Contents

Introduction

Alice laughed. "There's no use trying," she said:
"one *can't* believe impossible things."
"I daresay you haven't had much practice," said
the Queen. "When I was your age, I always did it
for half-an-hour a day. Why, sometimes I believed as many as
six impossible things before breakfast."
—Lewis Carroll, *Through the Looking-Glass*[1]

"NOTHING IS IMPOSSIBLE." This platitude is used as inspiration by parents, athletic coaches, motivational speakers, and politicians. The hyperbolic news media is constantly alerting us to individuals who have achieved the impossible. It is one of the tenets of the American dream.

In his valedictory high-school commencement speech of June 24, 1904, Robert Goddard, who would later invent the liquid-fueled rocket, said,[2]

> Just as in the sciences we have learned that we are too ignorant safely to pronounce anything impossible, so for the individual, since we cannot know just what are his limitations, we can hardly say with certainty that anything is necessarily within or beyond his grasp.... It has often proved true that the dream of yesterday is the hope of today and the reality of tomorrow.

However, some things *are* impossible, and mathematics can prove that they are. Some tasks cannot be accomplished, regardless of one's intellect, one's perseverance, or the time available. This book tells the story of four impossible problems, the so-called "problems of antiquity": trisecting an angle, doubling the cube, constructing every regular polygon, and squaring the circle. They are arguably the most famous problems in the history of mathematics.

In a geometry course, students are introduced to the Euclidean tools: a compass to draw circles and a straightedge to draw lines

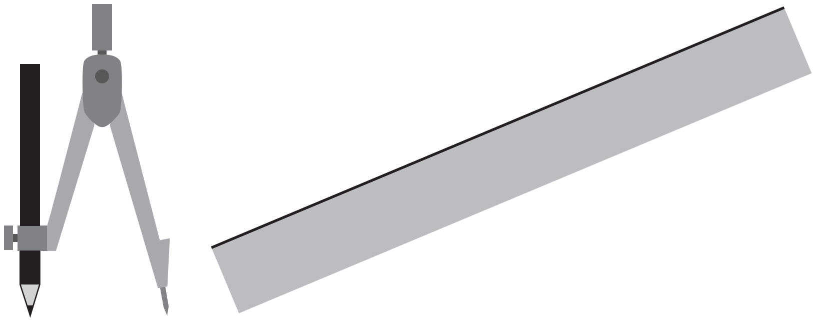

FIGURE I.1. A compass and straightedge.

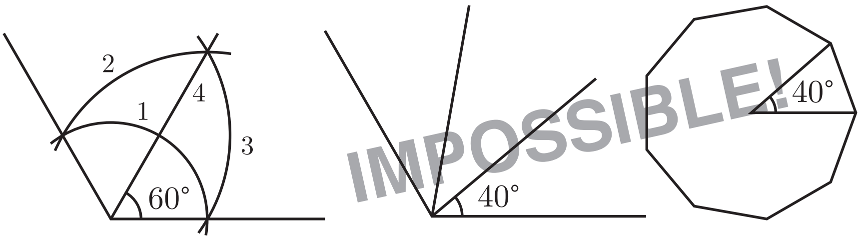

FIGURE I.2. It is possible to bisect a 120° angle using a compass and straightedge, but not to trisect it. Hence, it is impossible to construct a regular nonagon.

(see figure I.1). They learn a variety of basic constructions, such as how to bisect an angle, construct an equilateral triangle, and draw a perpendicular bisector. The problems of antiquity seem—at a glance—just as elementary as those exercises. But they are not. It takes three quick swipes of a compass and a trace along a straightedge to bisect an angle; figure I.2 shows a 120° angle split into two 60° angles. But it is impossible to use these same tools to draw the two rays that trisect a 120° angle; no matter how clever the geometer, it is impossible to construct a 40° angle. Thus, (1) it is impossible to trisect every angle. Moreover, the angle between the center of a regular 9-sided polygon, called a *nonagon*, and two adjacent vertices is $360°/9 = 40°$, so it is impossible to construct the polygon. Hence, (2) it is impossible to construct every regular polygon.

Likewise, (3) given a line segment *AB*, it is impossible to construct a line segment *CD* so that a cube with side length *CD* is twice the volume of a cube with side length *AB*; that is, it is impossible to double the cube. And finally, (4) it is impossible to square the circle: if we begin with any circle, it is impossible to construct a square with the same area.

It is important to point out that these four problems are not *practical* problems. The world is not waiting for a method of constructing a 40° angle or a regular nonagon. The same geometry students could use the protractors in their backpacks to draw a 40° angle. There are other tools that will allow a draftsman or a mathematician to solve each of these problems exactly, and there are numerous techniques that clever craftsmen have devised to get approximations as accurate as desired.

In fact, not only are these problems not practical problems, they are not physical problems at all. They are theoretical problems. More important than the constructions themselves are the *proofs* that they accomplish what they say they accomplish. How do we know that the angle bisection technique truly bisects the angle? For this we need theoretical mathematics. The primary text on geometry in the Greek era and for many centuries afterward is Euclid's 300 BCE masterwork, *Elements*. Euclid began *Elements* with five postulates, and from these he built up all of geometry. The first three postulates are the compass-and-straightedge postulates. The first postulate states that we can draw a line segment between any two points, and the second says that we can extend this line segment beyond its endpoints. The third says that we can draw a circle with a given center and a point on the circle. Euclid wrote,[3]

> Let the following be postulated:
>
> 1. To draw a straight line from any point to any point.
> 2. To produce a finite straight line continuously in a straight line.
> 3. To describe a circle with any center and any distance.

Thus his geometry was built from lines and circles, and to carry out the geometric techniques, one would use a compass and straightedge.

The problems of antiquity were known to be extremely challenging to the ancient Greeks. They were the subject of intense research by the leading mathematicians of the day. The historian of mathematics

Sir Thomas Heath called these problems "rallying points for [Greek] mathematicians during three centuries at least."[4]

The ancient Greeks knew that it was possible to solve the problems if they were allowed to change the rules. What if we have a compass, a straightedge, and a parabola? Or a hyperbola? Or a new mechanical drawing tool? And so on. For instance, Archimedes (ca. 287–212 BCE) proved that if the straightedge had two marks on it, he could trisect any angle. We will present many ingenious ways to solve these problems with an extended toolkit.

The problems were irresistible to mathematicians. For 2000 years, many of the major mathematical developments were directly or indirectly related to these problems. And the list of mathematicians who made contributions to the understanding of these problems is a who's who of the field. In 1913 Ernest Hobson wrote the following about the problem of squaring the circle, although the same could be said about all four problems:[5]

> When we look back, in the light of the completed history of the problem, we are able to appreciate the difficulties which in each age restricted the progress which could be made within limits which could not be surpassed by the means then available; we see how, when new weapons became available, a new race of thinkers turned to the further consideration of the problem with a new outlook.

Although these problems were actively studied for centuries, they were not proved impossible until the nineteenth century. There are several reasons it took over 2000 years for the proofs. First, mathematicians had to realize that the problems were impossible and not just very difficult. Second, they had to realize that it was possible to prove that a problem is impossible. This task is somewhat surprising—that we can use mathematics to prove that something is mathematically impossible. And lastly, the mathematicians had to invent the mathematical tools required to prove the impossibility. All four problems are geometric. The proofs of impossibility did not come from geometry, but from algebra and a deep understanding of the properties of numbers—not just the integers, but rational, irrational, algebraic, transcendental, and complex numbers. Algebra and a sufficient understanding of the real and complex numbers came long after the Greek period ended.

As algebra developed, mathematicians applied it to these problems. François Viète (1540–1603), René Descartes (1596–1650), and Carl Friedrich Gauss (1777–1855) made some headway toward understanding them. But the proofs of impossibility for three of the four—trisecting an angle, doubling the cube, and constructing regular polygons—were due to one man, a man who died young and, sadly, is not nearly as well known as these others: the French mathematician Pierre Wantzel (1814–1848). In his seven-page 1837 article he proved some preliminary results, applied them to the problems, and, in what may be the greatest single page in mathematics, proved all three problems impossible.

The fourth problem—squaring the circle—is somewhat unique among these problems. It is the most famous one, and it was the last to be proved impossible. While geometry and algebra are sufficient to prove that the other three are impossible, squaring the circle has one other wrinkle—it requires an understanding of the nature of π. If a circle has radius 1 cm, for instance, its area is $\pi \cdot 1^2 = \pi$ cm^2. Then to square the circle the geometer must construct a square with area π. Thus, much of our tale involves the history of this famous, enigmatic number. The problem of squaring the circle was eventually proved impossible by Ferdinand von Lindemann (1852–1939) in 1882; he used Wantzel's results and the fact—proved using calculus and complex analysis—that π is a transcendental number.

Because these four problems have been so famous for so long, a full treatment of their history would require multiple volumes. Moreover, even after the problems were proved impossible, their study continued, and they became subsumed into advanced fields such as abstract algebra and Galois theory. We chose to downplay this generalization—both because the mathematics required to understand it is too advanced for our intended audience and to keep the length of the book reasonable.

The book is organized as follows. The chapters follow the history of these fascinating problems—from the introduction of the problems by the Greeks through to the eventual solutions two millennia later. The chapters are roughly chronological. We give the colorful history of the problems, the history of other methods of solving them, and the histories of the fields that were created to finally resolve them.

There are also many interesting and entertaining side stories connected to these problems. So we inserted mini-chapters called Tangents in between each of the proper chapters. For instance, we discuss a bill passed by the Indiana legislature setting an incorrect value for π; we present a variety of unique methods of solving the problems, such as using origami, a drawing tool called a tomahawk, toothpicks, and a clock; we discuss Leonardo da Vinci's elegant contributions; and we present the two sides of the τ vs. π debate.

Index