Instability and overshoots of solutions for a class of homogeneous hybrid systems by Lyapunov-like analysis

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***Abstract*— For a class of homogeneous hybrid systems we present a generalization to the hybrid systems framework of Chetaev's theorem and we propose a set of local Lyapunov-like conditions for studying instability of the point $x_e = 0$ and overshoots of solutions (namely when the norm of the solution vector x at some time instant exceeds the norm of the initial condition of x). Based on these results, we design a sum of squares algorithm that constructs a suitable function to automatically fulfill such local conditions.**

I. INTRODUCTION

Hybrid systems combine continuous processes whose dynamics depends on differential equations, and discrete processes whose behavior depends on a specific transition relation. A mass subject to Coulomb friction, robots controlled by a finite state machine, electrical circuits that combine analog and digital components, are all examples of systems that combine continuous and discrete processes and that can be conveniently characterized within the hybrid systems framework.

Several models of hybrid systems can be found in the literature, [5], [7], [11], [16]. Here we consider the framework outlined in [8] for which several structural results have been developed [10], [23], [24] and partially summarized in [9]. Although several new phenomena arise from the interaction of continuous and discrete dynamics, important results on stability theory like Lyapunov-like tools, invariance principles and converse theorems, have been generalized to the hybrid systems framework, [1], [2], [3], [6], [10], [23].

Here we propose a "local" Lyapunov-like approach to the study of properties of solutions to a particular class of homogeneous hybrid systems [28] in the neighborhood of the point $x_e = 0$. We analyze the following cases:

1) Solutions that do not satisfy the classical (δ, ε)-argument of stability concepts, that is, solutions ξ for which there exists a $\varepsilon \in \mathbb{R}_{>0}$ and a set $\mathcal{U} \subset \mathbb{R}^n$, $x_e \in \mathcal{U}$, such that for each $\delta \in \mathbb{R}_{>0}$, if $\xi(0,0) \in \mathcal{U} \cap \delta\mathbb{B}$ then, for some $(T,J) \in \operatorname{dom}\xi$, $\xi(T,J) \notin \varepsilon\mathbb{B}$, no matter how small δ is.
2) Solutions that grows unbounded from a suitable subset of the state-space, that is, solutions ξ such that for any given $\varepsilon \in \mathbb{R}_{>0}$, there exists a set $\mathcal{U} \subset \mathbb{R}^n$, $x_e \in \mathcal{U}$, such that for each $\delta \in \mathbb{R}_{>0}$ if $\xi(0,0) \in \mathcal{U} \cap \delta\mathbb{B}$ then, for some $(T,J) \in \operatorname{dom}\xi$, $\xi(T,J) \notin \varepsilon\mathbb{B}$, no matter how big ε is.
3) Solutions that grows by a factor $\rho > 1$, that is, solutions ξ for which there exists a set $\mathcal{U} \subset \mathbb{R}^n$ and such that if $\xi(0,0)$ in $\mathcal{U}$ then then $|\xi(T,J)| > \rho|\xi(0,0)|$. Such behavior is denoted as *overshoot*.

Point 1 is analyzed by proposing a Chetaev-like theorem [12, Theorem 4.3] generalized to the hybrid systems framework. Points 2 and 3 are addressed by following a Lyapunov-like approach, that is, by defining a set of conditions whose satisfaction, in a suitable subset of the state-space, guarantees 2 or 3. Based on such results, we propose two sum of squares algorithms [19] that construct a suitable function to automatically fulfils such conditions.

The use of sum of squares algorithms in control and, in particular, the use of sum of squares algorithms to construct Lyapunov functions, is well developed. See for example [17], [21], [26], [27], [18]. A study of solutions behavior with sum of squares, not related to stability problems, can be found in [20], where safety problems are taken into account (namely problems in which solutions *must not enter* a given subset of the state space or they *must reach* some particular subset of the state space). A similar approach based on approximations of solutions with polyhedra is proposed in [4]. Here we propose an approach to study the behavior of solutions in the neighborhood of the point $x_e = 0$. Based on such analysis, if some solution either satisfies 1 or satisfies 2 then x_e is unstable. Intuitively, 3 is related to the properties of convergence of solutions to x_e. Note that overshoots with a large factor ρ do not necessarily indicate instability. They can be considered as a characterization of the convergence properties of a given point x_e. Indeed, large overshoots can be interpreted as a sign of poor performance and the sum of squares program presented below is a tool to check whether or not this kind of phenomena occurs.

The work is organized as follows: in Section II, the hybrid systems framework is briefly introduced and the class of hybrid system considered is defined. The main theoretical results, the sum of squares algorithm and an example are presented in Section III. Further analysis on sum of squares implementation is developed in Section IV. The conclusion follows. Proofs are in the Appendix.

Notation: The Euclidean norm of a vector and is denoted by $|\cdot|$. A continuous function $\alpha(\cdot) : [0,a) \to [0,+\infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$; it is said to belong to class $\mathcal{K}_\infty$ if $a = +\infty$ and $\lim_{r\to+\infty} \alpha(r) = +\infty$. For any given set $X \subset \mathbb{R}^n$, $\overline{\text{co}}X$ denotes the closed convex hull of points of X.

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II. HYBRID SYSTEM MODEL

We consider a model of hybrid systems given by the tuple (C, F, D, G), where $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ are, respectively, the *flow set* and the *jump set*, while $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ are set-valued mappings, respectively, the *flow map* and the *jump map*. F and G characterize the continuous and the discrete evolution of the system, that is, the motion of the state, while C and D characterize subsets of $\mathbb{R}^n$ where such evolution may occur. A hybrid system $\mathcal{H}$ can be conveniently represented as follows

$$\mathcal{H} = \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D. \end{cases} \tag{1}$$

Intuitively, the evolution of the state either continuously flows through C, by following the dynamic given by F, or it jumps from D, according to G. Such alternation of jumps and flow intervals can be conveniently characterized by using a generalized notion of time, called *hybrid time*. In what follows, we recall the notions of hybrid time and of solution to a hybrid system. For details, see [8], [9], [10].

Definition 1: A set $E \subseteq \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a *hybrid time domain* if it is the union of infinitely many intervals of the form $[t_j, t_{j+1}] \times \{j\}$ where $0 = t_0 \leq t_1 \leq t_2 \leq, \ldots$, or of finitely many such intervals, with the last one possibly of the form $[t_j, t_{j+1}] \times \{j\}$, $[t_j, t_{j+1}) \times \{j\}$, or $[t_j, \infty] \times \{j\}$.

Definition 2: A *hybrid arc* x is a map $x : \operatorname{dom} x \to \mathbb{R}^n$ such that (i) $\operatorname{dom} x$ is a hybrid time domain, and (ii) for each j, the function $t \mapsto x(t, j)$ is a locally absolutely continuous function on the interval $I_j = \{t : (t, j) \in \operatorname{dom} x\}$.

A hybrid arc $x : \operatorname{dom} x \to \mathbb{R}^n$ is a *solution to the hybrid system* $\mathcal{H}$ if $x(0, 0) \in C \cup D$ and

(i) for each $j \in \mathbb{N}$ such that I_j has a nonempty interior,

$$\begin{array}{ll} \dot{x}(t,j) \in F(x(t,j)) & \text{for almost all } t \in I_j \\ x(t,j) \in C & \text{for all } t \in [\min I_j, \sup I_j); \end{array} \tag{2}$$

(ii) for each $(t, j) \in \operatorname{dom} x$ such that $(t, j+1) \in \operatorname{dom} x$,

$$\begin{array}{rl} x(t, j+1) & \in G(x(t,j)) \\ x(t,j) & \in D. \end{array} \tag{3}$$

A solution ξ to a hybrid system $\mathcal{H}$ is *nontrivial* if $\operatorname{dom} \xi$ contains at least one point different from $(0, 0)$; *maximal* if it cannot be extended, that is, there are no solutions ξ' to $\mathcal{H}$ such that $\operatorname{dom} \xi$ is a proper subset of $\operatorname{dom} \xi'$ and $\xi'(t, j) = \xi(t, j)$ for each $(t, j) \in \operatorname{dom} \xi$; *complete* if $\operatorname{dom} \xi$ is unbounded.

In what follows we consider a particular class of hybrid systems in which flow set and jump set are defined as the union of closed polyhedral cones, and flow map and jump map are defined, respectively, as the convex hull and the union of several linear vector fields. For instance, let i be an index number in $\mathbb{N}$, and let $R^{(i)}$ be a closed set defined as follows

$$R^{(i)} = \left\{ x \,|\, \begin{bmatrix} m_1^{(i)} \\ \ldots \\ m_{r^{(i)}}^{(i)} \end{bmatrix} x \geq 0 \right\} \tag{4}$$

where $r^{(i)}$ belongs to $\mathbb{N}$ and $m_j^{(i)} \in \mathbb{R}^{1 \times n}$ is a row vector, for each $j = 1, \ldots, r^{(i)}$. Then, C and D can be defined as

$$C = \bigcup_{i \in I_C} R^{(i)} \qquad D = \bigcup_{i \in I_D} R^{(i)} \tag{5}$$

where I_C, I_D are disjoint and finite index sets. Note that C and D can overlap. Note also that it is possible to have $C \cup D \neq \mathbb{R}^n$.

In a similar way, consider set-valued mappings $F_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, for $i \in I_C$, and $G_i : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, for $i \in I_D$, defined as as follows. For each $i \in I_C$, $F_i(x)$ is a convex and closed set defined by

$$F_i(x) = \begin{cases} \overline{\text{co}}\{f \,|\, f = F_{ik}x \text{ for } k = 1 \ldots r_F\} & \text{if } x \in R^{(i)} \\ \emptyset & \text{otherwise} \end{cases} \tag{6}$$

where $F_{ik} \in \mathbb{R}^{n \times n}$ and $r_F \in \mathbb{N}$. For each $i \in I_D$, $G_i(x)$ is a set defined by

$$G_i(x) = \begin{cases} \{g \,|\, g = G_{ik}x \text{ for } k = 1 \ldots r_G\} & \text{if } x \in R^{(i)} \\ \emptyset & \text{otherwise} \end{cases} \tag{7}$$

where $G_{ik} \in \mathbb{R}^{n \times n}$ and $r_G \in \mathbb{N}$. Then, flow and jump mappings, $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$, can be defined as

$$F(x) = \overline{\text{co}} \bigcup_{i \in I_C} F_i(x) \qquad G(x) = \bigcup_{i \in I_D} G_i(x) \tag{8}$$

Note that $F(x)$ reduces to $F_i(x)$ when x belongs only to one cone $R^{(i)}$, for some $i \in I_C$. The same holds for $G(x)$.

Hybrid systems of the form (1),(4)-(8) satisfy the following *basic conditions*. Such conditions coincide with the *basic assumptions* of [9] and with the fundamental conditions of [10] (the proof of Claim 1 is omitted for lack of space).

Claim 1 (Basic Conditions): A hybrid system $\mathcal{H}$ of Equations (1),(4)-(8) satisfies the following properties:

1) $C \subseteq \mathbb{R}^n$ and $D \subseteq \mathbb{R}^n$ are closed sets in $\mathbb{R}^n$.
2) $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on C and, for each $x \in C$, $F(x)$ is nonempty and convex.
3) $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is an outer semicontinuous set-valued mapping, locally bounded on D and, for each $x \in D$, $G(x)$ is nonempty.

Proof: C and D are finite union of closed sets. Boundedness of F and G follows from the fact that they are constructed from linear vector fields. Convexity of $F(x)$, for each $x \in C$, follows from the use of the convex-hull operator. Finally, outer semicontinuity of F follows from the fact that its graph is closed. Thus, by [22, Theorem 5.7] F is outer semicontinuous. Analogously for G. ■

Remark 1: Hybrid systems that satisfy the conditions of Claim 1 exhibit a sort of regularity of solutions that leads to several important results. For example, for such systems, we have sequential compactness of the space of solutions holds, [10, Theorem 4.4 and Lemma 4.3] and outer semicontinuous dependence of solutions on initial conditions, [9, Theorem 5] and [10, Corollary 4.8]. Regularity of solutions has effects also on stability theory. See [10], [23] ⌟

Remark 2: Switched linear systems with state dependent switching policies, [13, Sections 3.3 and 3.4], and switched linear systems under arbitrary switching policies, [13, Section 2.1.4], can be characterized within the family of hybrid systems considered above. For example, for the case [13, Sections 3.3 and 3.4], consider the system

$$\dot{x} = A_i x \;\text{ if }\; x \in C_i, \qquad i = 1, \ldots, N.$$

where $N \in \mathbb{N}$ and, for each $i = 1, \dots, N$, $A_i \in \mathbb{R}^{n \times n}$ and C_i is a conic subset of $\mathbb{R}^n$. Such systems can be easily defined within the class of hybrid systems considered above, by defining $F_i(x) = A_i x$ if $x \in C_i$ and $F_i(x) = \emptyset$ otherwise, for each $i = 1, \dots, N$. ⌟

Finally, following [9], for a hybrid system $\mathcal{H}$, the point $x_e = 0$ is (i) *stable* if for each $\epsilon > 0$ there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0,0)| \leq \delta$ satisfies $|x(t,j)| \leq \epsilon$ for all $(t,j) \in \mathrm{dom}\, x$; (ii) *pre-attractive* if there exists $\delta > 0$ such that any solution x to $\mathcal{H}$ with $|x(0,0)| \leq \delta$ is bounded and $x(t,j) \to 0$ as $t + j \to 0$ whenever x is complete; (iii) *pre-asymptotically stable* if it is both pre-stable and pre-attractive. By assuming $\mathbb{R}^n \backslash (C \cup D) \subseteq \mathcal{B}_{x_e}$, if the basin of pre-attraction $\mathcal{B}_{x_e} = \mathbb{R}^n$ then x_e is *globally pre-asymptotically stable*. In such case we say that the system is globally pre-asymptotically stable. Finally, we say that x_e is *unstable* if it is not stable.

III. Overshoots and Instability

A. Main results

The following theorem is a generalization of Chetaev's Theorem [12, Theorem 4.3] to hybrid systems of Equations (1). Thus, it can be used to characterize the *instability* of $x_e = 0$ and it is related to *Case 1* of the introduction.

Theorem 1: (Chetaev-like theorem) Consider a hybrid system $\mathcal{H}$ of Equation (1) that satisfies 1), 2) and 3) of Claim 1, and define $x_e = 0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function in $C \cup D$. Choose $r \in \mathbb{R}_{>0}$ and define $U = \{x \in C \cup D \,|\, V(x) > 0,\, |x| \leq r\}$. Suppose that for each $\delta > 0, U \cap \delta\mathbb{B} \neq \emptyset$ and that

1) $\langle \nabla V(x), f \rangle > 0 \qquad \forall x \in C \cap U, \forall f \in F(x)$;
2) $V(g) - V(x) > 0 \qquad \forall x \in D \cap U, \forall g \in G(x)$;
3) Each maximal solution ξ to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$ is nontrivial.

Then x_e is unstable.

Proof: By using nontriviality of solutions, the argument of the proof of Theorem 1 can be developed as in the proof of [12, Theorem 4.3]. See appendix, Section A.1, for details. ■

Remark 3: If $C \cup D = \mathbb{R}^n$ then each solution to $\mathcal{H}$ is nontrivial. In such a case, condition 3) of Theorem 1 is automatically satisfied. In general, local existence of a solution from each point of U can be satisfying the conditions in [10, Proposition 2.4]. Note that Theorem 1 is not restricted to hybrid systems of Equations (1),(4)-(8) and it applies to general hybrid systems (1) that satisfy 1), 2) and 3) of Claim 1. ⌟

The following theorem defines a set of Lyapunov-like conditions for studying overshoots of solutions to hybrid systems of Equations (1),(4)-(8). The theorem is parameterized with respect to $c > 0$ and $\rho > 1$, that define the set $\{x \,|\, c \leq |x| \leq \rho c\}$ in which the conditions must be satisfied, and it guarantees that at least one solution ξ to $\mathcal{H}$ with initial condition in $c \leq |\xi(0,0)| \leq c + \varepsilon$ grows to $|\xi(T,J)| \geq \rho c$, for some given $(T,J) \in \mathrm{dom}\, \xi$. Theorem 2 is related to *Case 3* of the introduction.

Theorem 2: Consider a hybrid system $\mathcal{H}$ of Equations (1),(4)-(8). Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that for some $\ell \in \mathbb{R}_{>0}$, $c \in \mathbb{R}_{\geq 0}$

(1) $\max_{|x|=c} V(x) \leq \ell$;
(2) for a small $\varepsilon \in \mathbb{R}_{>0}$ and for some $x \in C \cup D$, $|x| = c + \varepsilon$ and $V(x) > \ell$

Choose $\rho \in \mathbb{R}_{>1}$ such that $\rho c > c + \varepsilon$, and define $U = \{x \in C \cup D \,|\, V(x) > \ell,\, c \leq |x| \leq \rho c\}$. Suppose

(3) $\langle \nabla V(x), f \rangle > 0 \qquad \forall x \in C \cap U, \forall f \in F(x)$;
(4) $V(g) - V(x) > 0 \qquad \forall x \in D \cap U, \forall g \in G(x)$;
(5) $|g| > c \qquad \forall x \in D \cap U, \forall g \in G(x)$;
(6) Each maximal solution ξ to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$ is nontrivial.

Then, for each $\lambda \in \mathbb{R}_{>0}$, there exists a solution ξ to $\mathcal{H}$ such that if $|\xi(0,0)| = \lambda(c + \varepsilon)$ then $|\xi(T,J)| \geq \lambda \rho c$, for some $(T,J) \in \mathrm{dom}\, \xi$.

Proof: See section A.2 ■

The meaning of the conditions of the theorem above can be explained by looking at Figure 1, where we considered the case of a planar hybrid system for which the conditions of Theorem 2 are satisfied. Conditions (1) and (2) guarantee that the level set ℓ of V is close to the circle of radius c, while conditions (3)-(6) guarantee that no solution can stay forever in the intersection of the grey colored set of Figure 1 with $c \leq |x| \leq \rho c$. Note that, despite the conditions of the theorem are local, parameterized with c and ρ, the conclusion defines an entire subspace of initial conditions from which the solutions grow of a factor ρ

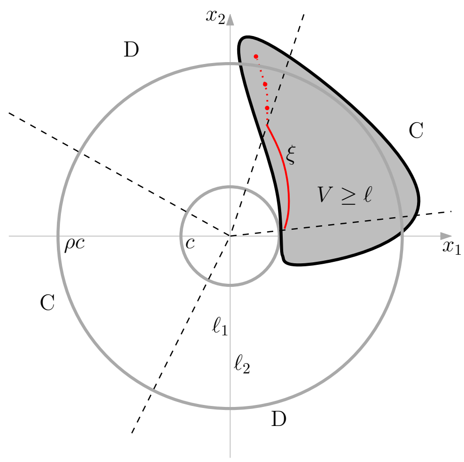


Fig. 1. A function V that satisfies the conditions of Theorem 2, for a planar hybrid system.

By adding a simple condition to Theorem 2 it is possible to characterize the instability of $x_e = 0$, as stated in the following corollary. The key point of such condition is that it guarantees that $\{x \,|\, |x| = \rho c\} \subseteq \{x \,|\, V(x) > \ell\}$. Thus, it is possible to show that there exists a solution that grows unbounded. Corollary 1 is related to *Case 2* of the introduction.

Corollary 1: Consider a hybrid system $\mathcal{H}$ of Equations (1),(4)-(8) and consider a point $x_e = 0$. Under the hypothesis of Theorem 2, if conditions (1)-(6) hold and the following condition is satisfied

(7) $\min_{|x|=\rho c} V(x) > \ell$,

then x_e is unstable.

Proof: See section A.3 ■

Remark 4: It is important to mention that Theorems 1 and 2 are conservative. In fact, both overshoot of solutions to a hybrid system $\mathcal{H}$ and instability properties of $x_e = 0$ are the results of the "behavior" of *one* solution to $\mathcal{H}$ only, while Theorem 1 and Theorem 2 require a particular "behavior" for an *entire set* of solutions. For instance, consider the following system with $x \in \mathbb{R}^2$ defined as $x = [\, x_1 \; x_2 \,]^T$.

$$\mathcal{H} = \begin{cases} \dot{x} = x & x \in \{x \,|\, x_1 \geq 0,\, x_2 = 0\} \\ x^+ = 0 & x \in \mathbb{R}^2. \end{cases}$$

In this case, for any given $\delta > 0$, the hybrid arc $\xi_1(t, 0) = [\, \delta e^t \;\; 0 \,]^T$, for each $t \in \mathbb{R}_{\geq 0}$, is a solution to $\mathcal{H}$ from the initial condition $[\, \delta \;\; 0 \,]^T$. Thus, x_e is unstable. Despite the instability of x_e, Theorem 1 does not apply. In fact, consider any given initial condition $x_0 \in \mathbb{R}^2$, then the hybrid arc $\xi_2(0, 0) = x_0$, $\xi_2(0, j) = 0$, for each $j \in \mathbb{Z}$, $j > 0$, is a solution to $\mathcal{H}$. Thus, Condition (2) of Theorem 1 cannot be satisfied. ⌟

Theorem 3: Consider a hybrid system $\mathcal{H}$ of Equations (1),(4)-(8). Under the hypothesis of Theorem 2, if Conditions (2)-(4), (6) of Theorem 2 are satisfied and Condition (1) is replaced by $\max_{|x| \leq c} V(x) \leq \ell$, then the conclusion of Theorem 2 still holds. Moreover, if Condition (7) of Corollary 1 is also verified, than the conclusion of Corollary 1 still holds.

Proof: The proof can be developed by following an argument similar to the one in Section A.2. In fact, by $\max_{|x| \leq c} V(x) \leq \ell$, each jump from $x \in U$ to some $g \in G(x)$, with $|g| < c$, would fall in $\{x \,|\, V(x) \leq \ell\}$, that is forbidden by Condition (3) of Theorem 2. ■

B. Sum of squares algorithms

Under the assumption $C \cup D = \mathbb{R}^n$, we can use the following algorithms to find functions V that fulfill the conditions of Theorem 2. Then, we will use one of such algorithms to construct functions V that fulfill also the conditions of Corollary 1.

Algorithm 1 is defined by a set of inequalities parameterized with respect to parameters: $(case, k_1, k_2)$. A solution to the set of inequalities is then computed by relaxing the satisfaction problem of such inequalities to a sum of squares decomposition problem. Then, if a *solution is found*, the algorithm ends. Otherwise, the algorithm runs on a new set of inequalities, constructed on a different selection of $(case, k_1, k_2)$, until each possible case of $(case, k_1, k_2)$ have been considered. In fact, by using a a parameterization with $(case, k_1, k_2)$, a non-convex search problem is reduced to several convex problems, suitable for sum-of-squares implementation. Therefore, by running Algorithm 1 several times, each of which on a different set of parameters, we explore a non-convex search-space, searching for a function V that fullfills the conditions of Theorem 2. At each run:

(i) the input of the algorithm is filled by the data of the hybrid system $\mathcal{H}$, by some parameters $\varepsilon, c, \rho, d_1$ and d_2, and by a selection of $(case, k_1, k_2)$, as stated in section `INPUT`.

(ii) A set of inequalities is then constructed, as stated in section `CONSTRAINTS`. Each inequality uses variables defined in `VARIABLES`.

(iii) A semidefinite program solver runs over such inequalities. A solution is computed by relaxing the satisfiability problem of the whole set of inequalities to a sum of squares decomposition problem. The sum of squares decomposition problem is then solved by using a semidefinite program solver.

(iv) If the solver finds a solution, the set of constraints is feasible and algorithm 1 has a positive output, as stated in `OUTPUT`.

The algorithm is based on the following polynomial $q(x)$

Definition 3: Let Q be a *symmetric* matrix in $\mathbb{R}^{n \times n}$, defined as follows.

$$Q = \begin{bmatrix} q_{11} & \dots & q_{1n} \\ \vdots & \ddots & \vdots \\ q_{n1} & \dots & q_{nn} \end{bmatrix} \tag{9}$$

Let $q(x)$ be a polynomial defined as follows.

$$q(x) = \begin{bmatrix} q_{01} & \dots & q_{0n} \end{bmatrix} x + x' Q x \tag{10}$$

where q_{0i} belongs to $\mathbb{R}$, for each $i \in \{1, \dots, n\}$.
By suitable conditions on elements of Q and on elements of $[q_{01} \; \dots \; q_{0n}]$, $q(x)$ can be interpreted as a function of x that is positive in some subset of $\mathbb{R}^n$. Specifically, the parameterization $(case, k_1, k_2)$ defines some specific conditions on Q and on $[q_{01} \; \dots \; q_{0n}]$ so that $q(x)$ is necessarily greater than zero in some subset of $\mathbb{R}^n$. For example, consider a planar space and assume $q_{11} + q_{22} > 0$ and $q_{12} = 0$. Then, $q(x) = q_{01}x_1 + q_{02}x_2 + q_{11}x_1^2 + q_{22}x_2^2$ is positive in a conic subset of $\mathbb{R}^2$. An example is summarized in Figure 2.

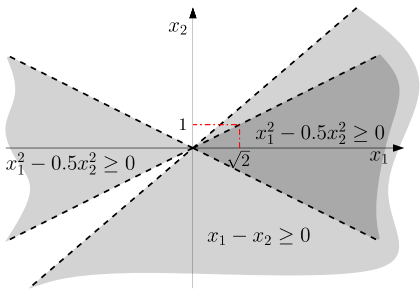


Fig. 2. Suppose $q(x) = x_1 - x_2 + x_1^2 - 0.5x_2^2$, then the intersection of $\{x \,|\, x_1 - x_2 \geq 0\}$ with $\{x \,|\, x_1^2 - 0.5x_2^2 \geq 0\}$ is a conic subset of $\{x \,|\, q(x) \geq 0\}$.

Also the following quantities are used in the algorithm.

Definition 4: For any given $i \in I_C \cup I_D$, the function $\Delta_2^{(i)}(x) : \mathbb{R}^n \to \mathbb{R}$ is defined as follows

$$\begin{aligned} \Delta_2^{(i)}(x) = & \sum_{j=1}^{r^{(i)}} p_j(x)\, m_j^{(i)} x + \sum_{j=1}^{r^{(i)}} \sum_{k=j+1}^{r^{(i)}} p_{jk}(x)\, m_j^{(i)} x\, m_k^{(i)} x \\ & + \sum_{j=1}^{r^{(i)}} \sum_{k=j+1}^{r^{(i)}} \sum_{h=k+1}^{r^{(i)}} p_{jkh}(x)\, m_j^{(i)} x\, m_k^{(i)} x\, m_h^{(i)} x \\ & + \dots + p_{1,2,\dots,r}(x) m_1 x\, m_2 x \cdot \dots \cdot m_{r^{(i)}} x \end{aligned}$$

where, for any given combination of indices j,k,..., $p_j, p_{jk}, \dots$ denote functions in $\mathbb{R}^n \to \mathbb{R}_{\geq 0}$, defined by non-negative polynomials of a given degree. We refer to the whole set of polynomials $p_j, p_{jk}, \ldots$ by using the name *slack polynomials*.

Definition 5: Let $\varepsilon_1, \varepsilon_2 \in \mathbb{R}_{\geq 0}$ be two constants and let $\Delta_1(\varepsilon_1, \varepsilon_2, \cdot) : \mathbb{R}^n \to \mathbb{R}$ be a map defined with respect to ε_1 and on ε_2 as follows:

$$\Delta_1(\varepsilon_1, \varepsilon_2, x) = -(|x|^2 - \varepsilon_1^2)(|x|^2 - \varepsilon_2^2).$$

Note that, for each $i \in I_C \cup I_D$, $\Delta_2^{(i)}(x)$ is positive for each x in $R^{(i)}$ while it is possibly negative for $x \notin R^{(i)}$, based on the particular configuration of slack polynomials. $\Delta_1(\varepsilon_1, \varepsilon_2, x)$ is positive for $\varepsilon_1 \leq |x| \leq \varepsilon_2$, and is strictly negative otherwise. A planar example of subset of $\mathbb{R}^n$ with positive Δ_1 and Δ_2 is in Figure 3. Δ_1 and Δ_2 are used in Algorithm 1 for relaxing the conditions on V to hold only in a subset of $\mathbb{R}^n$.

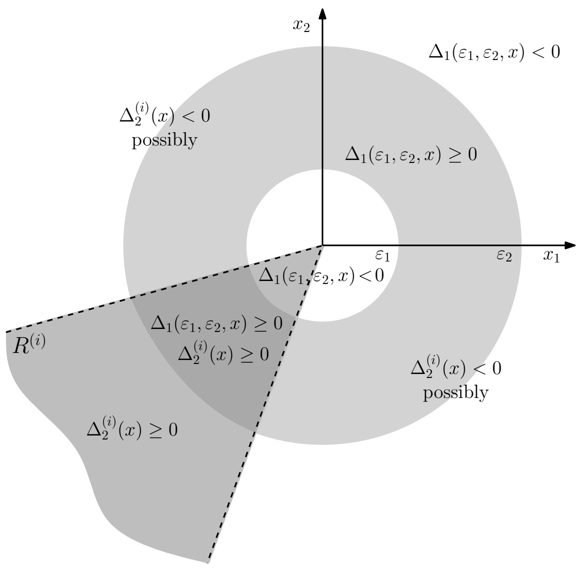


Fig. 3. Subsets of the state-space related to the sign of Δ_1 and Δ_2.

Algorithm 1:

`INPUT:` data $\langle F, G, C, D \rangle$ of the hybrid system $\mathcal{H}$;
constants $\varepsilon, c, \rho \in \mathbb{R}_{>0}$, with $\varepsilon \ll c$ and $\rho > 1$;
constants $d_1, d_2 \in \mathbb{N}$, $case \in \{1, 2, 3\}$ and, if $case \neq 1$, $k_1 \in \{1, \ldots n\}$, $k_2 \in \{k_1+1, \ldots, n\}$.

`OUTPUT`: feasibility of the sum of squares problem.

`VARIABLES:` scalars ℓ;
polynomials $V(x)$, $s_1^{(ik)}(x)$, for each i in I_C and $k \in \{1, \ldots, r_F\}$, $s_2^{(ik)}(x)$ for each i in I_D and $k \in \{1, \ldots, r_G\}$, $s_3(x)$, $s_4(x)$, and $s_5^{(i)}(x)$, $s_6^{(i)}(x)$, for each i in I_D, and all the slack polynomials.

`CONSTRAINTS:` let $V(x)$ be a polynomial of degree d_2. Let ϵ be a scalar variable.

- $\forall i \in I_C$, $\forall k \in \{1, \ldots, r_F\}$, let $s_1^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} \frac{\partial V}{\partial x}(x) F_{ik} x - \Delta_2^{(i)}(x) - s_1^{(ik)}(x) \Delta_1(c, \rho c, x) > 0 \\ s_1^{(ik)}(x) \geq 0 \end{aligned} \tag{11}$$

- $\forall i \in I_D$, $\forall k \in \{1, \ldots, r_G\}$, let $s_2^{(ik)}(x)$ be a polynomial of degree d_1

$$\begin{aligned} V(G_{ik} x) - V(x) - \Delta_2^{(i)}(x) - s_2^{(ik)}(x) \Delta_1(c, \rho c, x) > 0 \\ s_2^{(ik)}(x) \geq 0 \end{aligned} \tag{12}$$

- Assume $c + 3\varepsilon < \rho c$. Let $s_3(x)$, $s_4(x)$, $s_5^{(ik)}(x)$ and $s_6^{(ik)}(x)$ be polynomials of degree d_1

$$\begin{aligned} \ell - V(x) - s_3(x) \Delta_1(c, c+\varepsilon, x) \geq 0 \\ V(x) - \ell - s_4(x) \Delta_1(c + 2\varepsilon, c + 3\varepsilon, x) - q(x) \geq 0 \\ \ell \geq 0 \\ s_3(x), s_4(x) \geq 0 \\ \forall i \in I_D, \forall k \in \{1, \ldots, r_G\}, \\ \ell - V(x) - s_5^{(ik)}(x)(c^2 - x' G_{ik}' G_{ik} x) - \Delta_2^{(i)}(x) + \\ -s_6^{(ik)}(x) \Delta_1(c, \rho c, x) \geq 0 \\ s_5^{(ik)}(x), s_6^{(ik)}(x) \geq 0 \end{aligned} \tag{13}$$

- $q(x)$ satisfies the following inequalities:

$$\begin{array}{ll} \text{if } case = 1 & \sum_{i=1}^n q_{ii} > 0 \\ \text{if } case = 2 & \begin{cases} \forall i \in \{1 \ldots, n\}, \; q_{ii} \leq 0 \\ 2q_{k_1 k_2} + q_{k_1 k_1} + q_{k_2 k_2} > 0 \end{cases} \\ \text{if } case = 3 & \begin{cases} \forall i \in \{1 \ldots, n\}, \; q_{ii} \leq 0 \\ -2q_{k_1 k_2} + q_{k_1 k_1} + q_{k_2 k_2} > 0 \end{cases} \end{array} \tag{14}$$

- Each use of $cone^{(i)}(x)$ in inequalities (11),(12) and (13) requires a *new fresh* set of slack polynomials. Moreover for each slack polynomial, say $p(x)$, a new constraint $p(x) \geq 0$ is added.

Remark 5: The last bullet of Algorithm 1 requires a new set of slack polynomials for each use of $\Delta_2^{(i)}(x)$. For example, slack polynomials of $\Delta_2^{(i)}(x)$ used in an inequality that involves G_{ik_1} in (12) must not be confused with slack polynomials of $\Delta_2^{(i)}(x)$ used in an inequality that involves G_{ik_2} in (12), with $k_1 \neq k_2$. ⌟

Remark 6: Despite the number of indices, the algorithm is much more simple in practical cases. For example, if differential equations replace differential inclusions, for each cone, then $k = 1$ in (11) and (12). ⌟

Each inequality of Algorithm 1 can be divided into two parts: the first part defines some constraints on V while the second part uses Δ_1, Δ_2 and q to guarantee the satisfaction of such constraints only in a specific subset of $\mathbb{R}^n$. Suppose now to run Algorithm 1 and to find a feasible solution to the set of inequalities constructed by Algorithm 1, for some hybrid system $\mathcal{H}$ and for some selection of parameters $case$, k_1 and k_2. Then, the set of inequalities of Algorithm 1 guarantees the following properties. **(i)** By (11) and (12), the derivative of $V(x)$ is positive, for each $x \in C$ and each $f \in F(x)$ such that $c \leq |x| \leq \rho c$. The difference $V(g) - V(x)$ is positive, for each $x \in D$ and each $g \in G(x)$ such that $c \leq |x| \leq \rho c$. Inequalities (11) and (12) are related to Conditions (3) and (4) of Theorem 2. **(ii)** By (14), $q(x)$ is not a non-positive function. To see this, note that if Q is not negative semi-definite, then there exists a conic subset of $\mathbb{R}^n$ such that $q(x) > 0$. And so, inequalities (14) each break a necessary condition for negative semi-definiteness of Q. **(iii)** The first inequality of (13) guarantees that $V(x) \leq \ell$ for each $c \leq |x| \leq c + \varepsilon$. Thus, it is related to Condition (1) of Theorem 2. The second inequality of (13) guarantees that $V(x) > \ell$ for some $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$. Then, $V(x) = \ell$ in at least one point of $c + \varepsilon \leq |x| \leq c + 2\varepsilon$. Thus, it is related to Condition (2) of Theorem 2. **(iv)** If the system $\mathcal{H}$ jumps from a state x in $\{x \,|\, c \leq |x| \leq \rho c\}$ to a state g in $\{x \,|\, |x| \leq c\}$, then the penultimate inequality of (13) guarantees that $V(x) \leq \ell$. Therefore, the system cannot jump from the set $\{x \,|\, V(x) > \ell\} \cap \{c \leq |x| \leq \rho c\}$ to the set $\{x \,|\, |x| < c\}$. Thus, that inequality is related to Condition (5) of Theorem 2. It follows that a feasible solution to the set of constraints above produces a function V that satisfies Conditions (1)-(5) of Theorem 2.

Proposition 1: For any given hybrid system $\mathcal{H}$ defined by Equations (1),(4)-(8), if *(i)* the set of inequalities of Algorithm 1 has a feasible solution for some parameters $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}$, $\rho > 1$ and $(case, k_1, k_2)$, and *(ii)* each solution from $U = \{x \in C \cup D \,|\, V(x) > \ell,\ c \leq |x| \leq \rho c\}$ is non trivial, where V is the function constructed by Algorithm 1, then the conditions of Theorem 2 are satisfied with the same c and ρ.

Proof: See appendix, Section A.4. ■

The following modification to Algorithm 1 guarantees that the function $V(x)$ is greater than a costant $\overline{\ell} > \ell$ for each point x such that $|x| = \rho c$, as required by Corollary 1. For instance, replace the second inequality of (13) with

$$\begin{aligned} V(x) - \overline{\ell} - s_7(x)\Delta_1(\rho c - \varepsilon, \rho c, x) &\geq 0 \\ \overline{\ell} &> \ell \\ s_7(x) &\geq 0 \end{aligned} \tag{15}$$

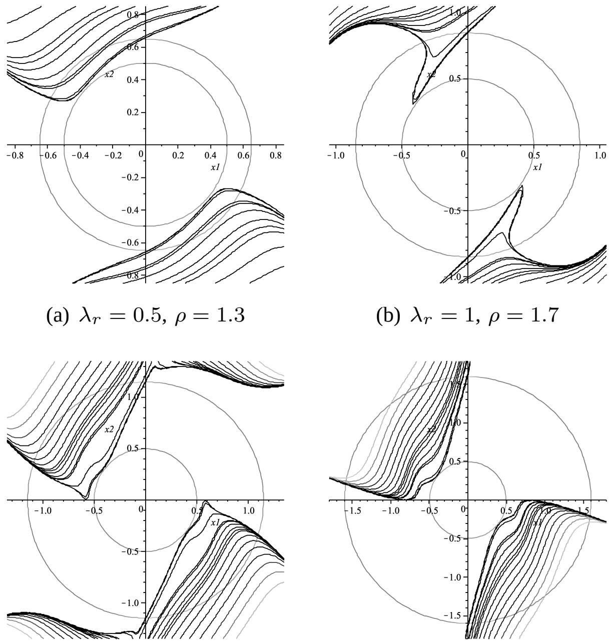
and delete (14). Then, the following proposition hold.

Proposition 2: For any given hybrid system $\mathcal{H}$ defined by Equations (1),(4)-(8), with $C \cup D = \mathbb{R}^n$ if *(i)* the modified set of inequalities of Algorithm 1 has a feasible solution for some parameters $c \in \mathbb{R}_{\geq 0}$, $\rho \in \mathbb{R}$, $\rho > 1$ and $(case, k_1, k_2)$, and *(ii)* each solution from $U = \{x \in C \cup D \,|\, V(x) > \ell,\ c \leq |x| \leq \rho c\}$ is non trivial, where V is the function constructed by Algorithm 1, then the conditions of Corollary 1 are satisfied with the same c and ρ.

Proof: Inequality (15) can be written as $V(x) \geq \overline{\ell} + s_7(x)\Delta_1(\rho c - \varepsilon, \rho c, x)$ from which $V(x) \geq \overline{\ell} > \ell$, for each $\rho c - \varepsilon \leq |x| \leq \rho c$. Therefore, $\min_{|x|=\rho c} V(x) \geq \overline{\ell} > \ell$, that satisfies Condition (7) of Corollary 1. ■

Remark 7: According to Theorem 3, Algorithm 1 still works if we replace $\ell - V(x) - s_3(x)\Delta_1(c, c+\varepsilon, x) \geq 0$ in (13) with $\ell - V(x) - s_3(x)((c+\varepsilon)^2 - x^T x) \geq 0$ and we delete the fifth inequality in (13). Note that this approach forces the function V to be lower than ℓ for $|x| \leq c$, while Algorithm 1 leaves V practically unconstrained near the origin. In fact, fifth inequality in (13) enforces a condition on V only if some jump $g \in G(x)$, $|g| \leq c$ from $c \leq |x| \leq \rho c$ occurs. ⌟

Remark 8: By (11), (12), Algorithm 1 searches for a function V whose directional derivative and increment are both positive in $c \leq |x| \leq \rho c$. According to Theorem 2, such conditions on V can be relaxed by requiring that both directional derivative and increment of V are positive only in a suitable subset of $\mathbb{R}^n$. It follows that Algorithm 1 is conservative. ⌟



(a) $\lambda_r = 0.5$, $\rho = 1.3$ (b) $\lambda_r = 1$, $\rho = 1.7$

(c) $\lambda_r = 2$, $\rho = 2.3$ (d) $\lambda_r = 4$, $\rho = 3.2$

Fig. 4. Level sets greater or equal than ℓ of the function $V(x)$ constructed by Algorithm 1, Example III-C. Note that the system is globally pre-asymptotically stable for $\lambda_r = 0.5$ and for $\lambda_r = 1$ while it is unstable for $\lambda_r = 2$ and for $\lambda_r = 4$.

C. Example

Let us consider the following hybrid system

$$\mathcal{H} = \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & \lambda_r \end{bmatrix} x & x \in C \\ x^+ = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} x & x \in D \end{cases}$$

where $\lambda_r \in \mathbb{R}$ is a parameter and C and D are defined as follows

$$C = \left\{ x \,\middle|\, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0 \text{ or } \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} x \geq 0 \right\}$$
$$D = \left\{ x \,\middle|\, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x \geq 0 \text{ or } \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} x \geq 0 \right\}.$$

We increase λ_r progressively so that the continuous dynamics of the hybrid system is characterized (i) by an asymptotic stable system, (ii) by a stable system and (iii) by an unstable system. We use Algorithm 1 to estimate the overshoot of $\mathcal{H}$. For instance, we study $\mathcal{H}$ for increasing values of λ_r and, for each λ_r, we run several times Algorithm 1 (for $c = 0.5$ and $d_2 = 10$) looking for the greatest values of ρ for which the set of constraints is still feasible. Some level sets of the function $V(x)$ constructed by Algorithm 1 are summarized in Figure 4.

IV. Notes on sum of squares implementation

The problem of finding a solution to the set of inequalities of each algorithm is addressed by replacing each inequality with a sum-of-squares decomposition. In fact, the left-hand side of each inequality involving polynomials is a polynomial, say $p(x)$. It follows that inequalities $p(x) \geq 0$ can be replaced by $p(x)$ *is a sum-of-squares* and each strict inequality $p(x) > 0$ can be considered as a non-strict inequality of the form $p(x) - \epsilon x^T x \geq 0$, with $\epsilon > 0$ variable of the problem, then replaced by $p(x) - \epsilon x^T x$ *is a sum-of-squares*.

Note that, even though (i) polynomial inequalities constructed by each algorithm are linear with respect to the set of variables and (ii) a sum of squares decomposition problem can be solved in polynomial time, the computational complexity of finding a solution to the set of inequalities grows rapidly with the dimension of the state-space of $\mathcal{H}$, with the degree of free polynomials used in the set of inequalities, with the number of disjoint cones of $C \cup D$, and with the number of matrices F_{ik}, G_{ik}.

It is worth to mention that a sum-of-squares decomposition is satisfied within the limits of numerical computation, therefore it cannot be exact. We need that, despite the numerical approximation errors, the polynomials constructed by sum of squares decomposition are still a feasible solution to the set of inequalities. By following [15], such goal can be achieved by considering a perturbed polynomial with a perturbation magnitude that depends on the numerical approximations errors of the decomposition (residuals). For instance, the sum of squares decomposition problem of a polynomial $p(x)$ is rewritten by YALMIP [14] as a SDP problem. The data of such formulation are stored in two matrices A and b. A solution P is computed by running the solver SeDuMi [25]. Then, the decomposition is $p(x) = v(x)'Pv(x)$, where $v(x)$ is a base of monomials. Using [15, Theorem 4], if the *test* $\lambda_{\min}(P) \geq M\| A(P) - b \|_\infty$ is verified, we have that $v(x)'Pv(x)$ is non-negative, that is, each inequality is certified.

V. Conclusions

We have shown a set of local conditions for studying solutions of a class of homogeneous hybrid systems in a neighborhood of the point $x_e = 0$. Such conditions can be used also to study the instability of $x_e = 0$. Based on such results we proposed a sum of squares algorithm to automatically fulfills such conditions. As a future work, it could be interesting to design an algorithm based on relaxed condition on derivative and on increment of Algorithm 1. In fact, according to Theorem 2, such conditions are conservative.

Appendix

A. Overshoots and instability proofs.

1) Proof of Theorem 1.: From the assumptions of Theorem 1, we have that for each $\delta > 0$, $U \cap \delta\mathbb{B} \neq \emptyset$ therefore by continuity of $V(x)$ in $C \cup D$, x_e belongs to the border of U. Let ξ be a solution to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$. By conditions (1), (2) and (3) of the theorem, ξ must leave U. In fact, suppose that there exists ξ with $\xi(0,0) \in U$ such that ξ remains in U for all $(t,j) \in \operatorname{dom}\xi$. Suppose also that $V(\xi(0,0)) = a$, for some $a \in \mathbb{R}_{>0}$. Then, by conditions (1) and (2) of the theorem, $V(\xi(t,j)) \geq a$ for each $(t,j) \in \operatorname{dom}\xi$, and the set $\{x \in U \,|\, V(x) \geq a\}$ is a compact subset of U. By the compactness of such set and Claim 1, we can say that there exist $\gamma_1, \gamma_2 \in \mathbb{R}_{>0}$ such that $\langle \nabla V(x), f\rangle > \gamma_1$, for each $x \in C \cap \{x \in U \,|\, V(x) \geq a\}$ and each $f \in F(x)$, and $V(g) - V(x) > \gamma_2$, for each $x \in D \cap \{x \in U \,|\, V(x) \geq a\}$ and each $g \in G(x)$. By Condition (3), it follows that, for each $(t,j) \in \operatorname{dom}\xi$, $V(\xi(t,j)) \geq a + \gamma(t+j)$, where $\gamma = \min\{\gamma_1, \gamma_2\}$. Then, by the fact that V has a maximum on $\{x \in U \,|\, V(x) \geq a\}$, ξ cannot stay forever in such compact set.
By (1) and (2) of the theorem, ξ cannot leave U by flowing across $\{x \in \mathbb{R}^n \,|\, V(x) = 0\}$ or by jumping to $\{x \in \mathbb{R}^n \,|\, V(x) \leq 0, |x| \leq r\}$, therefore it leaves U by flowing across $\{x \in U \,|\, |x| = r\}$ or by jumping to $\{x \in C \cup D \,|\, |x| > r\}$. Because this happens for points x arbitrarily close to 0, x_e is unstable. ∎

The following lemma will be used in the proof of Theorem 2 and of Corollary 1.

Lemma 1: Consider a hybrid system $\mathcal{H}$ of Equations (1),(4)-(8) and suppose ξ is a solution to $\mathcal{H}$. Then, for each $\lambda \in \mathbb{R}_{>0}$, $\lambda\xi$ is a solution to $\mathcal{H}$.

Proof: By (4), (5), for each $(t,j) \in \operatorname{dom}\xi$, if $\xi(t,j) \in \bigcap_{i \in I} R^{(i)}$, for some $I \subseteq I_D$ or some $I \subseteq I_C$, then $\lambda\xi(t,j) \in \bigcap_{i \in I} R^{(i)}$. By this fact and by Equations (6), (7) and (8), we can say that
- for each $(t,j) \in \operatorname{dom}\xi$ such that $(t,j+1) \in \operatorname{dom}\xi$, suppose $\xi(t,j+1) \in G(\xi(t,j))$. Then, $\lambda\xi(t,j+1) \in G(\lambda\xi(t,j))$;
- for each $\underline{t}, \overline{t} \in \mathbb{R}_{\geq 0}$ such that $[\underline{t}, \overline{t}] \times \{j\} \subseteq \operatorname{dom}\xi$. if $\dot{\xi}(t,j) \in F(\xi(t,j))$, for almost all $t \in [\underline{t}, \overline{t}]$, then $\lambda\dot{\xi}(t,j) \in F(\lambda\xi(t,j))$, for almost all $t \in [\underline{t}, \overline{t}]$.
It follows that $\lambda\xi$ is a solution to $\mathcal{H}$. ∎

2) Proof of Theorem 2.: By the continuity of V and from assumptions (1) and (2) of Theorem 2, we have that U is not empty and $V(x) = \ell$ for some point $x \in C \cup D$ such that $c \leq |x| \leq c + \varepsilon$. Let ξ be a solution to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$. Conditions (3)-(6) of Theorem 2 guarantee that ξ must leave U in finite time (this can be shown by following the argument of the proof of Theorem 1). Consider a solution ξ to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$. By Condition (3), such solution cannot leave U by flowing across $\{x \in \mathbb{R}^n \,|\, V(x) = \ell\}$, by Condition (4), it cannot leave U by jumping to $\{x \in \mathbb{R}^n \,|\, V(x) \leq \ell, c \leq |x| \leq \rho c\}$ and, by Condition (5), ξ cannot jump to $\{x \in \mathbb{R}^n \,|\, |x| \leq c\}$. It follows that ξ leaves U by flowing across $\{x \in \mathbb{R}^n \,|\, |x| = \rho c\}$ or by jumping to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$.
Consider now a solution ξ to $\mathcal{H}$ with initial condition $\xi(0,0) \in U$ and, by (2), $|\xi(0,0)| = c + \varepsilon$. Such solution leaves U in finite time, that is, $|\xi(T,J)| \geq \rho c$, for some $(T,J) \in \operatorname{dom}\xi$. Therefore, by Lemma 1, the result of the theorem follows. ∎

3) Proof of Corollary 1.: As stated in Theorem 2, each solution ξ to $\mathcal{H}$ leaves U in finite time. Note that, by Condition (7), each point $x \in C \cup D$ with $|x| = \rho c$ belongs to U. This implies that the set U surrounds the origin, therefore if a solution ξ leaves U, it cannot go back to the set $\{x \,|\, x \in (C \cup D), x \leq \rho c\}$ any more.

By the continuity of V and by Conditions (1) and (7), we can find two constants $\ell_1, \ell_2 \in \mathbb{R}$ such that $\min_{|x|=\rho c} V(x) = \ell_2$, $\ell < \ell_1 < \ell_2$ and the set $U_1 = \{x \in C \cup D \,|\, V(x) = \ell_1,\, |x| \leq \rho c\}$ surrounds the origin. It follows that (i) by continuity of V and by Conditions (1) and (7), U_1 is a subset of U, so that solutions ξ to $\mathcal{H}$ with $\xi(0,0) \in U_1$ escapes U in finite time by flowing or jumping to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$, and (ii) by Conditions (3)-(6) and by Lemma 1 we can use pieces of solutions to $\mathcal{H}$ from U_1 to $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ to construct a solution ξ to $\mathcal{H}$ that grows unbounded. Indeed, inductively, consider a solution ξ_i to $\mathcal{H}$ with initial condition $\xi_i(0,0) \in U_1$, where i is a positive integer (an index). Such solution enters the set $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time, say $(t_i, j_i) \in \operatorname{dom} \xi_i$. The point $\xi_i(t_i, j_i) \in \{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ can be scaled so that $\lambda_i \xi_i(t_i, j_i) \in U_1$, for some $\lambda_i \in \mathbb{R}_{>0}$. Then, consider a solution ξ_{i+1} to $\mathcal{H}$ with initial condition $\xi_{i+1}(t_{i+1}, j_{i+1}) = \lambda_i \xi_i(t_i, j_i)$. Also such solution enters the set $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time. Therefore, by using solutions ξ_i with $i \geq 0$ we can inductively define an unbounded solution ξ as follows:

Base case:

$$\xi(0,0) = \xi_0(0,0)$$
$$\xi(t,j) = \xi_0(t,j) \quad \forall (t,j) \in \operatorname{dom} \xi_0, (t,j) \leq (t_0, j_0).$$

Inductive case: for each $i > 0$

$$\xi(t + \sum_{k=0}^{(i-1)} t_k, j + \sum_{k=0}^{(i-1)} j_k) = \frac{1}{\lambda_i} \xi_i(t,j),$$
$$\forall (t,j) \in \operatorname{dom} \xi_i,\ (t + \sum_{k=0}^{(i-1)} t_k, j + \sum_{k=0}^{(i-1)} j_k) \leq (t_i, j_i).$$

ξ grows unbounded by the fact that each solution ξ_i begins from a U_1 that is a proper subset of $\{x \in \mathbb{R}^n \,|\, |x| < \rho c\}$ and enters $\{x \in \mathbb{R}^n \,|\, |x| \geq \rho c\}$ in finite time. Instability of x_e follows from Lemma 1. ∎

4) Proof of Proposition 1: V is a polynomial, therefore it is continuously differentiable.

(1) First inequality of (13) can be rewritten as $\ell - V(x) \geq s_3(x)\Delta_1(c, c+\varepsilon, x)$. Therefore, $\ell - V(x) \geq 0$ for each $c \leq |x| \leq c + \varepsilon$, that implies Condition (1) of Theorem 2.

(2) Rewrite the second inequality of (13) as $V(x) - \ell \geq s_4(x)\Delta_1(c + 2\varepsilon, c + 3\varepsilon, x) + q(x)$, then $V(x) - \ell \geq 0$ for $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$ and $q(x) \geq 0$. By (14), $q(x)$ is non-negative in a conic subset of $\mathbb{R}^n$, therefore $V(x) - \ell \geq 0$ in a subset of $c + 2\varepsilon \leq |x| \leq c + 3\varepsilon$, as required by Condition (2) of Theorem 2. In fact, denote $\varepsilon_{(Alg.1)}$ and $\varepsilon_{(Thm.2)}$ respectively the constants ε of Algorithm 1 and of Theorem 2, then V satisfies Condition (2) of Theorem 2 with $\varepsilon_{(Thm.2)} \geq 2\varepsilon_{(Alg.1)}$

(3,4) (11) and (12) imply conditions (3) and (4) of Theorem 2, respectively.

(5) The fifth inequality of (13) can be interpreted as $\ell - V(x) \geq s_5^{(ik)}(x)(c^2 - x'G_{ik}'G_{ik}x)$ for each $x \in R^{(i)}$ and $c \leq |x| \leq \rho c$, where $i \in I_D$. Therefore $\ell - V(x) \geq 0$ if $c - g \geq 0$, for each $x \in R^{(i)}$, $c \leq |x| \leq \rho c$ and each $g \in G(x)$. By negation, if $V(x) > \ell$ then $c - g < 0$, for each $x \in R^{(i)}$, $c \leq |x| \leq \rho c$ and each $g \in G(x)$, as required by Condition (5) of Theorem 2.

(6) Condition (6) is (ii) of Proposition 1 ∎

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