ECONOMICALLY SOLVING THE TOWER OF HANOI PUZZLE

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ABSTRACT. The rules of the Tower of Hanoi puzzle are not specific about which of the three pegs is the starting peg and which is the target peg. Obviously, it doesn't matter. Or does it? We show that, in terms of the total distance that the disks travel, the customary decision to start at the leftmost peg and move the disks to the rightmost peg is, in general, not optimal.

It is well known that the Tower of Hanoi puzzle with n disks has a unique solution consisting of $2^n - 1$ moves (e.g., [1, Theorem 2.1.]). Let a_n (resp. b_n) denote the total distance that n disks travel from peg 1 to peg 3 (resp. from peg 1 to peg 2) in the course of this solution. Clearly, $a_1 = 2$ and $b_1 = 1$. Let $A(x)$ (resp. $B(x)$) be the generating function of a_n (resp. b_n). The following relations between $A(x)$ and $B(x)$ hold:

$$A(x) = 2x + x\left(2B(x) + \frac{2}{1-x}\right),$$
$$B(x) = x + x\left(A(x) + \frac{1}{1-x} + B(x)\right),$$

Solving the system, we conclude that

$$A(x) = \frac{2x}{(1+x)(1-x)(1-2x)},$$
$$B(x) = \frac{x(2x+1)}{(1+x)(1-x)(1-2x)}.$$

It is well known that $J(x) = x/(1+x)(1-2x)$ is the generating function of the Jacobsthal numbers and that

$$J(x) = \frac{1}{3}\sum_{n=1}^{\infty}(2^n - (-1)^n)x^n$$

(e.g., A001045 in [2]). Thus,

$$A(x) = \frac{2}{1-x}J(x)$$

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$$= \frac{2}{3}\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\left(2^k-(-1)^k\right)\right)x^n$$
$$= \frac{2}{3}\sum_{n=1}^{\infty}\left(2^{n+1}-1-1_{n \text{ is even}}\right)x^n,$$

where $1_{n \text{ is even}}$ is 1 if n is even and 0 otherwise. It follows that

$$a_n = \begin{cases} \frac{2^{n+2}-4}{3}, & \text{if } n \text{ is even;} \\ \frac{2^{n+2}-2}{3}, & \text{otherwise,} \end{cases}$$

(giving A026644 in [2] a new interpretation). Similarly,

$$b_n = \begin{cases} a_n, & \text{if } n \text{ is even;} \\ a_n - 1, & \text{otherwise,} \end{cases}$$

(giving A084639 in [2] a new interpretation).

Conclusion. It follows from the above calculations that, starting at the leftmost or rightmost peg, it is more economical, in terms of the total distance that the disks travel, to move the disks to the central peg. An extension of the Tower of Hanoi puzzle to 4 pegs is known as The Reve's puzzle (e.g., [1, Chapter 5]) and it would be interesting (and probably harder) to find more economical starting and target pegs for this puzzle as well.

References

[1] A. M. Hinz, S. Klavžar, U. Milutinović, and C. Petr, *The Tower of Hanoi - Myths and Maths*, Springer, 2013.

[2] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, OEIS Foundation Inc., `https://oeis.org`.