Matrices with repeated columns - the generalised Appell groups

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1.1 The Appell group. Let $G(x) = 1 + g_1x + g_2x^2 + \cdots$ be an ordinary generating function (OGF) and consider the associated lower unitriangular array

$$\begin{pmatrix} 1 & & & \\ g_1 & 1 & & \\ g_2 & g_1 & 1 & \\ g_3 & g_2 & g_1 & \ddots \\ \vdots & g_3 & g_2 & \ddots \\ & \vdots & g_3 & \ddots \\ & & \vdots & \ddots \end{pmatrix}$$

having one repeated column formed from the coefficients of $G(x)$. The sequence of column OGFs of the array is

$$\left(G, xG, x^2G, x^3G, \cdots\right).$$

In the language of Riordan arrays this array is denoted by $(G(x), x)$. The set of Riordan arrays of this type form an abelian group under matrix multiplication called the (ordinary) *Appell group* [1, Section 3]. The group multiplication law is

$$(G(x), x) * (F(x), x) = (G(x)F(x), x)$$

with the group inverse

$$(G(x), x)^{-1} = \left(\frac{1}{G(x)}, x\right).$$

The purpose of these notes is to generalise the Appell group by considering lower unitriangular matrices with n repeating columns.

1.2 The generalised Appell groups. Let G_i, $i = 0..n-1$, be n power series, all with constant term 1. We shall use the double round brackets notation $((G_0, G_1, \cdots, G_{n-1}))$ to denote the lower unitriangular array with column generating functions

$$(G_0, xG_1, \cdots, x^{n-1}G_{n-1},\ x^nG_0, x^{n+1}G_1, \cdots, x^{2n-1}G_{n-1}, \cdots).$$

We denote the set of arrays of this type by $\mathcal{A}_n$.

Example 1. The array $\left(\left(\frac{1}{1-x}, \frac{1}{(1-x)^2}, \frac{1}{(1-x)^3}\right)\right)$ in $\mathcal{A}_3$ begins

$$\begin{pmatrix}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 1 & 1 \\
1 & 5 & 10 & 1 & 2 & 1 \\
\vdots & \vdots & \vdots & 1 & 3 & 3 & 1 \\
 & & & 1 & 4 & 6 & 1 & 1 \\
 & & & 1 & 5 & 10 & 1 & 2 & 1 \\
 & & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

with its first three columns repeated.

Clearly, $\mathcal{A}_1$ is the Appell group. In terms of the double bracket notation, the group operations in $\mathcal{A}_1$ read as

$$((G))\,((F)) = ((GF))$$

$$((G))^{-1} = \left(\left(\frac{1}{G}\right)\right).$$

The Appell group $\mathcal{A}_1$ is contained in the set $\mathcal{A}_{\mathrm{n}}$ for all n. More generally, if n divides m then $\mathcal{A}_{\mathrm{n}}$ is contained in $\mathcal{A}_{\mathrm{m}}$.

It is not difficult to show that $\mathcal{A}_{\mathrm{n}}$ is a group for $n > 1$. We prove this in the next section for the case $n = 2$; the proof can easily be extended to the general case. We expect this simple result is somewhere in the literature but a quick internet search didn't find anything relevant, hence these notes. It seems reasonable to refer to the groups $\mathcal{A}_{\mathrm{n}}$ as *generalised Appell groups.*

1.3 The group $\mathcal{A}_2$

We have

$$\mathcal{A}_2 = \left\{((G,F)) \mid \; G(x) = 1 + \sum_{k=1}^{\infty} g_k x^k, F(x) = 1 + \sum_{k=1}^{\infty} f_k x^k\right\}.$$

A typical element $((G,F))$ of $\mathcal{A}_2$ begins

$$\begin{pmatrix} 1 & & & & & & \\ g_1 & 1 & & & & & \\ g_2 & f_1 & 1 & & & & \\ g_3 & f_2 & g_1 & 1 & & & \\ \vdots & f_3 & g_2 & f_1 & 1 & & \\ & \vdots & g_3 & f_2 & \vdots & \ddots & \\ & & \vdots & f_3 & & & \\ & & & \vdots & & & \end{pmatrix},$$

with its first two columns repeated. Clearly, the identity element $((1,1)) \in \mathcal{A}_2$. Arrays in $\mathcal{A}_2$ are examples of double Riordan arrays introduced in [1]. In the notation of that paper, our array $((G,F))$ is the double Riordan array $\left(G; x\frac{F}{G}, x^2\frac{G}{F}\right)$.

It will be convenient in what follows to denote the even and odd parts of a power series $f(x)$ by f_{even} and f_{odd}:

$$f_{even} = \frac{f(x)+f(-x)}{2},\ f_{odd} = \frac{f(x)-f(-x)}{2}.$$

Theorem 1 *(i) The set of lower unitriangular arrays $\mathcal{A}_2$ forms a group under the associative operation of matrix multiplication. The multiplication operation is given by*

$$((g,f))\,((G,F)) = ((gG_{even} + fG_{odd}, fF_{even} + gF_{odd}))\,. \tag{1}$$

Equivalently,

$$((g,f))\,((G,F)) = ((h_1,h_2))$$

where

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} G_{even} & G_{odd} \\ F_{odd} & F_{even} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix} \tag{2}$$

(ii) the inverse of the array $((G,F))$ belongs to $\mathcal{A}_2$ and is given by

$$((G,F))^{-1} = \left(\left(\frac{F_{even} - G_{odd}}{\triangle}, \frac{G_{even} - F_{odd}}{\triangle}\right)\right), \tag{3}$$

where $\triangle(x) = (G(x)F(-x) + F(x)G(-x))/2$.

Proof. Consider the action of the array $((g,f))$ on column vectors. Suppose that $A = (a_0, a_1, ...)^{\mathrm{T}}$ and $B = (b_0, b_1, ...)^{\mathrm{T}}$ are column vectors with OGFs $A(x)$ and $B(x)$. Then

$$((g,f))\,A = \begin{pmatrix} \uparrow & 0 & 0 & 0 & \cdots \\ & \uparrow & 0 & 0 & \cdots \\ & & \uparrow & 0 & \cdots \\ & & & \uparrow & \\ g & f & g & f & \ddots \\ & & & & \\ \downarrow & \downarrow & \downarrow & \downarrow & \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ \\ \\ \\ \\ \end{pmatrix}$$

if and only if

$$\begin{aligned} B(x) &= a_0 g(x) + a_1 x f(x) + a_2 x^2 g(x) + a_3 x^3 f(x) + \cdots \\ &= g(x)\left(a_0 + a_2 x^2 + a_4 x^4 + \cdots\right) + f(x)\left(a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right) \\ &= g(x) A_{even}(x) + f(x) A_{odd}(x). \end{aligned} \tag{4}$$

With a small abuse of notation we can rewrite (4) as

$$B(x) = ((g,f)) * A(x) = g(x) A_{even}(x) + f(x) A_{odd}(x). \tag{5}$$

Now consider the matrix product $((g,f))\,((G,F))$ of two elements of $\mathcal{A}_2$. Since the first column of the array $((G,F))$ has the generating function $G(x)$, it follows from (4) that the first column of the product $((g,f))((G,F))$ has the generating function $gG_{even} + fG_{odd.}$. To identify the generating functions of the other columns of the matrix product observe that

$$\begin{aligned} ((g,f)) &= \left(\begin{array}{c|ccc} 1 & 0 & 0 & \cdots \\ \hline g_1 & & & \\ g_2 & & ((f,g)) & \\ \vdots & & & \\ & & & \end{array}\right) \\ &= \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & \cdots \\ g_1 & 1 & 0 & 0 & \cdots \\ \hline g_2 & f_1 & & & \\ \vdots & f_2 & & ((g,f)) & \\ & \vdots & & & \\ & & & & \\ & & & & \end{array}\right) \\ &= \cdots \end{aligned}$$

We see that the action of the array $((g,f))$ on the second column of the array $((G,F))$ is obtained from the action of the array $((f,g))$ on the column vector of coefficients of the power series F. Hence by (4), the second column of the product $((g,f))((G,F))$ has the generating function $fF_{even}+gF_{odd}$ (shifted by a factor of x).

Continuing in this way, we find that the even-indexed columns of the product $((g,f))((G,F))$ have the generating function $gG_{even}+fG_{odd}$ (shifted by some power of x), while the odd-indexed columns of the product have the generating function $fF_{even}+gF_{odd}$ (again shifted by some power of x).

Therefore, in terms of the double bracket notation, we have shown that

$$((g,f))\,((G,F)) = ((gG_{even}+fG_{odd,}fF_{even}+gF_{odd}))\,.$$

ii) To show that the inverse of $((G,F))\in\mathcal{A}_2$ also lies in $\mathcal{A}_2$ we need to find power series g and f such that $((g,f))\,((G,F)) = ((1,1))\,.$ By (2), this is equivalent to solving

$$\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}G_{even} & G_{odd}\\ F_{odd} & F_{even}\end{pmatrix}\begin{pmatrix}g\\f\end{pmatrix}. \tag{6}$$

The determinant $\triangle = G_{even}F_{even} - G_{odd}F_{odd}$ of the 2 x 2 array in (6) simplifies to $\triangle(x) = (G(x)F(-x)+F(x)G(-x))\,/2$. Note that $\triangle\neq 0$ since $\triangle(0) = 2F(0)G(0)/2 = 1$. Hence (6) has the solution

$$g = \frac{F_{even}-G_{odd}}{\triangle}, \quad f = \frac{G_{even}-F_{odd}}{\triangle}$$

such that

$$((G,F))^{-1} = ((g,f)) = \left(\left(\frac{F_{even}-G_{odd}}{\triangle}, \frac{G_{even}-F_{odd}}{\triangle}\right)\right)\in\mathcal{A}_2.$$

□

Examples of triangles in the OEIS in $\mathcal{A}_2$ include A070909 $=\left(\left(\frac{1}{1-x},1\right)\right)$, A106465 $=\left(\left(\frac{1}{1-x},\frac{1}{1-x^2}\right)\right)$, A177990 $=\left(\left(1,\frac{1}{1-x}\right)\right)$ and A177994 $=$ $\left(\left(\dfrac{1}{(1-x)(1-x^2)},\dfrac{1}{1-x}\right)\right)$.

As previously remarked, it is not difficult to extend the above proof that $\mathcal{A}_2$ is a group to show that $\mathcal{A}_n$ is a group for $n>2$. As an example, the multiplication rule in $\mathcal{A}_3$ is given by

$$((G_1,G_2,G_3))\,((F_1,F_2,F_3)) = ((h_1,h_2,h_3))\,,$$

where

$$\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} F_1^{(0)} & F_1^{(1)} & F_1^{(2)} \\ F_2^{(2)} & F_2^{(0)} & F_2^{(1)} \\ F_3^{(1)} & F_3^{(2)} & F_3^{(0)} \end{pmatrix} \begin{pmatrix} G_1 \\ G_2 \\ G_3 \end{pmatrix}$$

and $F^{(0)}$, $F^{(1)}$ and $F^{(2)}$ denote the trisections of a power series $F(x) = \sum_{k=0}^{\infty} f_k x^k$ defined as $F^{(0)}(x) = \sum_{k=0}^{\infty} f_{3k} x^{3k}$, $F^{(1)}(x) = \sum_{k=0}^{\infty} f_{3k+1} x^{3k+1}$ and $F^{(2)}(x) = \sum_{k=0}^{\infty} f_{3k+2} x^{3k+2}$.

We note one property of the group $\mathcal{A}_3$: every element $((G_1, G_2, G_3))$ in $\mathcal{A}_3$ factorises as $((G_1, G_2, G_3)) = ((F_1, 1, 1))\,((1, F_2, 1))\,((1, 1, F_3))$ for some uniquely determined power series F_1, F_2 and F_3.

1.4 Remarks. The following are easy consequences of Theorem 1.

1.4.1 For elements $((f, f))$ of the Appell group $\mathcal{A}_1$ sitting inside the group $\mathcal{A}_2$ we have

$$((f, f))\,((G, F)) = ((fG, fF))\,.$$

1.4.2 The set of arrays of the form $((\, g(x), g(-x)\,))$ is an abelian subgroup of $\mathcal{A}_2$ with the multiplication operation

$$((g(x), g(-x)))\,((G(x), G(-x))) = ((h(x), h(-x)))\,,$$

where

$$h(x) = G_{even}(x)g(x) + G_{odd}(x)g(-x).$$

1.4.3 The set of arrays of the form $\left(\left(g_{even}, \dfrac{1}{g_{even}}\right)\right)$ is an abelian subgroup of $\mathcal{A}_2$ with the multiplication operation

$$\left(\left(g_{even}, \frac{1}{g_{even}}\right)\right)\left(\left(G_{even}, \frac{1}{G_{even}}\right)\right) = \left(\left(g_{even}G_{even}, \frac{1}{g_{even}G_{even}}\right)\right)$$

1.4.4 (i) The set of arrays of the form $((g_{even}, f))$ is a subgroup of $\mathcal{A}_2$ with the multiplication operation

$$((g_{even}, f))\,((G_{even}, F)) = ((g_{even}G_{even}, g_{even}F_{odd} + fF_{even}))$$

(ii) The set of arrays of the form $((g, f_{even}))$ is a subgroup of $\mathcal{A}_2$ with the multiplication operation

$$((g, f_{even}))\,((G, F_{even})) = ((gG_{even} + f_{even}G_{odd}, f_{even}F_{even}))$$

1.4.5 (i) The set of arrays of the form $((g, 1))$ is a subgroup of $\mathcal{A}_2$ with the multiplication operation

$$((g, 1))\,((G, 1)) = ((gG_{even} + G_{odd}, 1))$$

and inverse operation

$$((g, 1))^{-1} = \left(\left(\frac{1 - g_{odd}}{g_{even}}, 1\right)\right).$$

(ii) The set of arrays of the form $((1, f))$ is a subgroup of $\mathcal{A}_2$ with the multiplication operation

$$((1, f))\,((1, F)) = ((1, fF_{even} + F_{odd}))$$

and inverse operation

$$((1, f))^{-1} = \left(\left(1, \frac{1 - f_{odd}}{f_{even}}\right)\right).$$

1.4.6 (i) Let $((G, F))) \in \mathcal{A}_2$. There exists unique power series g and f given by $g = G$, and $f = F_{even} + \dfrac{(1 - G_{odd})}{G_{even}} F_{odd}$ such that the array $((G, F))$ factorises as

$$((G, F)) = ((g, 1))\,((1, f))\,.$$

(ii) Let $((G, F))) \in \mathcal{A}_2$. There exists unique power series g and f given by and $g = G_{even} + \dfrac{(1 - F_{odd})}{F_{even}} G_{odd}$ and $f = F$ such that the array $((G, F))$ factorises as

$$((G, F)) = ((1, f))\,((g, 1))\,.$$

1.4.7 The operator * defined on the group $\mathcal{A}_2$ by $((g, f))^* = ((f, g))$ is an involutive automorphism of $\mathcal{A}_2$. We have

$$((g, f))\,((g, f))^* = ((1, 1))$$

$$\iff f = \frac{1 - g_{odd}g}{g_{even}} \iff g = \frac{1 - f_{odd}f}{f_{even}}.$$

2.1 The exponential Appell group Most of the results in the previous sections can be generalised by working with exponential generating functions rather than ordinary generating functions.

Let $G(x) = 1 + g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \cdots$ be an exponential generating function (EGF). We denote by $[[\, G \,]]$ the lower unitriangular array whose k-th column, $k = 0, 1, 2, \ldots$, has the EGF $\frac{x^k}{k!} G$. The array begins

$$\begin{pmatrix} 1 & & & & & & \\ g_1 & 1 & & & & & \\ g_2 & 2g_1 & 1 & & & & \\ g_3 & 3g_2 & 3g_1 & 1 & & & \\ g_4 & 4g_3 & 6g_2 & 4g_1 & 1 & & \\ g_5 & 5g_4 & 10g_3 & 10g_2 & 5g_1 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}. \tag{7}$$

The set of arrays of this type forms an abelian group under matrix multiplication called the exponential Appell group, which we denote by $\widetilde{\mathcal{A}}_1$.

Example 2. The array $[[\, \exp(x) \,]]$ is Pascal's triangle.

If we define the Hadamard product of arrays (a_{ij}) and(b_{ij}) - denoted by $(a_{ij}) \times_H (b_{ij})$ - to be the array $(a_{ij} b_{ij})$ then (7) is equal to the Hadamard product of Pascal's triangle with an element of the (ordinary) Appell group $\mathcal{A}_1$:

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times_H \begin{pmatrix} 1 & & & & & \\ g_1 & 1 & & & & \\ g_2 & g_1 & 1 & & & \\ g_3 & g_2 & g_1 & 1 & & \\ g_4 & g_3 & g_2 & g_1 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 & & & & & \\ g_1 & 1 & & & & \\ g_2 & 2g_1 & 1 & & & \\ g_3 & 3g_2 & 3g_1 & 1 & & \\ g_4 & 4g_3 & 6g_2 & 4g_1 & 1 & \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

2.2 The generalised exponential Appell group $\widetilde{\mathcal{A}}_2$. We associate to each pair of EGFs $G(x) = 1 + g_1 x + g_2 \frac{x^2}{2!} + g_3 \frac{x^3}{3!} + \cdots$ and $F(x) = 1 + f_1 x + f_2 \frac{x^2}{2!} + f_3 \frac{x^3}{3!} + \cdots$ a lower unitriangular array, denoted by $[[\, G, F \,]]$, whose sequence of column EGFs is given by

$$\left(G, xF, \frac{x^2}{2!} G, \frac{x^3}{3!} F, \frac{x^4}{4!} G, \frac{x^5}{5!} F, \cdots \right).$$

We denote the set of all arrays of this type by $\widetilde{\mathcal{A}}_2$. Clearly the exponential Appell group $\widetilde{\mathcal{A}}_1$ is contained in $\widetilde{\mathcal{A}}_2$.

The array $[[G,F]]$ in $\widetilde{\mathcal{A}}_2$ begins

$$\begin{pmatrix} 1 \\ g_1 & 1 \\ g_2 & 2f_1 & 1 \\ g_3 & 3f_2 & 3g_1 & 1 \\ g_4 & 4f_3 & 6g_2 & 4f_1 & 1 \\ g_5 & 5f_4 & 10g_3 & 10f_2 & 5g_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} \tag{8}$$

and is equal to the Hadamard product of Pascal's triangle with an element of the generalised Appell group $\mathcal{A}_2$:

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \times_H \begin{pmatrix} 1 \\ g_1 & 1 \\ g_2 & f_1 & 1 \\ g_3 & f_2 & g_1 & 1 \\ g_4 & f_3 & g_2 & f_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 1 \\ g_1 & 1 \\ g_2 & 2f_1 & 1 \\ g_3 & 3f_2 & 3g_1 & 1 \\ g_4 & 4f_3 & 6g_2 & 4f_1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Theorem 2 *(i) The set of lower unitriangular arrays*

$$\widetilde{\mathcal{A}}_2 = \left\{ [[G,F]] \mid G(x) = 1 + \sum_{k=1}^{\infty} g_k \frac{x^k}{k!}, F(x) = 1 + \sum_{k=1}^{\infty} f_k \frac{x^k}{k!} \right\}$$

is a group under the operation of matrix multiplication with identity element $[[1,1]]$. *The multiplication operation is given by*

$$[[g,f]] * [[G,F]] = [[gG_{even} + fG_{odd}, fF_{even} + gF_{odd}]].$$

Equivalently,

$$[[g,f]] * [[G,F]] = [[h_1,h_2]]$$
$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} G_{even} & G_{odd} \\ F_{odd} & F_{even} \end{pmatrix} \begin{pmatrix} g \\ f \end{pmatrix}$$

(ii) the inverse of the array $[[G,F]]$ *is*

$$[[G,F]]^{-1} = \left[\left[\frac{F_{even} - G_{odd}}{\triangle}, \frac{G_{even} - F_{odd}}{\triangle} \right]\right]$$

where $\triangle(x) = (G(x)F(-x) + F(x)G(-x))/2$.

Sketch proof. Let $g(x), f(x)$ be a pair of EGFs. Just as in Theorem 1, but with a little more work, one can show that the action of the array $[[\, g, f\,]]$ on the column vector $A = (a_0, a_1, ...)^{\mathrm{T}}$, and by extension its action on the corresponding EGF $A(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + \cdots$, is given by

$$[[\, g, f\,]] * A(x) = g(x)A_{even}(x) + f(x)A_{odd}(x). \tag{9}$$

By definition, the $(2k)^{th}$ column of the array $[[\, G, F\,]]$ has the EGF $\frac{x^{2k}}{(2k)!}G(x)$. Using (9), we see that the EGF of the $(2k)^{th}$ column in the product array $[[\, g, f\,]]\,[[\, G, F\,]]$ is

$$\begin{aligned} [[\, g, f\,]] * \frac{x^{2k}}{(2k)!}G(x) &= g(x)\left(\frac{x^{2k}}{(2k)!}G(x)\right)_{even} + f(x)\left(\frac{x^{2k}}{(2k)!}G(x)\right)_{odd} \\ &= g(x)\frac{x^{2k}}{(2k)!}G_{even}(x) + f(x)\frac{x^{2k}}{(2k)!}G_{odd}(x) \\ &= \frac{x^{2k}}{(2k)!}\left(g(x)G_{even}(x) + f(x)G_{odd}(x)\right). \end{aligned} \tag{10}$$

Similarly, the EGF of the $(2k+1)^{th}$ column in the product $[[\, g, f\,]]\,[[\, G, F\,]]$ is

$$\begin{aligned} [[\, g, f\,]] * \frac{x^{2k+1}}{(2k+1)!}F(x) &= g(x)\left(\frac{x^{2k+1}}{(2k+1)!}F(x)\right)_{even} + f(x)\left(\frac{x^{2k+1}}{(2k+1)!}F(x)\right)_{odd} \\ &= g(x)\frac{x^{2k+1}}{(2k+1)!}F_{odd}(x) + f(x)\frac{x^{2k+1}}{(2k+1)!}F_{even}(x) \\ &= \frac{x^{2k+1}}{(2k+1)!}\left(g(x)F_{odd}(x) + f(x)F_{even}(x)\right). \end{aligned} \tag{11}$$

It follows from (10) and (11) that the matrix product

$$[[\, g, f\,]]\,[[\, G, F\,]] = [[\, gG_{even} + fG_{odd}, fF_{even} + gF_{odd}\,]].$$

It is now an easy calculation to verify the claimed formula for the inverse array $[[\, G, F\,]]^{-1}$. □

2.3 The generalised exponential Appell group $\widetilde{\mathcal{A}}_n$. Let G_i, $i = 0..n-1$, be n EGFs, all with constant term 1. We use the double square brackets notation $[[\, G_0, G_1, \cdots, G_{n-1}\,]]$ to denote the lower unitriangular array with column EGFs generating functions

$$\left(G_0, xG_1, \frac{x^2}{2!}G_2, \cdots, \frac{x^{n-1}}{(n-1)!}G_{n-1}, \frac{x^n}{n!}G_0, \frac{x^{n+1}}{(n+1)!}G_1, \cdots, \frac{x^{2n-1}}{(2n-1)!}G_{n-1}, \cdots\right).$$

and denote the set of arrays of this type by $\widetilde{\mathcal{A}}_n$. A typical element of $\widetilde{\mathcal{A}}_n$ is the Hadamard product of Pascal's triangle with an element of the generalised Appell group $\mathcal{A}_{\mathrm{n}}$. It is straightforward to extend the proof sketched in Theorem 2 to show that $\widetilde{\mathcal{A}}_n$ for $n > 1$ is a group under matrix multiplication. We call the group $\widetilde{\mathcal{A}}_n$ a *generalised exponential Appel group.* The exponential Appell group $\widetilde{\mathcal{A}}_1$ is contained in the group $\widetilde{\mathcal{A}}_n$ for all n. More generally, if n divides m then $\widetilde{\mathcal{A}}_n$ is a subgroup of $\widetilde{\mathcal{A}}_m$.

References

[1] D. E. Davenport, L. W. Shapiro and L. C. Woodson, The Double Riordan Group, The Electronic Journal of Combinatorics, 18(2) (2012).