CAUSAL SIGNAL RECOVERY FROM U-INVARIANT SAMPLES

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ABSTRACT

Causal processing of a signal's samples is crucial in on-line applications such as audio rate conversion, compression, tracking and more. This paper addresses the problem of causally reconstructing continuous-time signals from their samples. We treat a rich variety of sampling mechanisms encountered in practice, namely in which each sampling function is obtained by applying a unitary operator on its predecessor. Examples include pointwise sampling at the output of an anti-aliasing filter and magnetic resonance imaging, which correspond respectively to the translation and modulation operators. Such sequences of functions were studied extensively in the context of stationary random processes. We thus utilize powerful tools from this discipline, to derive a causal interpolation method that best approximates the commonly used non-causal reconstruction formula.

Index Terms— Causality, sampling, stationary sequences.

1. INTRODUCTION

Sampling and reconstruction of signals play a crucial role in signal processing and communications. During the last several decades, sampling theory has enjoyed rapid development [1, 2] due in part to fruitful fertilizations from other disciplines, such as wavelet theory, approximation theory and optimization. While these recent developments found widespread use in image processing, their deployment in unidimensional applications, such as audio sampling-rate conversion, is less common. One reason for this seems to be the relatively few studies treating causality constraints within the above frameworks, which becomes crucial in on-line applications.

Causal recovery of signals from their samples was mainly addressed in the context of spline interpolation and uniform pointwise sampling. Several heuristic methods were developed and analyzed in [3, 4, 5] for modifying the non-causal pre-filter, which is at the heart of cubic spline interpolation, into a causal counterpart. In [6], an H^∞ optimization approach was proposed for approximating the non-causal pre-filter by a causal one. Causal interpolation was also studied in [7] from an approximation-theory perspective.

In modern sampling theory, sampling of a signal x is often described by an evaluation of inner products $c_n = \langle x, s_n \rangle$ with a set of sampling functions $\{s_n\}$. In this paper, we study causal sampling problems with a special type of structure of the sampling functions, which we term U-*invariance*. Specifically, we concentrate on situations in which the sampling functions are obtained from a single generator s as

$$s_0 = s \quad \text{and} \quad s_{n+1} = \mathrm{U} s_n \,, \qquad n \in \mathbb{Z} \tag{1}$$

where U is some unitary operator. Examples include the translation, modulation, and dilatation operators, which are used in classical shift-invariant (SI) sampling problems, in magnetic resonance imaging (MRI), in Gabor analysis, and in wavelet analysis. The special case in which U is a translation operator has been studied extensively in the sampling literature and corresponds to uniform sampling at the output of an anti-aliasing filter. For this scenario, a wide variety of non-causal recovery techniques have been developed. Non-causal recovery with an arbitrary U was studied in [8].

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2. MATHEMATICAL BACKGROUND

The complex conjugate of a scalar $a \in \mathbb{C}$ is denoted $\overline{a}$. Throughout the paper, $\mathcal{H}$ stands for a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The closed linear span of a set of vectors $\{s_n\}$ in $\mathcal{H}$ is written $\overline{\mathrm{span}}\{s_n\}$. The Hilbert space of complex square integrable functions on the real axis $\mathbb{R}$ is denoted by $L^2(\mathbb{R})$. The Fourier transform of $x \in L^2(\mathbb{R})$ is defined by

$$\widehat{x}(\omega) = \int_{\mathbb{R}} x(t)\, \mathrm{e}^{-\mathrm{i}\omega t}\, \mathrm{d}t \,, \quad \omega \in \mathbb{R}$$

and the discrete-time Fourier transform (DTFT) of a sequence c_n in ℓ^2 is defined by

$$\widehat{c}(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{-\mathrm{i}\theta n} \,, \quad \theta \in [-\pi, \pi] \,.$$

Moreover, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle in the complex plane $\mathbb{C}$, and for $1 \leq p \leq \infty$ the common spaces of Lebesgue integrable functions on $\mathbb{T}$ are denoted by $L^p(\mathbb{T})$.

Most of our development is based on the theory of stationary sequences (see, e.g., [9]). A sequence $\boldsymbol{s} = \{s_n\}_{n\in\mathbb{Z}}$ of vectors in $\mathcal{H}$ is called *stationary* if $\langle s_{m+k}, s_{n+k} \rangle = \langle s_m, s_n \rangle$ for all $m, n, k \in \mathbb{Z}$, i.e. if the inner product $\langle s_m, s_n \rangle$ depends only on the difference $m - n$. The sequence $r_{\boldsymbol{s}}(n) = \langle s_n, s_0 \rangle$, $n \in \mathbb{Z}$, is called the *covariance function* of $\boldsymbol{s}$ and its DTFT $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})$ is called the *spectral density* of $\boldsymbol{s}$. A sequence $\boldsymbol{s}$ is stationary if and only if there exists a unitary operator U on $\mathcal{H}$ and vector $s \in \mathcal{H}$, uniquely determined by $\boldsymbol{s}$, such that $s_n = \mathrm{U}^n s$ for every $n \in \mathbb{Z}$. We therefore say that $\boldsymbol{s}$ is generated by (U, s).

Two stationary sequences $\boldsymbol{s}$ and $\boldsymbol{w}$ are said to be *stationary correlated* if $\langle s_{m+k}, w_{n+k} \rangle = \langle s_m, w_n \rangle$ for all $m, n, k \in \mathbb{Z}$. In this case, the sequence $r_{\boldsymbol{s},\boldsymbol{w}}(n) = \langle s_n, w_0 \rangle$, $n \in \mathbb{Z}$, is called the *cross-covariance* function of $\boldsymbol{s}$ and $\boldsymbol{w}$ and its DTFT $\Phi_{\boldsymbol{s},\boldsymbol{w}}(\mathrm{e}^{\mathrm{i}\theta})$ is referred

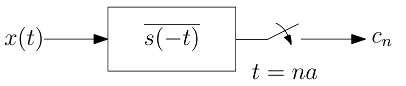

Fig. 1. Shift-invariant sampling with sampling period a.

to as the *cross-spectral density*. Two sequences $\boldsymbol{s}$ and $\boldsymbol{w}$ are stationary correlated if and only if they are generated by the same unitary operator U.

3. U-INVARIANT SAMPLING AND RECOVERY

A simple model for a practical sampling device comprises a prefilter $\overline{s(-t)}$ followed by an ideal sampler [10], as depicted in Fig. 1. Assuming that the signal x and the function s are both in $L^2(\mathbb{R})$, the nth sample is given by

$$c_n = \int_{\mathbb{R}} \overline{s(t-na)}\, x(t)\, \mathrm{d}t = \langle x, s_n \rangle \ ,$$

where $s_n(t) := s(t-na)$ is the nth *sampling function*. The closed linear span of the sampling functions $\{s_n\}_{n\in\mathbb{Z}}$ is termed the *sampling space* $\mathcal{S}$, which is, in this case, a shift-invariant (SI) space.

SI sampling can be considered a special case of a more general acquisition paradigm, where a signal x in an arbitrary Hilbert space $\mathcal{H}$ is measured by a sequence of sampling vectors $\{s_n\}_{n\in\mathbb{Z}}$ in $\mathcal{H}$ that are generated by successive application of some unitary operator U as in (1). Clearly, in the SI case U is the *translation operator* $\mathrm{T}_a : s(t) \mapsto s(t-a)$. Other frequently employed operators include *modulation* $\mathrm{M}_a \ : \ s(t) \mapsto s(t)\,\mathrm{e}^{\mathrm{i}at}$, which is used in MRI and in Gabor analysis and *dilatation* $\mathrm{D}_a : s(t) \mapsto |a|^{-1/2} s(t/|a|)$, which is used in wavelet analysis. We term this general setting U-*invariant sampling*. One may readily notice that the sampling functions associated with a U-invariant sampling device form a stationary sequence, as defined in Section 2.

A common task in signal processing is that of recovering a continuous-time signal $x \in \mathcal{H}$ from a sequence of its generalized samples $\boldsymbol{c} = \{c_n\}_{n\in\mathbb{Z}}$ taken with a U-invariant sampling device. In the SI setting the recovery problem is typically approached by employing a series of the form

$$\widetilde{x}(t) = \sum_{n\in\mathbb{Z}} d_n v(t-na) = \sum_{n\in\mathbb{Z}} d_n (\mathrm{T}_a^n v)(t), \qquad (2)$$

with some predefined reconstruction kernel $v(t)$ [1]. The expansion coefficients $\boldsymbol{d} = \{d_n\}_{n\in\mathbb{Z}}$ are often chosen such that the recovery $\widetilde{x}(t)$ is consistent with the measured samples in the sense that $\langle \widetilde{x}, s_n \rangle = c_n$ for all $n \in \mathbb{Z}$ [10].

In the more general setting of U-invariant sampling it is reasonable to replace the SI reconstruction formula (2) by its U-invariant generalization

$$\widetilde{x}(t) = \sum_{n\in\mathbb{Z}} d_n\, (\mathrm{U}^n v)(t)\,. \qquad (3)$$

The sequences $\boldsymbol{s} = \{\mathrm{U}^n s\}_{n\in\mathbb{Z}}$ and $\boldsymbol{v} = \{\mathrm{U}^n v\}_{n\in\mathbb{Z}}$ are stationary correlated in this case. In [8] it was shown that *consistent recovery* in this setting is unique if the cross-spectral density $\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{\mathrm{i}\theta})$ associated with $\boldsymbol{s}$ and $\boldsymbol{v}$ satisfies $|\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{\mathrm{i}\theta})| \geq A > 0$ for all $\theta \in [-\pi, \pi)$. In this case, d_n is obtained by filtering the samples with a linear filter whose transfer function is

$$\widehat{h}_{\mathrm{nc}}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\overline{\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{-\mathrm{i}\theta})}} \ . \qquad (4)$$

The overall recovery scheme, therefore, comprises a digital correction filter followed by a digital-to-analog reconstruction stage, as shown in Fig. 2 for the special case of SI sampling.

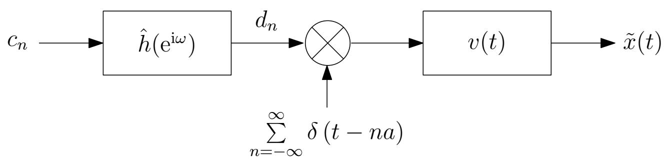

Fig. 2. Reconstruction after digital correction.

4. CAUSAL RECOVERY

A major drawback of the consistency approach is that the correction filter (4) is generally non-causal. Our goal is to design a filter $h(k)$ whose impulse response vanishes for every $k \leq n$, for some $n \in \mathbb{Z}$. More concretely, we would like to approximate the coefficient d_n of the consistent recovery by a coefficient $\widetilde{d}_n$, which is determined only by the past samples $\{c_k\}_{k\leq 0}$.

The coefficient d_n can be written as

$$\begin{aligned} d_n &= \sum_{k\in\mathbb{Z}} c_k\, h_{\mathrm{nc}}(n-k) \\ &= \sum_{k\in\mathbb{Z}} \langle x, s_k \rangle\, h_{\mathrm{nc}}(n-k) \\ &= \Big\langle x, \sum_{k\in\mathbb{Z}} \overline{h_{\mathrm{nc}}(n-k)} s_k \Big\rangle \\ &= \langle x, w_n \rangle, \end{aligned}$$

where $w_n = \mathrm{U}^n w$ with w given by

$$w = \sum_{k\in\mathbb{Z}} \overline{h_{\mathrm{nc}}(-k)}\, s_k \ . \qquad (5)$$

This representation allows a simple interpretation of our task. Given the past samples $c_k = \langle x, s_k \rangle$, $k \leq 0$, corresponding to a stationary sequence generated by (U, s), we would like to produce an estimate $\widetilde{d}_n$ of the generalized sample $d_n = \langle x, w_n \rangle$ for some $n \in \mathbb{Z}$, corresponding to the stationary sequence $\boldsymbol{w} = \{w_k\}_{k\in\mathbb{Z}}$ generated by (U, w) such that the error

$$|d_n - \widetilde{d}_n| = |\langle x, w_n \rangle - \widetilde{d}_n| \qquad (6)$$

is minimized.

4.1. Causal Estimation of Generalized Samples

Unfortunately, (6) depends on x, which is unknown. To eliminate this dependency, we instead seek an estimate $\widetilde{d}_n$ resulting in minimal worst-case error over the set

$$\mathcal{B} = \{x \in \mathcal{H} : \ \|x\| \leq L,\ \langle x, s_n \rangle = c_n \ \forall n \leq 0\}$$

of signals that could have generated the observed sequence of past samples. Here L is an arbitrary bound, which is placed merely to ensure that the error cannot grow indefinitely. As we will see, its value does not affect the solution.

Proposition 1: *Let $\boldsymbol{s}$ be a sequence in a Hilbert space $\mathcal{H}$. Then the unique solution of the problem*

$$\arg\min_{\widetilde{d}_n} \max_{x\in\mathcal{B}} |\langle x, w_n \rangle - \widetilde{d}_n| \qquad (7)$$

is given by

$$\widetilde{d}_n = \langle x, \mathrm{P}_{\mathcal{S}_0} w_n \rangle \ , \tag{8}$$

where $\mathrm{P}_{\mathcal{S}_0} w_n$ *is the orthogonal projection of* w_n *onto the past sampling space* $\mathcal{S}_0 := \overline{\mathrm{span}}\{s_k : k \leq 0\}$.

A proof for the case in which the past sampling functions $\{s_n\}_{n\leq 0}$ form a frame appears in [11]. A proof for the general case can be found in [12].

Note that (8) does not depend on the bound L. However, it seems to depend on x, which is unknown. Nevertheless, $\widetilde{d}_n$ of (8) can be written explicitly as a linear combination of the past samples, which are given. Indeed, by the definition of $\mathcal{S}_0$, we can write

$$\mathrm{P}_{\mathcal{S}_0} w_n = \textstyle\sum_{k=0}^{\infty} \gamma_k \, s_{-k} \tag{9}$$

for some sequence of coefficients $\boldsymbol{\gamma} = \{\gamma_k\}_{k=0}^{\infty}$ (which depends on n). Consequently,

$$\widetilde{d}_n = \left\langle x, \textstyle\sum_{k=0}^{\infty} \gamma_k \, s_{-k} \right\rangle = \textstyle\sum_{k=0}^{\infty} \overline{\gamma}_k \, c_{-k} \ . \tag{10}$$

Interestingly, the min-max estimator (10) is linear in the past samples, although we did not restrict ourselves to linear schemes in (7).

Since the orthogonal projection is self-adjoint, one obtains from (8) that $\widetilde{d}_n = \langle \mathrm{P}_{\mathcal{S}_0} x, w_n \rangle$. This allows the interpretation that the optimal estimate $\widetilde{d}_n$ is obtained by first approximating the signal x by its orthogonal projection onto the past sampling space $\mathcal{S}_0$, and then sampling this approximation with the sampling function w_n.

For general sequences $\boldsymbol{s}$ and $\boldsymbol{w}$, it might be complicated to obtain an explicit expression for the coefficients $\{\gamma_k\}_{k=0}^{\infty}$. However, if $\boldsymbol{s}$ and $\boldsymbol{w}$ are stationary correlated, as is the case in our setting, then the solution is well known from the theory of stationary stochastic sequences. Specifically, assume for simplicity that $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) \geq A > 0$ for almost all $\theta \in [-\pi, \pi)$. Then, the coefficients γ_k of (9) are given by the inverse DTFT of the causal Wiener filter (see e.g. [9])

$$\Gamma_n(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})} \left\{ \frac{\Phi_{\boldsymbol{w},\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}\theta n}}{\overline{\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})}} \right\}_{+} \ . \tag{11}$$

Here, $\Phi_{\boldsymbol{s}}^{+} \in L^2(\mathbb{T})$ denotes the spectral factor of the density $\Phi_{\boldsymbol{s}}$, which is the function satisfying $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = |\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})|^2$ and whose inverse DTFT is a causal stable sequence. The operator $\{\cdot\}_{+} : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ sets the past coefficients of its argument to zero in the time domain, i.e

$$\left\{\textstyle\sum_{k=-\infty}^{\infty} \alpha_k \mathrm{e}^{-\mathrm{i}\theta k}\right\}_{+} = \textstyle\sum_{k=0}^{\infty} \alpha_k \mathrm{e}^{-\mathrm{i}\theta k} \ .$$

To summarize, (10) and (11) imply that the optimal estimator of d_n is obtained by feeding the samples c_n into the filter $\overline{\Gamma_n(\mathrm{e}^{-\mathrm{i}\theta})}$ corresponding to (11).

4.2. Application to Causal Recovery

We now utilize (11) to approximate the noncausal consistent recovery method using a causal digital correction filter. To do that, we need to express $\Phi_{\boldsymbol{w},\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})$ in terms of the kernels s and v. Substituting (5), $r_{\boldsymbol{w},\boldsymbol{s}}(n)$ becomes

$$\begin{aligned} r_{\boldsymbol{w},\boldsymbol{s}}(n) &= \langle w_n, s_0 \rangle \\ &= \Big\langle \sum_{k\in\mathbb{Z}} \overline{h_{\mathrm{nc}}(n-k)}\, s_k, s_0 \Big\rangle \\ &= \sum_{k\in\mathbb{Z}} \overline{h_{\mathrm{nc}}(n-k)}\, r_{\boldsymbol{s}}(k) \ . \end{aligned}$$

In the Fourier domain, this relation is given by

$$\Phi_{\boldsymbol{w},\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = \overline{\widehat{h}_{\mathrm{nc}}(\mathrm{e}^{-\mathrm{i}\theta})}\, \Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = \frac{\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta})}{\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{\mathrm{i}\theta})} \tag{12}$$

where we used (4). Substituting (12) into (11), and using the fact that $\Phi_{\boldsymbol{s}}(\mathrm{e}^{\mathrm{i}\theta}) = |\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})|^2$, we conclude that the min-max causal correction filter is given by $\widehat{h}_{\mathrm{mx}}(\mathrm{e}^{\mathrm{i}\theta}) = \overline{H_n(\mathrm{e}^{-\mathrm{i}\theta})}$, where

$$H_n(\mathrm{e}^{\mathrm{i}\theta}) = \frac{1}{\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})} \left\{ \frac{\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{e}^{\mathrm{i}\theta n}}{\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{\mathrm{i}\theta})} \right\}_{+} \ . \tag{13}$$

This filter, in general, does not lead to a consistent recovery. However, among all causal filters, the expansion coefficients it produces are closest to those of the consistency approach for the worst-case signal x.

An interesting phenomenon occurs when the sampling functions $\{s_n\}$ are orthogonal. This happens, for example, in the SI setting, when $s(t)$ is the rectangular kernel $\mathrm{rect}(t/a)$ or the ideal low-pass filter $\mathrm{sinc}(t/a)$. In this situation, $\Phi_{\boldsymbol{s}}^{+}(\mathrm{e}^{\mathrm{i}\theta}) = A$ for some constant $A \neq 0$ and thus $H_n(\mathrm{e}^{\mathrm{i}\theta})$ reduces to $\{\mathrm{e}^{\mathrm{i}\theta n}/\Phi_{\boldsymbol{s},\boldsymbol{v}}(\mathrm{e}^{\mathrm{i}\theta})\}_{+}$, which corresponds to a simple truncation of the impulse response of the noncausal filter $\widehat{h}_{\mathrm{mx}}(\mathrm{e}^{\mathrm{i}\theta})$ of (4). However, when $\{s_n\}$ are not orthogonal, simple truncation is in general no longer optimal.

5. EXAMPLE: CAUSAL SPLINE INTERPOLATION

To demonstrate the causal recovery technique discussed above, we next apply it to causal spline reconstruction. A spline $x(t)$ of degree N is a piecewise polynomial with the pieces combined at knots, such that the function is continuously differentiable $N-1$ times. It can be shown that any spline of degree N with knots at the integers can be generated using (2), where $v(t) = \beta^N(t)$ is the B-spline function of degree N, defined by

$$\beta^N(t) = (\beta^{N-1} * \beta^0)(t) \tag{14}$$

and $\beta^0(t)$ is the unit square

$$\beta^0(t) = \begin{cases} 1 & -\frac{1}{2} < t < \frac{1}{2}; \\ 0 & \text{otherwise,} \end{cases}$$

In other words, $\beta^N(t)$ is obtained by the $(N+1)$-fold convolution of $\beta^0(t)$.

As a simple example, consider the SI setting of Fig. 1 with a sampling period of $a = 1$ and with the sampling filter $s(t) = \beta^1(t)$. The frequency response of $\beta^1(t)$ is given by $\widehat{\beta}^1(\omega) = \mathrm{sinc}^2(\omega/(2\pi))$. Therefore, $s(t)$ can be considered a nonideal anti-aliasing low-pass filter whose cutoff frequency is slightly smaller than the sampling rate $\omega = \pi$, as shown in Fig. 3. We would like to approximate the signal $x(t)$ using (2) with[1] $v(t) = \beta^0(t)$, where the coefficients d_n are obtained by causal processing of the samples c_n.

Using the convolution property (14) of B-splines, we have

$$r_{\boldsymbol{s},\boldsymbol{v}}(n) = \langle s_n, v_0 \rangle = \textstyle\int_{\mathbb{R}} \beta^1(t-n)\, \beta^0(t)\, \mathrm{d}t = \beta^2(n).$$

As shown in [13], the Z-transform of $\beta^2(n)$ is given by $(1 - z_2 z^{-1})(1 - z_2 z)/(-8 z_2)$, where $z_2 = \sqrt{8} - 3$. Therefore, the noncausal consistent filter (4) is given by

$$\widehat{h}_{\mathrm{nc}}(z) = \frac{1}{\Phi_{\boldsymbol{s},\boldsymbol{v}}(z)} = \frac{-8 z_2}{(1 - z_2 z^{-1})\,(1 - z_2 z)}, \tag{15}$$

[1] Reconstruction with $\beta^0(t)$ is a shifted-by-1/2 version of zero-order hold recovery, also known as nearest neighbor interpolation in image processing.

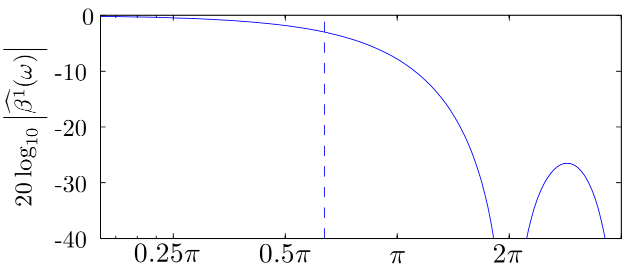

Fig. 3. Frequency response of the filter $\beta^1(t)$.

and corresponds to the impulse response

$$h_{\mathrm{nc}}(n) = \frac{-8z_2}{1 - z_2^2}\, z_2^{|n|}.$$

A naive solution for obtaining a causal correction filter follows from truncating the impulse response $h_{\mathrm{nc}}(n)$ [3, 4, 5]. This approach results in a filter whose Z-transform is

$$\widehat{h}_{\mathrm{tr}}(z) = \frac{-8z_2}{1 - z_2^2}\, \frac{1}{1 - z_2 z^{-1}}. \tag{16}$$

Our framework, however, dictates a different strategy. Specifically, we have that $\Phi_{\boldsymbol{s}}^{+}(z) = a_1(1 - z_1 z^{-1})$, where $z_1 = \sqrt{3} - 2$ and a_1 is some constant [13]. Substituting this expression into (13) leads to

$$\begin{aligned}\widehat{h}_{\mathrm{mx}}(z) &= \frac{-8z_2}{1 - z_1 z^{-1}} \left\{ \frac{1 - z_1 z^{-1}}{(1 - z_2 z^{-1})\,(1 - z_2 z)} \right\}_{+} \\ &= \frac{-8z_2}{1 - z_2^2}\, \frac{(1 - z_1 z_2) - z_1\left(1 - z_2^2\right) z^{-1}}{1 - (z_1 + z_2)\, z^{-1} + z_1 z_2 z^{-2}}. \end{aligned} \tag{17}$$

Figure 4 compares the above methods in the task of recovering a randomly-generated spline $x(t)$ of degree 2. In this example, the noncausal solution (15) attains a signal-to-noise ratio (SNR) of $20\log_{10}(\|x\|/\|x - \widetilde{x}_{\mathrm{nc}}\|) = 8.67$dB. The truncated filter (16) suffers from a significant degradation, attaining an SNR of 7.59dB. By contrast, our min-max solution (17) results in an SNR of 8.21dB, which is only slightly worse than the noncausal approach. Note that the SNR was computed for a long time segment, only a small portion of which is shown in Fig. 4.

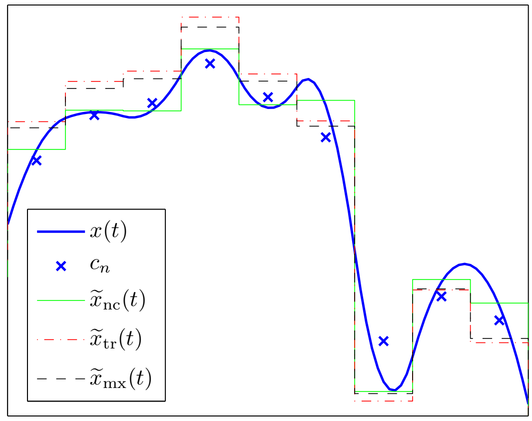

Fig. 4. Reconstruction of $x(t)$ from its samples c_n using the noncausal filter $\widehat{h}_{\mathrm{nc}}(z)$ of (15), the truncated filter $\widehat{h}_{\mathrm{tr}}(z)$ of (16) and the min-max optimal causal filter $\widehat{h}_{\mathrm{c}}(z)$ of (17).

6. CONCLUSIONS AND EXTENSIONS

In this paper we explored the use of the theory of stationary stochastic sequences to solve the problem of causal recovery of deterministic signals from their U-invariant samples. Thereby we focused on the consistency approach of signal recovery. However, the proposed methodology may be applied to other reconstruction techniques, as well. Moreover, the same framework can be harnessed to predict future samples of a signal based on its past or causally interpolating missing samples. Furthermore, well-known results from the theory of stochastic processes can be used to characterize those cases in which the proposed causal recovery technique yields zero estimation or prediction error for every possible signal [12].

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