f91

6531

ENUMERATION OF SEMIORDERS ON A FINITE SET

-:-

J.L. CHANDON

IAE
Université d'Aix-Marseille III
29, avenue R. Schuman
13617 AIX-EN-PROVENCE

J. LEMAIRE, J. POUGET

Université de Nice
41 boulevard Napoléon III
06041 NICE CEDEX

Published in : Mathématiques et Sciences Humaines
n° 62, 1978, pp. 61-80

Scan A6531

Chardon
Lemaire
Bouget
English summary of their French paper

5 pages

ABSTRACT

The aim of this paper is to compute the number of semiorders on a finite set X.

The semiorder model has been introduced by LUCE R.D. (1956) to deal with non transitive indifference judgements. According with SCOTT D. and SUPPES P. (1958), a semiorder on the finite set X is a relation R such that there exist :

- a numerical function u defined on X,
- a positive number σ ,

with :

$$x \, R \, y \iff u(x) \geq u(y) - \sigma$$

For example, a complete order and a complete quasiorder are semiorders.

Let n = card(X) and q_n, the number of semiorders defined on X. The following formula is established in the paper :

(1) $$q_n = \sum_{k=1}^{n} (-1)^{n-k} S(n,k) \ \{2k\}_{k-1}$$

with S(n,k), the stirling numbers of the second kind and $\{a\}_b = a(a-1)\dots(a-b+1)$.

For example one can obtain :

n	1	2	3	4	5	6	7
q_n	1	3	19	183	2371	38703	763099

new A6531

Neverless the previous formula is alternate which makes some trouble in the evaluation of q_n when n increases. In order to avoid this desagreement, the paper proposes another way for computing q_n. The basic formula is :

(2) $$q_n = \sum_{k=1}^{n} p_{n,k} \, c_k$$

Here $p_{n,k} = S(n,k)\,k!$ is the number of complete quasiorders on X with k classes. Using the recurrence relation of the Stirling number of the second kind we obtain :

$$p_{n,k} = k(p_{n-1,k-1} + p_{n-1,k})$$

With the initial conditions :

$$p_{n,1} = 1 \text{ and } p_{n,n} = n!$$

the calculation of the $p_{n,k}$ is then easy.

In order to define c_k, we introduce the complete quasiorder S_R associated to a semiorder R. (LUCE R.D. (1956)). This relation is defined by :

$$x\ S_R\ y \Longleftrightarrow \forall z : z\ R\ x \Longrightarrow z\ R\ y \quad \text{and } y\ R\ z \Longrightarrow x\ R\ z.$$

Then c_k is the number of semiorders defined on a set with k elements such that the associated quasiorders are orders. In the first part of the paper the formula (2) is established.

The main problem is the evaluation of c_k. To solve it, we define the notion of "subdiagonal scale of order (n,k)" (SDS of order (n,k)) : this is a double sequence of integers $(a_j, b_j)_{1 \le j \le q}$ such that :

$$0 < a_1 < \dots < a_q = n$$
$$0 < b_1 < \dots < b_q = k$$
$$a_j \le b_j \quad \text{for every } j$$

(A similar notion has been defined by KREWERAS G. (1970)).

A "strict subdiagonal scale of order (n,k)" (SSDS of order (n,k)) is a SDS of order (n,k) for which :

$$a_j < b_j \text{ for every } j < q.$$

Let $c_{n,k}$ and $c^*_{n,k}$ the numbers of SDS and SSDS of order (n,k).

Using a characterization of the semiorders proposed by MENUET J.(1974) and JACQUET-LAGREZE E. (1975), we first prove that :

(3) $$c_n = c_{n,n}$$

Secondly we prove that

(4) $$c_{n,n} = c^*_{n+1,n}$$

thirdly we make a connection between the $c^*_{n,k}$ and the generalized ballot numbers $d^*_{n,k}$ introduced by CARLITZ L. (1969) :

(5) $$d^*_{n,k} = c^*_{2k-n+1,k}$$

These last numbers can be evaluated by the following formulas :

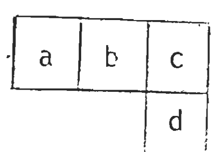
$$d^*_{n,k} = d^*_{n,k-1} + d^*_{n-1,k-1} + d^*_{n-2,k-1}$$

$$d^*_{n,k} = 0 \text{ if } n > k \text{ or } n=0$$

$$d^*_{1,k} = 1.$$

Example :

	n=1	2	3	4	5	6
k=1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	2	2	0	0	0
4	1	3	5	4	0	0
5	1	4	9	12	9	0
6	1	5	14	25	30	21



d = a+b+c

As a consequence of formulas (3) (4) and (5), the numbers c_n appear on the diagonal of the previous table.

BIBLIOGRAPHY

(1) AVERY P. "Semiorders and representable graphs. *Proc. 5th British combinatorial conf.*, pp. 5-9, 1975.

(2) BARBUT M., MONJARDET B. "Ordre et classification. Algèbre et combinatoire. *Hachette Université*. 1970.

(3) BERGE C. "Principes de combinatoire" *Dunod*, 1968.

(4) CARLITZ L. "Solution of certain recurrences". *Siam J. of Appl. Math.* Vol. 17, n°2, pp. 251-259, 1969.

(5) FISHBURN P.C. "Utility theory for decision making" *Wiley*, 1970.

(6) JACQUET-LAGREZE E. " La modélisation des préférences, préordres, quasi ordres et relations floues". *SEMA Diretion Scientifique Rapport de recherche n°80*, 1975.

(7) KREWERAS G. "Sur les éventails de segments". *Cahier du BURO n°15*, pp. 3-40, 1970.

(8) LUCE R.D. "Semi orders and a theory of utility discrimination econometrica. Vol. 24, pp. 178-191., 1956.

(9) MENUET J. "Quasi ordres et modelisation des préférences . *SEMA Direction scientifique. Note de travail n°197*.

(10) RIORDAN J. "Combinatorial identities" *Wiley* , 1968.

(11) SCOTT D. "Measurement structures and linear inequalities . *Journ. of Math. Psychology. 1.* pp. 233-247, 1964.

(12) SCOTT D., SUPPES P. "Foundational aspects of theories of measurement *The Journ. of symbolic logic*, 23, 2, pp. 113-128.

(13) SLOANE N.J.A. "A handbook of integer sequences" *Academic press*, 1973.