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Interconnecting a System Having a Single Input-to-State Gain With a System Having a Region-Dependent Input-to-State Gain

Humberto STEIN SHIROMOTO[1], Vincent ANDRIEU[2], Christophe PRIEUR[1]

Abstract— **For an ISS system, by analyzing local and non-local properties, it is obtained different input-to-state gains. The interconnection of a system having two input-to-state gains with a system having a single ISS gain is analyzed. By employing the Small Gain Theorem for the local (resp. non-local) gains composition, it is concluded about the local (resp. global) stability of the origin (resp. of a compact set). Additionally, if the region of local stability of the origin strictly includes the region attraction of the compact set, then it is shown that the origin is globally asymptotically stable. An example illustrates the approach.**

I. INTRODUCTION

The use of nonlinear input-output gains for the study of the stability of nonlinear was introduced in [22], [23] by considering a system as an input-output operator. The condition that ensures stability, called Small Gain Theorem, of the resulting interconnected system is based on the contraction principle ([23]).

The works [15] and [16] introduce a new concept of gain relating the input to system states. This notion of stability, called Input-to-State Stability (ISS), combines Zames and Lyapunov approaches ([18], [19]). Characterizations in terms of dissipation and Lyapunov functions are given in [20] and [21].

In [11], the contraction principle is used in the ISS notion to obtain an equivalent Small Gain Theorem. A formulation of this criteria in terms of Lyapunov functions may be found in [10] and [12].

Besides stability analysis, the Small Gain Theorem may also be used for the design of dynamic feedback laws satisfying robustness constraints. The interested reader may see [5], [6], [7] and [14] and references therein.

Other versions of the Small Gain theorem do exist in the literature, examples of which can be found in [3]. See also [2], [8] and [9] for the interconnection of possibly non-ISS systems.

In order to apply the Small Gain Theorem, it is required that the composition of the nonlinear gains is smaller than the argument for all of its positive values ([10], [12]). Such a condition, called Small Gain Condition, restricts the application of the Small Gain Theorem to a composition of well chosen gains.

In this work, it is made use of the Small Gain Theorem in a less conservative way. This new condition ensures the asymptotic stability of a system by showing that if there exist two different gains compositions such that they satisfy the Small Gain Condition, not for all values of the arguments, but in two different regions, and if these regions cover the set of all positive values, then the resulting interconnected system is globally asymptotically stable. Thus, this approach may be seen as a composition of two different small gain conditions that hold in different regions: a local and a global.

The use of a unifying approach is well known in the literature, see [1] for the combination of control Lyapunov functions and [4] for a stability concept uniting ISS and the integral variant of ISS (namely, iISS [17]) properties.

This paper is organized as follows. In Section II, the basic concepts of Input-to-State Stability and the Dini derivative are presented. Also, the system under consideration, the problem statement and a motivational example are presented. In Section III, the assumptions to solve the problem under consideration, as well as the main results are presented. Section IV presents an example that illustrates the assumptions and main results. Section V contains the proofs of the main results. Concluding remarks are given in Section VI. Finally, in Section VII, auxiliary results are stated. Due to space limitations, some of the proofs were omitted.

Notation. Let $\mathbf{S}$ be a subset of $\mathbb{R}^n$ containing the origin, the notation $\mathbf{S}_{\neq 0}$ stands for $\mathbf{S} \setminus \{0\}$. The closure of $\mathbf{S}$ is denoted by $\overline{\mathbf{S}}$. Let $x \in \mathbb{R}^n$, the notation $|x|$ stands for the Euclidean norm of x. A function $f : \mathbf{S} \to \mathbb{R}$ defined in a subset $\mathbf{S}$ of $\mathbb{R}^n$ containing 0 is *positive definite* if, $\forall x \in \mathbf{S}_{\neq 0}$, $f(x) > 0$ and $f(0) = 0$. It is *proper* if $f(|x|) \to \infty$ as $|x| \to \infty$. By $\mathcal{C}^k$ it is denoted the class of k-times continuously differentiable functions, by $\mathcal{K}$ it is denoted the class of continuous and strictly increasing functions $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\gamma(0) = 0$; it is denoted by $\mathcal{K}_\infty$ if, in addition, they are unbounded. Let $c \in \mathbb{R}_{>0}$, the notation $\Omega_c(f)$ stands for the subset of $\mathbb{R}^n$ defined by $\{x \in \mathbb{R}^n : f(x) < c\}$. Let $x, \bar{x} \in \mathbb{R}_{\geq 0}$, the notation $x \nearrow \bar{x}$ (resp. $x \searrow \bar{x}$) stands for $x \to \bar{x}$ with $x < \bar{x}$ (resp. $x > \bar{x}$).

II. BACKGROUND AND PROBLEM STATEMENT

Consider the system

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where, $\forall t \in \mathbb{R}_{\geq 0}$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$, for some positive integers n and m. The map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be continuous, locally Lipschitz on x and uniformly in u on compact sets. A solution of (1) with initial condition x, and input u at time t is denoted by $X(t, x, u)$. Assume that the origin is an equilibrium point for the system (1), i.e., $f(0, 0) = 0$.

Definition 1. Consider the function $\xi : [a, b) \to \mathbb{R}$, the limit

$$D^+\xi(t) = \limsup_{\tau \searrow 0} \frac{\xi(t+\tau) - \xi(t)}{\tau}$$

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(if it exists) is called *Dini derivative*. Let k be a positive integer. Consider the functions $\varphi : \mathbb{R}^k \to \mathbb{R}$ and $h : \mathbb{R}^k \to \mathbb{R}^k$, the limit

$$D_h^+\varphi(y) = \limsup_{\tau \searrow 0} \frac{\varphi(y+\tau h(y))-\varphi(y)}{\tau}.$$

(if it exists) is called *Dini derivative of φ in the h-direction at y.* •

Definition 2. A continuous locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS-Lyapunov function for system (1) if

• there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\overline{\alpha}$ such that, $\forall x \in \mathbb{R}^n$, $\underline{\alpha}(|x|) \leq V(x) \leq \overline{\alpha}(|x|)$;

• there exist a class $\mathcal{K}$ function α_x, called *ISS gain*, and a continuous positive definite function $\lambda_x : \mathbb{R}^n \to \mathbb{R}$ such that, $\forall (x,u) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$|x| \geq \alpha_x(|u|) \Rightarrow D_f^+V(x,u) \leq -\lambda_x(x) \tag{2}$$

holds. •

From now on, V will be assumed to be an ISS-Lyapunov function for (1).

Consider the system

$$\dot{z}(t) = g(v(t), z(t)), \tag{3}$$

where, $\forall t \in \mathbb{R}_{\geq 0}$, $v(t) \in \mathbb{R}^n$ and $z(t) \in \mathbb{R}^m$, for some positive integers n and m. The map $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is assumed to be continuous, locally Lipschitz on z and uniformly in v on compact sets. A solution of (3) with initial condition z, and input v at time t is denoted by $Z(t,z,v)$. Assume that the origin is an equilibrium point for the system (3), i.e., $g(0,0) = 0$. Consider also the following

Assumption 1. There exists a continuous locally Lipschitz function $W : \mathbb{R}^m \to \mathbb{R}$ that is an ISS-Lyapunov function for the z-subsystem. More precisely, there exist class $\mathcal{K}_\infty$ functions $\underline{\beta}$ and $\overline{\beta}$ satisfying, $\forall z \in \mathbb{R}^m$, $\underline{\beta}(|z|) \leq W(z) \leq \overline{\beta}(|z|)$. Furthermore, there exist a class $\mathcal{K}$ function δ and a continuous positive definite function $\lambda_z : \mathbb{R}^m \to \mathbb{R}$ such that, $\forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m$,

$$W(z) \geq \delta(V(x)) \Rightarrow D_g^+W(x,z) \leq -\lambda_z(z), \tag{4}$$

where V is the ISS-Lyapunov function of x-subsystem. ◦

System under consideration. Interconnecting systems (1) and (3) by linking the state of (1) with the input of (3) and vice versa leads to the following system

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,z) \\ \dot{z} & = & g(x,z). \end{array} \right. \tag{5}$$

Since $f(0,0) = 0$ and $g(0,0) = 0$, the origin is an equilibrium point for (5). Considering the ISS-Lyapunov inequalities, after the interconnection the following implications

$$\begin{array}{rcccccl} V(x) & \geq & \gamma(W(z)) & \Rightarrow & D_f^+V(x,z) & \leq & -\lambda_x(x), \\ W(z) & \geq & \delta(V(x)) & \Rightarrow & D_g^+W(x,z) & \leq & -\lambda_z(z) \end{array}$$

are obtained with suitable class $\mathcal{K}$ functions γ and δ.

A sufficient condition that ensures stability of (5) is given by the following

Theorem 1. *[10] If,*

$$\forall s \in \mathbb{R}_{>0}, \quad \gamma \circ \delta(s) < s. \tag{6}$$

Then, the origin is globally asymptotically stable for (5).

Problem statement. At this point, it is possible to explain the problems that are dealt with, in this work.

• *ISS gains computation*. Although the use of ISS gains renders the analysis of stability easy to work with, it is not a trivial task to compute those gains;

• *Small gain condition*. Since the ISS gain is not unique, it might not be an easy task to find two ISS gains: one for the x-subsystem of (5) and another for the z-subsystem of (5) such that their composition satisfies (6), for all positive values of the argument. An illustration of the problem that is dealt with is presented in the following

EXAMPLE 1. Let the functions $f, g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and consider the system

$$\left\{ \begin{array}{rcccl} \dot{x} & = & f(x,z) & = & -\rho(x) + z \\ \dot{z} & = & g(x,z) & = & -\operatorname{sign}(z)\tilde{\delta}(|z|) + x, \end{array} \right. \tag{7}$$

where $\tilde{\delta}$ will be defined below. Let, $\forall x \in \mathbb{R}$, $V(x) = |x|$, $\rho(x) = 5x/4 - 2x^2 + x^3$ and, $\forall z \in \mathbb{R}$, $W(z) = |z|$.

Taking the Dini derivative of V in the f-direction, $\forall (x,z) \in \mathbb{R} \times \mathbb{R}$, it yields

$$D_f^+V(x,z) \leq -\rho(V(x)) + W(z). \tag{8}$$

This implies that $\exists \varepsilon_x \in (0,1)$ such that, $\forall (x,z) \in \mathbb{R} \times \mathbb{R}$,

$$\rho(V(x)) \geq \frac{W(z)}{1-\varepsilon_x} \Rightarrow D_f^+V(x,z) \leq -\lambda_x(x), \tag{9}$$

where $\lambda_x(\cdot) := \varepsilon_x \rho(V(\cdot))$. From now on, let $\varepsilon_x = 0.05$. Note also that, in the interval $[1/2, 5/6]$, ρ is decreasing.

Consider the piecewise continuous function Γ defined by

$$\Gamma(s) = \left\{ \begin{array}{ll} \rho^{-1}\left(\frac{s}{0.95}\right), & \text{if} \quad s \in \left[0, 0.95\rho\left(\frac{5}{6}\right)\right), \\ \rho_+^{-1}\left(\frac{s}{0.95}\right), & \text{if} \quad s \in \left[0.95\rho\left(\frac{5}{6}\right), \infty\right), \end{array} \right. \tag{10}$$

where $[5/6, \infty) \ni s \mapsto \rho_+(s) = \rho(s) \in [\rho(5/6), \infty)$.

Remark 1. The function Γ can be viewed as a discontinuous input-to-state gain of the x-subsystem of (7). More precisely, $\forall (x,z) \in \mathbb{R} \times \mathbb{R}$, $V(x) \geq \Gamma(W(z)) \Rightarrow D_f^+V(x,z) \leq -\lambda_x(x)$. Furthermore, the function Γ is "optimal", in the sense that if there exist a function $\Gamma^* : \mathbb{R} \to \mathbb{R}$ and a value $s^* \in \mathbb{R}_{>0}$ such that $\Gamma^*(s^*) < \Gamma(s^*)$, then $\exists (x^*, z^*) \neq (0,0)$ such that $V(x^*) \geq \Gamma^*(W(z^*))$ and $D_f^+V(x^*, z^*) > 0$. ◦

It follows from Remark 1 that an ISS gain for the x-subsystem of (7) is any class $\mathcal{K}$ function γ such that, $\forall s \in \mathbb{R}_{>0}$, $\Gamma(s) \leq \gamma(s)$.

A local gain. Consider the function $[0, 1/2) \ni s \mapsto \rho_-(s) = \rho(s) \in [0, \rho(1/2))$. Since ρ_- is strictly increasing on its domain, it is invertible. Let γ_ℓ be a class $\mathcal{K}$ function such that, $\forall s \in [0, 0.95\rho(1/2))$,

$$\gamma_\ell(s) = \rho_-^{-1}\left(\frac{s}{0.95}\right). \tag{11}$$

Note that, γ_ℓ satisfies the following inequalities

$$\forall s \in \left[0, 0.95\rho\left(\tfrac{1}{2}\right)\right), \quad \gamma_\ell(s) \leq \Gamma(s),$$
$$\forall s \in \left(0.95\rho\left(\tfrac{5}{6}\right), 0.95\rho\left(\tfrac{1}{2}\right)\right), \quad \gamma_\ell(s) < \Gamma(s).$$

Moreover, $\forall (x,z) \in \Omega_{\rho(1/2)}(V) \times \mathbb{R}$,

$$V(x) \geq \gamma_\ell(W(z)) \Rightarrow D_f^+V(x,z) \leq -\lambda_x(x). \tag{12}$$

Let the constant values $M_\ell = 0.236$ and $M_g = 0.245$. At this point, it is possible to define the function $\tilde{\delta}$ of the z-subsystem of (7). It is a function of class $\mathcal{K}_\infty$ satisfying the following inequalities

$$\forall s \in (0, M_\ell], \qquad \gamma_\ell(s) < \tilde{\delta}(s), \tag{13}$$

$$\forall s \in [M_g, \infty), \qquad \Gamma(s) < \tilde{\delta}(s), \tag{14}$$

$$\forall s \in \left(\rho\left(\tfrac{5}{6}\right), M_\ell\right), \qquad \tilde{\delta}(s) < \Gamma(s). \tag{15}$$

Equations (13) and (14) correspond to two different small gain conditions, the first may be seen as a small gain condition for small values of the argument while the last as a small gain condition for large values of the argument. Note that (15) implies that Theorem 1 cannot be applied.[1]

Fig. 1 shows a plot of the functions ρ, id, Γ, γ_ℓ and $\tilde{\delta}$. •

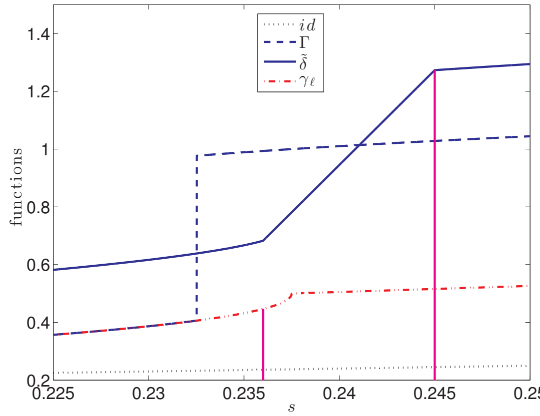

Fig. 1. Plot of the functions id (dotted black line), Γ (dashed blue line), γ_ℓ (dash dotted red line) and the continuous function $\tilde{\delta}$ (solid blue line), in the interval $[0.225, 0.25]$. The vertical lines are the values $M_\ell = 0.236$ and $M_g = 0.245$, respectively.

In this work, it will be shown that, if

• there exist two ISS gains γ_ℓ and γ_g, for the x-subsystem of (5);
• there exists one ISS gain δ, for the z-subsystem of (5);
• the compositions $\gamma_\ell \circ \delta$ and $\gamma_g \circ \delta$ satisfy the Small Gain Condition, not for all values of the arguments, but for two different intervals ($\mathbf{I}_\ell, \mathbf{I}_g \subset \mathbb{R}_{\geq 0}$). In other words, $\forall s \in \mathbf{I}_\ell \setminus \{0\}, \gamma_\ell \circ \delta(s) < s$ and, $\forall s \in \mathbf{I}_g \setminus \{0\}, \gamma_g \circ \delta(s) < s$;
• these intervals are such that $\mathbf{I}_\ell \cap \mathbf{I}_g \neq \emptyset$ and $\mathbf{I}_\ell \cup \mathbf{I}_g = \mathbb{R}_{\geq 0}$;

then, the origin is globally asymptotically stable for (5). See Theorem 2 below for a precise statement of this result.

III. Assumptions and main results

In this section, it is specified the assumptions on the system (5) necessary to solve the problem under consideration. The proof of the stabilization results are provided from Section V-A to Section V-C.

A. Local set of assumptions on the x-subsystem

In this section, it is introduced the set of assumptions to ensure that the origin is locally asymptotically stable for (5).

Assumption 2. There exist a class $\mathcal{K}$ function γ_ℓ and a strictly positive constant M_ℓ such that,

$$M_\ell < \lim_{s\to\infty} \gamma_\ell(s) = b_\ell. \tag{16}$$

Moreover, $\forall (x,z) \in \overline{\Omega_{M_\ell}(V)} \times \mathbb{R}^m$,

$$V(x) \geq \gamma_\ell(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x). \tag{17}$$

◦

Assumption 3. The composition of the functions γ_ℓ and δ is such that,

$$\forall s \in (0, M_\ell], \quad \gamma_\ell \circ \delta(s) < s. \tag{18}$$

◦

Proposition 1. *Under Assumptions 1, 2 and 3 the origin is locally asymptotically stable for system* (5).

[1]To see this fact, note that $M_\ell < 0.95\rho(1/2)$. Since $\tilde{\delta}$ is of class $\mathcal{K}_\infty$ and from (15), $\forall s \in (0.95\rho(5/6), M_\ell)$, $s < \tilde{\delta}^{-1} \circ \Gamma(s)$. Thus, there exists no class $\mathcal{K}$ function γ such that (6) holds.

B. Non-local set of assumptions on the x-subsystem

In this section, it is introduced the set of assumptions to ensure that a neighborhood of the origin is globally attractive for (5).

Assumption 4. There exist a class $\mathcal{K}$ function γ_g and a strictly positive constant M_g such that

$$M_g < \lim_{s\to\infty} \gamma_g(s) = b_g. \tag{19}$$

Moreover, $\forall (x,z) \in (\mathbb{R}^n \setminus \Omega_{M_g}(V)) \times \mathbb{R}^m$,

$$V(x) \geq \gamma_g(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x). \tag{20}$$

◦

Assumption 5. The composition of the functions γ_g and δ is such that,

$$\forall s \in [M_g, \infty), \quad \gamma_g \circ \delta(s) < s. \tag{21}$$

◦

Proposition 2. *Under Assumptions 1, 4 and 5, there exist a proper definite positive function U_g and a positive constant $\tilde{M}_g$ such that the set $\overline{\Omega_{\tilde{M}_g}(U_g)}$ is globally asymptotically stable for system* (5).

C. Main result

In this section, it is introduced the assumption to ensure that the origin is globally asymptotically stable for (5).

Assumption 6. The positive constants M_ℓ and M_g given, respectively, by Assumptions 2 and 4 satisfy $M_g < M_\ell$. ◦

Theorem 2. *Under Assumptions 1-6, the origin is globally asymptotically stable for system* (5).

IV. Illustration

Example 2. [Example 1 revisited.]

Verifying Assumption 1. Let the function δ be given by the inverse of $\tilde{\delta}$. It follows that, $\forall (x,z) \in \mathbb{R} \times \mathbb{R}$, $W(z) \geq \delta(V(x)) \Rightarrow D_g^+ W(z) \leq -\lambda_z(z)$, where for a given $\varepsilon_z \in (0,1)$ and $\forall z \in \mathbb{R}$, $\lambda_z(z) = \varepsilon_z W(z)$. Thus, Assumption 1 holds.

Verifying Assumption 2. The function γ_ℓ is given by (11) and $M_\ell = 0.236$. Moreover, it follows from (12) that Assumption 2 holds.

Verifying Assumption 3. It follows from inequality (13) that Assumption 3 holds.

From Proposition 1, it follows that the origin is locally asymptotically stable for (7). Figure 2 shows some solutions of (7). •

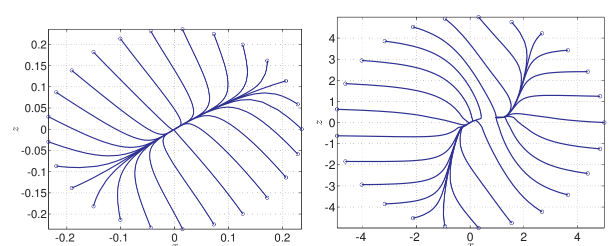

Fig. 2. Solutions to (7) for initial condition starting at a ball centered at the origin with radius, respectively, given by M_ℓ and 5.

Example 3. [Example 1 revisited.]

Verifying Assumption 4. Let a class $\mathcal{K}_\infty$ function γ_g be such that, $\forall s \in [0.95\rho(5/6), \infty)$, $\gamma_g(s) = \Gamma(s)$. Moreover,

$M_g = 0.245$. It follows from Remark 1 that, $\forall(x,z) \in (\mathbb{R} \setminus \Omega_{M_g}(V)) \times \mathbb{R}$, $V(x) \geq \gamma_g(W(z)) \Rightarrow D_f^+ V(x,z) \leq -\lambda_x(x)$. Thus, Assumption 4 holds.

Verifying Assumption 5. It follows from inequality (14) that Assumption 5 holds.

From Proposition 2, it follows that a neighborhood of the origin is globally asymptotically stable for (7). Figure 2 shows some solutions of (7). •

V. Proofs

A. Proof of Proposition 1

Proof. This proof is divided into three parts. In the first one, it is obtained a function σ_ℓ that is class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ with strictly positive derivative. This function is used in the second part, where a class $\mathcal{C}^0$ proper positive definite function U_ℓ is defined and its Dini derivative is studied. In the last part, it is shown that U_ℓ is locally Lipschitz and the local asymptotical stability of the origin is concluded by using Lemma 2.

First part. Consider the class $\mathcal{K}$ functions δ and γ_ℓ from Assumptions 1 and 2. Under Assumption 3, δ and γ_ℓ are such that, $\forall s \in (0, M_\ell]$, $\delta(s) < \gamma_\ell^{-1}(s)$.

Since γ_ℓ is class of $\mathcal{K}$, from Lemma 1, there exists a class $\mathcal{K}_\infty$ function $\tilde{\gamma}_\ell$ such that, $\forall s \in \mathbb{R}_{>0}$,

$$\delta(s) < \tilde{\gamma}_\ell(s) \tag{22}$$

and, $\forall s \in (0, M_\ell]$,

$$\tilde{\gamma}_\ell(s) < \gamma_\ell^{-1}(s). \tag{23}$$

Since δ is of class $\mathcal{K}$ and $\tilde{\gamma}_\ell$ is of class $\mathcal{K}_\infty$ satisfying, $\forall s \in \mathbb{R}_{>0}$, inequality (22), from Lemma [10, Lemma A.1], there exists a class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ function σ_ℓ whose derivative is strictly positive and satisfies, $\forall s \in \mathbb{R}_{>0}$,

$$\delta(s) < \sigma_\ell(s) < \tilde{\gamma}_\ell(s). \tag{24}$$

Second part. Let, $\forall(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, $U_\ell(x,z) := \max\{\sigma_\ell(V(x)), W(z)\}$. Note that the function U_ℓ is proper positive definite. Pick $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, one of three cases is possible: $\sigma_\ell(V(x)) < W(z)$, $W(z) < \sigma_\ell(V(x))$ or $W(z) = \sigma_\ell(V(x))$. The proof follows by showing that the Dini derivative of U_ℓ is negative. For each case, assume that $(x,z) \in \overline{\Omega_{M_\ell}(V)} \times \mathbb{R}^m$.

Case 1. Suppose that $\sigma_\ell(V(x)) < W(z)$. This implies that $U_\ell(x,z) = W(z)$ and $D_{f,g}^+ U_\ell(x,z) = D_g^+ W(x,z)$.

From (24), the following inequality $\delta(V(x)) < \sigma_\ell(V(x)) < W(z)$ holds. Together with (4), it follows that $D_g^+ W(x,z) \leq -\lambda_z(z)$. This concludes Case 1.

Case 2. Suppose that $W(z) < \sigma_\ell(V(x))$. This implies that $U_\ell(x,z) = \sigma_\ell(V(x))$ and $D_{f,g}^+ U_\ell(x,z) = \sigma_\ell'(V(x)) D_f^+ V(x,z)$. From (24), the following inequality $W(z) < \sigma_\ell(V(x)) < \tilde{\gamma}_\ell(V(x))$ holds. Since $V(x) \leq M_\ell$, it follows that

$$W(z) < \sigma_\ell(V(x)) < \tilde{\gamma}_\ell(V(x)) < \gamma_\ell^{-1}(V(x)), \tag{25}$$

where the last inequality follows from (23). Equation (17) together with (25) yields $D_f^+ V(x,z) \leq -\lambda_x(x)$.

Since, $\forall s \in \mathbb{R}_{>0}$, $\sigma_\ell'(s) > 0$, it follows that $D_{f,g}^+ U_\ell(x,z) = \sigma_\ell'(V(x)) D_f^+ V(x,z) \leq -\sigma_\ell'(V(x))\lambda_x(x)$. This concludes Case 2.

Case 3. Let $W(z) = \sigma_\ell(V(x)) := U_\ell^*(x,z)$. This implies

$$\begin{aligned} D_{f,g}^+ U_\ell^*(x,z) &= \limsup_{t \searrow 0} \tfrac{1}{t}(\max\{\sigma_\ell(V(X(x,z,t))), \\ &\qquad W(Z(z,x,t))\} - U_\ell^*(x,z)) \\ &= \limsup_{t \searrow 0} \max\left\{ \tfrac{\sigma_\ell(V(X(x,z,t))) - \sigma_\ell(V(x))}{t}, \tfrac{W(Z(z,x,t)) - W(z)}{t} \right\} \\ &= \max\{\sigma_\ell'(V(x)) D_f^+ V(x,z), D_g^+ W(x,z)\}. \end{aligned}$$

The analysis of $D_{f,g}^+ U_\ell^*$ is divided in two sub cases. In the first one, the function $D_g^+ W$ is analyzed while in the last, the function $D_f^+ V$ is analyzed.

Case 3.a. *The analysis of $D_g^+ W$.* From (24), the following inequality $\delta(V(x)) < \sigma_\ell(V(x)) = W(z)$ holds. Together with Equation (4), it yields $D_g^+ W(x,z) \leq -\lambda_z(z)$.

Case 3.b. *The analysis of $D_f^+ V$.* From (24), the following inequality $W(z) = \sigma_\ell(V(x)) < \tilde{\gamma}_\ell(V(x))$ holds. Since $V(x) \leq M_\ell$, it follows that

$$W(z) = \sigma_\ell(V(x)) < \tilde{\gamma}_\ell(V(x)) < \gamma_\ell^{-1}(V(x)), \tag{26}$$

where the last inequality is due to (23).

Equation (17) together with (26) yields $D_f^+ V(x,z) \leq -\lambda_x(x)$.

To conclude Case 3, $W(z) = \sigma_\ell(V(x)) \Rightarrow D_{f,g}^+ U_\ell^*(x,z) \leq -\min\{\sigma_\ell'(V(x))\lambda_x(x), \lambda_z(z)\}$ holds, since $(x,z) \in \overline{\Omega_{M_\ell}(V)} \times \mathbb{R}^m$.

Let $\tilde{M}_\ell := \max\{c \in \mathbb{R}_{>0} : \overline{\Omega_c(U_\ell)} \subset \overline{\Omega_{M_\ell}(V)} \times \{0\}$ and $\overline{\Omega_c(U_\ell)}$ is connected$\}$. To sum up all the above cases, $\forall(x,z) \in \overline{\Omega_{\tilde{M}_\ell}(U_\ell)}$,

$$U_\ell(x,z) \leq \tilde{M}_\ell \Rightarrow D_{f,g}^+ U_\ell(x,z) \leq -E_\ell(x,z), \tag{27}$$

where $E(\cdot,\cdot) := \min\{\sigma_\ell'(V(\cdot))\lambda_x(\cdot), \lambda_z(\cdot)\}$ is continuous and positive definite.

Third part. To conclude local asymptotical stability of the origin, it remains to show that U_ℓ is locally Lipschitz. Since $\sigma_\ell(V(\cdot))$ (resp. W) is locally Lipschitz, U_ℓ is locally Lipschitz in the region $W(\cdot) \leq \sigma_\ell(V(\cdot))$ (resp. $\sigma_\ell(V(\cdot)) \leq W(\cdot)$). Since the hypotheses of Lemma 2 (in Section VII) below are verified with $U(\cdot) = U_\ell(\cdot)$ and $E(\cdot) = E_\ell(\cdot)$, the origin is locally asymptotically stable for (5). This concludes the proof of Proposition 1. ■

B. Proof of Proposition 2

Proof. This proof is analogous to the proof of Proposition 1 and divided into three parts. In the first one, it is obtained a function σ_g that is class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ with strictly positive derivative. This function is used in the second part, where a class $\mathcal{C}^0$ proper and positive definite function U_g is defined and its Dini derivative is studied. In the last part, it is used Lemma 4 to show that the set $\overline{\Omega_{M_g}(V)} \times \{0\}$ is globally asymptotically stable.

First part. Consider the class $\mathcal{K}$ functions δ and γ_g from Assumptions 1 and 4. The function γ_g^{-1} is defined on $[0, b_g)$ and satisfies $\lim_{s \nearrow b_g} \gamma_g^{-1}(s) = \infty$. Assumption 5 implies that, $\forall s \in [M_g, b_g)$, $\delta(s) < \gamma_g^{-1}(s)$. Since γ_g is of class $\mathcal{K}$, from Lemma 1 (in Section VII), there exists a class $\mathcal{K}_\infty$ function $\tilde{\gamma}_g$ such that, $\forall s \in \mathbb{R}_{>0}$,

$$\delta(s) < \tilde{\gamma}_g(s) \tag{28}$$

and, $\forall s \in [M_g, b_g)$,

$$\tilde{\gamma}_g(s) < \gamma_g^{-1}(s). \tag{29}$$

Since δ is of class $\mathcal{K}$ and $\tilde{\gamma}_g$ is of class $\mathcal{K}_\infty$ satisfying, $\forall s \in \mathbb{R}_{>0}$, the inequality (28), from Lemma [10, Lemma A.1], there exists a function σ_g that is of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ whose derivative is strictly positive and satisfies, $\forall s \in \mathbb{R}_{>0}$,

$$\delta(s) < \sigma_g(s) < \tilde{\gamma}_g(s). \tag{30}$$

Second part. Let, $\forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, $U_g(x,z) := \max\{\sigma_g(V(x)), W(z)\}$. Note that the function U_g is proper positive definite. Pick $(x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, one of three cases is possible: $\sigma_g(V(x)) < W(z)$, $W(z) < \sigma_g(V(x))$ or $W(z) = \sigma_g(V(x))$. The proof follows by showing that the Dini derivative of U_g is negative. For each case, assume that $(x,z) \in (\mathbb{R}^n \setminus \Omega_{M_g}(V)) \times \mathbb{R}^m$.

Case 1. Suppose that $\sigma_g(V(x)) < W(z)$. Analogously to the Case 1 of proof of Proposition 1, $\sigma_g(V(x)) < W(z) \Rightarrow D^+_{f,g}U_g(x,z) \leq -\lambda_z(z)$. This concludes Case 1.

Case 2. Suppose that $W(z) < \sigma_g(V(x))$. This implies that $U_g(x,z) = \sigma_g(V(x))$ and $D^+_{f,g}U_g(x,z) = \sigma'_g(V(x))D^+_f V(x,z)$. From (30), the following inequality

$$W(z) < \sigma_g(V(x)) < \tilde{\gamma}_g(V(x)) \tag{31}$$

holds. At this point, two regions of x will be analyzed: $b_g \leq V(x)$ and $M_g \leq V(x) < b_g$.

Case 2.a. In the region where $b_g \leq V(x)$, Equation (20) together with (19) yields $D^+_f V(x,z) \leq -\lambda_x(x)$.

Case 2.b. In the region where $M_g \leq V(x) < b_g$, from (29) and (31), it yields

$$W(z) < \sigma_g(V(x)) < \tilde{\gamma}_g(V(x)) < \gamma_g^{-1}(V(x)). \tag{32}$$

Equation (20) together with (32) yields $D^+_f V(x,z) \leq -\lambda_x(x)$.

Since, $\forall s \in \mathbb{R}_{>0}$, $\sigma'_g(s) > 0$, it follows that $D^+_{f,g}U_g(x,z) = \sigma'_g(V(x))D^+_f V(x,z) \leq -\sigma'_g(s)\lambda_x(x)$. This concludes Case 2.

Case 3. Let $W(z) = \sigma_g(V(x)) := U^*_g(x,z)$. Analogously to the Case 3 of proof Proposition 1 and together with the analysis of Cases 1 and 2, the implication $W(z) = \sigma_g(V(x)) \Rightarrow D^+_{f,g}U_g(x,z) \leq -\min\{\sigma'_g(s)\lambda_x(x), \lambda_z(z)\}$ holds, since $(x,z) \in (\mathbb{R}^n \setminus \Omega_{M_g}(V)) \times \mathbb{R}^m$.

Let $\tilde{M}_g = \min\{c \in \mathbb{R}_{>0} : \overline{\Omega_{M_g}(V)} \times \{0\} \subset \overline{\Omega_c(U_g)}$ and $\overline{\Omega_c(U_g)}$ is connected$\}$. To sum up all the above cases, $\forall (x,z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \overline{\Omega_{\tilde{M}_g}(U_g)}$,

$$\tilde{M}_g < U_g(x,z) \Rightarrow D^+_{f,g}U_g(x,z) \leq -E_g(x,z), \tag{33}$$

where $E_g(\cdot,\cdot) = \min\{\sigma'_g(V(\cdot))\lambda_x(\cdot), \lambda_z(\cdot)\}$ is continuous and positive definite.

Third part. Analogously to the third part of the proof of Proposition 1, it follows that U_g is locally Lipschitz. From Lemma 3 and (33), it follows that, $\forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m$ and $\forall t \in \mathbb{R}_{\geq 0}$, along solutions of (5),

$$D^+U_g(X(t,x,z), Z(t,z,x)) = D^+_{f,g}U_g(X(t,x,z), Z(t,z,x)).$$

Since solutions of (5) are absolutely continuous functions and, along solutions of (5), E_g is a continuous positive definite function, from Lemma 4, $\forall (x,z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \overline{\Omega_{\tilde{M}_g}(U_g)}$ and $\forall t \in \mathbb{R}_{\geq 0}$, the function

$$t \mapsto U_g(X(t,x,z), Z(t,z,x)) \tag{34}$$

is strictly decreasing. Pick $(x,z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \overline{\Omega_{\tilde{M}_g}(U_g)}$, it will be proven that

$$U_g^\infty := \lim_{t \to \infty} U_g(X(t,x,z), Z(t,z,x)) \leq \tilde{M}_g.$$

To see the above suppose, by contradiction, that $U_g^\infty > \tilde{M}_g$. From the continuity of U_g, $\exists \varepsilon > 0$ such that $U_g^\infty - \varepsilon > \tilde{M}_g$ and $U_g^\infty - \varepsilon \leq U_g(x,z) \leq U_g^\infty + \varepsilon$. Since U_g is proper, the constant $\xi = \min\{E_g(x,z) > 0 : (x,z) \in U_g(x,z)$ and $U_g^\infty - \varepsilon \leq U_g(x,z) \leq U_g^\infty + \varepsilon\}$ exists. Recalling the definition of U_g, $\exists T > 0$ such that, $\forall t \geq T$, $U_g(X(t,x,z), Z(t,z,x)) - U_g^\infty < \varepsilon$. Moreover, from the definition of the constant ξ,

$$U_g(X(t,x,z), Z(t,z,x)) - U_g(X(T,x,z), Z(T,z,x)) = \int_T^t D^+U_g(X(s,x,z), Z(s,z,x))\, ds \leq -\xi \cdot (t - T).$$

Then,

$$\begin{aligned} U_g^\infty &= \lim_{t\to\infty} U_g(X(t,x,z), Z(t,z,x)) \\ &= U_g(X(T,x,z), Z(T,z,x)) \\ &+ \lim_{t\to\infty} \int_T^t D^+U_g(X(s,x,z), Z(s,z,x))\, ds \leq -\infty \end{aligned}$$

which contradicts the fact that U_g is positive definite. Therefore, $U_g^\infty \leq \tilde{M}_\ell$.

In summary, the following facts hold for the function U_g: 1) U_g is a proper positive definite function; 2) U_g decreases along solutions of (5) having initial conditions in $(\mathbb{R}^n \times \mathbb{R}^m) \setminus \overline{\Omega_{\tilde{M}_g}(U_g)}$. From facts 1) and 2), the set $\overline{\Omega_{\tilde{M}_g}(U_g)}$ is globally asymptotically stable for (5). This concludes the proof of Proposition 2. ■

C. Proof of Theorem 2

Proof. Under Assumption 6, $\exists M > 0$ such that $M_g < M < M_\ell$. Under Assumptions 1, 2, 3 and Proposition 1, it follows that the origin is locally asymptotically stable. From the proof of Proposition 1, there exists a proper positive definite function given, $\forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, by $U_\ell(x,z) = \max\{\sigma_\ell(V(x), W(z)\}$, where σ_ℓ is of class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ satisfying (24). Moreover, letting $\hat{M}_\ell := \max\{c \in \mathbb{R}_{>0} : c > M, \overline{\Omega_c(U_\ell)} \subset \overline{\Omega_{M_\ell}(V) \times \{0\}}$ with $\overline{\Omega_c(U_\ell)}$ connected$\}$ every solution starting in $\overline{\Omega_{\hat{M}_\ell}(U_\ell)}$ converges to the origin.

Together with Assumptions 1, 4, 5 and the proof of Proposition 2, it is possible to define, $\forall s \in \mathbb{R}_{\geq 0}$, a class $\mathcal{K}_\infty$ function $\hat{\gamma}_g(s) = \min\{\tilde{\gamma}_g(s), \sigma_\ell(s)\}$ satisfying (28) and (29). Then, it is obtained a class $\mathcal{K}_\infty$ and $\mathcal{C}^1$ function $\hat{\sigma}_g$ whose derivative is strictly positive and satisfies, $\forall s \in \mathbb{R}_{>0}$,

$$\delta(s) < \hat{\sigma}_g(s) < \hat{\gamma}_g(s). \tag{30.new}$$

Defining a proper positive definite function given, $\forall (x,z) \in \mathbb{R}^n \times \mathbb{R}^m$, by $\hat{U}_g(x,z) = \max\{\hat{\sigma}_g(V(x)), W(z)\}$ and the constant $\hat{M}_g = \min\{c \in \mathbb{R}_{>0} : c < M, \overline{\Omega_{M_g}(V)} \times \{0\} \subset \overline{\Omega_c(\hat{U}_g)}$ with $\overline{\Omega_c(\hat{U}_g)}$ connected$\}$, it follows from the proof of Proposition 2 that the set $\overline{\Omega_{\hat{M}_g}(\hat{U}_g)}$ is globally asymptotically stable.

Since, $\forall s \in \mathbb{R}_{>0}$, $\hat{\sigma}_g(s) < \sigma_\ell(s)$, it follows that, $\forall (x,z) \in (\mathbb{R}^n \times \mathbb{R}^m) \setminus \{(0,0)\}$, $\hat{U}_g(x,z) < U_\ell(x,z)$. This inequality implies that, $\forall c \in \mathbb{R}_{>0}$, $\Omega_c(\hat{U}_g) \subset \overline{\Omega_c(U_\ell)}$. Then, the following inclusion holds

$$\overline{\Omega_{\hat{M}_g}(\hat{U}_g)} \subset \overline{\Omega_M(\hat{U}_g)} \subset \overline{\Omega_M(U_\ell)} \subset \overline{\Omega_{\hat{M}_\ell}(U_\ell)}. \tag{35}$$

Thus, every solution of (5) starting in $(\mathbb{R}^n \times \mathbb{R}^m) \setminus \overline{\Omega_{\hat{M}_\ell}(U_\ell)}$ converges to $\overline{\Omega_{\hat{M}_g}(\hat{U}_g)}$, in finite time. Then, due to (35), $\overline{\Omega_{\hat{M}_g}(\hat{U}_g)} \subset \overline{\Omega_{\hat{M}_\ell}(U_\ell)}$ holds, and thus solutions will converge to the origin, as $t \to \infty$.

From the above, combining the local asymptotical stability of the origin with its global attractivitty it is concluded that the origin is globally asymptotically stable for (5). ■

VI. Conclusion and perspectives

In this work, the authors shown that it is possible to make use of local and non-local input-to-state properties of an ISS system, in order to derive an "optimal" ISS gain. As a result of such approach, it is possible to apply the Small Gain Theorem in a less conservative way by deriving local and non-local small gain conditions to ensure the stability of an interconnected system.

In a future work, the authors will generalize the above results for the case in which there exist four ISS gains: two for each subsystem. Moreover, the authors also intend to use the region-dependent gain condition to develop a methodology for the design of feedback laws under different gains constraints.

VII. Auxiliary results

Lemma 1. *Let β be a class $\mathcal{K}$ function with*

$$b = \lim_{s\to\infty} \beta(s). \tag{36}$$

Let also p, q be two constants and α be a class $\mathcal{K}$ function such that, $0 < p < q$ and, $\forall s \in [p, q]$,

$$\beta \circ \alpha(s) < s. \tag{37}$$

Then, the class $\mathcal{K}_\infty$ function $\tilde{\beta}$ given by

$$\tilde{\beta}(s) := \begin{cases} \alpha(s) + \min\{s, K\}, \text{if } p \neq 0 \text{ and } s \in [0, p), \\ \alpha(s) + \min\left\{s, \frac{\beta^{-1}(s)-\alpha(s)}{2}\right\}, \text{if } q + \varepsilon < b \\ \qquad\qquad\qquad\qquad \text{and } s \in [p, q], \\ A + B(s - q), \text{if } q + \varepsilon < b \text{ and } s \in [q, q + \varepsilon), \\ \alpha(s) + s, \text{if } q + \varepsilon \geq b \text{ or } s \in [q + \varepsilon, \infty), \end{cases} \tag{38}$$

is such that, $\forall s \in \mathbb{R}_{>0}$,

$$\alpha(s) < \tilde{\beta}(s). \tag{39}$$

Moreover, $\forall s \in [p, q]$, it also satisfies

$$\tilde{\beta}(s) < \beta^{-1}(s). \tag{40}$$

Due to space constraints, the proof of Lemma 1 is not provided in this paper.

Lemma 2. *[13, Théorème 2.133] Let $\mathbf{S} \subset \mathbb{R}^k$ be a be a neighborhood of the origin. Let also the class $\mathcal{C}^0$ function $h : \mathbb{R}^k \to \mathbb{R}^k$ and consider the system $\dot{y} = h(y)$. If there exist a positive definite and locally Lipschitz function $U : \mathbf{S} \to \mathbb{R}$ and a positive definite function $E : \mathbf{S} \to \mathbb{R}$ such that, $\forall y \in \mathbf{S}$, $D_h^+ U(y) \leq -E(y)$. Then, the origin is locally asymptotically stable for $\dot{y} = h(y)$.*

Lemma 3. *[13, Lemme 1.28] Let the measurable and essentially bounded function $d : \mathbb{R} \to \mathbb{R}^p$ and the class $\mathcal{C}^0$ function $h : \mathbb{R}^k \times \mathbb{R}^p \to \mathbb{R}^k$. If $U : \mathbb{R}^k \to \mathbb{R}$ is locally Lipschitz, then, for all maximal solutions $Y(t, y, d)$ of the system $\dot{y} = h(y, d(t))$ defined in the interval (t_-, t_+), the function $t \mapsto U(Y(t, y, d))$, defined over (t_-, t_+), is locally Lipschitz and, for almost every $t \in (t_-, t_+)$,*

$$\frac{\partial U(Y)}{\partial t}(t, y, d) = D^+ U(Y(t, y, d)) = D_h^+ U(Y(t, y, d)).$$

Moreover, if d is continuous, the above equality holds, $\forall t \in (t_-, t_+)$.

Lemma 4. *Let $Y : \mathbb{R} \to \mathbb{R}^k$ be an absolutely continuous function, $U : \mathbb{R}^k \to \mathbb{R}$ be a locally Lipschitz proper positive definite function and $E : \mathbb{R}^k \to \mathbb{R}$ be a continuous positive definite function. Define, $\forall t \in \mathbb{R}$, $U(t) = U \circ Y(t)$ and $E(t) = E \circ Y(t)$. If, $\forall t \in \mathbb{R}$, $D^+U(t) \leq -E(t)$, then, $\forall t \in \mathbb{R}$, $U(t)$ is strictly decreasing.*

Due to space constraints, the proof of Lemma 4 is not provided in this paper.

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