), 142-254.

9. Read, R.C., Contributions to the Cell Growth Problem, Canad. J. Math. 14(1962), 1-20.

10. Sloane, N.J.A., Handbook of Integer Sequences, Academic Press, New York 1973.

N1607 ✓ N1982 N2115

Table 1. Clusters roo[illegible] at an external edge.

x \ y	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		2	5	9	14	20	27	35	44	54	65	77	90	104	119
3			5	21	56	120	225	385	616	936	1365	1925	2640	3536	4641
4				14	84	300	825	1925	4004	7644	13650	23100	37400	58344	88179
5					42	330	1485	5005	14014	34398	76440	157080	302940	554268	969969
6						132	1287	7007	28028	91728	259896	659736	1534896	3325608	6789783
7							429	5005	32032	148512	556920	1790712	5116320	13302432	32008977
8								1430	19448	143208	755820	3197700	11511720	36581688	105172353
9									4862	75582	629850	3730650	17587350	70114902	245402157
10										16796	293930	2735810	17978180	93486536	409003595
11											58786	1144066	11767536	84987760	483367885
12												208012	4457400	50220040	395482815
13													742900	17383860	212952285
14														2674440	67863915
15															9694845

dull →

(f)

N1141.5 3441 N791.5 3442 N1257.5 3443

Table 2. Cell-rooted clusters without reflection.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7	8	9	10	11	12	13
3			3	11	24	46	75	117	168	236	315	415	528	666
4				10	48	150	368	770	1456	2550	4200	6600	9976	14586
5					30	208	825	2507	6370	14344	29400	56120	100980	173244
6						99	858	4205	15288	45864	119952	282744	613960	1247103
7							335	3507	20384	86656	299880	895416	2387628	5819964
8								1144	14144	95472	465120	1827258	6139584	18290844
9									3978	56698	436050	2398350	10552410	39439946
10										14000	226100	1954150	11985472	58429085
11											49742	898942	8629528	58429382
12												178296	3565920	37665030
13													643855	14124496
14														2340135

2nd differences too irregular

Table 3. Un-rooted clusters without reflection.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	1	2	2	3	3	4	4	5	5	6	6	7
3			1	4	8	16	25	40	56	80	105	140	176	224
4				4	12	40	93	197	364	646	1050	1660	2496	3660
5					6	43	165	505	1274	2878	5880	11240	20196	34677
6						19	143	712	2548	7672	19992	47184	102328	207963
7							49	504	2912	12400	42840	127968	341100	831552
8								150	1768	11976	58140	228558	767448	2286768
9									442	6310	48450	266550	1172490	4382495
10										1424	22610	195580	1198564	5843651
11											4522	81752	784504	5312032
12												14924	297160	3139396
13													49536	1086601
14														167367

(f)

3448 N1212.5

3446 NG31.5   3447 N701.5

N690.5 = 3452

Table 4. Cell-rooted clusters with reflection.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	2	3	4	5	6	7	8	9	10	11	12	13
3			2	7	15	28	45	69	98	136	180	235	297	373
4				6	26	81	194	404	756	1319	2160	3385	5098	7439
5					16	108	422	1276	3225	7246	14815	28245	50755	87012
6						52	434	2124	7682	23029	60132	141683	307446	6 524355
7							170	1765	10223	43418	150117	448106	1194493	2911282
8								579	7086	47813	232700	914072	3070538	9147213
9									1996	28384	218130	1199524	5276948	19721898
10										7021	113092	977354	5993252	29216476
11											24892	449582	4315130	29216032
12												89214	1783092	18833538
13													321994	7062611
14														1170282

(f)

3449 = N1076.5

3450 = N658.5

(N942 have)

3453 = N1004.5

Table 5.   Un-rooted clusters with reflection.

	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2		1	1	2	2	3	3	4	4	5	5	6	6	7
3			1	3	6	11	17	26	36	50	65	85	106	133
4				3	7	24	51	109	194	345	550	870	1293	1896
5					4	24	89	265	660	1477	3000	5710	10228	17521
6						12	74	371	1291	3891	10061	23747	51349	104349
7							27	259	1478	6249	21524	64183	170904	416385
8								82	891	6044	29133	114541	484035	1144304
9									228	3176	24302	133464	586696	2192206
10										733	11326	98000	599516	2923018
11											2282	40942	392528	2656742
12												7528	148646	1570490
13													24834	543515
14														83898

wrong

~~check~~ 16 (have) ↓ ✓ N1163 1003

3454 = N659.5 ✓ New ↓ ✓

3455 = N342.8 New ↓ ✓

3456 = N593.5 New ↓ ✓

N339 extended ↓ 1004 ✓

Table 6. Totals for the five series.

(column sums for tables 1 to 5)

N	U	F	G	f	g
3	1	1	1	1	1
4	3	2	2	2	2
5	11	6	3	5	3
6	45	25	11	17	9
7	197	107	29	62	20
8	903	509	122	275	75
9	4279	2468	479	1272	262
10	20793	12258	2113	6225	1117
11	103049	61797	9369	31075	4783
12	518859	315830	43392	158376	21971
13	2646723	1630770	203595	816229	102249
14	13648869	8498303	975563	4251412	489077
15	71039373	44629855	4736005	23319056	2370142
16	372603519	235974495	23296394	117998524	11654465
17	1968801519	1255105304	115811355	627573216	57916324
18	10463578353	6710883952	581324861	3355499036	290693391
19	55909013009	36050676617	2942579633	18025442261	1471341341
20	300159426963	194478962422	15008044522	97239773408	7504177738
21	1618362158587	1055120661726	77064865555	526560862829	38552692207
22	8759309660445	5722375202661	398150807179	2861189112867	199076194985
23	47574827600981	31191334491891	2068470765261	15595669996482	1034236705992
24	259215937709463	170504130213134	10800665952375	85252072993968	5400337050086
25	1416461675464871	934495666529377	56658467018644	467247847612316	28329240333758

I have (9) instead of (0) but he is wrong! ✓

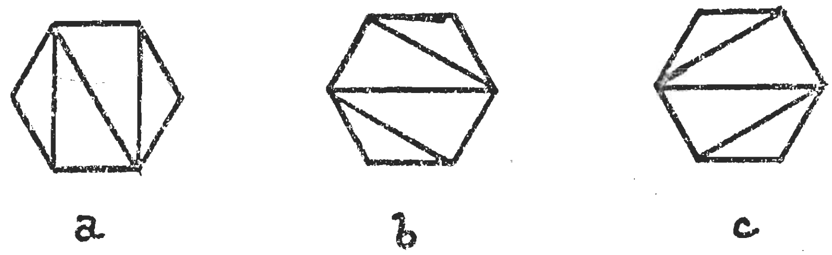


Fig. 1.

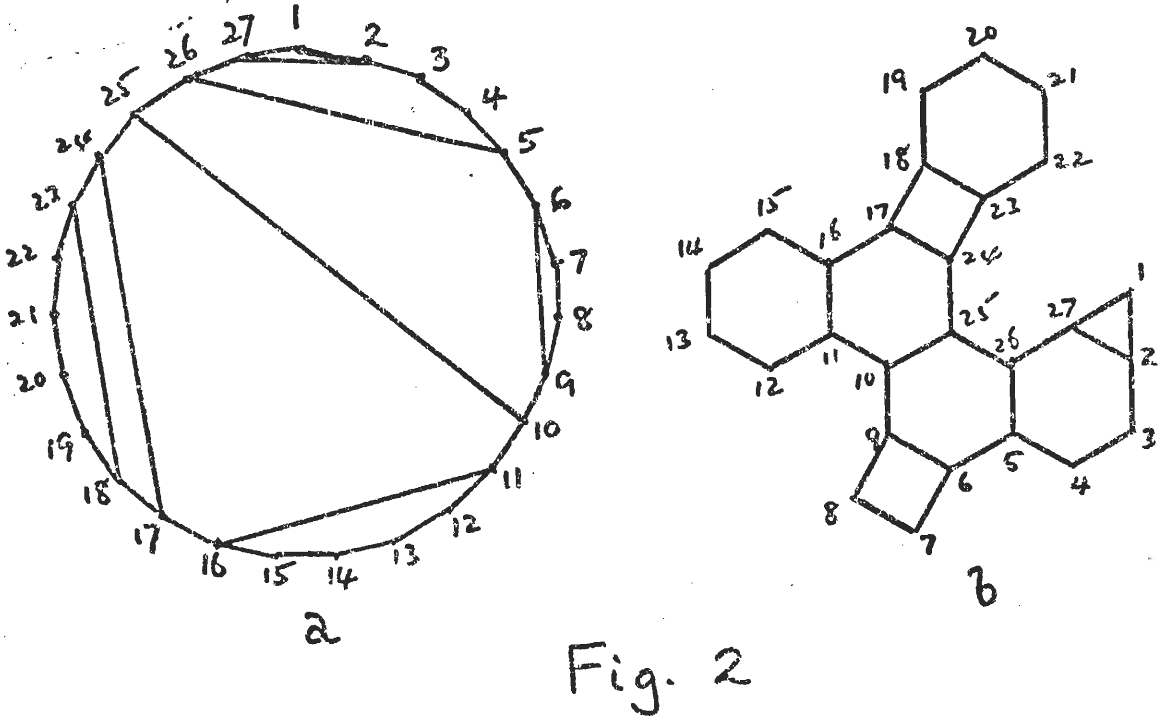
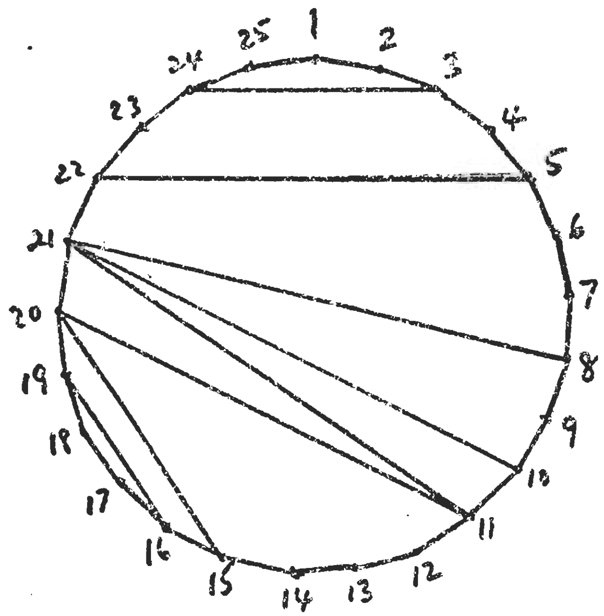
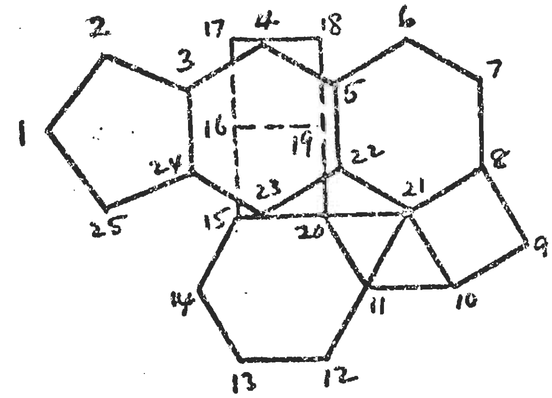


Fig. 2


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25

a


1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25

b

Fig. 3.

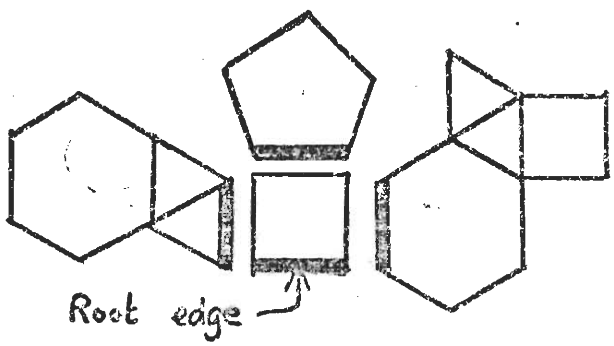

Root edge

Fig. 4.

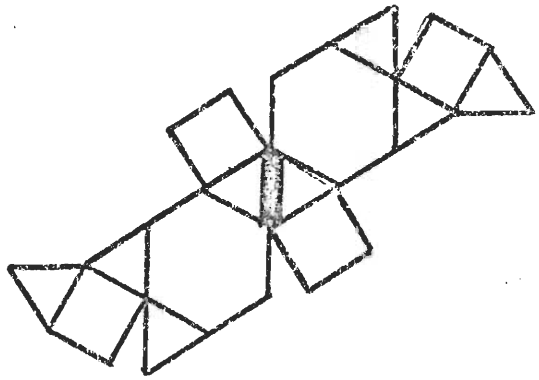


Fig. 5

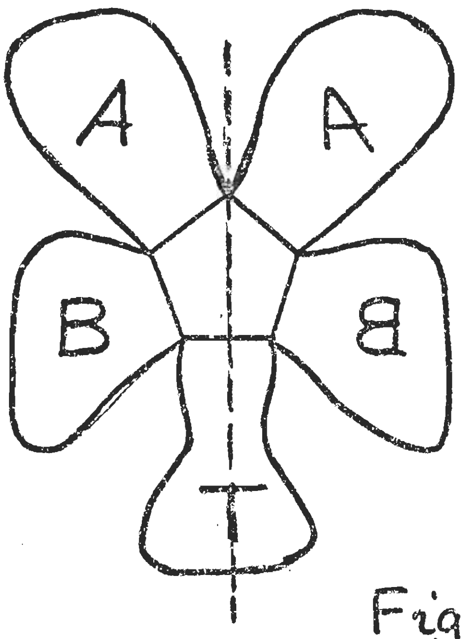


Fig. 6

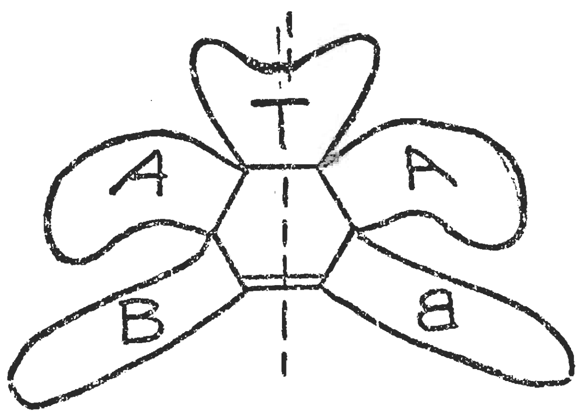


Fig. 7

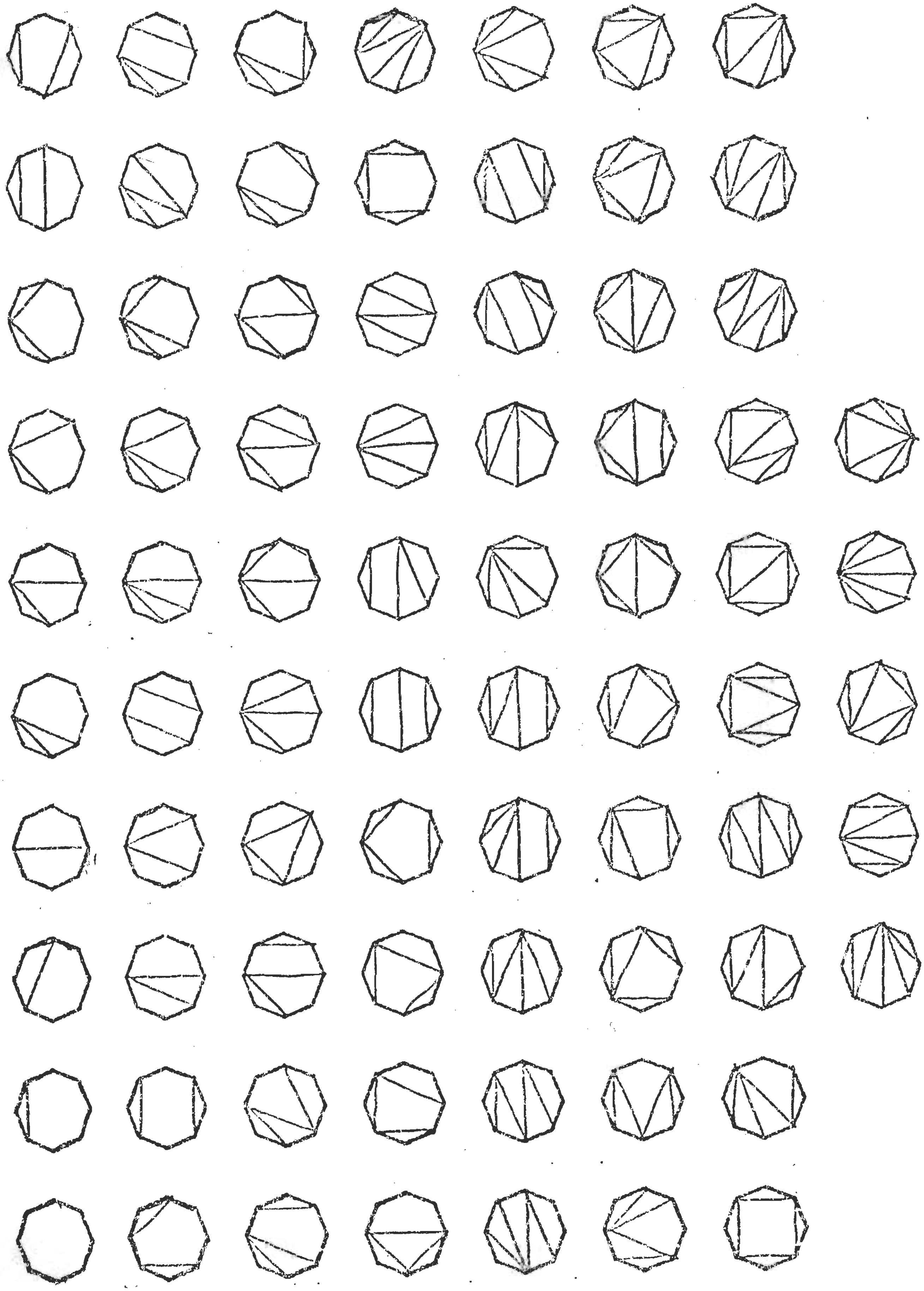


Fig. 8