Irregular Repetition Slotted ALOHA Over the Binary Adder Channel

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***Abstract*—We propose an irregular repetition slotted ALOHA (IRSA) based random-access protocol for the binary adder channel (BAC). The BAC captures important physical-layer concepts, such as packet generation, per-slot decoding, and information rate, which are neglected in the commonly considered collision channel model. We divide a frame into slots and let users generate a packet, to be transmitted over a slot, from a given codebook. In a state-of-the-art scheme proposed by Paolini *et al.* (2022), the codebook is constructed as the parity-check matrix of a BCH code. Here, we construct the codebook from independent and identically distributed binary symbols to obtain a random-coding achievability bound. Our per-slot decoder progressively discards incompatible codewords from a list of candidate codewords, and can be improved by shrinking this list across iterations. In a regime of practical interests, our scheme can resolve more colliding users in a slot and thus achieves a higher average sum rate than the scheme in Paolini *et al.* (2022).**

I. Introduction

In recent years, modern random-access protocols [1] have been developed based on packet-based coding at the users and successive interference cancellation (SIC) decoding at the receiver. In particular, irregular repetition slotted ALOHA (IRSA) [2] employs packet repetitions: active users transmit multiple copies of their packets in randomly chosen slots of a fixed-length frame. This creates time diversity and boosts the throughput with respect to slotted ALOHA. Modern random-access protocols have been adopted in commercial satellite communication systems, e.g., DVB-RCS2 [3]. They provide a practical mean to achieve uncoordinated medium access in internet-of-things (IoT) applications [4]. Under an "unsourced" model where all users use the same codebook and the receiver aims to return an unordered list of messages, slotted ALOHA and IRSA exhibit a large energy-efficiency gap to random-coding achievability bounds [5], [6]. This gap can be reduced by combining IRSA with more advanced multi-user code constructions and multi-user joint decoding; see, e.g., [7], [8].

Most works on IRSA assume a collision channel where successful decoding is achieved only in slots containing a single packet. This model enables a direct connection between SIC decoding of IRSA and graph-based iterative erasure decoding of low-density parity-check (LDPC) codes. This yields in turn a tractable analysis of the packet loss rate (PLR) and leads to the definition of a decoding threshold, i.e., the maximum average number of active users per slot such that vanishing PLR can be achieved as the number of slots per frame goes to infinity. However, this simple model does not capture some physical-layer concepts such as how a packet is generated, how packet decoding is performed in a slot, and the information rate that can be achieved with vanishing PLR.

The binary adder channel (BAC), where users transmit binary symbols and the receiver observes the noiseless sum of these symbols, provides a more realistic setting to analyze IRSA than the collision channel. The BAC resembles the interference-limited regime where noise is negligible, and captures the aforementioned physical-layer concepts. Insights on designing codes for the two-user unsourced BAC were reported in [9]. In [10], Bar-David, Plotnik, and Rom proposed a coding scheme, which we shall refer to as BPR, capable of correctly decoding T colliding codewords with T known *a priori*. The codebook therein is constructed as the parity-check matrix of a T-error-correcting binary Bose–Chaudhuri–Hocquenghem (BCH) code. In [11], Paolini *et al.* considered IRSA over the BAC and let users generate packets from the BPR codebook. A unit pilot symbol, prepended to each codeword, allows the receiver to infer the number of active users in a slot. The per-slot decoder can resolve collisions of up to T users. The decoding threshold of this scheme follows from that of IRSA with multi-packet reception (MPR) [12].

In this paper, we propose a new IRSA-based scheme for the BAC. As in [11], a unit pilot symbol is prepended to each codeword. However, different from [11], we generate the remaining symbols in an independent and identically distributed (i.i.d.) manner from a Bernoulli distribution. This results in a random-coding achievability bound. Our per-slot decoder progressively discards incompatible codewords from a candidate list. This leads to better MPR capability and higher asymptotic achievable average sum rate than the scheme in [11] in the regime where the number of bits per user increases slowly with the number of users and the total number of bits normalized by the slot length is close to 1. Our decoder can be further improved by shrinking the candidate list across iterations.

Notation: Lowercase boldface letters denote vectors. We denote random quantities with nonitalic sans-serif letters, e.g., a scalar x and a vector $\mathbf{\mathsf{x}}$, and deterministic quantities with italic letters, e.g., a scalar x and a vector $\boldsymbol{x}$. Calligraphic uppercase letters denote sets, $[m:n]$ stands for the set $\{m, m+1, \dots, n\}$, $[n] = [1:n]$, and $\mathbb{1}\{\cdot\}$ is the indicator function. The Binomial

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distribution with parameters (n, p) is denoted by $\mathrm{Bino}(n, p)$.

II. System Model

A. Random-Access Over the Binary Adder Channel

We consider a scenario with K users, each active with probability μ. Hence, the number of active users, $\mathsf{K}_{\rm a}$, follows a $\mathrm{Bino}(K, \mu)$ distribution. The active users send encoded messages to a receiver over n uses of a stationary memoryless BAC. Let $\mathbf{x}_k$ be the length-n binary coded sequence transmitted by user k. The received signal is the noiseless sum of the transmitted sequences, i.e., $\mathbf{y} = \sum_{k=1}^{\mathsf{K}_{\rm a}} \mathbf{x}_k$. The receiver does not know $\mathsf{K}_{\rm a}$ *a priori* and aims to recover all transmitted messages.

B. Irregular Repetition Slotted ALOHA

In IRSA, each length-n frame is divided into N slots and the users are frame- and slot-synchronous. A message is encoded in a packet (also called codeword) transmitted over a slot. An active user in a frame sends L identical replicas of its packet in L slots chosen uniformly without replacement. Here, L is called the degree of the transmitted packet or the transmitting user. It follows a probability distribution $\{\Lambda_L\}$ with $\Lambda_L = \mathbb{P}[\mathsf{L} = L]$, which we write using a polynomial notation as $\Lambda(x) = \sum_L \Lambda_L x^L$. IRSA was first proposed for the collision channel, in which slots containing a single packet (called singleton slots) always lead to successful decoding, whereas slots containing multiple packets lead to decoding failures [2]. For a packet $\boldsymbol{c}$, both the users and the receiver employ a common function $h(\boldsymbol{c})$ to generate the position of the replicas. See [11, App.] for details. The receiver employs a SIC decoder that operates as follows. It seeks a singleton slot, decodes the packet therein, then locates and removes its replicas. These steps are repeated until no more singleton slots can be found.

Consider a bipartite graph where variable nodes (VNs) represent users, check nodes (CNs) represent slots, and VN k is connected to CN s by an edge if user k transmits in slot s. Let λ_L be the probability that an edge is connected to a degree-L VN. We also use the polynomial notation $\lambda(x) = \sum_L \lambda_L x^{L-1}$. It holds that $\lambda(x) = \frac{\Lambda'(x)}{\Lambda'(1)}$ [13], where $\Lambda'(x)$ is the first-order derivative of $\Lambda(x)$. SIC decoding can be interpreted as erasure decoding on this bipartite graph: an edge connected to a VN is revealed if at least another edge connected to the same VN has been revealed; an edge connected to a CN is revealed if all other edges connected to the same CN have been revealed.

III. An IRSA-Based Scheme for the BAC

For the BAC, one can divide a length-n frame into N slots (assuming that n divides N) and directly apply the original IRSA protocol. Define $G = \mathbb{E}[\mathsf{K}_{\rm a}]/N = \mu K/N$ as the channel load. According to [2, Sec. IV], as $K \to \infty$ and $N \to \infty$ while G is fixed, the PLR drops at a certain threshold value as the channel load G increases. That is, all but a vanishing fraction of the transmitted messages are successfully decoded if the channel load is below a decoding threshold. The decoding threshold $G^*(\Lambda)$ is the maximum value of G such that $p > \lambda(1 - e^{-pG\Lambda'(1)}), \forall p \in (0, 1]$. We next propose a more advanced IRSA-based scheme based on random coding and MPR.

A. Encoder

Let $n_0 + 1 = n/N$ be the slot length. We generate a codebook $\{\boldsymbol{c}_m\}_{m=1}^M$ ($M \le 2^{n_0}$) where each codeword contains a unit pilot symbol followed by n_0 i.i.d. binary symbols, each being 1 with probability ν. (The choice of ν will be discussed in Remark 1.) An active user selects a codeword and transmits L copies, with $\mathsf{L} \sim \Lambda(x)$, of this codeword in L slots chosen from the N available slots. We next propose two SIC decoders.

B. Erasure Decoding With Multi-Packet Reception (ED-MPR)

This decoder can resolve collisions in nonsingleton slots. In each iteration, it goes through all slots. Consider slot s. The number of active users in this slot, K_s, is called the slot degree. In the first channel use, the channel output is identical to K_s, thus the receiver estimates perfectly K_s. The receiver forms a list $\mathcal{S}_s$ of candidate codewords containing all codewords that could have been transmitted in the slot.[1] That is, $\mathcal{S}_s = \{i \in [M] \colon s \in h(\boldsymbol{c}_i)\}$. Next, the receiver attempts to resolve the slot by using a per-slot decoder with MPR capability, which we describe in the next paragraph. If the slot is resolved, i.e., all transmitted codewords are decoded, the receiver removes the replicas of these codewords. These per-slot-decoding and interference-cancellation steps are repeated until no more resolvable slots can be found.

Per-Slot Decoder: If $\mathsf{K}_s > 0$, the receiver goes through the remaining channel uses with index $j \in [2 : n_0 + 1]$ of the slot and progressively discards incompatible codewords from the candidate list $\mathcal{S}_s$. Let $y(j)$ be the received signal in the jth channel use of the slot and $c(j)$ be the jth entry of codeword $\boldsymbol{c}$. For each channel use j, if $y(j) = \mathsf{K}_s$, the decoder discards from $\mathcal{S}_s$ all codewords $\boldsymbol{c}$ with $c(j) = 0$; similarly, if $y(j) = 0$, the decoder discards from $\mathcal{S}_s$ all codewords $\boldsymbol{c}$ with $c(j) = 1$. We use $\log_2 K$ bits of each codeword to represent the user's signature,[2] so that different users transmit different codewords. Therefore, after the discarding process, if K_s codewords survive, they must be the transmitted codewords and the slot is resolved. An illustration of this per-slot decoder is given in Fig. 1. Its MPR capability is analyzed in the following theorem.

[1]The candidate lists $\{\mathcal{S}_s\}_{s\in[N]}$ can be computed once and then stored at the receiver. Storing $\{\mathcal{S}_s\}_{s\in[N]}$, however, requires a large memory if M is large. Alternatively, the bipartite graph, and thus $\{\mathcal{S}_s\}_{s\in[N]}$, can be constructed by the receiver and communicated to the users as assumed in [14]. This is possible in systems where users go through an access procedure to join the network, after which the user population is known to the receiver. In this case, the receiver can impose a graph structure that allows for efficient storage.

[2]In the unsourced model, the user identification task is deferred to upper layer protocols that spend at least $\log_2 K$ bits in the overhead of each packet.

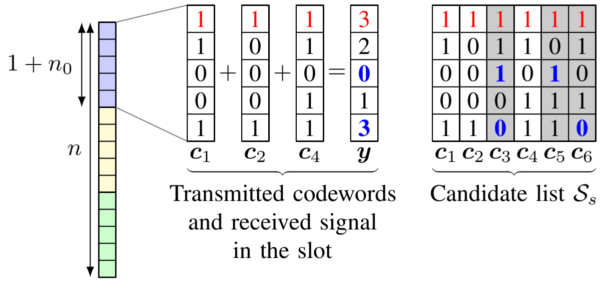


Fig. 1: An example of the transmission and decoding in a slot s with $n_0 = 4$, $\mathsf{K}_s = 3$, and $\mathcal{S}_s = \{\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \boldsymbol{c}_4, \boldsymbol{c}_5, \boldsymbol{c}_6\}$. The decoder reads K_s from the first (top) entry of the received signal $\boldsymbol{y}$. It discards the codewords $\boldsymbol{c}_3, \boldsymbol{c}_5$, and $\boldsymbol{c}_6$ from the candidate list $\mathcal{S}_s$ since they are incompatible with the third and/or the fifth entries of $\boldsymbol{y}$. Only $\mathsf{K}_s = 3$ codewords remain in $\mathcal{S}_s$. Thus, the slot is resolved.

Theorem 1 (MPR capability): The proposed per-slot decoder can resolve a degree-U slot with probability

$$\pi_U = \mathbb{E}\left[\left(1 - \frac{\Lambda'(1)}{N}\nu^{\mathsf{A}_U}(1-\nu)^{\mathsf{A}_0}\right)^{M-U}\right] \quad (1)$$

where A_0 and A_U are random nonnegative integers with joint probability mass function (PMF) given by

$$\mathbb{P}[\mathsf{A}_0 = A_0, \mathsf{A}_U = A_U] = \frac{n_0!}{A_0! A_U! (n_0 - A_0 - A_U)!} \cdot (1-\nu)^{U A_0} \nu^{U A_U} (1 - (1-\nu)^U - \nu^U)^{n_0 - A_0 - A_U} \quad (2)$$

for $A_0 + A_U \le n_0$. In particular, if $\nu = 0.5$, it holds that

$$\pi_U = \mathbb{E}\left[\left(1 - \frac{\Lambda'(1)}{N} 2^{-\mathsf{A}}\right)^{M-U}\right] \quad (3)$$

$$\ge \pi'_U \triangleq \left(1 - \frac{\Lambda'(1)}{N}(1 - 2^{-U})^{n_0}\right)^{M-U}. \quad (4)$$

with $\mathsf{A} \sim \mathrm{Bino}(n_0, 2^{1-U})$.

Proof: See Appendix A. ■

Remark 1: Due to symmetry, assume that $\nu \le 0.5$. Compared to $\nu = 0.5$, setting $\nu < 0.5$ increases the chance that all active users transmit 0 and thus creates more discarding instances. This increases π_U for large U. However, setting $\nu < 0.5$ also increases the probability $1 - \nu$ that a nontransmitted codeword is compatible with the channel output in a channel use where the channel output is 0. This reduces π_U if U is small. We shall illustrate this trade-off in Section V.

We are interested in the asymptotic regime where M, K, N, and n_0 all go to infinity and satisfy: i) $\mu K/N = G$ where G is the channel load as in conventional analyses of IRSA, ii) $n_0 = \beta \log_2 M$ where $\beta^{-1} = \frac{\log_2 M}{n_0} \le 1$ is the total number of bits transmitted per channel use in a slot, and iii) $M = DK^{1/\delta}$, $\delta \in (0,1)$, so that the per-user payload $\log_2(M/K)$ is split into a fixed amount of $\log_2 D$ bits and an increasing amount of $(1/\delta - 1)\log_2 K$ bits as K grows. We assume that the per-user payload increases with K because, if the users transmit a fixed number of bits over an infinite number of channel uses n_0, they achieve zero PLR but also zero information rate $\log_2(M/K)/n_0$. We refer to the defined regime as the (G, D, β, δ)-asymptotic regime. In this regime, we characterize the MPR capability of the per-slot decoder and the resulting decoding threshold for $\nu = 0.5$ as follows.

Theorem 2 (Asymptotic lower bound on MPR capability and decoding threshold): Let $\nu = 0.5$. In the (G, D, β, δ)-asymptotic regime, it holds that

$$\lim_{N\to\infty} \pi_U \ge \bar{\pi}_U = \begin{cases} 0, & \text{if } U > -\log_2(1 - 2^{-(1-\delta)/\beta}), \\ \frac{(1-2^{-U})^{\beta \log_2 D}}{\exp(\mu^{-1} D G \Lambda'(1))}, & \text{if } U = -\log_2(1 - 2^{-(1-\delta)/\beta}), \\ 1, & \text{if } U < -\log_2(1 - 2^{-(1-\delta)/\beta}). \end{cases} \quad (5)$$

Furthermore, the decoding threshold $G^*(\Lambda)$ is lower bounded by the largest value of G such that

$$p > \lambda\left(1 - e^{-G\Lambda'(1)p} \sum_{u=0}^{\bar{T}_{\text{ED-MPR}}-1} \frac{\bar{\pi}_{u+1}}{u!}(pG\Lambda'(1))^u\right), \forall p \in (0,1], \quad (6)$$

where

$$\bar{T}_{\text{ED-MPR}} = \left\lfloor -\log_2\left(1 - 2^{-(1-\delta)/\beta}\right)\right\rfloor \quad (7)$$

is a lower bound on the maximum number of resolvable users in a slot.

Proof: See Appendix B. ■

Remark 2: The regime of interest corresponds to a small β, so that the users collectively transmit close to 1 bit per channel use in a slot. Furthermore, aiming for massive IoT applications where a large number of devices transmit small data volumes [15], we are interested in δ close to 1, so that the per-user payload increases slowly with the number of users.

Remark 3: In [11], Paolini, Valentini, Tralli, and Chiani consider a similar IRSA-based scheme, which we shall refer to as PVTC. While our random-coding scheme uses an i.i.d. binary codebook, [11] employs the BPR codebook [10], which is constructed as the parity-check matrix of a BCH code. This construction applies to $D = 1$ and β being a positive integer. If β is not an integer, one can apply the PVTC scheme designed for $\lfloor\beta\rfloor$. This allows the per-slot decoder to resolve slots of degree up to $T_{\text{PVTC}} = \lfloor\beta\rfloor$, i.e., $\pi_U = \mathbb{1}\{U \le \lfloor\beta\rfloor\}$. Currently, there is no efficient decoder that attempts to resolve slots of degree higher than $\lfloor\beta\rfloor$ for this scheme. The capability to do so has not been discussed in [10], [11]. It is easy to show that

$$\begin{cases} \bar{T}_{\text{ED-MPR}} > T_{\text{PVTC}}, & \text{if } \delta > 1 + \beta\log_2(1 - 2^{-\lfloor\beta\rfloor - 1}), \\ \bar{T}_{\text{ED-MPR}} < T_{\text{PVTC}}, & \text{if } \delta < 1 + \beta\log_2(1 - 2^{-\lfloor\beta\rfloor}), \\ \bar{T}_{\text{ED-MPR}} = T_{\text{PVTC}}, & \text{otherwise.} \end{cases} \quad (8)$$

Therefore, if $\delta > 1 + \beta\log_2(1 - 2^{-\lfloor\beta\rfloor-1})$, our per-slot decoder resolves more colliding users than the PVTC scheme. This corresponds to the regime of interest noted in Remark 2, i.e., β is small and δ is close to 1, as we shall illustrate in Section V.

C. Erasure Decoding on the Full Graph (ED-FG)

In the decoder introduced in Section III-B, while decoding in a slot s, the candidate lists of all other slots remain unchanged. Notice that if a codeword $\boldsymbol{c}$ is incompatible with at least one of the slots in $h(\boldsymbol{c})$, it cannot have been transmitted, and thus can be removed from $\mathcal{S}_s, \forall s \in h(\boldsymbol{c})$. Based on this observation, we augment the per-slot discarding decoder as follows. For each slot, after the discarding process, the decoder removes each discarded codeword from the candidate lists for all associated slots. In this way, we shrink the candidate list in a slot, and thus increase the probability of resolving the slot in later iterations.

Consider a *full* graph that contains all M codewords as VNs. This is identical to the original graph if all codewords have been transmitted. The decoder has knowledge of this full graph but initially cannot distinguish between VNs corresponding to transmitted and nontransmitted codewords. The aforementioned improved decoder can be equivalently interpreted as erasure decoding on this full graph. Specifically, from the CN perspective, an edge is revealed if the corresponding codeword is either resolved (i.e., we know it has been transmitted) or it is incompatible with the channel output (i.e., we know it has not been transmitted). From the VN perspective, if an edge has been revealed, we know whether the corresponding codeword has been transmitted. Thus all other edges are also revealed.

IV. Information Rate

As $\log_2 K$ bits are allocated for the user's signature, the information rate of each user is $R = \frac{\log_2(M/K)}{n}$ bits per channel use. The average sum rate is thus given by $R_{\text{sum}} = \mathbb{E}[\mathsf{K}_\text{a}]\, R = \frac{\mu K \log_2(M/K)}{n}$. It follows from the frame structure that

$$R_{\text{sum}} = \frac{\mu K}{N} \cdot \frac{\log_2(M/K)}{1+n_0}. \tag{9}$$

The next theorem characterizes the asymptotic average sum rate achievable with vanishing PLR.

Theorem 3 (Achievable average sum rate): In the (G, D, β, δ)-asymptotic regime, an IRSA-based scheme with degree distribution $\Lambda(x)$ and decoding threshold $G^*(\Lambda)$ can achieve the following average sum rate with vanishing PLR

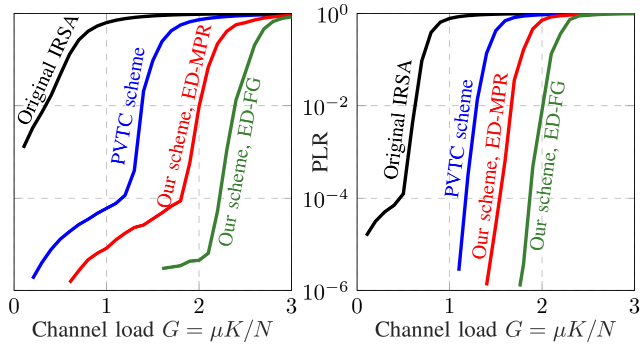
$$\lim_{K\to\infty} R_{\text{sum}} = \frac{1-\delta}{\beta} G, \tag{10}$$

for every $D > 0$, $\beta \geq 1$, $\delta \in (0, 1)$, and $G \leq G^*(\Lambda)$.

Proof: In the considered asymptotic regime, the average sum rate is $R_{\text{sum}} = G \cdot \frac{\log_2(M/K)}{1+n_0} = G \cdot \frac{(1/\delta - 1)\log_2 K + \log_2 D}{1 + \frac{\beta}{\delta}\log_2 K} \to \frac{1-\delta}{\beta} G$. By definition of the decoding threshold $G^*(\Lambda)$, for any channel load $G \leq G^*(\Lambda)$, the PLR vanishes. ■

V. Numerical Results

We numerically evaluate the random-coding bound on the performance of our proposed decoders and compare it with the original IRSA and the PVTC scheme [11]. We consider $D = 1$ to enable a comparison with the PVTC scheme. In Fig. 2, we plot the PLR (obtained via Monte-Carlo simulation) achieved with these schemes as a function of the channel load $G = \mu K/N$ for $\nu = 0.5$, $N = 200$, $\mu = 0.2$, and $\Lambda(x) \in \{x^2, x^3\}$. For our scheme, we implemented the per-slot discarding decoder. We set $M = K^{1/\delta}$, $n_0 = \beta \log_2 M$ with



(a) $\Lambda(x) = x^2$ (b) $\Lambda(x) = x^3$

Fig. 2: The PLR achieved with our scheme, original IRSA, and the PVTC scheme for $\nu = 0.5$, $N = 200$, $\mu = 0.2$, $\mu K/N = G$, $M = K^{1/\delta}$, and $n_0 = \beta \log_2 M$ with $(\beta, \delta) = (2, 0.9)$.

$(\beta, \delta) = (2, 0.9)$, and increase K. We observe that our proposed scheme with ED-MPR (Section III-B) significantly outperforms the original IRSA and the PVTC scheme. The PLR can be further reduced notably with ED-FG (Section III-C). For example, to achieve a PLR of 10^{-4} with $\Lambda(x) = x^2$, our scheme with ED-MPR and ED-FG can support around 52% and 91% more users per slot in average, respectively, than the PVTC scheme.

The advantage of our scheme over the PVTC scheme for the considered setting comes from the capability to resolve more colliding users. To illustrate this, we consider a similar setting as in Fig. 2(a) with $(\beta, \delta, \Lambda(x)) = (2, 0.9, x^2)$ and plot in Fig. 3 the probability of resolving degree-U slots, π_U, and its lower bound π'_U, given in Theorem 1. We assume that $\mu^{-1} G = 2.5$, i.e., $N = K/2.5$. The PVTC scheme resolves slots of degree up to $\beta = 2$. For finite K, our scheme with ED-MPR resolves most of degree-2 slots, and can further resolve slots of higher degrees. As K becomes large, with $\nu = 0.5$, π'_U converges to $\mathbb{1}\{U \leq \bar{T}_{\text{ED-MPR}}\}$ with $\bar{T}_{\text{ED-MPR}} = 4$ obtained using (7). This agrees with Theorem 2. With $\nu = 0.1$, our scheme resolves high-degree slots with higher probability but low-degree slots with lower probability. This is due to the trade-off mentioned in Remark 1. For $\Lambda(x) \in \{x^2, x^3\}$, the majority of the slots have low degree, thus $\nu = 0.5$ leads to lower PLR than $\nu \neq 0.5$.

Hereafter, we focus on the (G, D, β, δ)-asymptotic. In this regime, the PVTC scheme can resolve slots of degree up to $T_{\text{PVTC}} = \lfloor \beta \rfloor$, while our scheme with ED-MPR guarantees to resolve slots of degree up to $\bar{T}_{\text{ED-MPR}}$ given in (7). In Fig. 4, we depict the regions of (β, δ) where either T_{PVTC} or $\bar{T}_{\text{ED-MPR}}$ is larger, or they are equal, as characterized in (8), for $\nu = 0.5$. We see that our scheme with ED-MPR resolves more colliding users than the PVTC scheme if β is small and δ is close to 1. (We have seen in Fig. 3 that this is the case for $(\beta, \delta) = (2, 0.9)$.) As noted in Remark 2, this is the regime of interest.

Finally, we compare in Fig. 5 the average sum rate $\bar{R}_{\text{sum}} = \frac{1-\delta}{\beta} G^*(\Lambda)$ achievable with vanishing PLR in the (G, D, β, δ)-

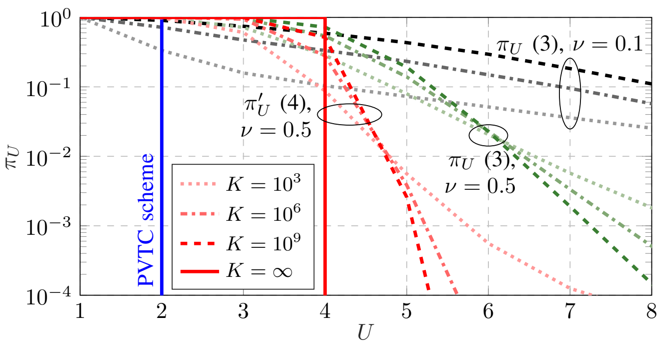


Fig. 3: The probability of resolving degree-U slots, π_U, achieved by our proposed scheme with ED-MPR and the PVTC scheme for $\Lambda(x) = x^2$, $\nu \in \{0.1, 0.5\}$, $K/N = \mu^{-1}G = 2.5$, $M = K^{1/\delta}$, and $n_0 = \beta \log_2 M$ with $(\beta, \delta) = (2, 0.9)$.

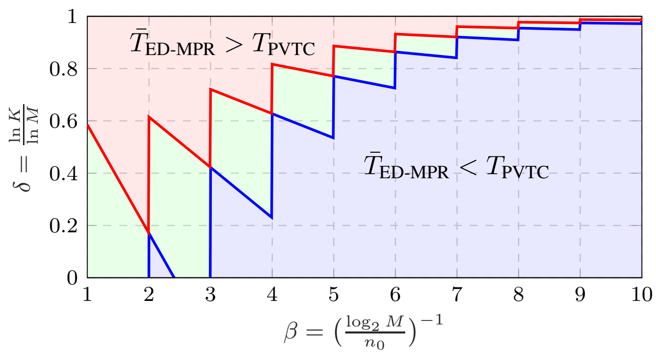


Fig. 4: The (β, δ) regions where either $\bar{T}_{\text{ED-MPR}}$ or T_{PVTC} is larger for $D = 1$ and $\nu = 0.5$. In the middle (green) region, $\bar{T}_{\text{ED-MPR}} = T_{\text{PVTC}}$. When $\bar{T}_{\text{ED-MPR}} > T_{\text{PVTC}}$ (depicted in red), our scheme with ED-MPR guarantees to resolve more colliding users than the PVTC scheme in the (G, D, β, δ)-asymptotic regime.

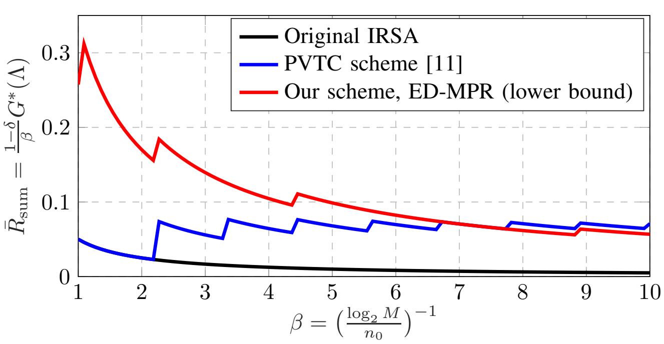


Fig. 5: The achievable average sum rate $\bar{R}_{\text{sum}} = \frac{1-\delta}{\beta} G^*(\Lambda)$ with vanishing PLR in the (G, D, β, δ)-asymptotic regime with $D = 1$, $\delta = 0.9$, $\nu = 0.5$, and $\Lambda(x) = x^2$.

asymptotic regime by the original IRSA, the PVTC scheme, and our scheme with ED-MPR. For our scheme, $G^*(\Lambda)$ is replaced by the lower bound in Theorem 2. We consider $\delta = 0.9$ and $\Lambda(x) = x^2$. As expected, the original IRSA with single-packet reception achieves the lowest sum rate. Our scheme significantly outperforms the PVTC scheme when β is small.

VI. Conclusion

We discussed an IRSA-based random-access scheme for the binary adder channel. In a slot, the active users transmit a pilot symbol followed by i.i.d. binary symbols. We proposed a per-slot decoder that discards incompatible codewords from a list of candidate codewords, and an improved decoder that shrinks this list across iterations. Our scheme resolves more colliding users per slot and thus achieves a higher sum rate than the state-of-the-art scheme proposed in [11], in the regime where the per-user payload increases slowly with the number of users, and the total number of bits transmitted per channel use in a slot is close to 1. A subject for future work is the asymptotic analysis of the improved decoder.

Appendix A
Proof of Theorem 1

Consider a degree-U slot. Let $\mathcal{A}_0$ and $\mathcal{A}_U$ be the set of channel uses in this slot where the channel output is 0 and U, respectively. Let $\mathsf{A}_0 = |\mathcal{A}_0|$ and $\mathsf{A}_U = |\mathcal{A}_U|$. After some manipulations, we can derive the joint PMF of A_0 and A_U as in (2).[3] Let L_0 be the number of nontransmitted codewords associated with the slot. The slot is resolved if none of these L_0 nontransmitted codewords survive the discarding process. A nontransmitted codeword survives if its entries in all positions in $\mathcal{A}_0 \cup \mathcal{A}_U$ agree with the channel outputs, i.e., the entries in positions in $\mathcal{A}_U$ are 1 and the entries in positions in $\mathcal{A}_0$ are 0. The probability of this event is $\nu^{\mathsf{A}_U}(1-\nu)^{\mathsf{A}_0}$. It follows that the probability π_U that the slot is resolved is

$$\pi_U = \mathbb{E}_{\mathsf{L}_0, \mathsf{A}_0, \mathsf{A}_U}\left[\left(1 - \nu^{\mathsf{A}_U}(1-\nu)^{\mathsf{A}_0}\right)^{\mathsf{L}_0}\right]. \tag{11}$$

To compute the PMF of L_0, notice that a generic codeword is associated with $\Lambda'(1)$ slots in average. Thus the probability that a generic codeword is associated with a given slot is $\Lambda'(1)/N$. Therefore, the probability that L out of $M-U$ nontransmitted codewords are associated with a given slot is given by

$$\mathbb{P}\left[\mathsf{L}_0 = L\right] = \binom{M-U}{L}\left(\frac{\Lambda'(1)}{N}\right)^L \left(1 - \frac{\Lambda'(1)}{N}\right)^{M-U-L}.$$

It follows that

$$\begin{aligned}\mathbb{E}\left[x^{\mathsf{L}_0}\right] &= \sum_L \binom{M-U}{L}\left(\frac{\Lambda'(1)}{N}x\right)^L \left(1 - \frac{\Lambda'(1)}{N}\right)^{M-U-L} \\ &= \left(1 - \frac{\Lambda'(1)}{N}(1-x)\right)^{M-U}. \end{aligned} \tag{12}$$

Next, by setting $x = 1 - \nu^{\mathsf{A}_U}(1-\nu)^{\mathsf{A}_0}$ in (12) and by using (11), we obtain (1). If $\nu = 0.5$, (3) follows directly from (1) by noting that $\mathsf{A}_0 + \mathsf{A}_U \sim \text{Bino}(n_0, 2^{1-U})$. Finally, by applying Jensen's inequality, we lower-bound π_U in (3) by $\pi'_U = \big(1 - \frac{\Lambda'(1)}{N}\mathbb{E}[2^{-\mathsf{A}}]\big)^{M-U} = \big(1 - \frac{\Lambda'(1)}{N}(1-2^{-U})^{n_0}\big)^{M-U}$.

[3] We treat the codewords as independent although they are required to be distinct. The probability of generating two identical codewords is $(\nu^2 + (1-\nu)^2)^{n_0}$, which is negligible for large n_0.

Appendix B
Proof of Theorem 2

We shall show that the lower bound π'_U converges to $\bar{\pi}_U$ in the (G, D, β, δ)-asymptotic regime. First, by applying the inequalities $\frac{z}{1+z} \le \ln(1+z) \le z, \forall z > -1$, with $z = -\frac{\Lambda'(1)}{N}x$, and after some manipulations, we obtain that

$$\exp\left(\frac{-\frac{M-U}{N}\Lambda'(1)x}{1-\Lambda'(1)x/N}\right) \le \left(1-\frac{\Lambda'(1)}{N}x\right)^{M-U} \le \exp\left(-\frac{M-U}{N}\Lambda'(1)x\right). \quad (13)$$

Next, let $\mu K/N = G$, $M = DK^{1/\delta}$, and $n_0 = \beta \log_2 M$. Setting $x = (1-2^{-U})^{n_0}$ in (13), we obtain

$$\exp\left(\frac{-\frac{DK^{1/\delta}-U}{\mu K/G}\Lambda'(1)(1-2^{-U})^{\frac{\beta}{\delta}\log_2 K+\beta\log_2 D}}{1-\Lambda'(1)(1-2^{-U})^{\frac{\beta}{\delta}\log_2 K+\beta\log_2 D}G/(\mu K)}\right) \le \pi'_U \le \exp\left(-\frac{DK^{1/\delta}-U}{\mu K/G}\Lambda'(1)(1-2^{-U})^{\frac{\beta}{\delta}\log_2 K+\beta\log_2 D}\right).$$

As $K \to \infty$, one can verify that both the lower bound and upper bound above converge to $\bar{\pi}_U$ given in (5). We conclude that $\pi'_U \to \bar{\pi}_U$, and thus $\pi_U \ge \bar{\pi}_U$, in the considered asymptotic regime.

It follows from Lemma 1 in Appendix C that for a scheme that resolves a collision of U users with probability π'_U, the decoding threshold is given by the largest value of G such that (6) holds. Our scheme resolves collisions with higher probability, thus achieves a lower PLR, and in turn a higher decoding threshold.

Appendix C
IRSA With Probabilistic MPR

The existing analyses of IRSA with MPR, such as [12], assume that the receiver can resolve a collision with probability 1 if the number of colliding users is at most T, and the collision is unresolved otherwise. With degree distribution $\Lambda(x) = x$, this is also called T-fold slotted ALOHA [5]. In this appendix, we extend the decoding threshold analysis of IRSA with MPR in [12] to a more general setting where a collision of U users is resolved with probability $\pi_U^{(N)}$ that depends on the number of slots per frame, N. This result can be of independent interest.

Lemma 1: Consider IRSA with N slots per frame, degree distribution $\Lambda(x)$, and MPR capability where a collision of U users is resolved with probability $\pi_U^{(N)}$. Assume that $\pi_U^{(N)} \to \tilde{\pi}_U$ as $N \to \infty$. The decoding threshold G^* of this scheme is the largest value of G such that

$$p > \lambda\left(1-e^{-G\Lambda'(1)p}\sum_{u=0}^{T-1}\frac{\tilde{\pi}_{u+1}}{u!}(pG\Lambda'(1))^u\right), \quad \forall p \in (0,1]. \quad (14)$$

where $\lambda(x) = \Lambda'(x)/\Lambda'(1)$ and $T = \max\{u\colon \tilde{\pi}_u > 0\}$.

Proof: Consider a degree-L VN. Let p be the probability that an edge is unknown, given that each of the other $L-1$ edges has been revealed with probability $1-q$ in the previous step. Since the edge is revealed whenever at least one of the other edges have been revealed, we have that $p = q^{L-1}$. Next, consider a degree-L CN. Here, q denotes the probability that an edge is unknown, given that each of the other $L-1$ edges has been revealed with probability $1-p$ in the previous step. By assumption, the edge is revealed with probability $\pi_{u+1}^{(N)}$ if the number of unrevealed edges among the other $L-1$ edges is u. Therefore, $1-q = \sum_{u=0}^{L-1}\pi_{u+1}^{(N)}\binom{L-1}{u}p^u(1-p)^{L-u-1}$. According to a tree analysis argument, by averaging the expressions of p and q over the edge distributions, we can derive the evolution of the average erasure probabilities during the i-th iteration as

$$p_i = \sum_L \lambda_L q_{i-1}^{L-1} = \lambda(q_{i-1}), \quad (15)$$

$$q_i = \sum_L \rho_L\left(1-\sum_{u=0}^{L-1}\pi_{u+1}^{(N)}\binom{L-1}{u}p_i^u(1-p_i)^{L-u-1}\right) = 1-\sum_{u\colon \pi_{u+1}^{(N)}>0}\frac{\pi_{u+1}^{(N)}}{u!}p_i^u\rho^{(u)}(1-p_i), \quad (16)$$

where ρ_L is the probability that an edge is connected to a degree-L CN, $\rho(x) = \sum_L \rho_L x^{L-1}$, and $\rho^{(u)}(x)$ is the uth order derivative of $\rho(x)$. As $N \to \infty$, $\rho(x) = e^{-G\Lambda'(1)(1-x)}$, and it follows that $\rho^{(u)}(x) = (G\Lambda'(1))^u e^{-G\Lambda'(1)(1-x)}$. Furthermore, $\pi_{u+1}^{(N)} \to \tilde{\pi}_{u+1}$. Therefore, with $T = \max\{u : \tilde{\pi}_u > 0\}$,

$$q_i = 1-\sum_{u=0}^{T-1}\frac{\tilde{\pi}_{u+1}}{u!}p_i^u(G\Lambda'(1))^u e^{-G\Lambda'(1)p_i} \quad (17)$$

$$= 1-e^{-G\Lambda'(1)p_i}\sum_{u=0}^{T-1}\frac{\tilde{\pi}_{u+1}}{u!}(p_iG\Lambda'(1))^u. \quad (18)$$

By combining (15) and (18), we obtain

$$p_i = \lambda\left(1-e^{-G\Lambda'(1)p_{i-1}}\sum_{u=0}^{T-1}\frac{\tilde{\pi}_{u+1}}{u!}(p_{i-1}G\Lambda'(1))^u\right). \quad (19)$$

The decoding threshold G^* is the largest value of G such that $\{p_i\}$ is a decreasing sequence, i.e., $p_{i-1} > p_i$ for all i. The inequality (14) ensures this property. ■

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