Exact solitary and periodic-wave solutions of the $K(2,2)$ equation (defocusing branch)

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Abstract

An auxiliary elliptic equation method is presented for constructing exact solitary and periodic travelling-wave solutions of the $K(2,2)$ equation (defocusing branch). Some known results in the literature are recovered more efficiently, and some new exact travelling-wave solutions are obtained. Also, new stationary-wave solutions are obtained.

Key words: Auxiliary elliptic equation method; Solitary-wave solutions; Periodic-wave solutions; $K(2,2)$ equation (defocusing branch).
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1 Introduction

In the 1990s Rosenau (see [1,2], for example) introduced the KdV-like $K(m,n)$ equation, namely

$$u_t \pm (u^m)_x + (u^n)_{xxx} = 0, \quad m,\ n > 1, \tag{1.1}$$

where m and n are integers. The $(+)$ case is known as the focusing branch and the $(-)$ case as the defocusing branch. Rosenau's motivation was to provide a prototypical model in order to try to understand the effect of full nonlinearity and nonlinear dispersion in the context of pattern formation. Of particular interest was

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the exploration of patterns of finite extent and weakly singular patterns where a singularity in the first derivative occurs in a finite (or countable, in the case of periodic waves) number of points, the dependent variable being continuous. The hallmark of the focusing branch is the possibility of compactons, namely solitary waves with compact support (see [1], for example), whereas for the defocusing branch there is the possibility of a variety of wave types including peaks and cusps (see [2], for example).

There appears to be some confusion in the literature regarding what is meant by a peak and a cusp. In this paper we use the terminology given by Lenells [3], namely that a continuous function u has a peak at $\xi = \xi_p$ if u is smooth locally on either side of ξ_p and

$$0 \neq \lim_{\xi \uparrow \xi_p} u_\xi = -\lim_{\xi \downarrow \xi_p} u_\xi \neq \pm\infty, \tag{1.2}$$

whereas u has a cusp at $\xi = \xi_c$ if u is smooth locally on either side of ξ_c and

$$\lim_{\xi \uparrow \xi_c} u_\xi = -\lim_{\xi \downarrow \xi_c} u_\xi = \pm\infty. \tag{1.3}$$

Wave profiles with peaks are called peaked waves or peakons, whereas wave profiles with cusps are cusped waves or cuspons. Boyd [4] used the alternative terminology corner waves for peaked waves.

In this paper attention is confined to the $K(2,2)$ equation (defocusing branch), namely

$$u_t - (u^2)_x + (u^2)_{xxx} = 0. \tag{1.4}$$

In [2], Rosenau used the analogy of a particle moving in a potential field (which is equivalent to phase-plane analysis) to identify some possible bounded travelling-wave solution types to (1.4). However, he gave only one explicit exact solution, namely the expression for a solitary peakon. Wazwaz [5] found explicit expressions for some unbounded solutions. Xu and Tian [6] investigated bounded travelling-wave solutions of the equation

$$u_t + (u^2)_x - (u^2)_{xxx} = 0. \tag{1.5}$$

They called this equation the '$K(2,2)$ equation with osmosis dispersion'. Actually, it is just Eq. (1.4) with $u \to -u$. By using the qualitative analysis methods of dynamical systems and drawing the phase-portrait bifurcation diagram for the travelling-wave system, they classified possible solution types. In two particular cases they obtained expressions for exact solutions of (1.5), namely a solitary peakon and periodic peakons, respectively. (In [6], the latter were referred to as periodic cusp waves.) Using a similar method, Zhou and Tian [7] found exact smooth solitary-wave solutions.

In this paper, we employ an auxiliary elliptic equation method to search for further travelling-wave solutions of the $K(2,2)$ equation (defocusing branch). The solution

procedure is the one that was used to find implicit periodic and solitary travelling-wave solutions of the Degasperis–Procesi (DP) equation in [8, Section 2]. An important feature of the method is that it delivers solutions in which the dependent and independent variable are given in terms of a parameter. Some solutions obtained in this way turn out to be multi-valued. Recently, in the context of some other nonlinear wave equations, Stepanyants [9] and Li [10] have described how such solutions may be interpreted and used to construct composite single-valued solutions.

Zhang and Qiao [11] have investigated the one-parameter families of cuspon and smooth-hump solitary-wave solutions of the DP equation. We have been able to show that that these solutions are equivalent to ones obtained by our method in [8]. In this paper we will demonstrate that the corresponding solutions for the $K(2,2)$ equation (defocusing branch) as obtained by our method can be expressed in an alternative form that could be obtained by the method in [11].

The rest of this paper is organized as follows. In Section 2, we present solutions of the appropriate auxiliary elliptic equation. In Section 3, some known results for the $K(2,2)$ equation (defocusing branch) are recovered and new exact travelling-wave solutions are obtained. In Section 4 we discuss the interpretation of multi-valued solutions. This leads to the construction of some composite single-valued solutions. In Section 5 we show that the inverted-cuspon and smooth-hump solitary-wave solutions may be expressed in an alternative form similar to the ones for the DP equation given in [11]. We also show that one member of the family of smooth-hump solutions may be expressed in explicit form. In Section 6 we obtain stationary composite solutions. Some conclusions are provided in Section 7.

2 The auxiliary elliptic equation

In this section we present solutions of the relevant auxiliary elliptic equation.

Consider the following elliptic equation

$$(\phi\phi_\xi)^2 = \varepsilon^2 f(\phi), \tag{2.1}$$

where

$$f(\phi) := (\phi-\phi_1)(\phi-\phi_2)(\phi_3-\phi)(\phi_4-\phi); \tag{2.2}$$

for the bounded solutions that we are seeking, ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are all real constants with $\phi_1 \le \phi_2 \le \phi \le \phi_3 \le \phi_4$.

Following [12] we introduce η defined by

$$\frac{d\xi}{d\eta} = \frac{\phi}{\varepsilon} \tag{2.3}$$

so that (2.1) becomes

$$\phi_\eta^2 = f(\phi). \tag{2.4}$$

As shown in [8,13], Eq. (2.4) has two possible forms of solution.

The first solution of Eq. (2.4) is

$$\phi = \frac{\phi_2 - \phi_1 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)} \quad \text{with} \quad n = \frac{\phi_3 - \phi_2}{\phi_3 - \phi_1}\,, \tag{2.5}$$

where

$$w = p\eta, \quad p = \frac{1}{2}\sqrt{(\phi_4 - \phi_2)(\phi_3 - \phi_1)} \quad \text{with} \quad m = \frac{(\phi_3 - \phi_2)(\phi_4 - \phi_1)}{(\phi_4 - \phi_2)(\phi_3 - \phi_1)}\,. \tag{2.6}$$

In (2.5), the notation $\mathrm{sn}(w|m)$ represents a Jacobian elliptic function [14]. From (2.3) and (2.5), we have

$$\xi = \frac{1}{\varepsilon p}[w\phi_1 + (\phi_2 - \phi_1)\Pi(n; w|m)], \tag{2.7}$$

where $\Pi(n; w|m)$ is the elliptic integral of the third kind [14]. The solution to Eq. (2.1) is given in parametric form by (2.5) and (2.7) with w as the parameter. With respect to w, ϕ in (2.5) is periodic with period $2K(m)$, where $K(m)$ is the complete elliptic integral of the first kind [14].

When $\phi_3 = \phi_4$, $m = 1$ and so (2.5) and (2.7) become

$$\phi = \frac{\phi_2 - \phi_1 n \tanh^2 w}{1 - n \tanh^2 w}\,, \quad \xi = \frac{1}{\varepsilon}\left[\frac{w\phi_3}{p} - 2\tanh^{-1}(\sqrt{n}\tanh w)\right]. \tag{2.8}$$

The second form of solution of Eq. (2.4) is

$$\phi = \frac{\phi_3 - \phi_4 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)} \quad \text{with} \quad n = \frac{\phi_3 - \phi_2}{\phi_4 - \phi_2}\,, \tag{2.9}$$

where w, p and m are as in (2.6). From (2.9) and (2.3), we have

$$\xi = \frac{1}{\varepsilon p}[w\phi_4 + (\phi_3 - \phi_4)\Pi(n; w|m)]. \tag{2.10}$$

The solution to Eq. (2.1) is given in parametric form by (2.9) and (2.10) with w as the parameter.

When $\phi_1 = \phi_2$, $m = 1$ and so (2.9) and (2.10) become

$$\phi = \frac{\phi_3 - \phi_4 n \tanh^2 w}{1 - n \tanh^2 w}\,, \quad \xi = \frac{1}{\varepsilon}\left[\frac{w\phi_2}{p} + 2\tanh^{-1}(\sqrt{n}\tanh w)\right]. \tag{2.11}$$

3 Exact travelling-wave solutions of the $K(2,2)$ equation (defocusing branch)

In this section, we seek travelling-wave solutions of Eq. (1.4). It is convenient to introduce a new dependent variable z defined by

$$z = \frac{u}{|v|} \tag{3.1}$$

and to assume that z is an implicit or explicit function of ξ, where

$$\xi = x - vt \tag{3.2}$$

and $v(\neq 0)$ is the arbitrary constant wave velocity. (Stationary solutions, for which $v = 0$, are discussed in Section 6.) Substitution of (3.1) into Eq. (1.4) yields

$$(zz_\xi)_{\xi\xi} = \left(z + \frac{c}{2}\right) z_\xi, \quad \text{where} \quad c := \frac{v}{|v|} = \pm 1. \tag{3.3}$$

After two integrations Eq. (3.3) gives

$$(zz_\xi)^2 = \frac{1}{4}(z^4 + \frac{4}{3}cz^3 + Az^2 + B), \tag{3.4}$$

where A and B are real constants.

Rosenau [2] identifed some possible types of solution to (3.4) for z by first writing (3.4) in the form

$$z_\xi^2 + P(z) = E, \tag{3.5}$$

where

$$P(z) := -\left(\frac{z^2}{4} + \frac{cz}{3} + \frac{Bz^{-2}}{4}\right), \quad E := \frac{A}{4}\,.$$

Equation (3.5) is equivalent to (18b) in [2]. Rosenau analysed (3.5) by using the analogy to a particle of total energy E moving in a potential field $P(z)$, where z is the displacement of the particle.

Here we adopt a different route. We introduce η defined by

$$\frac{d\xi}{d\eta} = 2z \tag{3.6}$$

so that (3.4) becomes

$$z_\eta^2 = f(z), \tag{3.7}$$

where

$$f(z) := z^4 + \frac{4}{3}cz^3 + Az^2 + B. \tag{3.8}$$

We apply the particle analogy to (3.7) with B regarded as the total energy. (This has the advantage that the 'potential' is a quartic and does not have the singularities which occur in $P(z)$.) In this way we may identify solution types for z regarded as a function of η. Then, by taking into consideration (3.6), we can identify the solution types for z regarded as a function of ξ.

Later in this Section we find actual solutions for the cases in which z as a function of η is bounded. In this case $f(z)$ may be written

$$f(z) = (z - z_1)(z - z_2)(z_3 - z)(z_4 - z), \tag{3.9}$$

where z_1, z_2, z_3 and z_4 are real constants with $z_1 \leq z_2 < z_3 \leq z_4$. Bounded solutions for z are such that $z_2 \leq z \leq z_3$ and may be found by using the results in Section 2 with $\varepsilon = 1/2$.

Note that $f(z)$ in (3.8) does not have a term linear in z. This is also the case for the $f(z)$ in (2.5) in [8] for the DP equation. Hence we can use a categorization procedure similar to the one described in [8, Section 2]. Accordingly, for convenience, we define the functions $g(z)$ and $h(z)$ by

$$g(z) = z^2 + \frac{4}{3}cz + A, \quad h(z) = 2z^2 + 2cz + A, \tag{3.10}$$

and then we have

$$f(z) = z^2 g(z) + B, \quad f'(z) = 2zh(z). \tag{3.11}$$

Moreover, we define z_L, z_M and B_L, B_M by

$$z_L := \frac{-c - \sqrt{c^2 - 2A}}{2}, \quad z_M := \frac{-c + \sqrt{c^2 - 2A}}{2}, \tag{3.12}$$

$$B_L := -z_L^2 g(z_L) = \frac{A^2}{4} - \frac{Ac^2}{2} + \frac{c^4}{6} + \frac{c}{6}(c^2 - 2A)\sqrt{c^2 - 2A}, \tag{3.13}$$

$$B_M := -z_M^2 g(z_M) = \frac{A^2}{4} - \frac{Ac^2}{2} + \frac{c^4}{6} - \frac{c}{6}(c^2 - 2A)\sqrt{c^2 - 2A}. \tag{3.14}$$

z_L and z_M are the roots of $h(z) = 0$. Provided that A is a nonzero constant that satisfies the condition $A < c^2/2$, $f(z)$ has three different stationary points that occur at $z = z_L$, z_M, 0, and comprise two minimums separated by a maximum.

Note that Eq. (3.3) is invariant under the transformation $z \to -z$, $c \to -c$; this corresponds to the transformation

$$u \to -u, \quad v \to -v. \tag{3.15}$$

Here we will seek the family of solutions of Eq. (3.3) for which $v > 0$ so that $c = 1$. However, as a help to the reader, we will display c in what follows.

Obviously, the types of solutions of Eq. (3.7) will depend on the values of the constants A and B. (The corresponding quantities in [2] are $E \equiv A/4$ and $\beta \equiv B/2$, respectively.) These solutions, and the corresponding ones for z as a function of ξ, may be categorized conveniently by the following four cases.

Case 3.1 $A < 0$

In this case, we have $z_L < 0 < z_M$ with $f(z_L) < f(z_M)$. For each value of A and B satisfying $A < 0$ and $0 < B < B_M$ respectively, $z_1 < z_2 < 0 < z_3 < z_4$. This can be seen in Fig. 1(a) where the $f(z)$ curves for $B = 0$, $B = B_M$ and $B = B_L$ are plotted for a specific value of A satisfying $A < 0$. Thus z as a function of η is a smooth periodic wave. However, because $d\xi/d\eta$ changes sign at $z = 0$ in the interval $[z_2, z_3]$, z regarded as a function of ξ comprises periodic (inverted) loops. From (2.5) and (2.7) the solution is given in parametric form by

$$z(w) = \frac{z_2 - z_1 n \,\mathrm{sn}^2(w|m)}{1 - n \,\mathrm{sn}^2(w|m)} \quad \text{with} \quad n = \frac{z_3 - z_2}{z_3 - z_1}\,, \tag{3.16}$$

$$\xi(w) = \frac{2}{p}[w z_1 + (z_2 - z_1)\Pi(n; w|m)], \tag{3.17}$$

where

$$w = p\eta, \quad p = \frac{1}{2}\sqrt{(z_4 - z_2)(z_3 - z_1)} \quad \text{and} \quad m = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_4 - z_2)(z_3 - z_1)}\,. \tag{3.18}$$

An example is illustrated in Fig. 2(a).

When $B = B_M$, we have $z_3 = z_4 = z_M$ and $m = 1$. Then z as a function of η is a smooth solitary well and z as a function of ξ is a solitary inverted loop. From (2.8) this is given in parametric form by

$$z(w) = \frac{z_2 - z_1 n \tanh^2 w}{1 - n \tanh^2 w}\,, \quad \xi(w) = \frac{2 w z_M}{p} - 4 \tanh^{-1}(\sqrt{n} \tanh w)\,, \tag{3.19}$$

where $z_2 \le z < z_M$ and

$$z_1 = \frac{-c - 3\sqrt{c^2 - 2A}}{6} - \frac{1}{3}\sqrt{c^2 + 3c\sqrt{c^2 - 2A}}\,,$$
$$z_2 = \frac{-c - 3\sqrt{c^2 - 2A}}{6} + \frac{1}{3}\sqrt{c^2 + 3c\sqrt{c^2 - 2A}}\,.$$

These solutions are a one-parameter family (with parameter A) of inverted loop-like solitary waves; see Fig. 3(a) for an example.

For $B_M < B < B_L$, z as a function of η is unbounded with $z_2 \le z$, but z as a function of ξ comprises periodic upward cusps with $z_2 \le z \le 0$. For $B = B_L$, $z_1 = z_2 = z_L$ and z as a function of ξ is a solitary upward cuspon with $z_L \le z \le 0$.

Case 3.2 $0 < A < 4c^2/9$

In this case, we have $z_L < z_M < 0$ and $f(z_L) < f(0)$. For each value of A and B satisfying $0 < A < 4c^2/9$ and $B_M < B < 0$ respectively, $z_1 < z_2 < z_3 < 0 < z_4$ (see Fig. 1(b)). Thus z as a function of η is a smooth periodic wave. Because $d\xi/d\eta$ does not change sign in the interval $[z_2, z_3]$, z regarded as a function of ξ is also a smooth periodic wave. These are given in the same form as (3.16)–(3.18). An example is illustrated in Fig. 2(b).

When $B = 0$, we have $z_3 = z_4 = 0$ and $m = 1$. Then z as a function of η is a solitary smooth well with $z_2 \le z \le 0$. However, z as a function of ξ is a compacton. From (2.8) these are given by the one-parameter family of solutions (with parameter A)

$$z(w) = \frac{z_2 - z_1 n \tanh^2 w}{1 - n \tanh^2 w}, \quad \xi(w) = -4 \tanh^{-1}(\sqrt{n} \tanh w), \tag{3.20}$$

where z_1, z_2 are the roots of $g(z) = 0$, namely

$$z_1 = -\frac{2}{3}c - \sqrt{\frac{4}{9}c^2 - A}, \quad z_2 = -\frac{2}{3}c + \sqrt{\frac{4}{9}c^2 - A}.$$

We may eliminate the parameter w from Eqs. (3.20) to give

$$\begin{aligned} z = \hat{z}(\xi) &:= [z_2 - z_1 \tanh^2(\xi/4)] \cosh^2(\xi/4) \\ &\equiv -\frac{2}{3}c + \left(\sqrt{\frac{4}{9}c^2 - A}\right) \cosh(\xi/2), \end{aligned} \tag{3.21}$$

where $-\xi_0 \le \xi \le \xi_0$, and $\xi_0 (> 0)$ is given by

$$\xi_0 = 2 \cosh^{-1}\left(\frac{2c/3}{\sqrt{\frac{4}{9}c^2 - A}}\right) \equiv 2\left[\ln\left|\frac{2}{3}c + \sqrt{A}\right| - \ln\left(\sqrt{\frac{4}{9}c^2 - A}\right)\right]. \tag{3.22}$$

We may construct a weak solution, namely the periodic upward corner-wave

$$z = \hat{z}(\xi - 2j\xi_0), \quad (2j-1)\xi_0 \le \xi \le (2j+1)\xi_0, \quad j = 0, \pm 1, \pm 2, \ldots. \tag{3.23}$$

With $\hat{z}(\xi)$ given by (3.21), the corners are located at $\xi = (2j+1)\xi_0$; see Fig. 3(b) for an example. However, this solution may be phase-shifted by ξ_0 so that the corners are located at $\xi = 2j\xi_0$. In this case $\hat{z}(\xi)$ is given by

$$\hat{z}(\xi) = -\frac{2}{3}c + \left(\sqrt{\frac{4}{9}c^2 - A}\right) \cosh[(|\xi| - \xi_0)/2] \tag{3.24}$$

$$\equiv -\frac{2}{3}c + \frac{1}{2}\left[\left(\frac{2}{3}c + \sqrt{A}\right) e^{-\frac{1}{2}|\xi|} + \left(\frac{2}{3}c - \sqrt{A}\right) e^{\frac{1}{2}|\xi|}\right]. \tag{3.25}$$

Equations (3.22), (3.23) and (3.25) are equivalent to the periodic waves given in Section 3.2 in [6]. (In [6], g and c correspond to our $-Av^2/2$ and v, respectively.)

In passing, we note that, when $B = 0$, (3.4) may be written

$$z_\xi^2 = \frac{1}{4}z^2 + \frac{1}{3}cz + \frac{1}{4}A\,, \quad z_2 \le z \le 0. \tag{3.26}$$

Integration of Eq. (3.26) leads to either (3.21) or (3.24) directly.

For $0 < B < B_L$, z as a function of η is unbounded with $z_2 \le z$, but z as a function of ξ comprises periodic upward cusps with $z_2 \le z \le 0$. For $B = B_L$, $z_1 = z_2 = z_L$ and z as a function of ξ is a solitary upward cuspon with $z_L \le z \le 0$.

Case 3.3 $A = 4c^2/9$

In this case, we have $z_L < z_M < 0$ and $f(z_L) = f(0) = B$. For each B such that $B_M = -c^4/81 < B < 0$, $z_1 < z_2 < z_3 < 0 < z_4$ (see Fig. 1(c)). Thus, as in Section 3.2, z regarded as a function of ξ is a smooth periodic wave. From (2.9) and (2.10), we may write the solution in parametric form as

$$z(w) = \frac{z_3 - z_4 n\,\mathrm{sn}^2(w|m)}{1 - n\,\mathrm{sn}^2(w|m)} \quad \text{with} \quad n = \frac{z_3 - z_2}{z_4 - z_2}\,, \tag{3.27}$$

and

$$\xi(w) = \frac{2}{p}[wz_4 + (z_3 - z_4)\Pi(n; w|m)], \tag{3.28}$$

where w, p and m are given by (3.18). An example is given in Fig. 2(c).

When $B = 0$, we obtain that $z_1 = z_2 = z_L = -2c/3$ and $z_3 = z_4 = 0$. In this case, Eq. (3.4) becomes

$$\frac{dz}{d\xi} = \pm\frac{1}{2}\left(z + \frac{2}{3}c\right). \tag{3.29}$$

Note that the bounded solution has $-2c/3 < z \le 0$, so that, on integrating Eq. (3.29) and setting $z = 0$ at $\xi = 0$, one gets the solitary-peakon solution

$$z(\xi) = -\frac{2}{3}\,c\left(1 - e^{-|\xi|/2}\right). \tag{3.30}$$

This is illustrated in Fig. 3(c). The expression in (3.30) is equivalent to (3.5) in [6] for the osmosis $K(2,2)$ equation. The inverted counterpart of the peakon given by (3.30), i.e. the one travelling in the negative x-direction, was given by Eq. (21c) in [2] and was called a dark peakon. There is a slight error in (21c); in the notation in [2] it should read

$$u = \frac{2\lambda}{3a}\left(1 - e^{-\sqrt{a}\,|x+\lambda t|/2}\right) \tag{3.31}$$

with $\lambda > 0$.

Case 3.4 $4c^2/9 < A < c^2/2$

In this case, we have $z_L < z_M < 0$ and $f(z_L) > f(0)$. For each value of A and B satisfying $4c^2/9 < A < c^2/2$ and $B_M < B < B_L$ respectively, $z_1 < z_2 < z_3 < 0 < z_4$ (see Fig. 1(d)). Thus, as in Section 3.2, z regarded as a function of ξ is a smooth periodic wave. The solution in parametric form is as in (3.27)–(3.28); see Fig. 2(d) for an example.

When $B = B_L$, we have $z_1 = z_2 = z_L$ and $m = 1$. Then z as a function of ξ is a solitary smooth hump with $z_L \le z \le z_3$. The solution is a one-parameter family (with parameter A) given by

$$z(w) = \frac{z_3 - z_4 n \tanh^2 w}{1 - n \tanh^2 w}, \quad \xi(w) = \frac{2 w z_L}{p} + 4 \tanh^{-1}(\sqrt{n} \tanh w), \tag{3.32}$$

where w and p are given by (3.18) and

$$z_3 = \frac{-c + 3\sqrt{c^2 - 2A}}{6} - \frac{1}{3}\sqrt{c^2 - 3c\sqrt{c^2 - 2A}},$$
$$z_4 = \frac{-c + 3\sqrt{c^2 - 2A}}{6} + \frac{1}{3}\sqrt{c^2 - 3c\sqrt{c^2 - 2A}}.$$

An example is given in Fig. 3(d).

Our solution (3.32) is considerably less complicated in form than the corresponding one in [7]. (In [7], g and c correspond to our $-Av^2/2$ and v, respectively.) Furthermore, in [7] the authors seem unaware of the significance of (3.15); however, their Examples 2.1 and 2.2 and the associated figures clearly demonstrate the consequences of (3.15).

4 An interpretation of multi-valued solutions

The wave profiles for z in Figs. 2(a) and 3(a) are multi-valued. Recently, Li [10] discussed the interpretation of similar solutions for other nonlinear wave equations. Here we will apply Li's ideas to the solitary wave illustrated in Fig. 3(a).

In the solution given by (3.19), z and ξ may be regarded as the limit as $m \to 1$ of Eqs. (3.16) and (3.17), respectively. Alternatively, they may be derived directly from

$$z_\eta^2 = (z - z_1)(z - z_2)(z_M - z)^2 \quad \text{and} \quad \frac{d\xi}{d\eta} = 2z \tag{4.1}$$

with $w = p\eta$. Note that ξ is not a monotonic function of the parameter w. Furthermore, the phase portrait of the solution in the (z, z_η)-plane is a single closed trajectory in the region $z_2 \le z \le z_M$ with a saddle point at $(z_M, 0)$. However,

the phase portrait in the (z, z_ξ)-plane consists of the stable and unstable manifolds through the saddle in the region $0 < z \le z_M$, and an open curve through $(z_2, 0)$ for which $z_2 \le z < 0$. Li's point of view is that each of these three trajectories corresponds to a different single-valued travelling-wave solution. The multi-valued solitary-wave solution illustrated in Fig. 3(a) may be regarded as a composite solution made up of these three single-valued solutions. (The notion of composite solutions for the DP equation is discussed in detail by Lenells in [15].) The three single-valued solutions may be combined in different ways so as to give a variety of composite single-valued solutions. For example, consider the case with z as given in (3.19) and ξ given by

$$\xi(w) = \begin{cases} \dfrac{2wz_M}{p} - 4\tanh^{-1}(\sqrt{n}\tanh w) - 2\xi_0, & w \in (-\infty, -w_0), \\ -\dfrac{2wz_M}{p} + 4\tanh^{-1}(\sqrt{n}\tanh w), & w \in [-w_0, w_0], \\ \dfrac{2wz_M}{p} - 4\tanh^{-1}(\sqrt{n}\tanh w) + 2\xi_0, & w \in (w_0, \infty), \end{cases} \tag{4.2}$$

where

$$w_0 = \tanh^{-1}\sqrt{\frac{z_2}{nz_1}}, \quad \xi_0 = -\frac{2w_0 z_M}{p} + 4\tanh^{-1}(\sqrt{n}\tanh w_0). \tag{4.3}$$

This gives a one-parameter family (with parameter A) of inverted-bell solitary waves. An example is illustrated in Fig. 4(a). (Rosenau [2] used the terminology tipon instead of bell.) Note that, in this solution, ξ is a monotonic increasing function of the parameter w. Similarly, with z as given in (3.19) and ξ given by

$$|\xi| = \frac{2wz_M}{p} - 4\tanh^{-1}(\sqrt{n}\tanh w) + \xi_0, \quad w \ge w_0 \tag{4.4}$$

we have a one-parameter family (with parameter A) of inverted-cuspon solitary waves. An example is illustrated in Fig. 4(b) (cf. Fig. 1(h) in [15]).

A composite one-parameter solution (with parameter A) comprising periodic upward cusps may be constructed by using the single-valued solution for which $z_2 \le z \le 0$ as follows:

$$z = z(w - 2jw_0), \quad \xi = \xi(w - 2jw_0) + 2jw_0, \tag{4.5}$$

with $j = 0, \pm 1, \pm 2, \ldots$, and $w \in [(2j-1)w_0, (2j+1)w_0]$, where $z(w)$ and $\xi(w)$ are as in (3.19). An example is illustrated in Fig. 4(c) (cf. Fig. 1(e) in [15]).

The composite two-parameter families (with parameters A and B) of periodic single-valued solutions corresponding to Figs. 4(a) and 4(b) may be constructed from the single-valued solutions making up the multi-valued periodic wave in Fig. 2(a). Similarly, a two-parameter family of periodic upward cusps may be constructed. (The solution given by (4.5) is the particular case for which $B = B_M$.)

5 Solitary waves: an alternative form of solution

In Section 3 we noted that Eq. (3.4) has the same structure as Eq. (2.4) in [8] for the DP equation. Zhang and Qiao [11] investigated cuspon and smooth solitary-wave solutions of the DP equation. In this section we demonstrate that, as may be expected, for the $K(2,2)$ equation (defocusing branch) there are results similar to the results in [11] for the DP equation.

First we consider the inverted-cuspon solution given in Section 4. With

$$X := \sqrt{n}\,\tanh w \quad (w > w_0), \quad a := \sqrt{n}\,, \quad r := z_M/2p, \tag{5.1}$$

z in (3.19) may be written

$$z = \frac{z_2 - z_1 X^2}{1 - X^2} \tag{5.2}$$

and (4.4) becomes

$$|\xi| - \xi_0 = 4r \tanh^{-1}(X/a) - 4 \tanh^{-1}(X). \tag{5.3}$$

By using the identity $2\tanh^{-1}\theta = \ln[(1+\theta)/(1-\theta)]$ $(0 \le \theta < 1)$, we may write (5.3) as

$$C_0\, e^{-|\xi|/2} = \left(\frac{a - X}{a + X}\right)^r \left(\frac{1 + X}{1 - X}\right), \quad C_0 = e^{|\xi_0|/2}, \tag{5.4}$$

where

$$0 < X_0 \le X < a < 1, \quad 0 < r < 1/2, \quad X_0 := \sqrt{n}\,\tanh w_0$$

and $X = X_0$ corresponds to $\xi = 0$. Equation (5.4) is equivalent to equation Eq. (5.4) in [11] for the DP equation. Equations (5.2) and (5.4) give an implicit solution for the cuspon in which z and ξ are given in terms of the parameter X.

In order to obtain (5.4) by the method in [11], we would have to change the dependent variable z in

$$(z z_\xi)^2 = \frac{1}{4}\,(z - z_1)(z - z_2)(z_M - z)^2 \tag{5.5}$$

to X by using the relation (5.2). Direct integration of the resulting differential equation would yield (5.4).

Now let us consider the smooth hump-like solitary wave given in Section 3.4. With

$$Y := \sqrt{n}\,\tanh w \quad (w > 0), \quad b := \sqrt{n}\,, \quad r := -z_L/2p, \tag{5.6}$$

(3.32) gives

$$z = \frac{z_3 - z_4 Y^2}{1 - Y^2} \tag{5.7}$$

and

$$e^{-|\xi|/2} = \left(\frac{b - Y}{b + Y}\right)^r \left(\frac{1 + Y}{1 - Y}\right), \quad 0 \le Y < b < 1, \quad 1 < r < \infty. \tag{5.8}$$

(With the notation $X := Y^{-1}$ and $a := b^{-1}$, Eq. (5.8) is equivalent to Eq. (5.7) in [11] for the DP equation.)

Following the observation regarding the DP equation in [11], we note that if $r = 2$ (which corresponds to $A = 112/225$) then

$$z = \frac{2c}{15}\left(\sqrt{5} - 1 - \frac{2\sqrt{5}}{1 - Y^2}\right) \tag{5.9}$$

and Eq. (5.8) yields a cubic equation, namely

$$Y^3 + (1 - 2b)kY^2 + b(b - 2)Y + b^2k = 0, \tag{5.10}$$

where

$$b = \frac{3 - \sqrt{5}}{2}, \quad k = \frac{1 - e^{-|\xi|/2}}{1 + e^{-|\xi|/2}}.$$

For each value of k ($0 \le k < 1$), the cubic equation (5.10) has three distinct real roots, Y_1, Y_2 and Y_3 such that $Y_1 < Y_2 < Y_3$ with $0 \le Y_2 < b$. Thus $Y = Y_2$ is the required root. Hence, for the smooth solitary wave for which $A = 112/225$, there is an explicit solution for z as a function of ξ, namely Eq. (5.9) with $Y = Y_2$.

6 Stationary solutions

In this section we seek stationary solutions to Eq. (1.5) in which u is assumed to be a function of x only.

After two integrations Eq. (1.5) gives

$$(uu_x)^2 = \frac{1}{4}(u^4 + Au^2 + B), \tag{6.1}$$

where A and B are real constants. With y defined by

$$\frac{dx}{dy} = 2u \tag{6.2}$$

Eq.(6.1) becomes

$$u_y^2 = u^4 + Au^2 + B. \tag{6.3}$$

For bounded solutions we require $A < 0$ and $0 < B \le A^2/4$. In this case (6.3) can be written

$$u_y^2 = (\alpha^2 - u^2)(\beta^2 - u^2), \tag{6.4}$$

where α and β are real positive constants given by

$$\alpha^2 = (-A - \sqrt{A^2 - 4B}\,)/2, \quad \beta^2 = (-A + \sqrt{A^2 - 4B}\,)/2 \tag{6.5}$$

so that $0 < \alpha \leq \beta$. The solution to Eq. (6.1) in parametric form is

$$u(w) = \alpha\,\mathrm{sn}(w|m), \quad x(w) = 2\ln\left[\frac{\mathrm{dn}(w|m) - \sqrt{m}\,\mathrm{cn}(w|m)}{1-\sqrt{m}}\right], \tag{6.6}$$

where

$$m = (\alpha/\beta)^2, \quad w = \beta y.$$

The expressions in (6.6) were obtained by using the results 219.00 and 310.01 in [14] in order to integrate Eqs. (6.4) and (6.2), respectively. We have chosen constants of integration so that $u(0) = 0$ and $x(0) = 0$.

Note that $x(w)$ is periodic with period $4K(m)$. It follows that the solution to Eq. (6.1) is a closed curve in the (x, u)-plane around the point $(x_0, 0)$, where

$$x_0 := x\left(K(m)\right) = \ln\left[\frac{1+\sqrt{m}}{1-\sqrt{m}}\right]. \tag{6.7}$$

This curve passes through the points $(0, 0)$, $(2x_0, 0)$, $(x_0, \pm\alpha)$. Composite solutions may be constructed from this curve such as periodic upward cusps, periodic inverted cusps and a periodic bell solution (cf. Fig. 3(c) in [8]).

When $B = A^2/4$, $\alpha = \beta$ and $m = 1$. In this case (6.6) becomes

$$u(w) = \alpha\tanh w, \quad x(w) = 2\ln(\cosh w) \tag{6.8}$$

so that

$$u^2 = \alpha^2(1 - e^{-x}). \tag{6.9}$$

From this we may construct the explicit stationary cuspon solutions

$$u = \pm\alpha\sqrt{1 - e^{-|x|}} \tag{6.10}$$

for example. The analogous result for the DP equation was derived in a different way in [11, Section 4].

7 Conclusions

We have used an auxiliary elliptic equation method to investigate the travelling-wave solutions of the $K(2, 2)$ equation (defocusing branch) that travel in the positive x-direction with speed v. By applying this method, we have obtained new exact solitary and periodic-wave solutions, and recovered some previous results. Because of the invariance of the governing equation under the transformation (3.15), for all such solutions that we have obtained, expressed with u as the dependent variable, there is a solution for u that is the mirror image in the x-axis and travels with the same speed but in the negative x-direction. We have also obtained stationary

solitary and periodic-wave solutions. We have shown how single-valued composite solutions may be obtained from our multi-valued solutions.

The auxilliary elliptic equation method described in this paper is simple and efficient. It has been used previously in the context of the DP equation [8]; we believe that it may be useful in seeking travelling-wave solutions for some other nonlinear evolution equations.

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Fig. 1.

The $f(z)$ curves with $B = 0$ (solid), $B = B_L$ (dot-dash), $B = B_M$ (dashed): (a) $A = -63/2$; (b) $A = 5/12$; (c) $A = 4/9$; (d) $A = 15/32$.

Fig. 2.

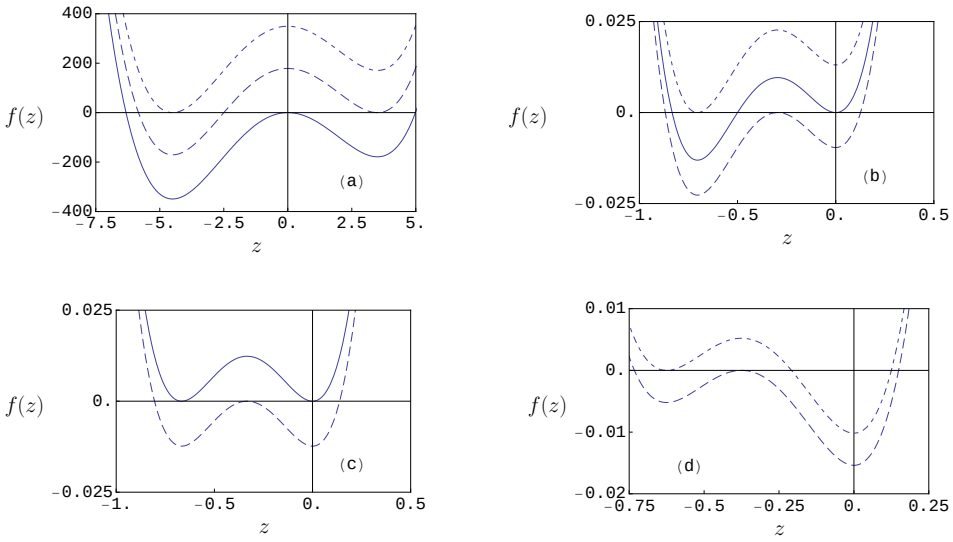
Periodic solutions with $c = 1$ and $0 < m < 1$: (a) $A = -63/2$, $B = 0.98B_M$ so $m = 0.9697$; (b) $A = 5/12$, $B = 0.4B_M$ so $m = 0.7928$; (c) $A = 4/9$, $B = 0.6B_M = -1/135$ so $m = 0.6911$; (d) $A = 15/32$, $B = 0.5(B_L + B_M)$ so $m = 0.7286$.

Fig. 3.

Solutions with $c = 1$ and $m = 1$: (a) $A = -63/2$, $B = B_M$; (b) $A = 5/12$, $B = 0$; (c) $A = 4/9$, $B = 0$; (d) $A = 15/32$, $B = B_L$.

Fig. 4.

Composite waves derived from the wave in Fig. 3(a): (a) an inverted-bell solitary wave; (b) an inverted-cuspon solitary wave; (c) periodic upward cusps.



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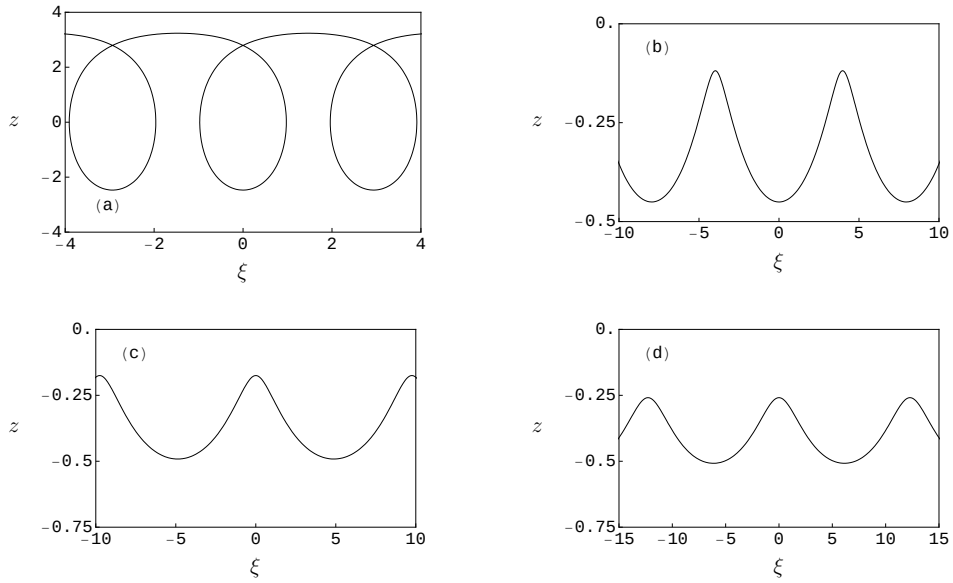

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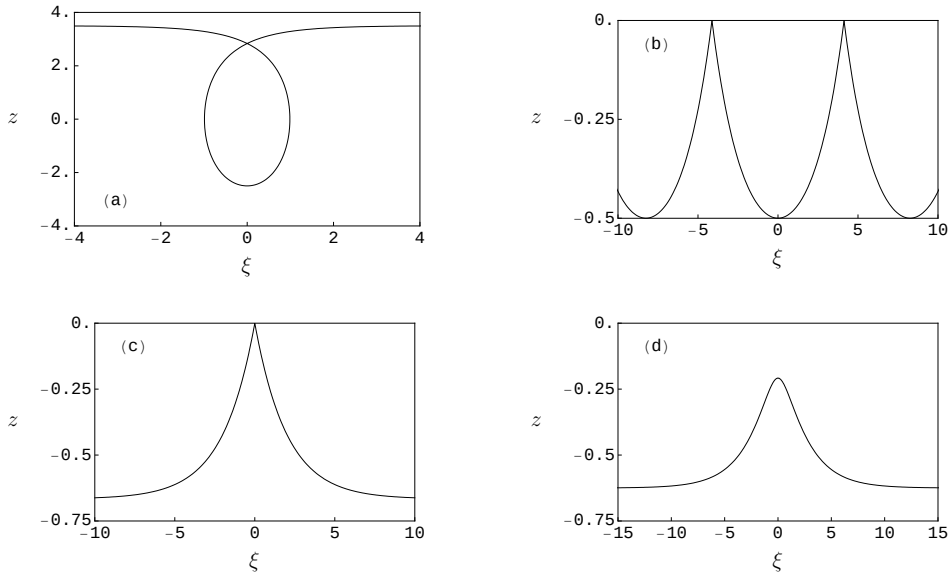

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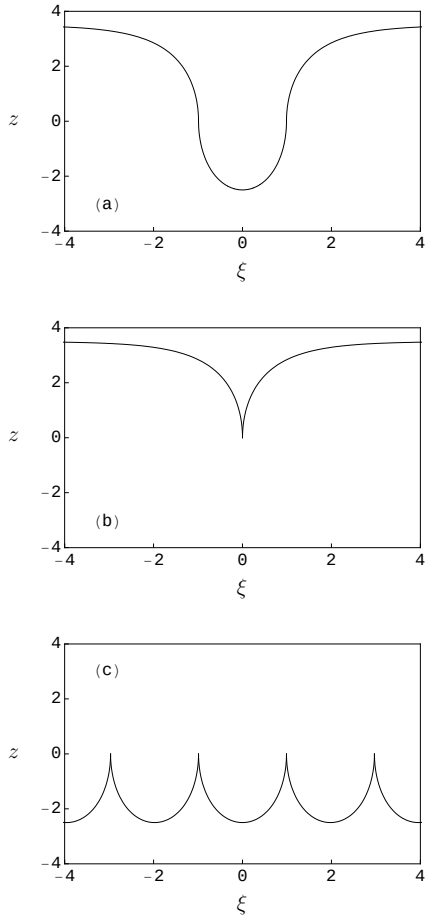

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