QUASI-ORDERINGS AND TOPOLOGIES ON FINITE SETS

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1. Throughout this paper S is the finite set $\{s_1, s_2, \cdots, s_n\}$, and if $\mathfrak{I}$ is a topology on S then A^- denotes the $\mathfrak{I}$-closure of the subset A of S. It is our purpose to investigate topologies on S and to answer a few combinatorial questions related to these topologies. The connection between T_0-topologies and partial orderings on finite sets (Theorem 7) already appears in several standard references [**1**, p. 28] and [**2**, p. 14]. That there is a one-to-one correspondence between the topologies on S and the quasi-orderings on S follows from the next paragraph.

For each set $A \subset S$, $A^- = \bigcup\{s_i\}^-$ over all $s_i \in A$, hence to identify a topology on S it suffices to display the closures of all singletons. For this purpose we choose the relation matrix

$$\begin{aligned} t_{ij} &= 1, \qquad \text{if } s_j \in \{s_i\}^-, \\ &= 0, \qquad \text{otherwise.} \end{aligned}$$

The Kuratowski closure axioms [**3**, p. 43] imply that $[t_{ij}]$ is reflexive ($A \subset A^-$) and transitive ($A^{--} = A^-$).

Let $T = [t_{ij}]$ be the matrix corresponding to a topology $\mathfrak{I}$ and let F_i and B_j be the subsets of S having characteristic functions $\{(s_1, t_{i1}), (s_2, t_{i2}), \cdots, (s_n, t_{in})\}$ and $\{(s_1, t_{1j}), (s_2, t_{2j}), \cdots, (s_n, t_{nj})\}$. Note that $s_j \in F_i$ iff $s_i \in B_j$. For each i, $F_i = \{s_i\}^-$ is the minimal closed set containing s_i.

THEOREM 1. *For each j, B_j is the minimal open set in $\mathfrak{I}$ containing s_j.*

PROOF. We show first that $S - B_j$ is closed. If $s_i \in S - B_j$ and if $s_k \in F_i$, then $t_{ij} = 0$ and $t_{ik} = 1$. Transitivity forbids $t_{kj} = 1$, hence $F_i \subset S - B_j$. To show that B_j is minimal, let U be any open set containing s_j. If $s_k \in S - U$ then $F_k \subset S - U$ and $s_j \notin F_k$. Hence $s_k \notin B_j$ and $S - U \subset S - B_j$.

COROLLARY. *The weight* [**1**, p. 7] *of any topology on S does not exceed $n+1$.*

Adjoining $\varnothing$ to the family of distinct minimal open sets B_j produces a basis for the topology which we call the *minimal basis.*

THEOREM 2. *If $i \neq j$, $t_{ij} = 1$ iff $B_i \subset B_j$.*

Received by the editors July 29, 1964.

The following example shows that the assumption that M is closed cannot be deleted. Consider the union of two disjoint closed discs in the plane together with the segment joining their centers. From each disc delete all the points of a diameter not parallel to the line of centers excepting the end points and the center itself. The set described is M and S_1 is the intersection of the line of centers with M. Then M is not closed, satisfies Valentine's condition and Condition A, but it is not the union of two star-shaped sets.

Reference

1. F. A. Valentine, *Convex sets*, McGraw-Hill, New York, 1964.

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PROOF. If $B_i \subset B_j$ then $s_i \in B_j$ and $t_{ij}=1$. On the other hand suppose $t_{ij}=1$. For each k if $t_{ki}=1$ then $t_{kj}=1$ and $B_i \subset B_j$.

COROLLARY. *If* $i \neq j$, $t_{ij}=t_{ji}=1$ *iff* $B_i=B_j$.

THEOREM 3. *If* $i \neq j$, $t_{ij}=1$ *iff* $F_j \subset F_i$.

The proof is like that of Theorem 2.

COROLLARY. *If* $i \neq j$, $t_{ij}=t_{ji}=1$ *iff* $F_j=F_i$.

THEOREM 4. *A reflexive,* $n \times n$, *zero-one matrix* T *corresponds to a topology on* S *iff* $T^2=T$.

PROOF. Matrix multiplication here involves Boolean arithmetic. The theorem follows from the fact that a reflexive relation ρ is transitive iff $\rho\rho=\rho$ [2, p. 209].

2. Let $\mathfrak{J}$ and $\mathfrak{J}^*$ be topologies on S with corresponding matrices $T=[t_{ij}]$ and $T^*=[t_{ij}^*]$. Then $\mathfrak{J}=\mathfrak{J}^*$ iff $t_{ij}=t_{ij}^*$ for each i and j. On the other hand $\mathfrak{J}$ and $\mathfrak{J}^*$ are topologically equivalent iff there exists a permutation $\pi(S)=S$ under which the minimal bases of $\mathfrak{J}$ and $\mathfrak{J}^*$ correspond. The matrices T and T^* are called *isomorphic* (*nonisomorphic*) if $\mathfrak{J}$ and $\mathfrak{J}^*$ are equivalent (nonequivalent) [5]. It follows that T and T^* are isomorphic iff there exists an $n \times n$ permutation matrix P such that $T^*=P'TP$, where P' is the transpose of P.

If $\mathfrak{J}$ is a topology on S then the family $\mathfrak{J}'$ of complements of members of $\mathfrak{J}$ also is a topology on S. We shall call $\mathfrak{J}'$ the *transpose* (or the *dual*) topology with respect to $\mathfrak{J}$.

THEOREM 5. *If* T *is the matrix corresponding to the topology* $\mathfrak{J}$ *then* T' (*the transpose of* T) *is the matrix corresponding to the topology* $\mathfrak{J}'$.

PROOF. We show first that $(T')^2=T'$. Let $T=[t_{ij}]$ and $T'=[t_{ji}]$. Then $(T')^2=[v_{ij}]$ where

$$v_{ij} = \sum_{k=1}^{n} t_{jk}t_{ki}.$$

But $T^2=T$, therefore $v_{ij}=t_{ji}$ and $(T')^2=T'$. By Theorem 4, T' corresponds to a topology on S, and the nonempty members of its minimal basis are the $\mathfrak{J}$-closures F_i. Hence the topology consists of the family of all unions $\bigcup F_i$; that is, of all $\mathfrak{J}$-closed sets.

THEOREM 6. *The topology* $\mathfrak{J}$ *is not connected iff for some* k, $0<k<n$, *both* T *and* T' *contain the same* $k \times (n-k)$ *zero submatrix.*

PROOF. A topology $\mathfrak{J}$ is not connected iff there exists a nonempty proper subset A of S such that $A \in \mathfrak{J}$ and $A \in \mathfrak{J}'$. This means that

$A = \bigcup B_i = \bigcup F_i$ over all i such that $s_i \in A$. But the complement, $S-A$, has the same property. Let k be the cardinal of A and the theorem follows.

In finite topological spaces the separation properties characterizing T_0-, T_1-, T_2-, etc., spaces are of limited help in the study of topological structure. The only interesting partition of topologies in this hierarchy occurs at the T_0 level. The theorem stated next formalizes the relation mentioned at the beginning of the paper.

THEOREM 7. *The topology $\mathfrak{J}$ on S is T_0 iff its matrix T is anti-symmetric (that is, T defines a partial ordering on S).*

COROLLARY. *The weight of a topology $\mathfrak{J}$ on S is $n+1$ iff $\mathfrak{J}$ is T_0.*

In general, the topologies $\mathfrak{J}$ and $\mathfrak{J}'$ are neither equal nor equivalent. In the event, however, that $\mathfrak{J}' = \mathfrak{J}$ the matrix T is symmetric and we call its corresponding topology *symmetric*. The symmetric topologies correspond to the equivalence relations on S. Theorems 6 and 7 imply that $\mathfrak{J}'$ is T_0 or connected iff $\mathfrak{J}$ is.

In the matrix T corresponding to the topology $\mathfrak{J}$, let $C(\mathfrak{F}) = (c_1, c_2, \cdots, c_n)$ be the *column sum vector* and let $R(\mathfrak{J}) = (r_1, r_2, \cdots, r_n)$ be the *row sum vector* [4, p. 61]. The class of vectors each of which is some permutation of the coordinates of C (or of R) is a topological invariant. Also, the sum, τ, of the entries in T is a topological invariant. These, unfortunately, are not topological characters; for the two matrices below describe nonequivalent topologies.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

In each matrix $C = (4, 3, 2, 1, 1, 1)$ and $R = (1, 1, 2, 2, 2, 4)$.

We shall call the matrix $T = [t_{ij}]$ *triangular* if $t_{ij} = 0$ for all $i < j$.

THEOREM 8. *The matrix T corresponding to a topology $\mathfrak{J}$ is isomorphic to a triangular matrix iff $\mathfrak{J}$ is T_0.*

PROOF. If T is isomorphic to a triangular matrix then $t_{ij} \cdot t_{ji} = 0$ for all $i \neq j$. Now assume that $\mathfrak{J}$ is T_0. There exists a permutation matrix P such that $T^* = P'TP$ has a monotone (nonincreasing) column sum vector. If T^* is not triangular, then for some $i < j$ $t^*_{ij} = 1$. By Theorem 2

$B_i^* \subset B_j^*$, and by the Corollary to Theorem 7 $B_i^* \neq B_j^*$, hence $c_i < c_j$ which is a contradiction.

THEOREM 9. *Let $\mathfrak{J}$ be a topology on S. There exists a topology $\mathfrak{J}^*$ equivalent to $\mathfrak{J}$ such that $C(\mathfrak{J}^*)$ and $R(\mathfrak{J}^*)$ each are monotone (nonincreasing) iff $\mathfrak{J}$ is symmetric.*

PROOF. Sufficiency is evident since $c_i = r_i$. If $\mathfrak{J}$ is not symmetric then for some $i \neq j$ $t_{ij} = 1$ while $t_{ji} = 0$. By Theorems 2 and 3 $c_i \leqq c_j$ and $r_i \geqq r_j$, but since $t_{ji} = 0$ strict inequality holds in each case.

THEOREM 10. *Among the symmetric topologies only the discrete is T_0 and only the indiscrete is connected.*

PROOF. If $t_{ij} = t_{ji} = 1$ and if $\mathfrak{J}$ is T_0 then by Theorem 7 $i = j$. To prove the latter statement, we may assume by Theorem 9 that the column sum and row sum vectors are monotone. The least coordinate in the column sum vector is c_n, and we assume that $c_n = k < n$. If $t_{in} = 1$ then $B_i = B_n$ and T contains k identical columns each with $n - k$ zero entries. By Theorem 6 T is not connected.

The following corollary refers to different, although possibly homeomorphic, topologies.

COROLLARY. *If $n > 1$ then the number of different T_0 topologies is odd, the number of different connected topologies is odd, and the number of connected T_0 topologies is even* [6].

3. If n is 3 the *trivial* topologies (discrete and indiscrete) correspond, respectively, to the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

It is evident that the extreme values of τ, in general, are n and n^2; but it is not the case that all intermediate values are possible.

THEOREM 11. *If $\mathfrak{J}$ is nontrivial then $n < \tau \leqq n^2 - n + 1$.*

PROOF. Only the right-hand part of the inequality is in question. Suppose for some $i \neq j$ $t_{ij} = 0$. Then for each k such that $k \neq i$ and $k \neq j$ either $t_{ik} = 0$ or $t_{kj} = 0$.

A little more than 10 years ago R. L. Davis published a formula (among others) for the number of nonisomorphic reflexive relations on S [5]. The author is not aware of a formula enumerating the subfamily of transitive relations. Such a formula, in addition to being of value in logic and combinatorics, would answer the question: how many nonequivalent topologies are there on a finite set?

For small n the preceding theory can be used to good advantage in the enumeration problem. Though the method lacks subtlety, it is not impossibly tedious for $n \leqq 5$, even without the assistance of a digital computer. In Table 1, "t" denotes the number of nonequivalent topologies on S, "tc" denotes the number that are connected, "to" denotes the number that are T_0, "tco" denotes the number that are both connected and T_0, and "ts" denotes the number that are symmetric. Figure 1 displays matrices corresponding to all nonequivalent topologies for $n=3$ and $n=4$.

(N1133)

TABLE 1

n	t	tc	to	tco	ts
2	3	2	2	1	2
3	9	6	5	3	3
4	33	21	16	10	5
5	139	94	63	44	7

FIGURE 1

100	100	100	110	100	100	100	110	111
010	110	110	110	010	110	111	110	111
001	001	101	001	111	111	111	111	111

1000	1000	1000	1000	1100	1000	1100
0100	1100	1100	0100	1100	0100	1100
0010	0010	1010	1110	0010	1010	0010
0001	0001	0001	0001	0001	0101	0011
1000	1000	1000	1000	1100	1000	1100
1100	0100	1100	0100	1100	1110	1100
1110	1110	1010	0010	0011	1110	1110
0001	1001	1001	1111	0011	0001	0001
1000	1000	1000	1000	1000	1000	1000
0100	1100	1100	1100	1100	0100	1110
1110	1110	0010	1010	1110	1110	1110
1101	1001	1111	1111	1101	1111	1001
1100	1110	1000	1100	1000	1000	1000
1100	1110	1100	1100	0100	1110	1100
0010	1110	1110	1110	1111	1110	1111
1111	0001	1111	1101	1111	1111	1111
	1100	1100	1000	1110	1111	
	1100	1100	1111	1110	1111	
	1110	1111	1111	1110	1111	
	1111	1111	1111	1111	1111	

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A CHARACTERIZATION OF THE DIFFERENTIABLE SUBMANIFOLDS OF R^n

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1. **Introduction.** It is known [3, p. 49] that any class C^1 differentiable submanifold of R^n is a (class C^1) differentiable neighborhood retract. In this paper we prove that the subsets of R^n which are class C^1 neighborhood retracts (of connected open sets) are precisely the class C^1 differentiable submanifolds of R^n. In particular, Theorem 1 shows that the range of a class C^1 retraction is a class C^1 submanifold.

2. If S is a linear transformation on R^n, then rank (S) is the dimension of the range space of S.

LEMMA 1. *If C is a connected set of idempotent linear transformations (i.e. projections) on R^n and if $S, T \in C$, then rank $(S) =$ rank (T).*

PROOF. Let Mn denote the set of all real $n \times n$ matrices and let Tr: $Mn \to R$ be the trace operator, i.e., $\mathrm{Tr}(A) = \sum_{i=1}^{n} a_{ii}$ where $A = (a_{ij})$. It is easily verified that Tr is continuous on Mn and an invariant of similarity class [2, p. 96]. Suppose that $A \in Mn$ is an idempotent. Then A is similar to a matrix $B = (b_{ij})$ such that $b_{ii} = 1$ for $1 \leqq i \leqq$ rank (A) and $b_{ij} = 0$ otherwise. Hence, $\mathrm{Tr}(A) = \mathrm{Tr}(B) = \sum_{i=1}^{\mathrm{rank}\,(A)} b_{ii} =$ rank (A). Letting the trace of a linear operator be the trace of any matrix representation, it follows that the trace of any member of C is its rank and, therefore, that Tr is constant on C. Hence, $\mathrm{Tr}(S) = \mathrm{Tr}(T)$, i.e., rank $(S) =$ rank (T).

If S and T are projections having the same rank r, then there is an arc of projections of rank r joining S and T. It now follows from Lemma 1 that, in the space of linear transformations on R^n, there are precisely $n+1$ components of idempotent linear transformations, two idempotents being in the same component if and only if they have the same rank.

The proof of the next lemma may be found in [1, pp. 273–276].

LEMMA 2 (RANK THEOREM). *Let E be an n-dimensional space, F an m-dimensional space, A an open neighborhood of a point $a \in E$, f a continuously differentiable mapping of A into F, such that in A the rank of $f'(x)$ is a constant number p. Then there exists*

1. *an open neighborhood $U \subset A$ of a, and a homeomorphism μ of U onto the open unit n-cube $I^n = \{(x_1, x_2, \cdots, x_n) \in R^n : |x_i| < 1$ for*

Received by the editors December 6, 1965.