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ASYMPTOTIC EXPANSIONS OF THE GAMMA FUNCTION ASSOCIATED WITH THE WINDSCHITL AND SMITH FORMULAS

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Abstract. In this paper, we develop the Windschitl and Smith formulas for the gamma function to complete asymptotic expansions and provide explicit formulas for determining the coefficients of these asymptotic expansions. Furthermore, we establish new asymptotic expansions for the ratio of gamma functions $\Gamma(x+1)/\Gamma(x+\frac{1}{2})$.

1. Introduction

Stirling's formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad n \in \mathbb{N} := \{1, 2, \ldots\} \tag{1.1}$$

has many applications in statistical physics, probability theory and number theory. Actually it was first discovered in 1733 by the French mathematician Abraham de Moivre (1667–1754) in the form

$$n! \sim constant \cdot \sqrt{n}(n/e)^n$$

when he was studying the Gaussian distribution and the central limit theorem. Afterwards, the Scottish mathematician James Stirling (1692–1770) found the missing constant $\sqrt{2\pi}$ when he was trying to give the normal approximation of the binomial distribution.

Stirling's series for the gamma function is given (see [1, p. 257, Eq. (6.1.40)]) by

$$\begin{aligned}\Gamma(x+1) &\sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\sum_{m=1}^{\infty}\frac{B_{2m}}{2m(2m-1)x^{2m-1}}\right) \qquad (1.2)\\ &= \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \exp\left(\frac{1}{12x}-\frac{1}{360x^3}+\frac{1}{1260x^5}-\frac{1}{1680x^7}+\cdots\right)\end{aligned}$$

as $x \to \infty$, where B_n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$) are the Bernoulli numbers (see, for example, [**?**, Section 1.7]). The following asymptotic formula is due to Laplace

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(1+\frac{1}{12x}+\frac{1}{288x^2}-\frac{139}{51\,840x^3}-\frac{571}{2\,488\,320x^4}+\cdots\right) \tag{1.3}$$

as $x \to \infty$ (see [1, p. 257, Eq. (6.1.37)]). The expression (1.3) is sometimes incorrectly called Stirling's series (see [10, pp. 2–3]). Stirling's formula is in fact the first approximation to the

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asymptotic formula (1.3). Stirling's formula has attracted much interest of many mathematicians and has motivated a large number of research papers concerning various generalizations and improvements (see [4, 5, 7, 11, 12, 13, 14, 17, 18, 19, 21, 22] and the references cited therein). See also an overview at [16].

Windschitl (see [3, p. 128], [23] and [24]) presented that

$$\Gamma(x+1)=\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+O\left(\frac{1}{x^5}\right)\right),\qquad x\to\infty. \tag{1.4}$$

Inspired by (1.4), Alzer [2] proved that for all $x>0$,

$$\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\alpha}{x^5}\right)<\Gamma(x+1)<\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\frac{\beta}{x^5}\right) \tag{1.5}$$

with the best possible constants $\alpha=0$ and $\beta=1/1620$.

Very recently, Lu *et al.* [12] extended Windschitl's formula as follows:

$$\Gamma(n+1)\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(n\sinh\left(\frac{1}{n}+\frac{a_7}{n^7}+\frac{a_9}{n^9}+\frac{a_{11}}{n^{11}}+\cdots\right)\right)^{n/2}, \tag{1.6}$$

where

$$a_7=\frac{1}{810},\quad a_9=-\frac{67}{42\,525},\quad a_{11}=\frac{19}{8505},\ldots. \tag{1.7}$$

However, the authors did not give the general formula for the coefficients a_j $(j\geq 7)$ in (1.6). Subsequently, Chen [5] gave a recurrence relation formula for determining the coefficient of $\frac{1}{n^j}$ $(j\in\mathbb{N})$ in (1.6). Also in [5], Chen developed Windschitl's approximation formula to a new asymptotic expansion:

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2+\sum_{j=0}^{\infty}r_jx^{-j}},\qquad x\to\infty, \tag{1.8}$$

and provided a recurrence relation for determining the coefficients r_j in (1.8).

Smith [23, Eq. (43)] presented the following analogous result to (1.4):

$$\Gamma\left(x+\frac{1}{2}\right)=\sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1+O\left(\frac{1}{x^5}\right)\right),\qquad x\to\infty. \tag{1.9}$$

The first aim of this paper is to develop the Windschitl and Smith formulas for the gamma function to complete asymptotic expansions and provide explicit formulas for determining the coefficients of these asymptotic expansions. More precisely, we provide explicit formulas for determining the coefficients λ_j,μ_j,α_j, and β_j $(j\in\mathbb{N})$ such that

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x\left(x\sinh\frac{1}{x}\right)^{x/2}\exp\left(\sum_{j=1}^{\infty}\frac{\lambda_j}{x^j}\right),$$

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x\sinh\frac{1}{x}\right)^{x/2}\left(1+\sum_{j=1}^{\infty}\frac{\mu_j}{x^j}\right),$$

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\exp\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{x^j}\right)$$

and

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x\tanh\frac{1}{2x}\right)^{x/2}\left(1+\sum_{j=1}^{\infty}\frac{\beta_j}{x^j}\right)$$

as $x \to \infty$.

The problem of finding new inequalities and asymptotic formulas for the gamma function Γ and in particular about the Wallis ratio

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} = \frac{1}{\sqrt{\pi}} \cdot \frac{(2n)!!}{(2n-1)!!} \quad \text{for} \quad n \in \mathbb{N} \tag{1.10}$$

has attracted the attention of many researchers (see [8, 9, 20] and references therein).

From (1.4) and (1.9), we derive

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \left(1+O\left(\frac{1}{x^5}\right)\right), \qquad x \to \infty.$$

This fact motivated us to establish new asymptotic expansions for the ratio of gamma functions $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$, which is the second aim of this paper. More precisely, we provide explicit formulas for determining the coefficients θ_j and ϑ_j such that

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \exp\left(\sum_{j=1}^{\infty}\frac{\theta_j}{x^j}\right)$$

and

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \left(1+\sum_{j=1}^{\infty}\frac{\vartheta_j}{x^j}\right)$$

as $x \to \infty$.

2. Lemmas

The following lemmas are required in our present investigation.

Lemma 1 (see [6]). *Let $a_0 = 1$ and*

$$g(x) \sim \sum_{n=0}^{\infty} a_n x^{-n}$$

be a given asymptotic expansion. Then the composition $\ln(g(x))$ *has asymptotic expansion of the following form*

$$\ln(g(x)) \sim \sum_{n=1}^{\infty} b_n x^{-n},$$

where

$$b_n = a_n - \frac{1}{n}\sum_{k=1}^{n-1} k b_k a_{n-k}, \qquad n \in \mathbb{N}.$$

Lemma 2 (see [6])**.** *Let*

$$g(x) \sim \sum_{n=1}^{\infty} a_n x^{-n}, \qquad x \to \infty.$$

be a given asymptotic expansion. Then the composition $\exp(g(x))$ *has asymptotic expansion of the following form*

$$\exp(g(x)) \sim \sum_{n=0}^{\infty} b_n x^{-n}, \qquad x \to \infty,$$

where

$$b_0 = 1, \quad b_n = \frac{1}{n}\sum_{k=1}^{n} k a_k b_{n-k}, \qquad n \in \mathbb{N}.$$

3. Main results

Theorem 1. *The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \exp\left(\sum_{j=1}^{\infty} \frac{\lambda_j}{x^j}\right), \qquad x \to \infty \tag{3.1}$$

with the coefficients λ_j *given by*

$$\lambda_j = \frac{B_{j+1}}{j(j+1)} - \frac{q_{j+1}}{2}, \qquad j \in \mathbb{N}, \tag{3.2}$$

where

$$q_j = c_j - \frac{1}{j}\sum_{k=1}^{j-1} k q_k c_{j-k}, \qquad j \in \mathbb{N}$$

with

$$c_{2j} = \frac{1}{(2j+1)!} \quad \textit{and} \quad c_{2j+1} = 0, \qquad j \in \mathbb{N}_0.$$

Here, B_n *are the Bernoulli numbers, and an empty sum (as usual) is understood to be nil.*

Proof. Write (3.1) as

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)-\frac{x}{2}\ln\left(x\sinh\frac{1}{x}\right)\sim\sum_{j=1}^{\infty}\frac{\lambda_j}{x^j},\qquad x\to\infty. \tag{3.3}$$

It follows from (1.2) that

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{2\pi x}(x/e)^x}\right)\sim\sum_{j=1}^{\infty}\frac{B_{j+1}}{j(j+1)}\frac{1}{x^j},\qquad x\to\infty. \tag{3.4}$$

It is well-known (see [1, p. 85, Equation (4.5.62)]) that

$$\sinh z=\sum_{j=0}^{\infty}\frac{z^{2j+1}}{(2j+1)!},\qquad |z|<\infty. \tag{3.5}$$

Let the sequence (c_j) be defined by

$$c_{2j}=\frac{1}{(2j+1)!}\quad\text{and}\quad c_{2j+1}=0,\qquad j\in\mathbb{N}_0.$$

Then, the formula (3.5) can be written as

$$\sinh z=\sum_{j=0}^{\infty}c_j z^{j+1},\qquad |z|<\infty.$$

By Lemma 1, we have

$$\ln\left(x\sinh\frac{1}{x}\right)=\ln\left(1+\sum_{j=1}^{\infty}\frac{c_j}{x^j}\right)\sim\sum_{j=1}^{\infty}\frac{q_j}{x^j},\qquad x\to\infty, \tag{3.6}$$

with

$$q_j=c_j-\frac{1}{j}\sum_{k=1}^{j-1}kq_kc_{j-k},\qquad j\geq 1,$$

where an empty sum (as usual) is understood to be nil. Substituting (3.4) and (3.6) into (3.3) yields

$$\sum_{j=1}^{\infty}\frac{B_{j+1}}{j(j+1)}\frac{1}{x^j}-\sum_{j=0}^{\infty}\frac{q_{j+1}}{2x^j}\sim\sum_{j=1}^{\infty}\frac{\lambda_j}{x^j},\qquad x\to\infty. \tag{3.7}$$

Noting that $q_1=c_1=0$, it follows from (3.7) that

$$\sum_{j=1}^{\infty}\left(\frac{B_{j+1}}{j(j+1)}-\frac{q_{j+1}}{2}\right)\frac{1}{x^j}\sim\sum_{j=1}^{\infty}\frac{\lambda_j}{x^j},\qquad x\to\infty. \tag{3.8}$$

Equating coefficients of the term x^{-j} on both sides of (3.8) yields

$$\lambda_j=\frac{B_{j+1}}{j(j+1)}-\frac{q_{j+1}}{2},\qquad j\in\mathbb{N}.$$

The proof of Theorem 1 is complete. □

Here, from (3.1), we give the following explicit asymptotic expansion:

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \times \exp\left(\frac{1}{1620x^5} - \frac{11}{18900x^7} + \frac{143}{170100x^9} - \frac{2260261}{1178793000x^{11}} + \cdots\right) \tag{3.9}$$

as $x \to \infty$.

Using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, from (3.9) we deduce that

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \times \left(1 + \frac{1}{1620x^5} - \frac{11}{18900x^7} + \frac{143}{170100x^9} + \frac{1}{5248800x^{10}} - \cdots\right) \tag{3.10}$$

as $x \to \infty$.

Even though as many coefficients as we please in the right-hand side of (3.10) can be obtained by using Mathematica, here we aim at giving a formula for determining these coefficients. Using Lemma 2 and Theorem 1, we immediately obtain the following

Theorem 2. *The gamma function has the following asymptotic expansion:*

$$\Gamma(x+1) \sim \sqrt{2\pi x}\left(\frac{x}{e}\right)^x \left(x \sinh\frac{1}{x}\right)^{x/2} \left(\sum_{j=0}^{\infty} \frac{\mu_j}{x^j}\right), \qquad x \to \infty \tag{3.11}$$

with the coefficients μ_j given by

$$\mu_0 = 1, \quad \mu_j = \frac{1}{j}\sum_{k=1}^{j} k\lambda_k \mu_{j-k}, \qquad j \in \mathbb{N}, \tag{3.12}$$

where λ_j $(j \in \mathbb{N})$ are defined by (3.2).

In 2014, Lu *et al.* [12] showed by numerical computations that Windschitl's approximation formula

$$\Gamma(n+1) \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(n \sinh\left(\frac{1}{n} + \frac{1}{810n^7}\right)\right)^{n/2} := \nu_n \quad \text{(see [3, p. 128])} \tag{3.13}$$

is stronger than other known formulas such as:

$$n! \approx \sqrt{2\pi n}e^{-n}\left(n + \frac{1}{12n - \frac{1}{10n + \frac{-\frac{2369}{252}}{n}}}\right)^n \quad \text{(Mortici [17])}, \tag{3.14}$$

$$n! \approx \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^{n+\frac{1}{2}} \exp\left(\frac{1}{12n} - \frac{1}{12n^2} + \frac{29}{360n^3} - \frac{3}{40n^4}\right) \quad \text{(Mortici [21])}, \tag{3.15}$$

$$n! \approx \sqrt{\frac{2\pi}{e}}\left(\frac{n+1}{e}\right)^n \left(n^3 + \frac{5}{4}n^2 + \frac{17}{32}n + \frac{172}{1920}\right)^{1/6} \quad \text{(Mortici [18])}, \tag{3.16}$$

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n^2-\frac{1}{10}}\right)^n \quad \text{(Nemes [22])}. \tag{3.17}$$

Very recently, Chen [5] showed by numerical computations that the following approximation formula:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(n\sinh\frac{1}{n}\right)^{\frac{n}{2}+\frac{1}{270n^3}} := \lambda_n \quad \text{(Chen [5])} \tag{3.18}$$

is a little stronger than the formula (3.13).

It follows from (3.9) and (3.10) that

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(n\sinh\frac{1}{n}\right)^{n/2}\exp\left(\frac{1}{1620n^5}-\frac{11}{18900n^7}\right) := u_n \tag{3.19}$$

and

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(n\sinh\frac{1}{n}\right)^{n/2}\left(1+\frac{1}{1620n^5}-\frac{11}{18900n^7}\right) := v_n. \tag{3.20}$$

It is observed from Table 1 that, for $n \geq 2$, the formulas (3.19) and (3.20) are stronger than the formula (3.18).

Table 1. Comparison among approximation formulas (3.18) to (3.20).

n	$\frac{n!-\lambda_n}{n!}$	$\frac{n!-u_n}{n!}$	$\frac{n!-v_n}{n!}$
2	3.29×10^{-6}	1.0981761×10^{-6}	1.0982847×10^{-6}
10	5.532×10^{-11}	$8.22120727 \times 10^{-13}$	$8.22139422 \times 10^{-13}$
100	5.613×10^{-18}	$8.40490272 \times 10^{-22}$	$8.40492177 \times 10^{-22}$
1000	5.433×10^{-25}	$8.40680034 \times 10^{-31}$	$8.40680224 \times 10^{-31}$

Theorem 3. *The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x\left(2x\tanh\frac{1}{2x}\right)^{x/2}\exp\left(\sum_{j=1}^{\infty}\frac{\alpha_j}{x^j}\right), \qquad x \to \infty \tag{3.21}$$

with the coefficients α_j ($j \in \mathbb{N}$) given by

$$\alpha_j = -\frac{(1-2^{-j})B_{j+1}}{j(j+1)} - \frac{p_{j+1}}{2}, \qquad j \in \mathbb{N}, \tag{3.22}$$

where

$$p_j = \omega_j - \frac{1}{j}\sum_{k=1}^{j-1} k p_k \omega_{j-k}, \qquad j \in \mathbb{N} \tag{3.23}$$

with

$$\omega_j = \frac{4(2^{j+2}-1)B_{j+2}}{(j+2)!}, \qquad j \in \mathbb{N}. \tag{3.24}$$

Here, B_n are the Bernoulli numbers, and an empty sum (as usual) is understood to be nil.

Proof. Write (3.21) as

$$\ln\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)-\frac{x}{2}\ln\left(2x\tanh\frac{1}{2x}\right)\sim\sum_{j=1}^{\infty}\frac{\alpha_j}{x^j},\qquad x\to\infty. \tag{3.25}$$

The logarithm of gamma function has asymptotic expansion (see [15, p. 32]):

$$\ln\Gamma(x+t)\sim\left(x+t-\frac{1}{2}\right)\ln x-x+\frac{1}{2}\ln(2\pi)+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}B_{n+1}(t)}{n(n+1)}\frac{1}{x^n} \tag{3.26}$$

as $x\to\infty$, where $B_n(t)$ denotes the Bernoulli polynomials defined by the following generating function:

$$\frac{xe^{tx}}{e^x-1}=\sum_{n=0}^{\infty}B_n(t)\frac{x^n}{n!}. \tag{3.27}$$

Note that the Bernoulli numbers B_n $(for\ n\in\mathbb{N}_0)$ are defined by (3.27) for $t=0$. It is well-known (see [1, p. 805]) that

$$B_n(\tfrac{1}{2})=-(1-2^{1-n})B_n,\qquad n\in\mathbb{N}_0.$$

Setting $t=\frac{1}{2}$ in (3.26) yields

$$\ln\left(\frac{\Gamma(x+\frac{1}{2})}{\sqrt{2\pi}\,(x/e)^x}\right)\sim\sum_{j=2}^{\infty}\frac{-(1-2^{1-j})B_j}{j(j-1)x^{j-1}},\qquad x\to\infty. \tag{3.28}$$

The Maclaurin series of $\tanh(z)$ (see [1, p. 85, Equation (4.5.64)])

$$\tanh z=\sum_{j=2}^{\infty}\frac{2^j(2^j-1)B_j}{j!}z^{j-1},\qquad |z|<\frac{\pi}{2}$$

yields

$$2x\tanh\frac{1}{2x}\sim 1+\sum_{j=1}^{\infty}\frac{\omega_j}{x^j},\qquad x\to\infty,$$

where

$$\omega_j=\frac{4(2^{j+2}-1)B_{j+2}}{(j+2)!},\qquad j\in\mathbb{N}.$$

By Lemma 1, we have

$$\ln\left(2x\tanh\frac{1}{2x}\right)\sim\ln\left(1+\sum_{j=1}^{\infty}\frac{\omega_j}{x^j}\right)\sim\sum_{j=1}^{\infty}\frac{p_j}{x^j},\qquad x\to\infty, \tag{3.29}$$

with

$$p_j=\omega_j-\frac{1}{j}\sum_{k=1}^{j-1}kp_k\omega_{j-k},\qquad j\in\mathbb{N},$$

where an empty sum (as usual) is understood to be nil. Substituting (3.28) and (3.29) into (3.40) yields

$$\sum_{j=2}^{\infty} \frac{-(1-2^{1-j})B_j}{j(j-1)x^{j-1}} - \sum_{j=1}^{\infty} \frac{p_j}{2x^{j-1}} \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{x^j}, \qquad x \to \infty. \tag{3.30}$$

Noting that $p_1 = \omega_1 = 0$, it follows from (3.30) that

$$\sum_{j=1}^{\infty} \left(\frac{-(1-2^{-j})B_{j+1}}{j(j+1)} - \frac{p_{j+1}}{2} \right) \frac{1}{x^j} \sim \sum_{j=1}^{\infty} \frac{\alpha_j}{x^j}, \qquad x \to \infty. \tag{3.31}$$

Equating coefficients of the term x^{-j} on both sides of (3.31) yields

$$\alpha_j = -\frac{(1-2^{-j})B_{j+1}}{j(j+1)} - \frac{p_{j+1}}{2}, \qquad j \in \mathbb{N}.$$

The proof of Theorem 3 is complete. □

Here, from (3.21), we give the following explicit asymptotic expansion:

$$\begin{aligned} \Gamma\left(x+\frac{1}{2}\right) &\sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \\ &\quad \times \exp\left(-\frac{31}{51840x^5} + \frac{1397}{2419200x^7} - \frac{10439}{12441600x^9} \right. \\ &\qquad \left. + \frac{4626754267}{2414168064000x^{11}} - \frac{26820782411}{4184557977600x^{13}} + \cdots \right) \end{aligned} \tag{3.32}$$

as $x \to \infty$.

Using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, from (3.32) we deduce that

$$\begin{aligned} \Gamma\left(x+\frac{1}{2}\right) &\sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \\ &\quad \times \left(1 -\frac{31}{51840x^5} + \frac{1397}{2419200x^7} - \frac{10439}{12441600x^9} \right. \\ &\qquad \left. + \frac{961}{5374771200x^{10}} + \frac{4626754267}{2414168064000x^{11}} - \cdots \right) \end{aligned} \tag{3.33}$$

as $x \to \infty$.

Using Lemma 2 and Theorem 3, we immediately obtain Theorem 4 below. Theorem 4 gives a recurrence relation for determining the coefficients in (3.33).

Theorem 4. *The gamma function has the following asymptotic expansion:*

$$\Gamma\left(x+\frac{1}{2}\right) \sim \sqrt{2\pi}\left(\frac{x}{e}\right)^x \left(2x \tanh \frac{1}{2x}\right)^{x/2} \left(\sum_{j=0}^{\infty} \frac{\beta_j}{x^j} \right), \qquad x \to \infty \tag{3.34}$$

with the coefficients β_j ($j \in \mathbb{N}$) given by

$$\beta_0 = 1, \quad \beta_j = \frac{1}{j} \sum_{k=1}^{j} k\alpha_k \beta_{j-k}, \qquad j \in \mathbb{N}, \tag{3.35}$$

where α_j $(j\in\mathbb{N})$ are defined by (3.22).

Theorem 5. *The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}\sim\sqrt{x}\left(\cosh\frac{1}{2x}\right)^{x}\exp\left(\sum_{j=1}^{\infty}\frac{\theta_j}{x^j}\right) \tag{3.36}$$

with the coefficients θ_j $(j\in\mathbb{N})$ given by

$$\theta_j=\frac{\big(1-(-1)^{j+1}(2^{-j}-1)\big)B_{j+1}}{j(j+1)}-r_{j+1},\qquad j\in\mathbb{N}, \tag{3.37}$$

where

$$r_j=d_j-\frac{1}{j}\sum_{k=1}^{j-1}kr_kd_{j-k},\qquad j\in\mathbb{N} \tag{3.38}$$

with

$$d_{2j}=\frac{1}{2^{2j}(2j)!}\quad\text{and}\quad d_{2j+1}=0,\qquad j\in\mathbb{N}_0. \tag{3.39}$$

Here, B_n are the Bernoulli numbers, and an empty sum (as usual) is understood to be nil.

Proof. Write (3.36) as

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)-x\ln\left(\cosh\frac{1}{2x}\right)\sim\sum_{j=1}^{\infty}\frac{\theta_j}{x^j},\qquad x\to\infty. \tag{3.40}$$

From (3.26), we obtain, as $x\to\infty$,

$$\left[\frac{\Gamma(x+t)}{\Gamma(x+s)}\right]^{1/(t-s)}\sim x\exp\left(\frac{1}{t-s}\sum_{j=1}^{\infty}\frac{(-1)^{j+1}\big(B_{j+1}(t)-B_{j+1}(s)\big)}{j(j+1)}\frac{1}{x^j}\right). \tag{3.41}$$

Setting $(s,t)=(\frac{1}{2},1)$ and noting that

$$B_n(0)=(-1)^nB_n(1)=B_n\quad\text{and}\quad B_n\left(\tfrac{1}{2}\right)=(2^{1-n}-1)B_n\quad\text{for}\quad n\in\mathbb{N}_0$$

(see [1, p. 805]), from (3.41) we obtain, as $x\to\infty$,

$$\ln\left(\frac{\Gamma(x+1)}{\sqrt{x}\Gamma(x+\frac{1}{2})}\right)\sim\sum_{j=1}^{\infty}\frac{\big(1-(-1)^{j+1}(2^{-j}-1)\big)B_{j+1}}{j(j+1)}\frac{1}{x^j}. \tag{3.42}$$

The Maclaurin series of $\cosh z$ (see [1, p. 85, Equation (4.5.63)])

$$\cosh z=\sum_{j=0}^{\infty}\frac{z^{2j}}{(2j)!},\qquad |z|<\infty$$

yields

$$\cosh\frac{1}{2x}=\sum_{j=0}^{\infty}\frac{1}{2^{2j}(2j)!x^{2j}},\qquad x\neq 0. \tag{3.43}$$

Let the sequence (d_j) be defined by

$$d_{2j} = \frac{1}{2^{2j}(2j)!} \quad \text{and} \quad d_{2j+1} = 0, \qquad j \in \mathbb{N}_0.$$

Then, the formula (3.43) can be written as

$$\cosh \frac{1}{2x} = \sum_{j=0}^{\infty} \frac{d_j}{x^j}, \qquad x \neq 0. \tag{3.44}$$

By Lemma 1, we have

$$\ln \left(\cosh \frac{1}{2x} \right) = \ln \left(1 + \sum_{j=1}^{\infty} \frac{d_j}{x^j} \right) \sim \sum_{j=1}^{\infty} \frac{r_j}{x^j}, \qquad x \to \infty, \tag{3.45}$$

where

$$r_j = d_j - \frac{1}{j} \sum_{k=1}^{j-1} k r_k d_{j-k}, \qquad j \in \mathbb{N}.$$

Substituting (3.42) and (3.45) into (3.40) yields

$$\sum_{j=1}^{\infty} \frac{\big(1 - (-1)^{j+1}(2^{-j} - 1)\big) B_{j+1}}{j(j+1)} \frac{1}{x^j} - \sum_{j=1}^{\infty} \frac{r_j}{x^{j-1}} \sim \sum_{j=1}^{\infty} \frac{\theta_j}{x^j}, \qquad x \to \infty. \tag{3.46}$$

Noting that $r_1 = d_1 = 0$, it follows from (3.46) that

$$\sum_{j=1}^{\infty} \left(\frac{\big(1 - (-1)^{j+1}(2^{-j} - 1)\big) B_{j+1}}{j(j+1)} - r_{j+1} \right) \frac{1}{x^j} \sim \sum_{j=1}^{\infty} \frac{\theta_j}{x^j}, \qquad x \to \infty. \tag{3.47}$$

Equating coefficients of the term x^{-j} on both sides of (3.47) yields

$$\theta_j = \frac{\big(1 - (-1)^{j+1}(2^{-j} - 1)\big) B_{j+1}}{j(j+1)} - r_{j+1}, \qquad j \in \mathbb{N}.$$

The proof of Theorem 5 is complete. □

Here, from (3.36), we give the following explicit asymptotic expansion:

$$\frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \sim \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \times \exp \left(\frac{7}{5760x^5} - \frac{187}{161280x^7} + \frac{48763}{29030400x^9} - \frac{29383393}{7664025600x^{11}} + \cdots \right) \tag{3.48}$$

as $x \to \infty$.

Using $e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$, from (3.48) we deduce that

$$\frac{\Gamma(x+1)}{\Gamma\left(x + \frac{1}{2}\right)} \sim \sqrt{x} \left(\cosh \frac{1}{2x} \right)^x \times \left(1 + \frac{7}{5760x^5} - \frac{187}{161280x^7} + \frac{48763}{29030400x^9} + \frac{49}{66355200x^{10}} - \cdots \right) \tag{3.49}$$

as $x \to \infty$.

Using Lemma 2 and Theorem 5, we immediately obtain Theorem 6 below. Theorem 6 gives a recurrence relation for determining the coefficients in (3.49).

Theorem 6. *The following asymptotic expansion holds:*

$$\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \sim \sqrt{x}\left(\cosh\frac{1}{2x}\right)^x \left(\sum_{j=0}^{\infty}\frac{\vartheta_j}{x^j}\right) \tag{3.50}$$

with the coefficients ϑ_j $(j \in \mathbb{N})$ *given by*

$$\vartheta_0 = 1, \quad \vartheta_j = \frac{1}{j}\sum_{k=1}^{j} k\theta_k\vartheta_{j-k}, \qquad j \in \mathbb{N}, \tag{3.51}$$

where θ_j $(j \in \mathbb{N})$ *are defined by* (3.37).

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