arXiv:1402.3321v2 [quant-ph] 9 Apr 2015

On the entanglement of formation of two-mode Gaussian states: a compact form

Y. Akbari-Kourbolagh and **H. Alijanzadeh-Boura**

Department of Physics, Azarbaijan Shahid Madani University,
Tabriz 53741-161, Iran
yakbari@azaruniv.edu, h.alijanzadeh@umz.ac.ir

Abstract

We define an EPR-like uncertainty by using the Duan et al.'s inequality which gives a necessary and sufficient condition for the separability of any two-mode Gaussian state. We show that for a given amount of entanglement, the uncertainty is minimized by pure two-mode squeezed states. Using this fact, we write the optimal pure-state decomposition and derive a compact form for the entanglement of formation of two-mode Gaussian states. For illustration purposes, we consider symmetric and squeezed thermal states as special cases and evaluate their entanglement of formation explicitly. For the symmetric states, our result is in agreement with the Giedke et al.'s one. To our knowledge, our work is the first one which gives the exact entanglement of formation of two-mode squeezed thermal states explicitly.

PACS numbers: 03.67.Mn, 03.65.Ud, 42.50.Dv
Key Words: Gaussian state, entanglement of formation, EPR-like uncertainty, optimal pure-state decomposition, squeezed states

1 Introduction

One of the main goals of quantum information science is to quantify the entanglement or inseparability of quantum states. For a pure bipartite state $|\psi\rangle$, it is well known that a convenient measure of entanglement is the von Neumann entropy [1, 2, 3]

$$E(\psi) = -\mathrm{Tr}(\rho_A \log_2 \rho_A) = -\mathrm{Tr}(\rho_B \log_2 \rho_B),$$

where $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ are its reduced states. $E(\psi)$ is called the entropy of entanglement or the entanglement for simplicity. However, there exists no unique measure of entanglement in the case of mixed bipartite states and several measures of entanglement have been introduced in this case [4].

The entanglement of formation (EOF) is one of the measures with an attractive physical motivation. For a bipartite mixed state, Bennet et al. [5] have defined this measure as the minimal amount of average entanglement for any ensemble of bipartite pure states realizing the state. Explicitly, the EOF of a mixed bipartite state ρ is defined as

$$E_F(\rho) := \inf\{\sum_k p_k E(|\psi_k\rangle\langle\psi_k|) : \quad \rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|\},$$

where the infimum is taken over all possible pure-state convex decompositions of ρ.

In quantum information with continuous variables, Gaussian states play an important role because they can be created relatively easily and can be used in quantum cryptography and quantum teleportation tasks [6, 7, 8]. The first calculation of the exact EOF in an infinite-dimensional Hilbert space has been performed in the Giedke et al.'s remarkable work [9] where they have evaluated the exact EOF of a symmetric two-mode Gaussian state by establishing a connection between its EPR-like uncertainty and entanglement. In [10], Wolf et al. introduced a Gaussian version of the EOF for bipartite Gaussian states by considering merely their decompositions into pure Gaussian states and showed that for symmetric two-mode Gaussian states, the Gaussian EOF coincides with the exact EOF. In [11], Adesso et al. computed Gaussian EOF for two important families of non-symmetric two-mode Gaussian states with extremal negativities at fixed global and local purities. For an arbitrary two-mode Gaussian state (TMGS), J. S. Ivan and R. Simon [12] have computed the EOF based on a conjecture. Marians [13] have shown that the EOF of a TMGS coincides with its Gaussian EOF and developed an insightful approach of evaluating the exact EOF. Rigolin et al [14] have derived two lower bounds on the EOF of arbitrary mixed TMGSs. Oliveira et al. [15] have established tight upper and lower bounds for the EOF of an arbitrary TMGS employing the necessary properties of Gaussian channels. In the recent solution of the Gaussian optimizer problem, Giovannetti et al. [16] have computed the EOF for a family of non-symmetric TMGSs and shown that it coincides with the Gaussian EoF.

Here, we derive a compact form for the EOF of arbitrary TMGS in terms of four parameters which specify the standard form of its covariance matrix (CM). We also obtain explicit results in the special cases of symmetric TMGS and squeezed thermal states. For the symmetric states, our result is in agreement with the Giedke et al.'s one. To our knowledge, our work is the first one which gives the exact entanglement of formation of two-mode squeezed thermal states explicitly. To achieve this compact form, we define an EPR-like uncertainty based on Duan et al.'s inequality [17]. Then we connect the entanglement of pure states with their EPR-like uncertainties and show that pure two-mode squeezed states are the least entangled states for a given uncertainty of this type. Ultimately, we show that the pure-state decomposition which leads to the EOF is established by a two-mode squeezed state together with its displaced versions. The advantage of our work is that it reduces the evaluation of EOF for the TMGSs to the solution of two simple algebraic equations and provides a simple method for determining the exact EOF of any TMGS.

This paper is structured as follows: In Section 2 we provide a brief description of the two-mode Gaussian states, including the standard form of their covariance matrices and Duan et al.'s inequality. In Section 3 we derive a compact form for the EOF of an arbitrary TMGS. In Section 4 we present two examples to illustrate the topic and compare our results with other ones. The paper is ended with a brief conclusion in section 5.

2 Gaussian states

Let us consider a Gaussian state σ of two single modes A and B described by the amplitude operators $\hat{a}_A = \frac{\hat{x}_A + i\hat{p}_A}{\sqrt{2}}$ and $\hat{a}_B = \frac{\hat{x}_B + i\hat{p}_B}{\sqrt{2}}$, respectively, in which the canonical quadrature operators $\hat{x}_k$, $\hat{p}_l$ have the commutators $[\hat{x}_k, \hat{p}_l] = i\,\delta_{kl}$ for all $k, l = A, B$. Introducing the notation $\hat{\xi} = (\hat{x}_A,\, \hat{p}_A,\, \hat{x}_B,\, \hat{p}_B)$, the commutators can be expressed as

$$[\hat{\xi}_k, \hat{\xi}_l] = i\Omega_{k,l} I \quad , \quad k, l = 1, 2, 3, 4 \tag{1}$$

where I is the identity operator and

$$\Omega = \bigoplus_{i=1}^{2} J \quad with \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

The state σ can also be specified by its characteristic function (CF)

$$\chi_\sigma(\xi) = \mathrm{Tr}(\sigma \hat{D}(\xi)), \tag{2}$$

where $\xi = (x_A, p_A, x_B, p_B)^T$ is a real vector and $\hat{D}(\xi)$ is a two-mode Weyl displacement operator

$$\hat{D}(\xi) = \exp(i\xi^T \Omega \, \hat{\xi}). \tag{3}$$

in which T means transposition. The CF of TMGS σ has the following form

$$\chi_\sigma(\xi) = \exp(i\xi^T \Omega d - \frac{1}{4}\xi^T \Omega^T \gamma_\sigma \Omega \xi), \tag{4}$$

in which d is its real displacement vector and γ_σ is its covariance matrix (CM) defined by

$$d_k = \mathrm{Tr}(\hat{\xi}_k \sigma) \quad , \quad \gamma_{\sigma_{k,l}} = \mathrm{Tr}[(\hat{\xi}_k \hat{\xi}_l + \hat{\xi}_l \hat{\xi}_k)\sigma] - 2\mathrm{Tr}(\hat{\xi}_k \sigma)\, \mathrm{Tr}(\hat{\xi}_l \sigma). \tag{5}$$

By definition, the CM γ is a real, symmetric and positive 4×4 matrix. It turns out that a matrix γ is a bona fide CM iff it satisfies the uncertainty relation [18]

$$\gamma + i\Omega \geq 0. \tag{6}$$

Note that the displacement vector d can always be shifted to zero by a sequence of local unitary operations. Hence, it is irrelevant for the study of entanglement and without lose of generality we may take it to be zero and work only with TMGS of zero displacement vector which is completely characterized by its CM.

In lemma 2 of [17] it has been shown that the CM of a TMGS can be transformed to the following standard form

$$\gamma = \begin{pmatrix} nr_1 & 0 & \sqrt{r_1 r_2}k_x & 0 \\ 0 & n/r_1 & 0 & k_p/\sqrt{r_1 r_2} \\ \sqrt{r_1 r_2}k_x & 0 & mr_2 & 0 \\ 0 & k_p/\sqrt{r_1 r_2} & 0 & m/r_2 \end{pmatrix}. \tag{7}$$

Here r_1, r_2 are arbitrary non-negative one-mode squeezing factors, $n, m \geq 1$, $k_x \geq -k_p > 0$ and we have

$$\frac{nr_1 - 1}{mr_2 - 1} = \frac{n/r_1 - 1}{m/r_2 - 1}, \tag{8}$$

$$|\sqrt{r_1 r_2}k_x| - |k_p/\sqrt{r_1 r_2}| = \sqrt{(nr_1 - 1)(mr_2 - 1)} - \sqrt{(n/r_1 - 1)(m/r_2 - 1)}. \tag{9}$$

It has been proved that Eqs. (8) and (9) have at least one solution $r_1, r_2 \geq 1$ for a given set of parameters n, m, k_x and k_p [17]. Therefore, the CM of a Gaussian state can be completely described by these four parameters. For example, a standard two-mode squeezed vacuum state $|\psi_r\rangle$ with squeezing parameter $r > 0$ is a TMGS for which $r_1 = r_2 = 1$ and the CM parameters are $n = m = \cosh 2r$ and $k_x = -k_p = \sinh 2r$.

For a given bipartite pure state $|\psi\rangle$, let us write its Schmidt decomposition as

$$|\psi\rangle = \sum_{N=0}^{\infty} c_N |u_N\rangle_A \otimes |v_N\rangle_B, \tag{10}$$

where $c = (c_0, c_1, ...)$ is the set of non-negative Schmidt coefficients in decreasing order with $\|c\|^2 := \sum_{N=0}^{\infty} c_N^2 = 1$ and $\{|u_N\rangle_A\}$ and $\{|v_N\rangle_B\}$ are orthonormal bases in the Hilbert spaces of modes A and B, respectively. Then the entropy of entanglement is expressed as

$$E(\psi) = e(c) := -\sum_{N=0}^{\infty} c_N^2 \log_2(c_N^2). \tag{11}$$

For example, the state $|\psi_r\rangle$ has the Schmidt form [9]

$$|\psi_r\rangle = \sum_{N=0}^{\infty} c_N |N\rangle_A \otimes |N\rangle_B \quad , \quad c_N = \frac{\tanh^N r}{\cosh r}, \tag{12}$$

where $\{|N\rangle_A\}$ ($\{|N\rangle_B\}$) is the standard Fock basis for the mode A (B) such that $\hat{a}_A^\dagger \hat{a}_A |N\rangle_A = N|N\rangle_A$ and $\hat{a}_B^\dagger \hat{a}_B |N\rangle_B = N|N\rangle_B$. The entropy of entanglement for $|\psi_r\rangle$ is calculated to be

$$E(\psi_r) = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r). \tag{13}$$

Duan et al. [17] have shown that any bipartite separable quantum state ϱ satisfies the inequality

$$\langle(\Delta\hat{u})^2\rangle_\varrho + \langle(\Delta\hat{v})^2\rangle_\varrho \geq a^2 + \frac{1}{a^2}, \tag{14}$$

where $\hat{u} = |a|\,\hat{x}_A + \frac{\hat{x}_B}{a}$ and $\hat{v} = |a|\,\hat{p}_A - \frac{\hat{p}_B}{a}$ are EPR-like operators, a is a real nonzero parameter and $\langle(\Delta\hat{u})^2\rangle_\varrho = \mathrm{Tr}(\varrho\hat{u}^2) - [\mathrm{Tr}(\varrho\hat{u})]^2$ is the variance of $\hat{u}$. Furthermore, in proposition 2 they have proved that inequality (14) is a necessary and sufficient condition for the separability of TMGSs provided that $a = -a_0$ with $a_0^2 = \sqrt{\frac{mr_2-1}{nr_1-1}} = \sqrt{\frac{m/r_2-1}{n/r_1-1}}$. For inseparable states, however, the uncertainty relation $\langle(\Delta\hat{u})^2\rangle_\varrho + \langle(\Delta\hat{v})^2\rangle_\varrho \geq |\langle[\hat{u},\hat{v}]\rangle_\varrho|$ requires that the total variance of operators $\hat{u}$ and $\hat{v}$ to be larger than or equal to $|a^2 - \frac{1}{a^2}|$, which reduces to zero for the case $|a| = 1$.

3 Entanglement of formation of Gaussian states

For any bipartite quantum state ρ, inequality (14) allows us to define an EPR-like uncertainty as follows

$$\Delta(\rho) := \min\{1, \frac{\langle(\Delta\hat{u})^2\rangle_\rho + \langle(\Delta\hat{v})^2\rangle_\rho}{a^2 + \frac{1}{a^2}}\}. \tag{15}$$

It is obvious from the definition and the foregoing facts that $\Delta(\rho) \in [b, 1]$ with $b := |a^2 - \frac{1}{a^2}|/(a^2 + \frac{1}{a^2}) = \sqrt{1 - \frac{4}{(a^2+\frac{1}{a^2})^2}}$. By inequality (14), $b \leq \Delta(\rho) < 1$ is an evidence for the inseparability of ρ. It is interesting to note that by introducing

$$\sin\theta := \frac{|a|}{\sqrt{a^2 + \frac{1}{a^2}}} \quad , \quad \cos\theta := \frac{1}{|a|\sqrt{a^2 + \frac{1}{a^2}}},$$

our definition of $\Delta(\rho)$ in Eq. (15) takes the same form of the generalized EPR correlation Λ_θ introduced in [12].

For a bipartite pure state $|\psi\rangle$ with $b \leq \Delta(\psi) < 1$ and with Schmidt decomposition (10), Eq. (15) gives

$$\begin{array}{c}\Delta(\psi) = 1 + \frac{2a^4}{1+a^4}\sum_{N=0}^{\infty} c_N^2 \langle u_N|\hat{a}_A^\dagger \hat{a}_A|u_N\rangle + \frac{2}{1+a^4}\sum_{N=0}^{\infty} c_N^2 \langle v_N|\hat{a}_B^\dagger \hat{a}_B|v_N\rangle \\ -\frac{2}{a^2+\frac{1}{a^2}}\sum_{N,M=0}^{\infty} c_N c_M \{\langle u_M|\hat{a}_A|u_N\rangle\langle v_M|\hat{a}_B|v_N\rangle + c.c.\}.\end{array} \tag{16}$$

Also for a TMGS σ for which the displacement vector is zero and the CM is given by Eq. (7), we have

$$\Delta(\sigma) = \min\{1, \frac{a^2\frac{nr_1+n/r_1}{2} + \frac{mr_2+m/r_2}{2a^2} + \frac{|a|}{a}(\sqrt{r_1r_2}k_x - \frac{k_p}{\sqrt{r_1r_2}})}{a^2 + \frac{1}{a^2}}\}. \tag{17}$$

In the case of $|\psi_r\rangle$ this reduces to

$$\Delta(\psi_r) = \min\{1, \cosh(2r) + \frac{2|a|}{a}\frac{\sinh(2r)}{a^2 + \frac{1}{a^2}}\}. \tag{18}$$

This reveals the entanglement of $|\psi_r\rangle$ when $\Delta(\psi_r) \in [b, 1)$, i.e.,

$$0 < \tanh r < -2\frac{|a|}{a}\frac{1}{a^2 + \frac{1}{a^2}} \ . \tag{19}$$

Since $0 < \tanh r < 1$, Eq. (19) implies that $a < 0$. Therefor, we have

$$\Delta(\psi_r) = \cosh(2r) - \frac{2}{a^2 + \frac{1}{a^2}}\sinh(2r) \quad , \quad a < 0 \quad , \quad \tanh r < \frac{2}{a^2 + \frac{1}{a^2}}. \tag{20}$$

The EPR-like uncertainty $\Delta(\psi_r)$ has the minimum value b. Hereafter, we consider only the case of $a < 0$.

For a fixed value of the parameter a, any value of $\Delta \in [b, 1)$ can be achieved by a two-mode squeezed state. To show this fact, we set $\Delta(\psi_r) = \Delta$ in Eq. (20) and solve it for r. It is easy to see that this equation has two solutions for r provided that $\Delta \geq b$. To determine which solution gives the desired two-mode squeezed state, we plot $\Delta(\psi_r)$

versus r in the range $(0, \tanh^{-1} \frac{2}{a^2+1/a^2})$ (Figure 1). As the figure shows, for small r, $\Delta(\psi_r)$ decreases with r, whereas for large r, it increases. We will show in the Lemma 2 below that the smaller solution has the right behavior and hence it is the relevant one. Denoting this solution by $r_{\Delta'}$, we have:

$$e^{-2r_{\Delta'}} = \frac{\Delta + \sqrt{\Delta^2 - b^2}}{1 + \sqrt{1 - b^2}} := \Delta'. \tag{21}$$

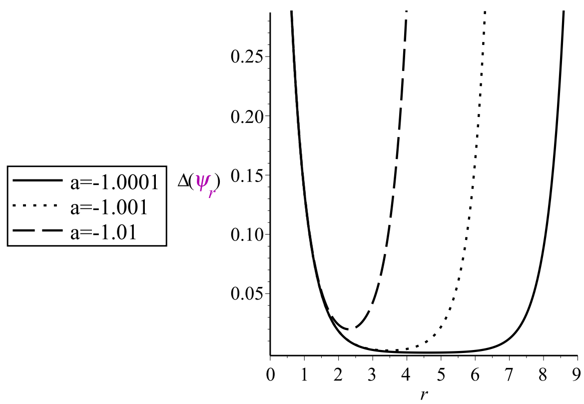

Figure 1: $\Delta(\psi_r)$ in terms of r for various values of a.

Given the above definitions and facts, we are now in a position to state the following proposition which is a generalized form of the proposition 1 of Giedke et al.'s work [9].

Proposition 1: For all pure states $|\psi\rangle$ of a two-mode system with $\Delta(\psi) \in [b, 1]$ we have $E(\psi) \geq E\left(\psi_{r_{\Delta'(\psi)}}\right)$, where

$$\Delta'(\psi) := \frac{\Delta(\psi) + \sqrt{\Delta^2(\psi) - b^2}}{1 + \sqrt{1 - b^2}} \quad , \quad e^{-2r_{\Delta'(\psi)}} := \Delta'(\psi). \tag{22}$$

Like Giedke et al., we prove this proposition by two lemmas and one definition. For a given $c = (c_0, c_1, ...)$, we define

$$\delta(c) = 1 + 2\sum_{N=0}^{\infty}(c_N^2 - \frac{2}{a^2 + \frac{1}{a^2}}\, c_N c_{N-1})N. \tag{23}$$

It is obvious that $\delta(c) \leq 1$ when $c_N \leq \frac{2}{a^2+1/a^2} c_{N-1}$ for all N, and by Eq. (16) we have $\delta(c) = \Delta(\psi)$ whenever in Eq. (10) the Schmidt basis coincides with Fock basis.

Lemma 1: For all $|\psi\rangle$ with Schmidt decomposition (10) which have the same set of Schmidt coefficients satisfying the constraints $c_N \leq \frac{2}{a^2+1/a^2}c_{N-1}$ for all N, we have $\Delta(\psi) \geq \delta(c)$ and hence $\Delta'(\psi) \geq \delta'(c)$, where

$$\delta'(c) := \frac{\delta(c) + \sqrt{\delta^2(c) - b^2}}{1 + \sqrt{1 - b^2}}. \tag{24}$$

Proof: Since in this case $\delta(c) \leq 1$ we only need to consider $|\psi\rangle$ with $b \leq \Delta(\psi) < 1$. From Eq. (16), we note that $\Delta(\psi) \geq \min\{Z(u), Z(v)\} := Z$, where

$$Z = 1 + 2\sum_{N=0}^{\infty} c_N^2 \langle u_N|\hat{a}_A^\dagger \hat{a}_A|u_N\rangle - \frac{4}{a^2 + \frac{1}{a^2}} \sum_{N,M=0}^{\infty} c_N c_M |\langle u_N|\hat{a}_A^\dagger|u_M\rangle|^2. \tag{25}$$

Let us now define $P := \sum_{N=0}^{\infty} c_N^2 |u_N\rangle\langle u_N|$ which is a density operator with eigenvalues c_N^2 and eigenvectors $|u_N\rangle$. By this, Eq. (25) takes the form

$$Z = 1 + 2\,\mathrm{Tr}(P\hat{a}_A^\dagger \hat{a}_A) - \frac{4}{a^2 + \frac{1}{a^2}} \mathrm{Tr}(\sqrt{P}\,\hat{a}_A^\dagger \sqrt{P}\,\hat{a}_A). \tag{26}$$

Since the trace operation is basis independent, we take the first trace in the Fock basis and get

$$Z = 1 + 2\sum_{N=0}^{\infty} N\langle N|P|N\rangle - \frac{4}{a^2 + \frac{1}{a^2}} \mathrm{Tr}(\sqrt{P}\,\hat{a}_A^\dagger \sqrt{P}\,\hat{a}_A). \tag{27}$$

Our aim is to show that $Z \geq \delta(c)$, i. e.

$$\begin{aligned} 1 + 2\textstyle\sum_{N=0}^{\infty} N\langle N|P|N\rangle - \frac{4}{a^2+\frac{1}{a^2}} \mathrm{Tr}(\sqrt{P}\,\hat{a}_A^\dagger \sqrt{P}\,\hat{a}_A) \\ \geq 1 + 2\textstyle\sum_{N=0}^{\infty} (c_N^2 - \frac{2}{a^2+\frac{1}{a^2}}\, c_N c_{N-1})N. \end{aligned} \tag{28}$$

This inequality can be rewritten as

$$\frac{2}{a^2 + \frac{1}{a^2}} \left(\mathrm{Tr}(\sqrt{P}\,\hat{a}_A^\dagger \sqrt{P}\,\hat{a}_A) - \sum_{N=0}^{\infty} c_N c_{N-1} N \right) \leq \sum_{N=0}^{\infty} N\langle N|P|N\rangle - \sum_{N=0}^{\infty} c_N^2 N. \tag{29}$$

As it was shown in [9], the inequality (29) is valid for the case of $a = -1$. Since the factor $2/(a^2 + \frac{1}{a^2})$ is positive and less than or equal to one, it is obvious that the inequality will also be valid for other values of a provided that we prove the non-negativity of the right hand side. For this purpose, we use the Schur-Horn theorem [19, 20]. Based on this theorem, if $\mathcal{A}$ be a self-adjoint operator in a finite dimensional Hilbert space or a positive compact operator in an infinite-dimensional Hilbert space with eigenvalues vector $\lambda(\mathcal{A}) = (\lambda_1 \geq \lambda_2 \geq \cdots)$ and if $\mathrm{diag}(\mathcal{A}) = (\mu_1 \geq \mu_2 \geq \cdots)$ denotes a vector whose components are the diagonal entries of $\mathcal{A}$ with respect to some orthonormal basis, then we have

$$\sum_{i=1}^{n}\mu_i \leq \sum_{i=1}^{n}\lambda_i\ ,\quad n=1,2,\cdots;\qquad \sum_i \mu_i = \sum_i \lambda_i.$$

In our discussion, P is a positive compact operator with eigenvalues $(c_0^2 \geq c_1^2 \geq \cdots)$, the Fock basis is an orthonormal basis and $\mathrm{diag(P)} = (\langle 0|\mathrm{P}|0\rangle \geq \langle 1|\mathrm{P}|1\rangle \geq \cdots)$. Hence, by the Schur-Horn theorem we have

$$\langle 0|P|0\rangle \leq c_0^2\ ,\ \langle 0|P|0\rangle + \langle 1|P|1\rangle \leq c_0^2 + c_1^2\ ,\ \cdots\ ,\ \sum_{N=0}^{\infty}\langle N|P|N\rangle = \sum_{N=0}^{\infty} c_N^2 = 1. \qquad (30)$$

To reach the final result, we rewrite the right hand side of the inequality (29) as follows:

$$\begin{aligned}&(\textstyle\sum_{N=1}^{\infty}\langle N|P|N\rangle - \sum_{N=1}^{\infty}c_N^2) + (\sum_{N=2}^{\infty}\langle N|P|N\rangle - \sum_{N=2}^{\infty}c_N^2) + \cdots\\ &= (c_0^2 - \langle 0|P|0\rangle) + (c_0^2 + c_1^2 - (\langle 0|P|0\rangle + \langle 1|P|1\rangle)) + \cdots.\end{aligned}$$

By Eq. (30), all terms in the brackets are non-negative and this gives rise to the non-negativity of the right hand side of inequality (29). In this way, we obtain the required result

$$\Delta(\psi) \geq Z \geq \delta(c).$$

Lemma 1 indicates that for a given set of Schmidt coefficients c, that is, for a given amount of entanglement, EPR-like uncertainties are minimized if the Schmidt vectors are Fock states in the right order, i.e. $|u_N\rangle = |v_N\rangle = |N\rangle$ for all N.

Lemma 2: For a given $\Delta \in [b,1)$ and any sequence c with $\|c\| = 1$ and $\delta(c) = \Delta$, we have $e(c) \geq e(c^{(\Delta)})$, where $c^{(\Delta)}$ is the unique geometric sequence with $\|c^{(\Delta)}\| = 1$ and $\delta(c^{(\Delta)}) = \Delta$.

Proof: As in [9], the method of Lagrange multipliers is used. The Lagrange functional is

$$F(c,\lambda,\mu) := e(c) + \frac{\lambda}{2\ln(2)}[\delta(c) - \Delta] + \frac{\mu+1}{\ln(2)}(\|c\| - 1), \qquad (31)$$

where $\frac{\lambda}{2\ln(2)}$ and $\frac{\mu+1}{\ln(2)}$ are positive Lagrange multipliers designed to simplify the subsequent expressions. Putting from (11) and (23) and setting to zero the derivative of Lagrange functional $F(c,\lambda,\mu)$ with repect to c_N, we obtain

$$c_N[N\lambda + \mu - \ln(c_N^2)] = \frac{\lambda}{a^2 + \frac{1}{a^2}}\,[Nc_{N-1} + (N+1)c_{N+1}], \qquad (32)$$

where we have defined $c_{-1} = 1$. It is clear from Eq. (32) that $c_N \neq 0$. Thus we can divide Eq. (32) by c_N and then subtract the same expression with N replaced by $N+1$.

Defining $x_N := \frac{c_{N+1}}{\frac{2}{a^2+1/a^2} c_N} := \exp(-2\kappa_N) \in (0,1]$ for $N = 0, 1, ...,$ and $\frac{2}{a^2+\frac{1}{a^2}} := e^{-2\beta}$, we find

$$x_{N+1} = x_N - (A_N + B_N), \tag{33}$$

where

$$A_N = \frac{4e^{4\beta}}{N+2}\left[e^{-2\beta}\sinh^2(\kappa_N+\beta) - e^{-\beta}\sinh\beta - \frac{2}{\lambda}(\kappa_N+\beta)\right], \tag{34}$$

$$B_N = \frac{Ne^{4\beta}}{N+2}\left[\frac{1}{x_N} - \frac{1}{x_{N-1}}\right]. \tag{35}$$

Clearly $B_0 = 0$ and the value of A_0 is fixed for given values of $\lambda > 0$ and x_0. There exist three possibilities: (i) $A_0 > 0$. Then, Eq. (33) gives $x_1 = x_0 - A_0 < x_0$ and hence $\kappa_1 > \kappa_0$. This yields $B_1 > 0$. We now show that $A_0 > 0$ also requires that $A_1 > 0$. When $A_0 > 0$, then we have

$$\frac{2}{\lambda} < \frac{e^{-2\beta}\sinh^2(\kappa_0+\beta)}{\kappa_0+\beta} - \frac{e^{-\beta}\sinh\beta}{\kappa_0+\beta}, \tag{36}$$

and therefore A_1 satisfies

$$\frac{3A_1}{4e^{4\beta}(\kappa_1+\beta)} = \left[e^{-2\beta}\frac{\sinh^2(\kappa_1+\beta)}{\kappa_1+\beta} - \frac{e^{-\beta}\sinh\beta}{\kappa_1+\beta} - \frac{2}{\lambda}\right]$$

$$> \left[e^{-2\beta}\left(\frac{\sinh^2(\kappa_1+\beta)}{\kappa_1+\beta} - \frac{\sinh^2(\kappa_0+\beta)}{\kappa_0+\beta}\right) + e^{-\beta}\left(\frac{1}{\kappa_0+\beta} - \frac{1}{\kappa_1+\beta}\right)\sinh\beta\right],$$

where in the last line we utilized Eq. (36). It can easily be checked that for $\kappa_1 > \kappa_0$, the right hand side of the above inequality is positive and hence $A_1 > 0$. Therefore, $x_2 = x_1 - (A_1 + B_1) < x_1$. In this manner, we conclude that x_N is decreasing. So x_N can achieve a negative value for some finite N, which is impossible. (ii) $A_0 < 0$. Then a similar argument as in the first case shows that x_N is increasing. So the normalization condition of c cannot be fulfilled. Hence, the only possibility is that (iii) $A_0 = 0$. This leads to the equality of all x_N and hence all κ_N. Denoting the common values of κ_N by α, we have $x_N = e^{-2\alpha}$ and hence by iteration $c_N = c_0 e^{-2N(\alpha+\beta)}$ for all N. To make $A_0 = 0$, we have to take the Lagrange multiplier λ to be

$$\lambda = \frac{2e^{2\beta}(\alpha+\beta)}{\sinh^2(\alpha+\beta) - e^{\beta}\sinh\beta}.$$

By applying $\sum_{N=0}^{\infty} c_N^2 = 1$, we get

$$c_0^2 = 1 - e^{-4(\alpha+\beta)}. \tag{37}$$

Putting the expression of c_N in Eq. (23), gives

$$\Delta = \delta(c) = 1 - \frac{2(1-e^{2\alpha})}{1-e^{4(\alpha+\beta)}}. \tag{38}$$

On the other hand, we know that the same value of $\Delta = \delta(c)$ can also be achieved by the two-mode squeezed state $|\psi_{r_{\Delta'}}\rangle$ with $\Delta' = \delta'(c)$. Therefore, by Eq. (20) we can also write

$$\Delta = \Delta(\psi_{r_{\Delta'}}) = \cosh(2r_{\Delta'}) - \frac{2}{a^2 + \frac{1}{a^2}} \sinh(2r_{\Delta'}). \tag{39}$$

By equating two expressions (38) and (39) of Δ, we obtain

$$e^{-2(\alpha+\beta)} = \tanh(r_{\Delta'}). \tag{40}$$

Hence, Eq. (37) gives $c_0 = \frac{1}{\cosh r_{\Delta'}}$. Finally, we have

$$c_N^{(\Delta)} = c_0 e^{-2N(\alpha+\beta)} = \frac{\tanh^N r_{\Delta'}}{\cosh r_{\Delta'}}.$$

As we have mentioned before and it is also clear from Figure 1, there exist two different $r_{\Delta'}$ which give rise to the same value of Δ. To decide which of them minimizes $e(c)$, let us calculate $e(c^{(\Delta)})$ as

$$e(c^{(\Delta)}) = -\ln(1-x) - x\ln x\ , \quad x := \exp[-4(\alpha+\beta)].$$

Using the facts that the function $e(c^{(\Delta)})$ is increasing in x and the smaller value of $r_{\Delta'}$ corresponds to the smaller value of x by Eq. (40), we conclude that for a given Δ, the smaller $r_{\Delta'}$ minimizes $e(c)$ and it is the desired solution. Therefore, among states with the same amount of EPR-like uncertainty, the squeezed state $|\psi_{r_{\Delta'}}\rangle$ with smaller $r_{\Delta'}$ has minimal entanglement.

Finally, with the help of Lemmas 1 and 2, Proposition 1 can be proved as in [9]. However, we include their proof here for completeness.

Proof of proposition 1 : Given a two-mode state $|\psi\rangle$, the proposition is trivial in the case of $\Delta(\psi) = 1$. Since, in this case Eq. (21) gives $r_{\Delta'(\psi)} = 0$ and hence by Eq. (13) we have $E(\psi_{r_{\Delta'(\psi)}=0}) = 0$. For the case of $\Delta(\psi) \in [b, 1)$, using Eq. (11) and Lemma 2, we have

$$E(\psi) = e(c) \geq e(c^{(\Delta)}) = E(\psi_{r_{\delta'(c)}}).$$

Furthermore, it follows from Lemma 1 and Eq. (21) that $r_{\Delta'(\psi)} \leq r_{\delta'(c)}$. As $E(\psi_r)$ increases in a monotonic manner with r, we have

$$E(\psi_{r_{\delta'(c)}}) \geq E(\psi_{r_{\Delta'(\psi)}})$$

which completes the proof.

So far, we have shown that for a given value of the EPR-like uncertainty in the range $[b, 1)$, the pure two-mode squeezed state with smaller squeezing parameter has minimal entanglement among all two-mode pure states, or equivalently, that for a given amount of entanglement, the pure two-mode squeezed state has minimal EPR-like uncertainty. This equivalence follows from Lemma 1 and the fact that for a pure two-mode squeezed state, the entanglement and EPR-like uncertainty behave quite oppositely versus squeezing parameter. Consequently, for a TMGS σ with the EPR-like uncertainty $\Delta(\sigma)$ given by Eq. (17), it is expected and will be proved in Proposition 2 that the optimal pure-state decomposition would consist of the standard two-mode squeezed state with squeezing parameter $r_{\Delta'(\sigma)}$ and all of its displaced versions

$$\sigma = \int d\xi g(\xi)\hat{D}(\xi)|\psi_{r_{\Delta'(\sigma)}}\rangle\langle\psi_{r_{\Delta'(\sigma)}}|\hat{D}^{\dagger}(\xi). \tag{41}$$

Here $g(\xi)$ is a weight function. To calculate $g(\xi)$, we multiply both sides of Eq. (41) by $\hat{D}(\eta)$ and then take the trace

$$\mathrm{Tr}\left(\hat{D}(\eta)\sigma\right) = \int d\xi g(\xi)\mathrm{Tr}\left(\hat{D}(\eta)\hat{D}(\xi)|\psi_{r_{\Delta'(\sigma)}}\rangle\langle\psi_{r_{\Delta'(\sigma)}}|\hat{D}^{\dagger}(\xi)\right). \tag{42}$$

Using the identity $\hat{D}(\eta)\hat{D}(\xi) = \hat{D}(\xi)\hat{D}(\eta)\exp\left(-i\eta^T\Omega\xi\right)$ and the cyclic property of trace operation, we have

$$\mathrm{Tr}\left(\hat{D}(\eta)\sigma\right) = \int d\xi g(\xi)\exp\left(-i\eta^T\Omega\xi\right)\mathrm{Tr}(\hat{D}(\eta)|\psi_{r_{\Delta'(\sigma)}}\rangle\langle\psi_{r_{\Delta'(\sigma)}}|). \tag{43}$$

Inserting

$$\mathrm{Tr}\left(\hat{D}(\eta)\sigma\right) = \chi_\sigma(\eta) = \exp(-\frac{1}{4}\eta^T\Omega^T\gamma_\sigma\Omega\eta),$$

and

$$\mathrm{Tr}(\hat{D}(\eta)|\psi_{r_{\Delta'(\sigma)}}\rangle\langle\psi_{r_{\Delta'(\sigma)}}|) = \chi_{\psi_{r_{\Delta'(\sigma)}}}(\eta) = \exp(-\frac{1}{4}\eta^T\Omega^T\gamma_{\psi_{r_{\Delta'(\sigma)}}}\Omega\eta)$$

we get

$$\exp\left[-\frac{1}{4}\eta^T\Omega^T(\gamma_\sigma - \gamma_{\psi_{r_{\Delta'(\sigma)}}})\Omega\eta\right] = \int d\xi g(\xi)\exp(-i\eta^T\Omega\xi), \tag{44}$$

in which $\gamma_\sigma - \gamma_{\psi_{r_{\Delta'(\sigma)}}} \geq 0$. Multiplying both sides by $\exp(i\eta^T\Omega\xi')$, taking the integral over η and using the following integral identities [21]

$$\begin{aligned} \int d^{2n}\lambda\exp(-\tfrac{1}{2}\lambda^TQ\lambda + i\lambda^Tx) &= \frac{(2\pi)^n\exp(-\frac{1}{2}x^TQ^{-1}x)}{\sqrt{\det(Q)}}, \\ \int d^{2n}\eta\exp\left[i\eta^T\Omega(\xi-\xi')\right] &= (2\pi)^{2n}\delta^{2n}(\xi-\xi'), \end{aligned} \tag{45}$$

with Q a real positive definite symmetric matrix, finally $g(\xi)$ is calculated to be

$$g(\xi) = \frac{1}{\pi^2\sqrt{\det(\gamma_\sigma - \gamma_{\psi_{r_{\Delta'(\sigma)}}})}}\exp\left[-\xi^T(\gamma_\sigma - \gamma_{\psi_{r_{\Delta'(\sigma)}}})^{-1}\xi\right]. \tag{46}$$

Since $\hat{D}(\xi)$ are local unitary operators, the average entanglement of the decomposition (41) is equal to $E[\psi_{r_{\Delta'(\sigma)}}]$. Introducing the auxiliary function $f:\ (b,1] \to [0,\infty)$

$$f(\Delta) = c_+(\Delta)\log_2[c_+(\Delta)] - c_-(\Delta)\log_2[c_-(\Delta)] \tag{47}$$

as in [9], where $c_\pm(\Delta) = \frac{(\Delta^{-1/2} \pm \Delta^{1/2})^2}{4}$ and f is a convex and decreasing function of Δ, we can write

$$E(\psi_{r_{\Delta'}}) = f(\Delta'). \tag{48}$$

Up to now, the parameter a was assumed to be arbitrary. As mentioned before, for the special value $a = -a_0$ the inequality (14) is a necessary and sufficient condition for the separability of any TMGS. Hence, for this value we have $\Delta(\sigma) < 1$ iff the TMGS σ is entangled. From now on, let us set $a = -a_0$ and denote by $\Delta_0(\sigma)$ the EPR-like uncertainty of an entangled TMGS σ for this value. Then, from Eq. (17) we have

$$\Delta_0(\sigma) = \frac{a_0^2\frac{nr_1+n/r_1}{2} + \frac{mr_2+m/r_2}{2a_0^2} - (\sqrt{r_1r_2}k_x - k_p/\sqrt{r_1r_2})}{a_0^2 + \frac{1}{a_0^2}}. \tag{49}$$

Proposition 2: Let σ be a TMGS with EPR-like uncertainty $\Delta_0(\sigma) \in [b_0, 1)$ where $\Delta_0(\sigma)$ is given by Eq. (49) and $b_0 := \sqrt{1 - \frac{4}{(a_0^2+\frac{1}{a_0^2})^2}}$. Then, we have

$$E_F(\sigma) = f(\Delta_0'(\sigma)), \tag{50}$$

where

$$\Delta_0'(\sigma) = \frac{\Delta_0(\sigma) + \sqrt{\Delta_0^2(\sigma) - b_0^2}}{1 + \sqrt{1 - b_0^2}}. \tag{51}$$

The proof of Proposition 2 is the same as the one presented in [9]. However, we restate the proof for completeness.

Proof: Let D be an arbitrary pure-state decomposition of σ as $\sigma = \sum_k p_k|\psi_k\rangle\langle\psi_k|$ and D_0 be its decomposition given by Eq. (41). Average entanglements of the decompositions D and D_0 are $\bar{E}(D) = \sum_k p_k E(\psi_k)$ and $\bar{E}(D_0) = E(\psi_{r_{\Delta_0'(\sigma)}}) = f(\Delta_0'(\sigma))$, respectively. Now it is enough to prove that $\bar{E}(D) \geq f(\Delta_0'(\sigma))$ for any decomposition D since the decomposition D_0 already saturates this bound. As a consequence of Lemma 1 and Eq. (48), we have

$$E(\psi_k) \geq E(\psi_{r_{\Delta_0'(\psi_k)}}) = f(\Delta_0'(\psi_k)).$$

So, we can write

$$\bar{E}(D) \geq \sum_k p_k f(\Delta_0'(\psi_k)) \geq f(\sum_k p_k \Delta_0'(\psi_k)),$$

where the second inequality follows from convexity of the function f. As a simple result of the definition of EPR-like uncertainty by Eq. (15), we have the inequality $\Delta_0(\sigma) \geq \sum_k p_k \Delta_0(\psi_k)$ and hence $\Delta'_0(\sigma) \geq \sum_k p_k \Delta'_0(\psi_k)$. The latter inequality together with the fact that f is a decreasing function of its argument give $f(\sum_k p_k \Delta'_0(\psi_k)) \geq f(\Delta'_0(\sigma))$ which completes the proof.

4 Discussions and examples

a. Symmetric TMGS

For an entangled symmetric TMGS $\tilde{\sigma}$, we have $n = m$ and $k_x \geq -k_p > 0$. Setting these values in Eqs. (8) and (9) and solving them for r_1 and r_2, give

$$r_1 = r_2 = \sqrt{\frac{n + k_p}{n - k_x}}.$$

In this case, we obtain $a_0 = 1$ and hence $b_0 = 0$. Using these values, Eqs. (49) and (51) yield

$$\Delta'_0(\tilde{\sigma}) = \Delta_0(\tilde{\sigma}) = \sqrt{(n - k_x)(n + k_p)}.$$

Finally, Eq. (50) gives

$$E_F(\tilde{\sigma}) = f\left(\sqrt{(n - k_x)(n + k_p)}\right)$$

which is in agreement with the result of the Giedke et al.'s work [9].

As a confirmation of the result (50), it is now an opportunity to make another proof for the Theorem 1 of [14] based on this result.

Theorem 1 of [14]: The EOF of any TMGS with a given EPR-like uncertainty is greater than or equal to the EOF of a mixed symmetric TMGS with the same EPR-like uncertainty.

Proof: Let σ be an arbitrary TMGS with EPR-like uncertainty $\Delta_0(\sigma)$ and $\tilde{\sigma}$ be a symmetric TMGS with the same EPR-like uncertainty $\Delta_0(\tilde{\sigma}) = \Delta_0(\sigma)$. Using the fact that the function f is a decreasing function of its argument, the problem is reduced to the verification of the validity of inequality $\Delta'_0(\tilde{\sigma}) \geq \Delta'_0(\sigma)$. We have $\Delta'_0(\tilde{\sigma}) = \Delta_0(\tilde{\sigma}) = \Delta_0(\sigma)$. Setting this in Eq. (50), gives

$$\Delta'_0(\sigma) = \frac{\Delta'_0(\tilde{\sigma}) + \sqrt{\Delta'^2_0(\tilde{\sigma}) - b_0^2}}{1 + \sqrt{1 - b_0^2}}.$$

Rewriting this equation as

$$\Delta_0'(\sigma) = \Delta_0'(\tilde{\sigma}) \frac{1 + \sqrt{1 - \frac{b_0^2}{\Delta_0'^2(\tilde{\sigma})}}}{1 + \sqrt{1 - b_0^2}},$$

and using the fact that $\Delta_0'^2(\tilde{\sigma}) \leq 1$, verify the validity of the inequality and hence complete the proof.

b. Two-mode squeezed thermal states

The class of two-mode squeezed thermal states $\check{\sigma}$ is a significant class of TMGSs. For such a state, we have $n \geq m$ and $k_x = -k_p > 0$. In this case, solving Eqs. (8) and (9) gives $r_1 = r_2 = 1$. With these values, we obtain $b_0 = \frac{n-m}{n+m-2}$ and Eqs. (49) and (51) yield

$$\Delta_0(\check{\sigma}) = \frac{n\tilde{m} + m\tilde{n} - 2k_x\sqrt{\tilde{n}\tilde{m}}}{\tilde{n} + \tilde{m}},$$

$$\Delta_0'(\check{\sigma}) = \left[\frac{\sqrt{n\tilde{m} - k_x\sqrt{\tilde{n}\tilde{m}}} + \sqrt{m\tilde{n} - k_x\sqrt{\tilde{n}\tilde{m}}}}{\sqrt{\tilde{n}} + \sqrt{\tilde{m}}}\right]^2,$$

where we have made the definitions $\tilde{n} := n - 1$ and $\tilde{m} := m - 1$.

In [16], Giovannetti et al. determined the EOF for a family of two-mode thermal states with parameters $n = 2(\bar{n} + 1)\kappa - 1$, $m = 2(\bar{n} + 1)\kappa - (2\bar{n} + 1)$ and $k_x = -k_p = 2(\bar{n} + 1)\sqrt{\kappa(\kappa - 1)}$ where $\kappa \in [1, \infty)$ is the gain parameter of an amplifier channel and $\bar{n} \in [0, \infty)$ is the average photon number. Their EOF is independent of $\bar{n}$ and its value is given by $g(\kappa) := \kappa \log_2[\kappa] - (\kappa - 1) \log_2[\kappa - 1]$. However, our method shows that for a given value of κ, the EOF is $\bar{n}$ dependent, its values are always less than $g(\kappa)$ and approaches $g(\kappa)$ as $\bar{n}$ gets large.

c. Our method versus the Gaussian EOF and the lower and upper bounds of EOF

The Gaussian EOF was introduced by Wolf et al. [10] as a version of the EOF for bipartite Gaussian states in which only decompositions into pure Gaussian states are considered. In the case of a symmetric TMGS, this measure coincides with the exact EOF while for a general TMGS it provides an upper bound. For a general TMGS ρ with standard-form parameters (n, m, k_x, k_p), if we define

$$C_x := \begin{pmatrix} n & k_x \\ k_x & n \end{pmatrix} \quad , \quad C_p := \begin{pmatrix} m & k_p \\ k_p & m \end{pmatrix},$$

then, as shown in [10, 11], its Gaussian EOF $E_{GF}(\rho)$ is given by minimum value of the entropy of entanglement of a pure gaussian state ρ_P whose CM is of the form $\gamma_P = \Gamma \oplus \Gamma^{-1}$. Here, Γ is a real positive 2×2 matrix as

$$\Gamma = \begin{pmatrix} x_0 + x_3 & x_1 \\ x_1 & x_0 - x_3 \end{pmatrix},$$

where x_0, x_1 and x_3 are real parameters satisfying

$$\det(\mathrm{C_x} - \Gamma) = \det(\Gamma - \mathrm{C_p^{-1}}) = 0. \tag{52}$$

The reduced covariance matrix $\gamma_P^{(A)}$ of the state ρ_P is

$$\gamma_P^{(A)} = \begin{pmatrix} x_0 + x_3 & 0 \\ 0 & (x_0 - x_3)/\det\Gamma \end{pmatrix}. \tag{53}$$

It was found that the Gaussian EOF $E_{GF}(\rho)$ is given by

$$E_{GF}(\rho) = f[(m_{opt})^{1/2} - (m_{opt} - 1)^{1/2}], \tag{54}$$

in which f is the function defined by Eq. (47) and m_{opt} is the minimum value of the determinant of $\gamma_P^{(A)}$:

$$\det\gamma_{\mathrm{P}}^{(\mathrm{A})} = 1 + \frac{\mathrm{x}_1^2}{\det\Gamma}.$$

To obtain m_{opt}, it is enough to minimize $\det\gamma_{\mathrm{P}}^{(\mathrm{A})}$ under the constraints of Eq. (52).

In the work [14], Rigolin et al have derived a lower bound on the EOF of a general mixed TMGS with CM as in Eq. (7). This lower bound is the EOF of a symmetric TMGS $\tilde{\sigma}$ whose standard-form parameters are

$$n_{\tilde{\sigma}} = m_{\tilde{\sigma}} = \frac{n+m}{2}, \quad (k_x)_{\tilde{\sigma}} = k_x, \quad (k_p)_{\tilde{\sigma}} = k_p,$$

$$(r_1)_{\tilde{\sigma}} = (r_2)_{\tilde{\sigma}} = \left[\frac{(n+m)/2 + k_p}{(n+m)/2 - k_x}\right]^{1/2}.$$

Oliveira et al. [15] established a tight upper bound for the EOF of a general TMGS employing the necessary properties of Gaussian channels. This upper bound is the EOF of a symmetric TMGS $\breve{\sigma}$ provided that a matrix with parameters

$$n_{\breve{\sigma}} = m_{\breve{\sigma}} = m, \quad (k_x)_{\breve{\sigma}} = k_x, \quad (k_p)_{\breve{\sigma}} = k_p,$$

$$(r_1)_{\breve{\sigma}} = (r_2)_{\breve{\sigma}} = \sqrt{\frac{n+k_p}{n-k_x}}.$$

constitutes a bona fide CM for $\breve{\sigma}$. Also they argued that the EOF of a general TMGS ρ with CM as in Eq. (7), must satisfy

$$E_F(\tilde{\sigma}) \le E_F(\rho) \le E_F(\breve{\sigma}) \tag{55}$$

$\mathrm{n, m, k_x, k_p}$	$E_F(\tilde{\sigma})$	Marians $\mathrm{E_F}(\rho)$	our $\mathrm{E_F}(\rho)$	$E_F(\breve{\sigma})$	$\mathrm{E_{GF}}(\rho)$
$2, 1.5, 1.2, -1$	0.28919	0.3836537397	0.3784745926	$--$	0.3836537389
$2, 1.5, 1, -1$	0.14672	0.2027415462	0.2022298409	0.56616	0.2027415477
$3, 2, 1.8, -1.2$	0.00681	0.04851229950	0.04850819279	$--$	0.04851230013
$2.6, 1.7, 1.3, -0.9$	0	0.01198094416	0.01198079698	0.40946	0.01198094462
$3, 2, 1.7, -1.2$	0.00142	0.01398144359	0.01398132663	$--$	0.01398144137
$2.5, 2, 1.3, -1.2$	0.00001	0.002510512206	0.002510511701	0.14838	0.002510512809

Table 1: The first column shows the parameters of the CM in its standard form. The next columns show the values of $E_F(\tilde{\sigma})$ given by [14], $E_F(\rho)$ based on Marians and our approach, $E_F(\breve{\sigma})$ and Gaussian EOF $E_{GF}(\rho)$, respectively. States on rows 1, 3 and 5 do not give a physical state $\breve{\sigma}$.

Also, Marians [13] developed an approach for evaluating the exact EOF of a general TMGS.

In the Table 1, we give upper and lower bounds together with Gaussian EOF and the exact EOFs computed based on Marians and our approach for the six mixed TMGSs given in [14]. As implied by the table, our values of $E_F(\rho)$ fall inside the valid region between upper and lower bounds and are obviously less than Marians values. Also, our values are less than the values of $E_{GF}(\rho)$ as they should be.

5 Conclusion

We have defined an EPR-like uncertainty by using Duan et al.'s inequality and showed that among pure states with a given amount of entanglement, pure squeezed states have the least entanglement. Then, for a TMGS we attained the optimal pure-state decomposition which leads to its EOF and provided a simple method for the evaluation of the EOF. For the two special and important cases of symmetric TMGSs and squeezed thermal states we have determined the EOF explicitly. We expect that this work will provide new insight into the subject of Gaussian states entanglement.

Acknowledgment: We acknowledge very valuable discussions with M. A. Jafarizadeh and also thanks M. A. Fasihi for interesting and useful discussions. This research was supported by a research fund No. 401.22 from Azarbaijan Shahid Madani University.

References

[1] Charles H. Bennett, Gilles Brassard, Sandu Popescu, Benjamin Schumacher, John A. Smolin, and William K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996).

[2] William K. Wootters, *Phys. Rev. Lett.* **80** 2245 (1998).

[3] V. Vedral and M. B. Plenio, *Phys. Rev. A* **57**, 1619 (1998).

[4] M. B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007).

[5] Charles H. Bennett, David P. DiVincenzo, John A. Smolin, and William K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).

[6] A. Ferraro, S. Olivares, and M. G. A. Paris, *Gaussian States in Quantum Information* (Bibliopolis, Napoli, 2005).

[7] J. Eisert and M. B. Plenio, *Int. J. Quantum Inform.* **1**, 479 (2003).

[8] S. L. Braunstein and P. Van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).

[9] G. Giedke, M. M. Wolf, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. Lett.* **91**, 107901 (2003).

[10] M. M. Wolf, G. Giedke, O. Krüger, R. F. Werner, and J. I. Cirac, *Phys. Rev. A* **69**, 052320 (2004).

[11] G. Adesso and F. Illuminati, *Phys. Rev. A* **72**, 032334 (2005).

[12] J. Solomon Ivan and R. Simon, arXiv: 0808.1658v1.

[13] Paulina Marian and Tudor A. Marian, *Phys. Rev. Lett.* **101**, 220403 (2008).

[14] G. Rigolin and C.O. Escobar, *Phys. Rev. A* **69**, 012307 (2004).

[15] F. Nicacio and M. C. de Oliveira, *Phys. Rev. A* **89**, 012336 (2014).

[16] V. Giovannetti, R. Garcia-Patron, N. J. Cerf, and A.S. Holevo, *Nature Photonics* **8**, 796-800 (2014); arXiv:1312.6225.

[17] Lu-Ming Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).

[18] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).

[19] William Arveson, Richard V. Kadison, *Contemporary Mathematics*, **414** no.1, 247-263 (2006).

[20] Victor Kaftal, Gary Weiss, *J. Funct. Anal.*, **259** no. 12, 3115-3162 (2010).

[21] S. Olivares, *Eur. Phys. J. Special Topics*, **203**, 3 (2012).