

# Problem of optimal control for bilinear systems with endpoint constraint

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## Abstract

*In this work, we will investigate the question of optimal control for bilinear systems with constrained endpoint. The optimal control will be characterized through a set of unconstrained minimization problems that approximate the former. Then a class of bilinear systems for which the optimal control can be expressed as a time-varying feedback law will be identified. Finally, applications to parabolic and hyperbolic partial differential equations are provided.*

**Key words:** Quadratic cost, optimal control, endpoint constraint, bilinear systems.

## I. INTRODUCTION AND THE PROBLEM STATEMENT

Linear systems are usually preferable when approximating nonlinear dynamical processes for their simplicity. However, there are many other practical situations for which bilinear models are more appropriate (see [6, 8, 15, 19, 23, 28] and the references therein). In general, a problem of control aims to achieve a certain degree of performance for the system at hand using suitable control laws among available options. If this is indeed feasible, then one usually aims to achieve this performance while optimizing a certain criterion. A problem of optimal control is an optimization problem on a reasonable set described by dynamic constraints. As an interesting example, the question of describing the best control among those that allow to reach a desired state with minimal cost or energy. Such problems arise in various applications, such as the optimization of hydrothermal systems and non-smooth modeling in mechanics and engineering, etc. (see e.g. [4, 5, 12, 13, 22]). The problem of optimal control for bilinear and semi-linear systems with unconstrained endpoint has been treated by many authors (see [8, 10, 15, 20, 21, 29, 30]). The question of optimal control with endpoint constraint has been treated in the context of linear and semi-linear systems with additive controls (see [16, 20] and the references therein). The approach is based on the Pontryagin's maximum principle. The main goal of this paper is to study a quadratic optimization problem with a restricted endpoint state. In the case of a bounded set of admissible control, we will characterize the optimal control either for exactly or approximately attainable states. This problem can be formulated as an optimization problem with endpoint constraint, which can also be approximated by a set of unconstrained problems. Moreover, if the steering

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control is scalar valued, then the optimal control can be expressed as a time-varying feedback law. Let us consider the following system

$$\begin{cases} \dot{y}(t) = Ay(t) + \mathcal{B}(u(t), y(t)) \\ y(0) = y_0 \in X \end{cases} \quad (1)$$

where

- $A : D(A) \subset X \mapsto X$  is the infinitesimal generator of a linear  $C_0$ - semi-group  $S(t)$  on a real Hilbert space  $X$  whose inner product and corresponding norm are denoted respectively by  $\langle \cdot, \cdot \rangle$  and  $\|\cdot\|$ ,
- $u \in L^2(0, T; U)$ , where  $U$  is a real Hilbert space equipped with inner product  $\langle \cdot, \cdot \rangle_U$  and the corresponding norm  $\|\cdot\|_U$ , and  $y$  is the corresponding mild solution to the control  $u$ ,
- $\mathcal{B} : U \times X \rightarrow X$  is a bounded bilinear operator.

Let us now consider the following assumptions:

- (a) For all  $y \in X$  the mapping  $u \mapsto \mathcal{B}(u, y)$  is compact,  
(b)  $A$  is the infinitesimal generator of a linear compact  $C_0$ - semigroup  $S(t)$ .

Note that assumption (b) is systematically satisfied for  $U = \mathbb{R}$ .

The quadratic cost function  $J$  to be minimized is defined by

$$J(u) = \int_0^T \|y(t)\|^2 dt + \frac{r}{2} \int_0^T \|u(t)\|_U^2 dt. \quad (2)$$

Here,  $r > 0$  and  $u$  belongs to the set of admissible control

$$U_{ad} = \{u \in V \mid y(T) = y_d\},$$

where  $V$  is a closed convex subset of  $L^2(0, T; U)$  and  $y_d \in X$  is the desired state.

The optimal control problem may be stated as follows

$$(P) \quad \begin{cases} \min J(u) \\ u \in U_{ad} \end{cases}$$

In order to solve the problem (P), let us introduce the following auxiliary cost function

$$J_\epsilon(u) = \|y(T) - y_d\|^2 + \epsilon J(u),$$

where  $\epsilon > 0$ , and let us consider the following optimal control problem

$$(P_\epsilon) \quad \begin{cases} \min J_\epsilon(u) \\ u \in V \end{cases}$$

This paper is organized as follows: In Section 2, we will first provide a solution to the auxiliary problem  $(P_\epsilon)$ . This result is then applied to build a solution of the problem (P). We will further provide sufficient conditions on the operators  $A$  and  $B$  under which the solution of the problem (P) can be expressed as a time-varying feedback law. Section 3 is devoted to examples and simulations.

## II. CHARACTERISATION OF THE OPTIMAL CONTROL

### i. Preliminary

Let us recall the notion of attainability.

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**Definition 1**

- A target state  $y_d \in X$  is *approximately attainable* for the system (1), if for all  $\varepsilon > 0$  there exists  $u_\varepsilon \in V$  such that  $\|y_{u_\varepsilon}(T) - y_d\| \leq \varepsilon$ .
- A target state  $y_d \in X$  is *exactly attainable* for the system (1), if there exists  $u \in V$  such that  $y_u(T) = y_d$ .

The following lemma provides a continuity property of the solution  $y$  with respect to the control  $u$ .

**Lemma 2** *If one of the assumption (a) or (b) hold, then for any sequence  $(u_n) \subset L^2(0, T; U)$  such that  $u_n \rightharpoonup u$  in  $L^2(0, T; U)$ , we have*

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t \leq T} \|y_n(t) - y(t)\| = 0,$$

where  $y_n$  and  $y$  are the mild solutions of the system (1) respectively corresponding to  $u_n$  and  $u$ .

**Proof**

First, let us recall that for all  $u \in L^2(0, T; U)$ , the system (1) has a unique mild solution corresponding to  $u$ , which is given by the following variation of constants formula (see e.g. [20], p. 66):

$$y(t) = S(t)y_0 + \int_0^t S(t-s)\mathcal{B}(u(s), y(s))ds.$$

Thus, the solutions  $y_n$  and  $y$  of the system (1) respectively corresponding to  $u_n$  and  $u$  satisfy the following formula for  $t \in [0, T]$

$$y_n(t) - y(t) = \int_0^t S(t-s) \left( \mathcal{B}(u_n(s), y_n(s)) - \mathcal{B}(u(s), y(s)) \right) ds.$$

Then, for all  $t \in [0, T]$  we have

$$\|y_n(t) - y(t)\| \leq \left\| \int_0^t S(t-s)\mathcal{B}(u_n(s) - u(s), y(s))ds \right\| + \|\mathcal{B}\| \int_0^t \|S(t-s)\| \|u_n(s)\|_U \|y_n(s) - y(s)\| ds$$

Applying the Gronwall lemma (see Theorem 1 in [17]) yields

$$\|y_n(t) - y(t)\| \leq \sup_{t \in [0, T]} \left( \left\| \int_0^t S(t-s)\mathcal{B}(u_n(s) - u(s), y(s))ds \right\| \right) \exp \left( \|\mathcal{B}\| \int_0^t \|S(t-s)\| \|u_n(s)\|_U ds \right) \quad (3)$$

Using the weak convergence of  $u_n$  in  $L^2(0, T; U)$  and the fact that the semi-group  $S(t)$  is bounded on the entire finite interval  $[0, T]$ , we have for some  $M > 0$

$$\exp \left( \|\mathcal{B}\| \int_0^t \|S(t-s)\| \|u_n(s)\|_U ds \right) \leq M, \quad \forall t \in [0, T]. \quad (4)$$

**1<sup>st</sup> case : Assume that (a) holds.**

The weak convergence of  $u_n \rightharpoonup u$  in  $L^2(0, T; U)$  implies that  $\mathcal{B}(u_n(\cdot), y(\cdot))$  strongly converge to  $\mathcal{B}(u(\cdot), y(\cdot))$  in  $L^2(0, T; X)$ .

Then, we conclude that

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s)\mathcal{B}(u_n(s) - u(s), y(s))ds \right\| = 0. \quad (5)$$

It follows from (3), (4) and (5) that

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t \leq T} \|y_n(s) - y(s)\| = 0.$$

**2<sup>nd</sup> case : Assume that (b) holds.**

According to Theorem 3.9 in [9], the weak convergence :  $u_n \rightharpoonup u$  in  $L^2(0, T; U)$  implies the following weak convergence :  $\mathcal{B}(u_n(\cdot), y(\cdot)) \rightharpoonup \mathcal{B}u(\cdot), y(\cdot)$  in  $L^2(0, T; X)$ .

Moreover, the weak convergence of  $\mathcal{B}(u_n(\cdot), y(\cdot)) \rightharpoonup \mathcal{B}u(\cdot), y(\cdot)$  in  $L^2(0, T; X)$  gives (see Corollary 3.3 of [20]):

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-s) \mathcal{B}(u_n(s) - u(s), y(s)) ds \right\| = 0. \quad (6)$$

It follows from (3), (4) and (6) that

$$\lim_{n \rightarrow +\infty} \sup_{0 \leq t \leq T} \|y_n(s) - y(s)\| = 0.$$

## ii. Optimal control for the problem $P_\epsilon$

The following result discusses the existence of the optimal control related to the auxiliary problem ( $P_\epsilon$ ).

### Theorem 3

Let one of the assumptions (a) or (b) hold.

- If  $V = \{u \in L^2(0, T; U) / \|u\|_U \leq M\}$  for some  $M > 0$ , then there exists an optimal control for the problem ( $P_\epsilon$ ), which satisfies the following formula:

$$u^*(t) = - \left( \frac{\|\epsilon r u^*(t) + (\mathcal{B}(\cdot, y^*(t)))^* \phi(t)\|_U}{M} + \epsilon r \right)^{-1} \mathcal{B}(\cdot, y^*(t))^* \phi(t),$$

where  $\phi$  is the mild solution of the following adjoint system

$$\begin{cases} \dot{\phi}(t) = -A^* \phi(t) - \mathcal{B}^*(u^*(t), \phi(t)) - 2\epsilon y(t) \\ \phi(T) = 2(y(T) - y_d) \end{cases} \quad (7)$$

$\mathcal{B}^*(u^*(t), \cdot)$  being the adjoint of the operator  $\mathcal{B}(u^*(t), \cdot)$ .

- If  $V = L^2(0, T; U)$ , then the control defined by

$$u^*(t) = -\frac{1}{\epsilon r} (\mathcal{B}(\cdot, y^*(t)))^* \phi(t)$$

is a solution of the problem ( $P_\epsilon$ ), where  $\phi$  is the mild solution of the adjoint system (7).

### Proof:

First let us show the existence of a solution of the problem ( $P_\epsilon$ ).

Since the set  $\{J_\epsilon(u) / u \in V\} \subset \mathbb{R}^+$  is not empty and bounded from below, it admits a lower bound  $J^*$ . Let  $(u_n)_{n \in \mathbb{N}}$  be a minimizing sequence such that  $J_\epsilon(u_n) \rightarrow J^*$ .

Then the sequence  $(u_n)$  is bounded, so it admits a sub-sequence still denoted by  $(u_n)$ , which

weakly converges to  $u^* \in V$ .

Let  $y_n$  and  $y^*$  be the solutions of (1) respectively corresponding to  $u_n$  and  $u^*$ .

From Lemma 2 we have

$$\lim_{n \rightarrow +\infty} \|y_n(t) - y^*(t)\| = 0, \quad \forall t \in [0, T]. \quad (8)$$

Since the norm  $\|\cdot\|$  is lower semi-continuous, it follows from (8) that for all  $t \in [0, T]$

$$\|y^*(t)\|^2 = \lim_{n \rightarrow +\infty} \inf \|y_n(t)\|^2.$$

Applying Fatou's lemma we get

$$\int_0^T \|y^*(t)\|^2 dt = \lim_{n \rightarrow +\infty} \inf \int_0^T \|y_n(t)\|^2 dt. \quad (9)$$

Since  $R : u \mapsto \int_0^T \|u(t)\|_U^2 dt$  is convex and lower semi-continuous with respect to weak topology, we have (see Corollary III.8 of [9])

$$R(u^*) \leq \lim_{n \rightarrow +\infty} \inf R(u_n). \quad (10)$$

Combining the formulas (8), (9) and (10) we deduce that

$$\begin{aligned} J_\epsilon(u^*) &= \|y^*(T) - y_d\|^2 + \epsilon \int_0^T \|y(t)\|^2 dt + \frac{\epsilon r}{2} \int_0^T \|u^*(t)\|_U^2 dt \\ &\leq \lim_{n \rightarrow +\infty} \inf \|y_n(T) - y_d\|^2 + \epsilon \lim_{n \rightarrow +\infty} \inf \int_0^T \|y_n(t)\|^2 dt + \frac{\epsilon r}{2} \lim_{n \rightarrow +\infty} \inf \int_0^T \|u_n(t)\|_U^2 dt \\ &\leq \lim_{n \rightarrow +\infty} \inf J_\epsilon(u_n) \\ &\leq J^*. \end{aligned}$$

We conclude that  $J_\epsilon(u^*) = J^*$  and so  $u^*$  is a solution of the problem  $(P_\epsilon)$ .

Let us proceed to the characterisation of the optimal control.

1. **The case**  $V = \{u \in L^2(0, T; U) / \|u\|_{L^2(0, T, U)} \leq M\}$ .

Let  $f_0 : X \times U \mapsto \mathbb{R}$  be defined by

$$f_0(y, u) = \epsilon \left( \|y\|^2 + \frac{r}{2} \|u\|_U^2 \right), \quad \forall (y, u) \in X \times U.$$

Then, the cost function  $J_\epsilon$  takes the form

$$J_\epsilon(u) = \|y(T) - y_d\|^2 + \int_0^T f_0(y(t), u(t)) dt.$$

Since  $V$  is bounded, by application of Pontryagin's maximum principle (see Theorem 5.2 p. 258 in [20] and Theorem 6.1 p. 162 in [10]), we find that for any solution  $u^*$  of the problem  $(P_\epsilon)$  there exists a function  $\phi$  solution of the following adjoint system

$$\begin{cases} \dot{\phi}(t) = -A^* \phi(t) - \mathcal{B}^*(u^*(t), \phi(t)) - 2\epsilon y^*(t) \\ \phi(T) = 2(y^*(T) - y_d) \end{cases}$$

and satisfies the following condition

$$H(t, u^*(t), y^*(t), \phi(t)) = \min_{u \in V} H(t, u(t), y^*(t), \phi(t)), \quad (11)$$

where

$$H(t, u(t), y^*(t), \phi(t)) = f_0(u(t), y^*(t)) + \langle \phi(t), \mathcal{B}(u(t), y^*(t)) \rangle.$$

By differentiating the function  $u \mapsto H(u) = H(t, u(t), y^*(t), \phi(t))$ , we have

$$H'(u)(t) = \epsilon r u(t) + \mathcal{B}(\cdot, y^*(t))^* \phi(t),$$

where  $(\mathcal{B}(\cdot, y^*(t)))^* : X \mapsto U$  is the adjoint of the operator  $\mathcal{B}(\cdot, y^*(t))$ .

If  $\|u^*\|_{L^2(0,T;U)} < M$ , then we conclude that

$$u^*(t) = -\frac{1}{\epsilon r} \mathcal{B}(\cdot, y^*(t))^* \phi(t). \quad (12)$$

If  $\|u^*\|_{L^2(0,T;U)} = M$ , we can distinguish two cases, if  $H'(u^*) = 0$  then the control is given by (12)

and if  $H'(u^*) \neq 0$ , then we proceed as follows:

Let  $v_1(t) = \frac{1}{M} u^*(t)$  and  $v_2(t) = -\frac{1}{\|H'(u^*)\|_{L^2(0,T;U)}} H'(u^*)(t)$ . We will show that  $v_1 = v_2$ .

For all  $u \in V$  we have

$$\langle v_1, u \rangle_{L^2(0,T;U)} \leq \|v_1\|_{L^2(0,T;U)} \|u\|_{L^2(0,T;U)} \leq M \quad \text{and} \quad \langle v_1, u^* \rangle_{L^2(0,T;U)} = M.$$

So we conclude that

$$\forall u \in V, \quad \langle v_1, u \rangle_{L^2(0,T;U)} \leq \langle v_1, u^* \rangle_{L^2(0,T;U)}.$$

Moreover, the fact that  $V$  is convex, implies

$$\forall u \in V, \quad \forall \lambda \in [0, 1], \quad u^* + \lambda(u - u^*) \in V.$$

Then since  $u^*$  is a solution of the problem  $(P_\epsilon)$ , we derive from (11)

$$\begin{aligned} H(u^*) &\leq H(u^* + \lambda(u - u^*)) \\ &\leq H(u^*) + \langle H'(u^*), \lambda(u - u^*) \rangle_{L^2(0,T;U)} \\ &+ \lambda \|u^* - u\|_{L^2(0,T;U)} \theta(\lambda \|u^* - u\|_{L^2(0,T;U)}), \quad \forall \lambda \in [0, 1], \quad \forall u \in V \end{aligned} \quad (13)$$

where the function  $\theta$  is such that

$$\lim_{\lambda \rightarrow 0^+} \theta(\lambda \|u^* - u\|_{L^2(0,T;U)}) = 0. \quad (14)$$

From (13) and (14) it comes

$$\langle H'(u^*), u \rangle_{L^2(0,T;U)} \geq \langle H'(u^*), u^* \rangle_{L^2(0,T;U)}.$$

So, we conclude that

$$\forall u \in V_{ad}, \quad \langle v_2, u \rangle_{L^2(0,T;U)} \leq \langle v_2, u^* \rangle_{L^2(0,T;U)}.$$

Taking into account that  $\sup_{u \in V} \langle v_2, u \rangle_{L^2(0,T;U)} = M$ , we deduce that  $\langle v_2, u^* \rangle_{L^2(0,T;U)} = M$  and that

$$\frac{1}{2} \langle v_1 + v_2, u^* \rangle_{L^2(0,T;U)} = \frac{1}{2} \langle v_1, u^* \rangle_{L^2(0,T;U)} + \frac{1}{2} \langle v_2, u^* \rangle_{L^2(0,T;U)} = M,$$

then

$$\left\| \frac{1}{2}(v_1 + v_2) \right\|_{L^2(0,T;U)} \geq 1.$$

It follows that

$$\|(v_1 + v_2)\|_{L^2(0,T;U)} = \|v_1\|_{L^2(0,T;U)} + \|v_2\|_{L^2(0,T;U)}$$

and that  $v_1 = v_2$ .

Furthermore, we have

$$\frac{1}{M}u^*(t) = -\frac{1}{\|H'(u^*)\|_{L^2(0,T;U)}}H'(u^*)(t). \quad (15)$$

According to (12) and (15) we have

$$u^*(t) = \frac{-1}{\frac{\|H'(u^*)\|_{L^2(0,T;U)}}{M} + \epsilon r} \mathcal{B}(\cdot, y^*(t))^* \phi(t),$$

where

$$H'(u)(t) = \epsilon r u(t) + \mathcal{B}(\cdot, y^*(t))^* \phi(t).$$

## 2. The case $V = L^2(0, T; U)$ .

From the first part of the proof, there exists a solution  $u^*$  of the problem  $(P_\epsilon)$ .

Let us consider the closed convex space

$$V^* = \{u \in L^2(0, T; U) / \|u\|_{L^2(0,T;U)} \leq \|u^*\|_{L^2(0,T;U)} + 1\}.$$

It is clear that  $u^*(t) \in \hat{V}^*$ , then from the first case, we have  $H'(u^*) = 0$ , which leads to

$$u^*(t) = -\frac{1}{\epsilon r} \mathcal{B}(\cdot, y^*(t))^* \phi(t),$$

where  $\phi$  is the mild solution of the adjoint system (7).

This achieves the proof of Theorem 3.

### iii. Sequential characterization of the solution of the problem $(P)$

In the sequel, we take a decreasing sequence  $(\epsilon_n)$  such that  $\epsilon_n \rightarrow 0$  with corresponding sequence of controls  $(u_n^*)$  solutions of problems  $(P_{\epsilon_n})$ .

**Theorem 4** *Assume that  $V$  is bounded and let  $y_d$  be an approximately attainable state by a control from  $V$ . Then the problem  $(P)$  posses a solution. Moreover any weak limit value of  $(u_n^*)$  in  $L^2(0, T, U)$  is a solution of  $(P)$ .*

**Proof:**

Since  $V$  is bounded, we deduce that the sequence  $(u_n^*)$  is bounded, so it admits a weakly converging subsequence, denoted by  $(u_n^*)$  as well. Let  $u^*$  be a weak limit value of  $(u_n^*)$  in  $V$ .

The remainder of the proof is divided into three steps

**Step 1:  $y_d$  is exactly attainable**  $U_{ad} \neq \emptyset$ .

Let us consider the following problem

$$\begin{cases} \min \|y_u(T) - y_d\|^2 \\ u \in V \end{cases} \quad (16)$$

The set  $\{\|y_u(T) - y_d\|^2 / u \in V\} \subset \mathbb{R}^+$  is not empty and bounded from below, so it admits a lower bound  $J_d$ .

Let  $(v_n)_{n \in \mathbb{N}}$  be a minimizing sequence such that  $\|y_{v_n}(T) - y_d\|^2 \xrightarrow{n \rightarrow +\infty} J_d$ .

Since  $V$  is bounded, we deduce that the sequence  $(v_n)$  is bounded, so it admits a weakly converging subsequence to  $v \in V$  still denoted by  $(v_n)$ .

By Lemma 2, we have for all  $t \in [0, T]$

$$\lim_{n \rightarrow +\infty} \|y_{v_n}(t) - y_v(t)\| = 0$$

then, we conclude that

$$\|y_v(T) - y_d\|^2 = \lim_{n \rightarrow +\infty} \|y_{v_n}(T) - y_d\|^2 = J_d = \min_{u \in V} \|y_u(T) - y_d\|^2 \quad (17)$$

So the control  $v$  is a solution of the problem (16).

Since the system (1) is approximately attainable, we have

$$\forall \varepsilon > 0, \exists v_\varepsilon \in V / \|y_{v_\varepsilon}(T) - y_d\| \leq \varepsilon \quad (18)$$

According to (17) and (18), we get

$$\forall \varepsilon > 0, \exists v_\varepsilon \in V, \|y_v(T) - y_d\| \leq \|y_{v_\varepsilon}(T) - y_d\| \leq \varepsilon$$

So we conclude that  $\|y_v(T) - y_d\| = 0$  and hence  $v \in U_{ad}$ .

**Step 2:**  $\forall v \in U_{ad}, J(u^*) \leq J(v)$ .

Taking into account that  $u_n^*$  is a solution of the problem  $(P_{\varepsilon_n})$  and  $y_n^*$  is the corresponding solution of the system (1), we get for all  $v \in U_{ad}$

$$J_{\varepsilon_n}(u_n^*) = \|y_n^*(T) - y_d\|^2 + \varepsilon_n J(u_n^*) \leq J_{\varepsilon_n}(v)$$

from which, it comes

$$\begin{aligned} \varepsilon_n J(u_n^*) &\leq J_{\varepsilon_n}(v) - \|y_n^*(T) - y_d\|^2 \\ &\leq \varepsilon_n J(v) \end{aligned}$$

So we find

$$J(u_n^*) \leq J(v) \text{ for all } v \in U_{ad}. \quad (19)$$

Let  $y^*$  be the solution of system (1) corresponding to  $u^*$ .

Since  $u_n \rightarrow u^*$  in  $L^2(0, T; U)$ , we have by Lemma 2

$$\lim_{n \rightarrow +\infty} \|y_n^*(t) - y^*(t)\| = 0, \forall t \in [0, T]. \quad (20)$$

The norm  $\|\cdot\|$  is lower semi-continuous, it follows that for all  $t \geq 0$  we have

$$\|y^*(t)\|^2 = \lim_{n \rightarrow +\infty} \inf \|y_n^*(t)\|^2.$$

Applying Fatou's lemma we get

$$\int_0^T \|y^*(t)\|^2 dt = \lim_{n \rightarrow +\infty} \inf \int_0^T \|y_n^*(t)\|^2 dt. \quad (21)$$



The function  $R$  is lower semi-continuous and convex, it follows from [9] that

$$R(u^*) \leq \liminf_{n \rightarrow +\infty} R(u_n^*). \quad (22)$$

By the inequalities (21) and (22) we deduce that

$$J(u^*) \leq \liminf_{n \rightarrow +\infty} J(u_n^*). \quad (23)$$

Combining (19) and (23) we deduce that

$$J(u^*) \leq J(v).$$

**Step 3 :**  $u^* \in U_{ad}$ .

According to the inequality (19), we deduce that  $J(u_n^*)$  is bounded and

$$\lim_{n \rightarrow +\infty} \|y_n^*(T) - y_d\|^2 = \lim_{n \rightarrow +\infty} J_{\epsilon_n}(u_n^*) \leq \lim_{n \rightarrow +\infty} J_{\epsilon_n}(v) = \|y_v(T) - y_d\|^2 = 0.$$

Then, taking into account the formula (20), we derive via the continuity of the norm that

$$\lim_{n \rightarrow +\infty} \|y_n^*(T) - y_d\| = \|y^*(T) - y_d\| \leq \|y_v(T) - y_d\| = 0.$$

Consequently,  $y^*(T) = y_d$  and the control  $u^*$  is a solution of problem (P).

**Theorem 5** *If  $U_{ad} \neq \emptyset$ , then there exists a solution  $u^*$  of the problem (P). Furthermore, any weak limit value of the solution  $(u_n^*)$  of  $(P_{\epsilon_n})$  in  $L^2(0, T; U)$  is a solution of (P).*

**Proof:**

Let  $v \in U_{ad}$ . Then keeping in mind that  $u_n^*$  is the solution of the problem  $(P_{\epsilon_n})$  corresponding to  $\epsilon_n$ , we can see that

$$J_{\epsilon_n}(u_n^*) \leq J_{\epsilon_n}(v) = \epsilon_n J(v)$$

It follows that

$$\epsilon_n J(u_n^*) = J_{\epsilon_n}(u_n^*) - \|y_n^*(T) - y_d\|^2 \leq J_{\epsilon_n}(u_n^*) \leq \epsilon_n J(v)$$

Using the definition of the cost  $J$  given by (2), the last equality gives

$$r \int_0^T \|u_n^*(t)\|_U^2 dt \leq J(u_n^*) \leq J(v). \quad (24)$$

We deduce that the sequence  $(u_n^*)$  is bounded, so it admits a weakly converging subsequence in  $V$ , also denoted by  $(u_n^*)$ . Let  $u^*$  be a weak limit value of  $(u_n^*)$  in  $V$  and let  $y^*$  be the solution of system (1) corresponding to  $u^*$ .

Since  $u_n \rightharpoonup u^*$  in  $L^2(0, T; U)$ , we have by Lemma 2

$$\lim_{n \rightarrow +\infty} \|y_n^*(t) - y^*(t)\| = 0, \quad \forall t \in [0, T].$$

Similarly to the proof of Theorem 4 we can show that

$$J(u^*) \leq J(v).$$

According to the inequality (24), we deduce that  $J(u_n^*)$  is bounded and

$$\lim_{n \rightarrow +\infty} J_{\epsilon_n}(u_n^*) = \lim_{n \rightarrow +\infty} \|y_n^*(T) - y_d\|^2 \leq \|y_v(T) - y_d\|^2.$$

Hence

$$\lim_{n \rightarrow +\infty} \|y_n^*(T) - y_d\| = \|y^*(T) - y_d\| \leq \|y_v(T) - y_d\| = 0$$

We conclude that  $u^* \in U_{ad}$ .

#### iv. Optimal feedback control

In this part we will try to express the optimal control  $u^*$  of the problem (P) as a time-varying feedback law for the class of commutative bilinear systems with scalar control [15, 28].

Assume that  $U = \mathbb{R}$ , then we can write the system (1) as follows

$$\begin{cases} \dot{y}(t) = Ay(t) + u(t)By(t) \\ y(0) = y_0 \in X \end{cases}$$

where  $A : D(A) \subset X \mapsto X$  is the infinitesimal generator of a linear  $C_0$ - semi-group  $S(t)$ ,  $B$  is a bounded linear operator and  $u \in V := L^2(0, T)$ .

**Theorem 6** *Assume that  $A$  and  $B$  commute with each other and that  $U_{ad} \neq \emptyset$ . Let  $v \in U_{ad}$  and let  $y_0 \in X$  be such that  $S(T)y_0 \notin \text{Ker}(B)$ . Then for any solution  $u^*$  of the problem (P), we have the following formula*

$$u^*(t) = \frac{1}{T} \int_0^T v(s)ds + \frac{2}{Tr} \int_0^T \int_\alpha^T \langle y^*(s), By^*(s) \rangle ds d\alpha - \frac{2}{r} \int_t^T \langle y^*(s), By^*(s) \rangle ds$$

**Proof:**

Let us consider the system (1) in the time horizon  $[0, T]$ , and let  $A_k = kA(kI - A)^{-1}$  be the Yosida approximation of the operator  $A$ . Let  $y_k$  and  $\phi_k$  be the respective solutions to (1) and (7) with  $A_k$  instead of  $A$ . For  $u \in L^2(0, T)$ , since  $A_k$  is bounded, we have  $y_k, \phi_k \in H^1(0, T)$  and

$$\begin{aligned} \langle \dot{\phi}_k(t), By_k(t) \rangle + \langle \phi_k(t), B\dot{y}_k(t) \rangle &= \langle -A_k^* \phi_k(t) - u(t)B^* \phi_k(t) - 2\epsilon y_k(t), By_k(t) \rangle \\ &+ \langle B^* \phi_k(t), A_k y_k(t) + u(t)By_k(t) \rangle \\ &= \langle \phi_k(t), BA_k y_k(t) - A_k By_k(t) \rangle - 2\epsilon \langle y_k(t), By_k(t) \rangle. \end{aligned}$$

Thus

$$\langle \dot{\phi}_k(t), By_k(t) \rangle + \langle \phi_k(t), B\dot{y}_k(t) \rangle = \langle \phi_k(t), [B, A_k]y_k(t) \rangle - 2\epsilon \langle y_k(t), By_k(t) \rangle \quad (25)$$

where  $[B, A_k] := BA_k - A_k B$ .

Integrating (25) over  $[t, T]$ , we get

$$\langle \phi_k(t), By_k(t) \rangle = 2 \langle y_k(T) - y_d, By_k(T) \rangle - \int_t^T \left( \langle \phi_k(s), [B, A_k]y_k(s) \rangle - 2\epsilon \langle y_k(s), By_k(s) \rangle \right) ds$$

Since  $\phi_k \rightarrow \phi$  and  $y_k \rightarrow y$  strongly, we obtain by letting  $k \rightarrow +\infty$

$$\langle \phi(t), By(t) \rangle = 2 \langle y(T) - y_d, By(T) \rangle + 2\epsilon \int_t^T \langle y(s), By(s) \rangle ds.$$

So, by Theorem 3, we conclude that the solution of the problem  $(P_{\epsilon_n})$  corresponding to  $\epsilon_n$ , is given by

$$u_n^*(t) = -\frac{1}{\epsilon_n r} \langle \phi_n(t), By_n^*(t) \rangle = -\frac{2}{\epsilon_n r} \langle y_n^*(T) - y_d, By_n^*(T) \rangle - \frac{2}{r} \int_t^T \langle y_n^*(s), By_n^*(s) \rangle ds. \quad (26)$$

Let  $v \in U_{ad}$ . By Theorem 5, any limit value  $u^*$  of  $u_n^*$  in  $L^2(0, T)$  is a solution of the problem (P). Since  $A$  and  $B$  commute, we have the following formulas

$$y_v(t) = S(t) \exp\left(B \int_0^t v(s)ds\right) y_0$$

and

$$y^*(t) = S(t) \exp(B \int_0^t u^*(s) ds) y_0.$$

Using the fact that  $v, u^* \in U_{ad}$  and  $\lim_{n \rightarrow +\infty} y_n^*(T) = y_d$ , we obtain

$$\lim_{n \rightarrow +\infty} y_n^*(T) = y_u^*(T) = y_v(T) = y_d.$$

Hence

$$\lim_{n \rightarrow +\infty} S(T) \exp(B \int_0^T u_n^*(t) dt) y_0 = S(T) \exp(B \int_0^T v(t) dt) y_0 = S(T) \exp(B \int_0^T u^*(t) dt) y_0.$$

From the assumption  $S(T)y_0 \notin \text{Ker}(B)$ , we deduce from the last inequalities that

$$\lim_{n \rightarrow +\infty} \int_0^T u_n^*(t) dt = \int_0^T v(t) dt = \int_0^T u^*(t) dt.$$

Moreover, we deduce from the formula (26), that

$$\begin{aligned} \lim_{n \rightarrow +\infty} \int_0^T u_n^*(t) dt &= \lim_{n \rightarrow +\infty} \int_0^T \left( -\frac{2}{\epsilon_n r} \langle y_n^*(T) - y_d, B y_n^*(T) \rangle - \frac{2}{r} \int_t^T \langle y_n^*(s), B y_n^*(s) \rangle ds \right) dt \\ &= \lim_{n \rightarrow +\infty} -\frac{2T}{\epsilon_n r} \langle y_n^*(T) - y_d, B y_n^*(T) \rangle - \frac{2}{r} \int_0^T \int_t^T \langle y^*(s), B y^*(s) \rangle ds dt \end{aligned}$$

from which, we derive

$$\lim_{n \rightarrow +\infty} -\frac{2T}{\epsilon_n r} \langle y_n^*(T) - y_d, B y_n^*(T) \rangle = \int_0^T v(t) dt + \frac{2}{r} \int_0^T \int_t^T \langle y^*(s), B y^*(s) \rangle ds dt. \quad (27)$$

By (26) and (27) we deduce that  $u_n^*(t) \rightarrow u^*(t)$  for all  $t \in [0, T]$  and

$$\begin{aligned} \lim_{n \rightarrow +\infty} u_n^*(t) &= \lim_{n \rightarrow +\infty} -\frac{2}{\epsilon_n r} \langle y_n^*(T) - y_d, B y_n^*(T) \rangle - \frac{2}{r} \int_t^T \langle y_n^*(s), B y_n^*(s) \rangle ds \\ &= \frac{1}{T} \int_0^T v(s) ds + \frac{2}{Tr} \int_0^T \int_\alpha^T \langle y^*(s), B y^*(s) \rangle ds d\alpha - \frac{2}{r} \int_t^T \langle y^*(s), B y^*(s) \rangle ds \\ &= u^*(t). \end{aligned}$$

We conclude that

$$u^*(t) = \frac{1}{T} \int_0^T v(s) ds + \frac{2}{Tr} \int_0^T \int_\alpha^T \langle y^*(s), B y^*(s) \rangle ds d\alpha - \frac{2}{r} \int_t^T \langle y^*(s), B y^*(s) \rangle ds.$$

**Remark 7** In the case where  $S(t_1)$  is one to one for some  $t_1 > 0$  and  $y_0 \notin \text{Ker}(B)$ , the assumption  $S(T)y_0 \notin \text{Ker}(B)$  in Theorem 6 is satisfied.

### III. EXAMPLES

#### i. Wave equation

Let us consider the following wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} z(t, x) &= \Delta z(t, x) + u(t, x)z(t, x), & t \in [0, T] \text{ and } x \in \Omega = (0, 1) \\ z(t, 0) &= z(t, 1) = 0, & t \in [0, T] \\ z(0, x) &= z_0(x), & x \in \Omega \end{cases}$$

where

- $u \in L^2(0, T, L^2(\Omega))$ ,
- $T > 4 \max_{x \in \Omega} |x - x_0|$  for some  $x_0 \in \mathbb{R} \setminus [0, 1]$ ,
- the desired state  $z_d \in H_0^1(\Omega) \cap H^2(\Omega)$  is such that  $\frac{\Delta z_d}{z_d} \mathbb{1}_{(z_d \neq 0)} \in L^\infty(\Omega)$ , where  $\mathbb{1}_{(z_d \neq 0)}$  indicates the characteristic function of the set  $(z_d \neq 0) := \{x \in \Omega / z_d(x) \neq 0\}$ .

This system has the form of the system (1) if we take  $y(t) = (z(t), \dot{z}(t))$ ,  $X = H_0^1(\Omega) \times L^2(\Omega)$  with  $\langle (y_1, z_1), (y_2, z_2) \rangle_X = \langle y_1, y_2 \rangle_{H_0^1(\Omega)} + \langle z_1, z_2 \rangle_{L^2(\Omega)}$  and

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix} \text{ with } D(A) = H_0^1(\Omega) \cap H^2(\Omega) \times H_0^1(\Omega) \text{ and } B = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

Here  $B$  is a compact linear bounded operator on  $X$  and  $A$  is the infinitesimal generator of a linear  $C_0$ - semi-group  $S(t)$  of isometries (see [3], p.176).

The quadratic cost function is given by

$$J(u) = \int_0^T (\|z(t)\|_{H_0^1(\Omega)}^2 + \|\dot{z}(t)\|_{L^2(\Omega)}^2) dt + \frac{r}{2} \int_0^T \|u(t)\|_{L^2(\Omega)}^2 dt,$$

where  $u(t) := u(t, \cdot)$  and  $z(t) := z(t, \cdot)$ .

According to [25], there exists a control  $v \in L^2(0, T; L^2(\Omega))$  such that the corresponding solution  $z_v$  of the system (1) verifies  $z_v(T) = z_d$ . Then, according to Theorem 5 there exists a control  $u^* \in L^2(0, T, \mathbb{R})$ , which guarantees the exact attainability of  $z_d$  at time  $T$ , and is a solution of the problem (P) with  $U_{ad} = \{u \in L^2(0, T, L^2(\Omega)) / z(T) = z_d\}$ .

**Remark 8** *The optimal control of the bilinear wave equation has been considered in [21, 30] in the context of unconstrained endpoint.*

#### ii. Heat equation

In this part we study the optimal exact attainability for the reaction-diffusion equation.

Let us consider the following system

$$\begin{cases} \frac{\partial}{\partial t} y(t, x) = \Delta y(t, x) + u(t, x)y(t, x), & \text{in } Q = \Omega \times (0, T), T > 0 \\ y(t, 0) = y(t, 1) = 0, & \text{on } (0, T) \\ y(0) = y_0 & \text{in } \Omega \end{cases} \quad (28)$$

where  $\Omega = (0, 1)$  and  $u \in L^2(0, T, U)$  is a control function.

**Case 1: Distributed control** ( $U = L^2(\Omega)$ )

Assume that  $y_0, y_d \in L^2(\Omega)$  are such that

- for a.e.  $x \in \Omega$ ,  $y_d y_0 \geq 0$ ,
- for a.e.  $x \in \Omega$ ,  $y_0(x) = 0 \iff y_d(x) = 0$ ,
- $a := \ln\left(\frac{y_d}{y_0}\right) \mathbf{1}_{(y_0 \neq 0)} \in L^\infty(\Omega)$ , where  $\mathbf{1}_{(y_0 \neq 0)}$  indicates the characteristic function of the set  $(y_0 \neq 0) := \{x \in \Omega / y_0(x) \neq 0\}$ .
- $\frac{\Delta y_d}{y_d} \mathbf{1}_{(y_d \neq 0)} \in L^\infty(\Omega)$ ,
- $|y_d| > 0$  a.e. on some nonempty open subset  $O$  of  $\Omega$ .

According to Theorem 2 in [24], there is a time  $T$  for which  $y_d$  is exactly attainable for the system (28) using a control  $v \in L^2(0, T, L^2(\Omega))$ , so  $U_{ad} \neq \emptyset$ . Then, according to Theorem 5, there exists a control  $u^*$  which guarantees the exact attainability of  $y_d$  at time  $T$ , and is solution of the following problem

$$\begin{cases} \min J(u) \\ u \in U_{ad} = \{u \in L^2(0, T, L^2(\Omega)) / y_u(T) = y_d\} \end{cases} \quad (29)$$

More precisely any weak limit of  $u_n^*$  given by Theorem 3 corresponding to sequence  $(\epsilon_n)$  gives a optimal control  $u^*$  for (29).

### Case 2: Scalar control ( $U = \mathbb{R}$ )

Here, we have  $u(t, x) = u(t) \in \mathbb{R}$ .

Assume that  $y_0, y_d \in L^2(\Omega)$  are such that  $y_d = \lambda y_0$  with  $\lambda > 1$  and  $y_0 > 0$ , a.e in  $\Omega$ . According to Theorem II 4 and Remark 4 in [26], there is a time  $T$  for which  $y_d$  is exactly attainable for the system (28) using the control  $v(t) = \frac{\lambda-1}{T+(\lambda-1)t} \in L^2(0, T, \mathbb{R})$ , so  $U_{ad} \neq \emptyset$ .

By Theorem 6, there exists a feedback control  $u^* \in L^2(0, T, \mathbb{R})$  which guarantees the exact attainability of  $y_d$  at time  $T$ , and is solution of the problem (P) with  $U_{ad} = \{u \in L^2(0, T, \mathbb{R}) / y^*(T) = y_d\}$ , and satisfies the following formula

$$u^*(t) = \frac{1}{T} \ln(\lambda) + \frac{2}{Tr} \int_0^T \int_\alpha^T \|y^*(s)\|^2 ds d\alpha - \frac{2}{r} \int_t^T \|y^*(s)\|^2 ds.$$

### iii. Transport equation

Let us consider the following transport problem

$$\begin{cases} \frac{\partial}{\partial t} y(t, x) = -\frac{\partial}{\partial x} y(t, x) + u(t)y(t, x), & t \in (0, T), x \in \Omega = (0, +\infty) \\ y(t, 0) = 0, & t \in (0, T) \\ y(0, x) = y_0(x), & x \in \Omega \end{cases} \quad (30)$$

where  $u \in L^2(0, T)$ . Here the operator  $A = -\frac{\partial}{\partial x}$  with the domain  $D(A) = H_0^1(\Omega)$  generates a  $C_0$ -semi-group of isometries  $S(t)$  in  $X = L^2(\Omega)$ . Below, we will develop numerical simulation for the example (30). For this end, we take  $r = 2$ ,  $T = 9$ ,  $y_0 = x \exp(-x)$  and

$$y_d(x) = \begin{cases} 0, & \text{if } x \leq 9 \\ (x-9) \exp(9-x), & \text{if } x \geq 9 \end{cases}$$

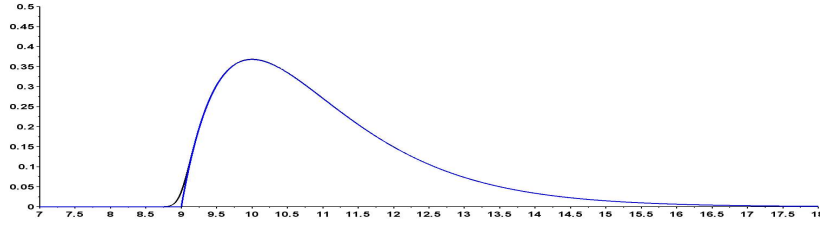
then the control  $v = 0 \in U_{ad} = \{u \in L^2(0, T) / y^*(T) = y_d\}$ . By Theorem 6, there exists a feedback control  $u^* \in L^2(0, T)$  which guarantees the exact attainability of  $y_d$  at time  $T$ . Moreover

$u^*$  is the solution of the problem (P) and satisfies the following formula

$$u^*(t) = \frac{1}{T} \int_0^T \int_{\alpha}^T \|y^*(s)\|^2 ds d\alpha - \int_t^T \|y^*(s)\|^2 ds. \quad (31)$$

In the Figure 1, we compare numerically the two controls  $u^*$  and  $v = 0$  in term of the state at the finite time  $T = 9$ . Moreover, we find  $J(u^*) = 1.2442$  and  $J(v) = 2.25 \approx 2J(u^*)$ .

We observe that the desired state is exactly attainable either by using the optimal control  $u^*$  or



**Figure 1:** The state  $y_{u^*}(T)$  (black line), and the desired state  $y_d$  (blue line)

the control  $v = 0$ . However, the control  $u^*$  leads to a lower cost than the zero control.

**Remark 9** Unlike the case of linear systems, the uniqueness of the optimal control of the quadratic cost (2) is not guaranteed in general when dealing with bilinear systems, which is due to the lack of convexity of the state w.r.t control. For instance, if we assume that  $r = 0$  and that  $A = B$  is a skew-adjoint matrix, we can see that the cost function is constant so we have an infinity of optimal controls. However, in the case of the quadratic cost function  $J(u) = \int_0^T u^2(t) dt$ , the uniqueness of the optimal control is assured by the strict convexity of the cost  $J$  (see [28]). Moreover, in the case of a cost function  $J$  of the form (2), one can prove the uniqueness of the optimal bilinear control under some constraint relating  $T$  and  $y_0$  [8, 29, 30].

## IV. CONCLUSION

In this work, we studied the question of quadratic optimal control with endpoint constraint for bilinear systems. The optimal control is characterized via a set of unconstrained minimization problems, then it is expressed as a time varying feedback for commutative bilinear systems. The obtained results are applied to parabolic and hyperbolic PDE. As an interesting continuation of the present work, one can consider the same questions for unbounded control operators, such as the case of Fokker Planck equation [1].

### Conflict of interest statement.

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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