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5981
etc

Paul Stein
Letter to NJAS
March 17, 1975

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STEIN

A5981 -5983 and A259452

UNIVERSITY OF CALIFORNIA
LOS ALAMOS SCIENTIFIC LABORATORY
(CONTRACT W-7405-ENG-36)
P. O. BOX 1663
LOS ALAMOS, NEW MEXICO 87544

[ST3]

IN REPLY REFER TO: T-7
MAIL STOP: 262

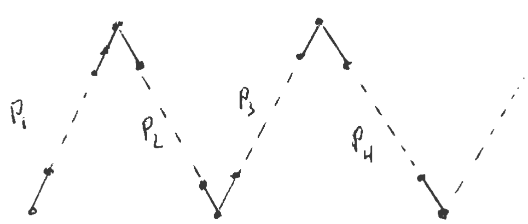
March 17, 1975

Dr. Neil J.A. Sloane
Bell Laboratories
Murray Hill, New Jersey 07974

Dear Neil:

Thanks very much for the copy of "Supplement I" to your handbook. I was anxious to see whether or not the supplement contained any of the new combinatorial sequences I discovered recently. Since it does not, you may be interested in learning about them.

Consider the Hasse diagram of a particularly simple class of partial orderings; I call them "sawtooths":



Here the i^{th} linear chain contains P_i linearly ordered points, including the maximal and minimal elements. Thus (since the maximal and minimal elements are counted twice), r chains of this diagram will contain $N_r = \sum_i^r P_i - r + 1$ points. (Clearly, $P_i \geq 2$ because of this convention). Suppose there are, in fact, just r chains. The problem is then to calculate the number of linear orderings consistent with the given partial ordering. If we label the points in some arbitrary but fixed fashion, this amounts to asking how many of the $N_r!$ permutations are not excluded by the ordinary restrictions of the diagram.

Suppose all the $P_i = 2$. As r increases we just get the well-known tangent-secant numbers as a function of r - a result some 100 years old. But now suppose all the $P_i = 3$. Using my algorithm (to be described below), one finds, writing A_r for the number of "allowed" permutations:

do!

"up-up-down-down..." perms"

5981

	r	A_r	
3			
5	2	6	1
7	3	71	2
	4	1456	4
	5	45541	5
	6	2020656	7
	7	120686411	9
	8	9336345856, etc.	10

length 2n+1 letters.

%0 1,2

Similarly, if all the $P_i = 4$:

length 3n+1 —

r	A_r
2	20
3	1301
4	202840
5	61889101
6	32676403052
7	27418828825961
8	34361404413755056, and so forth.

5982 "up up up down^3 — ..." perms"

%0 1,2

Of course, the P_i need not be all the same; they can be any set of ordered integers ($\geqslant 2$). Some simple examples I have looked at are 4,3,4,3,4,3,... and 5,2,5,2,5,2,... (I have also examined, exhaustively, various finite cases where the $\{P_i\}$ run over all compositions of some given integer). One amusing sequence - which I do not list here - is gotten by taking $P_i = i + 1$, $i = 1,2,3,...$ One might call this the "sawtooth factorial".

The iterative scheme for calculating A_r is as follows:

Given $\{P_1, P_2, \ldots P_r \ldots\}$, $P_i \geqslant 2$

$$N_r = \sum_{i=1}^{r} P_i + 1 - r$$

Let $F_1^{(1)} = 1$

$$L^{(2)}_{N_2-m} = \binom{P_2 - 2 + P_1 - 1 - m}{P_1 - 1 - m}, \quad 0 \leqslant m \leqslant P_1 - 1$$

"2 up 2down perms of lgth 2n+1"

$$F_m^{(k)} = \sum_{i=\max(P_{k-1},m)}^{N_{k-1}} L_i^{(k-1)} \cdot \binom{P_k - 2 + i - m}{i - m}, \quad 1 \leqslant m \leqslant N_{k-1} \text{ (k odd)}$$

$$L_{N_j - m}^{(j)} = \sum_{i-1}^{\min(N_{j-1} + 1 - P_{j-1},\ N_{j-1} - m)} F_i^{(j-1)} \cdot \binom{P_j - 2 + N_{j-1} - i - m}{N_{j-1} - i - m} \quad 0 \leqslant m \leqslant N_{j-1} - 1 \text{ (j even)}$$

$\vec{F}^{(k)}$ and $\vec{L}^{(j)}$ are vectors with integer components, $\vec{F}$ corresponding to the odd chains, $\vec{L}$ to the even chains. $A_r = \sum F_i^{(r)}$ or $\sum L_i^{(r)}$ according as r is odd or even. This may look complicated, but it is quite trivial on a computer. Naturally, one could write A_r as a multiple sum, but this would hardly be helpful. I have shown this scheme to various people, but no one has had any suggestions as to how one might get simple generating functions out of it - if, indeed, such exist. Stanley has concerned himself with the general problem of "stretching" partial orderings, but so far as I can see, his results are little more than a reformulation of the problem; of course, he considers more general situations than the sawtooth (which, as Rota remarked, is the simplest generalization of "shuffling" à la Ree).

If you are interested in including any of these sequences in the next supplement, I can supply as many as you want.

Best regards,

Paul

Paul R. Stein

PRS:bwm

Bell Laboratories

600 Mountain Avenue
Murray Hill, New Jersey 07974
Phone (201) 582-3000

March 26, 1975

Dr. Paul R. Stein
University of California
Los Alamos Scientific Laboratory
P.O. Box 1663
Los Alamos, New Mexico 87544

Dear Paul:

Thank you very much for your letter of March 17, and the enclosed sequences. They are certainly new to me, although they remind me of another problem which I will describe later.

Some of these sequences ought to go in the next Supplement. Which ones are your favorites? The two you sent, I suppose, i.e., with all the $P_i = 3$ and all the $P_i = 4$. Do you think you could send me some further terms of these two, and any others you like? A total of 150 digits for each sequence is enough to fill two lines. They should be listed as a private communication from you? (There is nothing published yet, I assume.) How would you like them to be called? And the initial value is $A_1 = 1$, presumably.

Have you ever looked at this problem? Suppose we have n objects $a_1, a_2, \ldots, a_n$, having weights $w(a_1), w(a_2), \ldots, w(a_n)$. Any subset $\{a_r, a_s, \ldots\}$ then weights $w(a_r) + w(a_s)$ .... Arrange the 2^n subsets of $\{a_1, \ldots, a_n\}$ in order of increasing weight (and assume no two subsets have the same weight). How many of the $2^n!$ possible orderings of the subsets can occur in this way? Assume $w(a_1) < w(a_2) < w(a_3) < \ldots$ for simplicity, which divides the number by n! For n = 2, the answer is 1, as follows:

Assume $w(a_1) < w(a_2)$. Then

$$w(\emptyset) < w(a_1) < w(a_2) < w(a_1+a_2)$$

is forced. For n = 3 there are two possibilities, assuming that

$$w(a_1) < w(a_2) < w(a_3),$$

namely

$$w(\emptyset) < w(a_1) < w(a_2) < w(a_3) < w(a_1+a_2) < \ldots$$

and

$$w(\emptyset) < w(a_1) < w(a_2) < w(a_1+a_2) < w(a_3) < \ldots$$

I think I have two or three more terms in this sequence in an old file somewhere, but could never find a nice way to look at the problem. Any ideas?

Yours sincerely,

MH-1216-NJAS-mv

N. J. A. Sloane

16 Apr 75 0812+10 S-Poset Problem

1985

n		a_n
2	3 3	6
3	3 3 3	71
4	3 3 3 3	1456
5	3 3 3 3 3	45541
6	3 3 3 3 3 3	2020656
7	3 3 3 3 3 3 3	120686411
8	3 3 3 3 3 3 3 3	9336345856
9	3 3 3 3 3 3 3 3 3	900138776681
10	3 3 3 3 3 3 3 3 3 3	108480272749056
11	3 3 3 3 3 3 3 3 3 3 3	15611712012050351
12	3 3 3 3 3 3 3 3 3 3 3 3	2664103110372192256
13	3 3 3 3 3 3 3 3 3 3 3 3 3	531909061958526321421
14	3 3 3 3 3 3 3 3 3 3 3 3 3 3	122840808510269863827456
15	3 3 3 3 3 3 3 3 3 3 3 3 3 3 3	32491881630252866646683891

16 Apr 75 0812+57 S-Poset Problem

5982

n		g_n
2	4 4	20
3	4 4 4	1301
4	4 4 4 4	202840
5	4 4 4 4 4	61889101
6	4 4 4 4 4 4	32676403052
7	4 4 4 4 4 4 4	27418828825961
8	4 4 4 4 4 4 4 4	34361404413755056
9	4 4 4 4 4 4 4 4 4	61335081309931829401
10	4 4 4 4 4 4 4 4 4 4	150221740688275657957940
11	4 4 4 4 4 4 4 4 4 4 4	489799709605132718770274141
12	4 4 4 4 4 4 4 4 4 4 4 4	2073641570051429601078643837960
13	4 4 4 4 4 4 4 4 4 4 4 4 4	1116309918606408410068710786325381
14	4 4 4 4 4 4 4 4 4 4 4 4 4 4	750635818911632881095350565885210274 52
15	4 4 4 4 4 4 4 4 4 4 4 4 4 4 4	62086217826009856798261374024407289433 8161

/a

16 Apr 75 0819+41 S-Poset Problem

n		a_n
2	5 5	70
3	5 5 5	26599
4	5 5 5 5	33757360
5	5 5 5 5 5	107709888805
6	5 5 5 5 5 5	726401013530416
7	5 5 5 5 5 5 5	9107888739246870571
8	5 5 5 5 5 5 5 5	20065663143869477105792O
9	5 5 5 5 5 5 5 5 5	706518300623233421587236O169
10	5 5 5 5 5 5 5 5 5 5	38144688404828693990329879311616O
11	5 5 5 5 5 5 5 5 5 5 5	3029951047847385035108711977447528 2895
12	5 5 5 5 5 5 5 5 5 5 5 5	34225296824160457610052605464630281512181 76
13	5 5 5 5 5 5 5 5 5 5 5 5 5	534300057849762938380919185569639876763713908941
14	5 5 5 5 5 5 5 5 5 5 5 5 5 5	11251829883153626316854101804941566755617736809290752O
15	5 5 5 5 5 5 5 5 5 5 5 5 5 5 5	3130421749257713330083315904615214660200590007856833107873 9

18 Apr 75 0708+55 S-Poset Problem

A259452

n		a_n
2	6 6	252
3	6 6 6	578005
4	6 6 6 6	6190034816
5	6 6 6 6 6	214265281290061
6	6 6 6 6 6 6	19157603395806362772
7	6 6 6 6 6 6 6	3800502511986185228829385
8	6 6 6 6 6 6 6 6	1498722661993096106927612109936
9	6 6 6 6 6 6 6 6 6	1081056808393919319749313795137642521
10	6 6 6 6 6 6 6 6 6 6	1336319624105519211256870506149168604698792
11	6 6 6 6 6 6 6 6 6 6 6	2686089689454807592037879534110673084561941532765
12	6 6 6 6 6 6 6 6 6 6 6 6	8408972461133782834101442837671862586056494037643207256
13	6 6 6 6 6 6 6 6 6 6 6 6 6	39545667402618849330626441245283926578222182547461215419190981
14	6 6 6 6 6 6 6 6 6 6 6 6 6 6	270940905810245475093815290821183448898411505670375146767822664321812
15	6 6 6 6 6 6 6 6 6 6 6 6 6 6	263404375237583825592227653916393424483098859352971655046494831617915493214

5

last digit is 5

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Omit

UNIVERSITY OF CALIFORNIA
LOS ALAMOS SCIENTIFIC LABORATORY
(CONTRACT W-7405-ENG-36)
P. O. BOX 1663
LOS ALAMOS, NEW MEXICO 87544

IN REPLY
REFER TO: T-7
MAIL STOP: 233

April 18, 1975

Dr. Neil J.A. Sloane
Bell Laboratories
600 Mountain Avenue
Murray Hill, New Jersey 07947

Dear Neil:

I enclose four sequences counting the number of linear orderings consistent with my "saw-tooth" partial orderings. For simplicity, I have chosen cases in which all the p_i are the same in each linear chain, namely the cases p_i = 3,4,5,6. In each case the number of chains goes from 2 through 15. a_1 = 1 (not shown) in all cases.

None of this has yet been published. I don't know what you should call these sequences. Perhaps, following Stanley, we should call these "extensions"; he would say that the a's are the "number of ways of extending a given partial order to a total order". You might say, "number of extensions to total order of a certain partial order". In any case, nobody will know exactly what it is unless they inquire. The a_n are also, in a sense, generalizations of the Euler numbers, but that might be a misleading characterization.

Hope you can use these sequences; any number of further ones will be furnished on request. The problem you mention does not appear, at first glance, to be directly related to mine.

Yours sincerely,

Paul

Paul R. Stein

PRS:bwm
encl.