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DIVISIBILITY PROPERTIES OF SOME FIBONACCI-TYPE SEQUENCES

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A generalized Fibonacci-type sequence is defined from a fourth order homogeneous linear recurrence relation, and various divisibility properties are developed. In particular, the notion of a proper divisor is modified to develop formulas for proper divisors in terms of the general terms of the recurrence sequences and various arithmetic functions.

1. THE SEQUENCE $\{A_n(x)\}$

The main sequence of interest is $\{A_n(x)\}$ defined by

$$(1.1)\quad \begin{cases} A_0(x) = 0,\ A_1(x) = 1,\ A_2(x) = 1,\ A_3(x) = x + 1 \\ \text{and} \\ A_n(x) = x\,A_{n-2}(x) - A_{n-4}(x) \qquad \text{for } n \geq 4. \end{cases}$$

For unrestricted n, it follows from (1.1) that

$$(1.2)\quad A_{-n}(x) = -A_n(x)\ .$$

The auxiliary equation for $\{A_n(x)\}$ is $r^4 - x\,r^2 + 1 = 0$ which has roots $\pm s, \pm t$ given by

$$(1.3)\quad \begin{cases} s^2 = \tfrac{1}{2}(x + \sqrt{x^2 - 4}) \\ t^2 = \tfrac{1}{2}(x - \sqrt{x^2 - 4}) \end{cases} \qquad \text{so that } s^2 + t^2 = x,\ s^2t^2 = 1$$

whence

$$(1.4)\quad \begin{cases} s = \tfrac{1}{2}(\sqrt{x - 2} + \sqrt{x + 2}) \\ t = \tfrac{1}{2}(\sqrt{x - 2} - \sqrt{x + 2}) \end{cases} \qquad \text{so that } s + t = \sqrt{x - 2},\ st = -1.$$

From the initial conditions in (1.1) we derive

$$(1.5)\quad A_{2n}(x) = \frac{s^{2n} - t^{2n}}{s^2 - t^2}\ .$$

Mathematical induction and the recurrence relation (1.1) lead to

$$(1.6)\quad A_{2n+2}(x) = A_{2n+1}(x) - A_{2n}(x)$$

whence, by (1.5),

$$A_{2n+1}(x) = \frac{s^{2n}(s^2+1) - t^{2n}(t^2+1)}{s^2 - t^2} \quad (1.7)$$

The generating function for $\{A_n(x)\}$ is given by

$$\sum_{n=1}^{\infty} A_n(x)\, t^n = \frac{t + t^2 + t^3}{1 - xt^2 + t^4} \quad (1.8)$$

Put $x = 3$ and let α, β be the roots of $r^2 - r - 1 = 0$. Then (1.3) yields

$$s^2 = \alpha^2,\ t^2 = \beta^2,\ s^2 - t^2 = \alpha - \beta = \sqrt{5},\ s^2 + 1 = \sqrt{5}\alpha,\ t^2 + 1 = -\sqrt{5}\beta\ . \quad (1.9)$$

From (1.5), (1.7) and (1.9), we have

$$\begin{cases} A_{2n}(3) = \dfrac{\alpha^{2n} - \beta^{2n}}{\alpha - \beta} = F_{2n} \\ A_{2n+1}(3) = \alpha^{2n+1} + \beta^{2n+1} = L_{2n+1} \end{cases} \quad (1.10)$$

in which F_{2n}, L_{2n+1} are the 2n-th Fibonacci number and (2n+1)-th Lucas number respectively. Equation (1.6) is by (1.2) an instance of the well-known result: $F_{k+2} + F_k = L_{k+1}$ which is true for all k. The generating function for $\{A_n(3)\}$ follows immediately from (1.8).

Table 1 shows the first 18 terms of $\{A_n(3)\}$, alternately the Lucas and Fibonacci numbers $L_1, F_2, L_3, F_4, L_5, F_6, L_7, F_8, L_9, \ldots$.

Other examples of $\{A_n(x)\}$ include

$$\left.\begin{aligned} |A_{2n}(-2)| &= n \in \mathbb{N} \\ |A_{2n+1}(-2)| &= 1 \end{aligned}\right\}$$, obtained directly from (1.1).

Further information about the sequence $\{A_n(x)\}$ may be found in Shannon, Horadam, and Loh [6] where a different notation is used.

2. PROPER DIVISORS

Vorob'ev [7] in the concluding chapter of his book on Fibonacci numbers refers to the notion of a *proper divisor*. We extend this idea a little as follows:

Definition. For any sequence $\{u_n\}$, $n \geq 1$, where $u_n \in \mathbb{Z}$ or $u_n(x) \in \mathbb{Z}(x)$, the *proper divisor* w_n is the quantity implicitly defined, for $n \geq 1$, by $w_1 = u_1$ and $w_n = \max\{d: d|u_n$ and g.c.d.$(d,w_m) = 1$ for every $m < n\}$.

(Strictly speaking, the second equation is all that is necessary here, since for $n = 1$ its g.c.d. condition is vacuous and so $w_1 = u_1$ follows.)

Proper divisors w_n for the sequence of integers $\{A_n(3)\}$ are the integers listed in Table 1. (Recall (1.10).)

A5013

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$A_n(3)$	1	1	4	3	11	8	29	21	76	55	199	144	521	377	1364	987	3571	2584
w_n	1	1	4	3	11	-	29	7	19	5	199	-	521	13	31	47	3571	17

Table 1. Proper divisors for $\{A_n(3)\}$

Proper divisors for the sequence of polynomials $\{A_n(x)\}$ are shown in Table 2. These proper divisors $w_n(x)$ are monic polynomials (over the integers).

n	$A_n(x)$
0	0
1	1
2	1

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n	$A_n(x)$	$w_n(x)$
3	$x+1$	$x+1$
4	x	x
5	x^2+x-1	x^2+x-1
6	x^2-1	$x-1$
7	x^3+x^2-2x-1	x^3+x^2-2x-1
8	x^3-2x	x^2-2
9	$x^4+x^3-3x^2-2x+1$	x^3-3x+1
10	x^4-3x^2+1	x^2-x-1
11	$x^5+x^4-4x^3-3x^2+3x+1$	$x^5+x^4-4x^3-3x^2+3x+1$
12	x^5-4x^3+3x	x^2-3

Table 2. Proper divisors for $\{A_n(x)\}$

From the definition of proper divisors we have that for $\{A_n(x)\}$:

$$A_p(x) = w_p(x)\, w_1(x)$$

$$A_{2p}(x) = w_{2p}(x)\, w_p(x)\, w_2(x)\, w_1(x)$$

$$A_{3p}(x) = w_{3p}(x)\, w_p(x)\, w_3(x)\, w_1(x)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

in which $w_2(x) = w_1(x) = 1$ for notational convenience.

Hence,

(2.1) $$A_n(x) = \prod_{d|n} w_d(x)$$

<u>Theorem 1</u>. $w_n(x) = \prod_{d|n} (A_d(x))^{\mu(n/d)}$ *where* μ *is the Möbius function.*

<u>Proof</u>. Taking logarithms in (2.1) we obtain

$$\ln A_n(x) = \sum_{d|n} \ln w_d(x)$$

which becomes, with the Möbius inversion formula,

$$\ln w_n(x) = \sum_{d|n} \mu(n/d) \ln A_d(x)$$

i.e. $$w_n(x) = \prod_{d|n} (A_d(x))^{\mu(n/d)}$$ as required.

As an example,

$$\begin{aligned} w_{12}(x) &= (A_3(x))^{\mu(4)} (A_4(x))^{\mu(3)} (A_6(x))^{\mu(2)} (A_{12}(x))^{\mu(1)} \\ &= (x+1)^0\, x^{-1}(x^2-1)^{-1}(x^5-4x^3+3x)^1 \\ &= x^2 - 3. \end{aligned}$$

Another approach to Theorem 1 is through cyclotomic polynomials.

3. PROPERTIES OF PROPER DIVISORS

Let $n = \prod_{i=1}^{m} p_i^{\alpha_i}$ where p_i are distinct primes, and let $\nu(n) = \sum_{i=1}^{m} \alpha_i$ be the number of prime factors of n, counted with multiplicity. Further, let

$$\varepsilon(n) = (-1)^{\nu(n)}.$$

Then

Theorem 2. $w_n(x) = \left(\prod_{d\in S_0} A_d(x) \Big/ \prod_{d\in S_1} A_d(x)\right)^{\varepsilon(n)}$, *where the sets* S_i *comprise all positive divisors* d *of* n *such that* n/d *is squarefree, and* $\nu(d) \equiv i \bmod(2)$ *for* $i = 0, 1$.

Proof. Write $d = \prod_{i=1}^{m} p_i^{\beta_i}$ $(\beta_i \le \alpha_i)$.

Then $$n/d = \prod_{i=1}^{m} p_i^{\alpha_i-\beta_i}$$

and $$\mu(n/d) = (-1)^{\Sigma(\alpha_i-\beta_i)} \qquad (0 \le \alpha_i - \beta_i \le 1),$$

since $$\mu(n/d) = 0 \qquad \text{if } \alpha_i - \beta_i \ge 2.$$

From Theorem 1 we have, using the specified notation,

$$\begin{aligned} w_n(x) &= \prod_{d|n} (A_d(x))^{(-1)^{\Sigma(\alpha_i-\beta_i)}} \\ &= \prod_{d|n} (A_d(x))^{\varepsilon(n)\cdot(-1)^{\Sigma\beta_i}} \\ &= \prod_{d|n} (A_d(x))^{\varepsilon(n)\varepsilon(d)} \\ &= \left(\prod_{d\in S_0} A_d(x) \Big/ \prod_{d\in S_1} A_d(x)\right)^{\varepsilon(n)} \end{aligned}$$

since $\varepsilon(d)$ will be positive or negative according as $d \in S_0$ or $d \in S_1$, respectively.

For example, $$w_{60}(x) = \frac{A_4(x)\ A_6(x)\ A_{10}(x)\ A_{60}(x)}{A_2(x)\ A_{12}(x)\ A_{20}(x)\ A_{30}(x)}$$

since $\nu(60) = 4$ (i.e. $\varepsilon(60) = 1$)

with $S_0 = \{4, 6, 10, 60\}$,

and $S_1 = \{2, 12, 20, 30\}$.

Note that $\mu(60/d) = 0$ for $d = 3, 5, 15$.

Corollary 1. $w_{p^n}(x) = A_{p^n}(x)/A_{p^{n-1}}(x)$.

Corollary 2. $w^2_{2^k}(x) - w_{2^{k+1}}(x) = 2 \qquad (k \geq 2)$.

The proof of Corollary 2 uses Corollary 1 and (1.5).

Corollary 2 may be illustrated by choosing $k = 2$, whence, by Table 2,

$$w_4^2(x) - w_8(x) = x^2 - (x^2 - 2) = 2.$$

Other results, such as

Corollary 3. $w^2_{2^{k+1}}(x) - w_{2^k\cdot 6}(x) = 3$

can be similarly proved. Putting $k = 1$ in Corollary 3 and using Table 2, we see that

$$w_4^2(x) - w_{12}(x) = x^2 - (x^2 - 3) = 3.$$

4. GENERALIZED PELL NUMBERS

Consider the *forward shift operator* E:

(4.1) $$E\,A_n(x) = A_{n+1}(x).$$

Then (1.1), along with (1.3), can be written as

$$(E^2 - s^2)(E^2 - t^2)\,A_n(x) = 0$$

i.e. $$(E - s)(E - t)(E + s)(E + t)\,A_n(x) = 0$$

so

(4.2) $$(E - s)(E - t)\,\phi_n(x) = 0$$

where

$$\phi_n(x) = (E^2 + (s + t)E + st)\,A_n(x)$$

i.e.

(4.3) $$\phi_n(x) = A_{n+2}(x) + M\,A_{n+1}(x) - A_n(x)$$

where, by (1.4), $M = s + t = \sqrt{x - 2}$.

Equation (4.2) can, with (1.4), be rewritten as

(4.4) $$\phi_{n+2}(x) = M\,\phi_{n+1}(x) + \phi_n(x)$$

which is the usual form of the Pellian recurrence relation. Consequently, we may call $\{\phi_n(x)\}$ a *generalized Pell sequence* and $\phi_n(x)$ *generalized Pell numbers.*

Equation (4.3) relates *generalized Pell numbers* to numbers of the sequence $\{A_n(x)\}$.

When $M = 2$ in (4.4) (i.e., $x = 6$, $s = 1 + \sqrt{2}$, $t = 1 - \sqrt{2}$ in (1.4)), we have the most common form which is used to generate the ordinary Pell sequence of numbers $\{P_n\} = \{1,2,5,12,29,70,169,408,...\}$, $n \geq 1$, defined (Horadam [5]) by

(4.5) $$P_{n+2} = 2P_{n+1} + P_n \quad \text{with} \quad P_0 = 0, \quad P_1 = 1.$$

Furthermore, the explicit form of P_n is

(4.6) $$P_n = \frac{s^n - t^n}{s - t} \qquad (s - t = 2\sqrt{2} \text{ from } (1.4)).$$

For unrestricted n, (4.6) yields

(4.7) $$P_{-n} = (-1)^{n+1} P_n \quad .$$

Elements of the sequences $\{A_n(x)\}$, $\{\phi_n(x)\}$ and $\{P_n\}$ are related thus, as may be demonstrated:

(4.8) $$\begin{cases} \phi_{2n+1}(x) = P_{2n+1} + A_{2n+3}(x) \\ \phi_{2n}(x) = P_{2n} - 2P_{2n+2} + A_{2n+3}(x). \end{cases}$$

Table 3 shows the first few numbers in the sequences $\{A_n(6)\}$ and $\{\phi_n(6)\}$ which are obtained from recurrence relations (1.1) and (4.3), and are confirmed by the recurrence relation (4.4).

n	1	2	3	4	5	6	7
$A_n(6)$	1	1	7	6	41	35	239
$\phi_n(6)$	8	19	46	111	268	647	1562

Table 3. $A_n(6)$ and $\phi_n(6)$

Observe that $\phi_0(6) = P_0 - 2P_2 + A_3(6) = 3$ by (4.5), (4.8) and Table 3.

Values of $\phi_{-n}(6)$ may by calculated from (4.8) in conjunction with (1.1), (1.2), (4.7) and (4.8).

<u>Theorem 3.</u> *If, in (4.4),* $M = 2N$ *(even),* $n \geq 0$, $\phi_0(N) = 1$, $\phi_1(N) = N$, *then* $(\phi_{2n}^2(N) - 1)/(N^2 + 1)$ *is a perfect square.*

<u>Proof.</u> The explicit form of $\phi_n(N)$ is, by the usual method,

$$\phi_n(N) = \tfrac{1}{2}(c^n + d^n)$$

where $c = N + \sqrt{N^2 + 1}$ and $d = N - \sqrt{N^2 + 1}$, so $c - d = 2\sqrt{N^2 + 1}$ and $cd = -1$.

Hence
$$\frac{\phi_{2n}^2(N) - 1}{N^2 + 1} = \frac{(c^{2n} + d^{2n})^2 - 4}{4(N^2 + 1)} = \left(\frac{c^{2n} - d^{2n}}{c - d}\right)^2$$

and $(c^{2n} - d^{2n})/(c - d)$ is an integer for non-negative integer n.

For example $\dfrac{\phi_4^2(N) - 1}{N^2 + 1} = [4N(2N^2 + 1)]^2$ when $n = 2$.

When $N = 1$ in Theorem 3, we have the sequence $\{\phi_n\}$ say:

ϕ_0	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	...
1	1	3	7	17	41	99	239	577	1393	...

A1333

whence

$$\begin{cases} \phi_{2n+1} = A_{2n+1}(6) \\ \phi_{2n} = P_{2n} + P_{2n-1} \end{cases}$$

and

$$\frac{\phi_{2n}^2 - 1}{2} = P_{2n}^2 \quad \text{(by (4.6) since } c = s,\ d = t \text{ when } N = 1\text{)}.$$

In the illustrative example above, when $n = 2$ we have $\dfrac{\phi_4^2 - 1}{2} = \dfrac{289-1}{2} = 144 = 12^2 = P_4^2$.

5. A FIBONACCI-TYPE SEQUENCE

Consider the sequence $\{Q_n(N)\}$ where $N \neq 2$ is an integer:

(5.1)
$$\begin{cases} Q_{n+2}(N) = N\,Q_{n+1}(N) - Q_n(N) & (n \geq 1) \\ Q_1(N) = Q_2(N) = 1. \end{cases}$$

The first few numbers of this sequence are given in Table 4:

n	$Q_n(N)$
1	1
2	1
3	$N - 1$
4	$N^2 - N - 1$
5	$N^3 - N^2 - 2N + 1$
6	$N^4 - N^3 - 3N^2 + 2N + 1$
7	$N^5 - N^4 - 4N^3 + 3N^2 + 3N - 1$
8	$N^6 - N^5 - 5N^4 + 4N^3 + 6N^2 - 3N - 1$
9	$N^7 - N^6 - 6N^5 + 5N^4 + 10N^3 - 6N^2 - 4N + 1$

Δ A65941

Table 4.

An interesting factorization result arises from these number, namely:

Theorem 4. $Q_{n+1}(N) - 1 = (N-2)\, A_n(N)\, A_{n-1}(N) \qquad (n \geq 2,\ N \neq 2).$

Proof. We use induction. The result is obvious when $n = 2,3$. Assume the result is true for $n = 4, 5, \ldots, k+1$. Then

$$\begin{aligned}
Q_{k+2}(N) &= N\, Q_{k+1}(N) - Q_k(N) && \text{by (5.1)}\\
&= N\, Q_{k+1}(N) - N\, Q_{k-1}(N) + Q_{k-2}(N) && \text{by (5.1)}\\
&= N(N-2)\, A_k(N)\, A_{k-1}(N) + N - N(N-2)\, A_{k-2}(N)\, A_{k-3}(N)\\
&\quad - N + (N-2)\, A_{k-3}(N)\, A_{k-4}(N) + 1 && \text{by hypothesis}\\
&= N(N-2)\, A_k(N)\, A_{k-1}(N) + (N-2)\, A_{k-3}(N)\left(A_{k-4}(N) - N\, A_{k-2}(N)\right) + 1\\
&= N(N-2)\, A_k(N)\, A_{k-1}(N) - (N-2)\, A_{k-3}(N)\, A_k(N) + 1 && \text{by (1.1)}\\
&= (N-2)\, A_k(N)\left(N\, A_{k-1}(N) - A_{k-3}(N)\right) + 1\\
&= (N-2)\, A_k(N)\, A_{k+1}(N) + 1 && \text{by (1.1)}
\end{aligned}$$

as required.

For example, $n = 7$ in Theorem 4 gives, with the help of Table 2,

$$Q_8(N) - 1 = (N-2)\, A_7(N)\, A_6(N) = (N-2)(N^2-1)(N^3+N^2-2N-1).$$

Further, $N = 3$ in this example yields, with (1.10),

$$F_{13} - 1 = L_7\, F_6 \quad (= 232 = 29 \times 8).$$

since the numbers $Q_n(3)$ are certain Fibonacci numbers.

Induction also leads to the result

$$(5.2) \qquad Q_n(N) = \sum_{j=0}^{[(n-2)/2]} (-1)^j \binom{n-j-2}{j} N^{n-2j-2} - \sum_{j=0}^{[(n-3)/2]} \binom{n-j-3}{j} N^{n-2j-3}$$

which is a generalization of (2.8) of Barakat [1].

When $N = 3$, (5.2) reduces to

$$Q_n(3) = F_{2n-2} - F_{2n-4}.$$

For example, $Q_5(3) = 13 = F_8 - F_6$ $(= 21 - 8)$.

A basic relationship amongst the elements of $\{Q_n(N)\}$ is

$$(5.3) \qquad Q_{n+1}(N)\, Q_{n-1}(N) - Q_n^2(N) = N - 2$$

which is a particular case of the general result for the sequence $\{w_n(a,b;p,q)\}$ in Horadam [4] where $a = 1$, $b = 1$, $p = N$, $q = 1$. (Equation (5.3) is the analogue for $\{Q_n(N)\}$ of the well-known Simson result for Fibonacci numbers: $F_{n+1}\, F_{n-1} - F_n^2 = (-1)^n$,

$n \geq 1$.)

Following the approach of Hoggatt and Bicknell [3] for Fibonacci polynomials and letting e^z and e^{-z} be the roots of the auxiliary equation $r^2 - Nr + 1 = 0$ associated with the recurrence relation (5.1), we obtain

(5.4) $$Q_{n+1}(N) = \frac{\cosh \frac{1}{2}(2n-1)z}{\cosh \frac{1}{2}z}$$

where $2 \cosh z = N$, $2 \sinh z = \sqrt{N^2 - 4}$.

Clearly, further results may be developed involving the sequences under consideration, e.g.

(5.5) $$Q_n(6) = P_{2n-3} \quad (n \geq 2),$$

and

(5.6) $$Q'_n(6) = P_{2n-2} \quad (n \geq 1)$$

if we define $Q'_n(N)$ as for $Q_n(N)$ in (5.1) but with the initial conditions $Q'_1(N) = 0$, $Q'_2(N) = 2$.

The theory for $Q_n(N)$ and $Q'_n(N)$ extends to negative values of n.

6. CONCLUSION

It is of interest to note ways in which this work can be further extended. When $M = 2$, the Pell equation can be readily related to the Diophantine equations

$$x^2 - 2y^2 = \pm 1$$

because of the simple continued fraction expansion of $\sqrt{2}$, namely,

$$\sqrt{2} = [1, \dot{2}] .$$

Bernstein [2] has shown how it can be further developed by considering the surd

$$\sqrt{m} = [b_0, \dot{b}_1, b_2, \ldots, b_{n-1}, 2\dot{b}_0]$$

and the recurrence relation

$$P_{j+2} = b_j P_{j+1} + P_j \qquad (j > 0),$$

where b_j is the j^{th} partial quotient of the continued fraction, and with suitable initial conditions. Bernstein has generalized this further by using the Jacobi-Perron algorithm to accommodate linear recurrence relations of order higher than two.

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