Index Coding – An Interference Alignment Perspective

Hamed Maleki, Viveck R. Cadambe, Syed A. Jafar

Abstract

The index coding problem is studied from an interference alignment perspective providing new results as well as new insights into, and generalizations of, previously known results. An equivalence is established between the capacity of the multiple unicast index coding (where each message is desired by exactly one receiver), and groupcast index coding (where a message can be desired by multiple receivers), which settles the heretofore open question of insufficiency of linear codes for the multiple unicast index coding problem by equivalence with groupcast settings where this question has previously been answered. Necessary and sufficient conditions for the achievability of rate half per message in the index coding problem are shown to be a natural consequence of interference alignment constraints, and generalizations to feasibility of rate $\frac{1}{L+1}$ per message when each destination desires at least L messages, are similarly obtained. Finally, capacity optimal solutions are presented to a series of symmetric index coding problems inspired by the local connectivity and local interference characteristics of wireless networks. The solutions are based on vector linear coding.

1 Introduction

Much progress in network information theory can be attributed to the pursuit of the capacity of simple-to-describe canonical network communication models. Simplicity in the network communication models often affords a clear formulation of techniques involved in the communication system. The focus of this paper is the *index coding* problem which is arguably the simplest multiuser capacity problem because it is a communication network that has *only one link with finite capacity*. Yet, this turns out to be the proverbial case where appearances can be quite deceiving. More than a decade after it was introduced by Birk and Kol in [1, 2], the index coding problem not only remains open, but also has been shown to include as special cases a number of difficult problems in both wired and wireless settings — such as the general multiple unicast problem with linear network coding [3], multi-way relay networks [4], and the topological interference management problem in wireless networks [5, 6], to name a few. Remarkably, the index coding problem is also the origin of the fundamental idea of interference alignment [2], which was re-discovered, extensively studied and developed in a variety of forms in wireless networks [7, 8, 9]. Interference alignment has recently found applications in network coding problems such as the distributed data storage exact repair problem [10, 11] and the 3 unicast problem [12, 13]. In this paper, we attempt to bring this idea "home", by applying the understanding of the principles of interference alignment, into the original setting — the index coding problem.

The essence of the index coding problem lies in its focus on a single bottleneck network. Having only one link with finite capacity concentrates the challenge of network coding in one place, highlighting some of the most fundamental, challenging, and surprising aspects of the network coding problem. Understanding the role of a single bottleneck edge in a network when the rest of the network is composed only of trivial links (of infinite capacity), is a natural stepping stone toward a broader understanding of communication networks[1]. We start with a discussion of similarly motivated single-bottleneck settings for both wired and wireless networks.

1.1 Single Bottleneck Wired Networks – Index Coding

Consider a general network coding setting shown in Figure 1(a) where the source nodes on the left communicate with the destination nodes on the right through a network (possibly containing cycles) of intermediate nodes connected via directed, orthogonal, noiseless, capacitated links. If only one link (shown in black) in the intermediate network has finite (unit) capacity, and all the other links have infinite capacity, then the remaining problem is the index coding problem. Clearly, the only non-trivial message flows are those for which every path between the source and the desired destination(s) *must* pass through the finite capacity link. All other messages have either rate zero or infinity, and can be eliminated. The remaining network contains three kinds of infinite capacity links in addition to the finite capacity link. First, the remaining source nodes must connect to the transmitter of the finite capacity link via infinite capacity links. Second, the destination nodes must connect to the receiver of the finite capacity link through infinite capacity links. Third, if there are infinite capacity paths between any sources and their non-desired destinations, those paths are replaced with infinite capacity links known as "antidote" links. An example is shown in Figure 1(b) where 5 messages originate at the sources on the left and are desired by the destination nodes on the right. The bottleneck link (finite capacity link) is shown at the top of the figure in black and carries a sequence of symbols from a finite alphabet $S^n \in \mathcal{S}^n$, which are chosen with full knowledge of all messages. The antidote links are shown in red. Clearly, the best use of the antidote links, which have infinite capacity, is to convey all the information, i.e., the messages, from the transmitters to the receivers of the antidote links. These antidotes comprise the side information that makes the problem interesting and quite challenging in general. Each destination must

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[1]The edge-removal problem introduced in [14, 15] is another intriguing open problem that seeks to understand the role of a single edge in a network, and has been shown to be closely related to the general question of whether zero-error capacity and ϵ-error capacity are the same for general network coding instances[16].

[3]It is worth noting that in some related works, the index coding problem is pictorially depicted using the side information graph where the bottleneck link is not explicitly shown (See, for example, [17]). In contrast, our graphs explicitly show the bottleneck link since we interpret the index coding problem as a special case of a network coding problem on a directed acyclic graph.
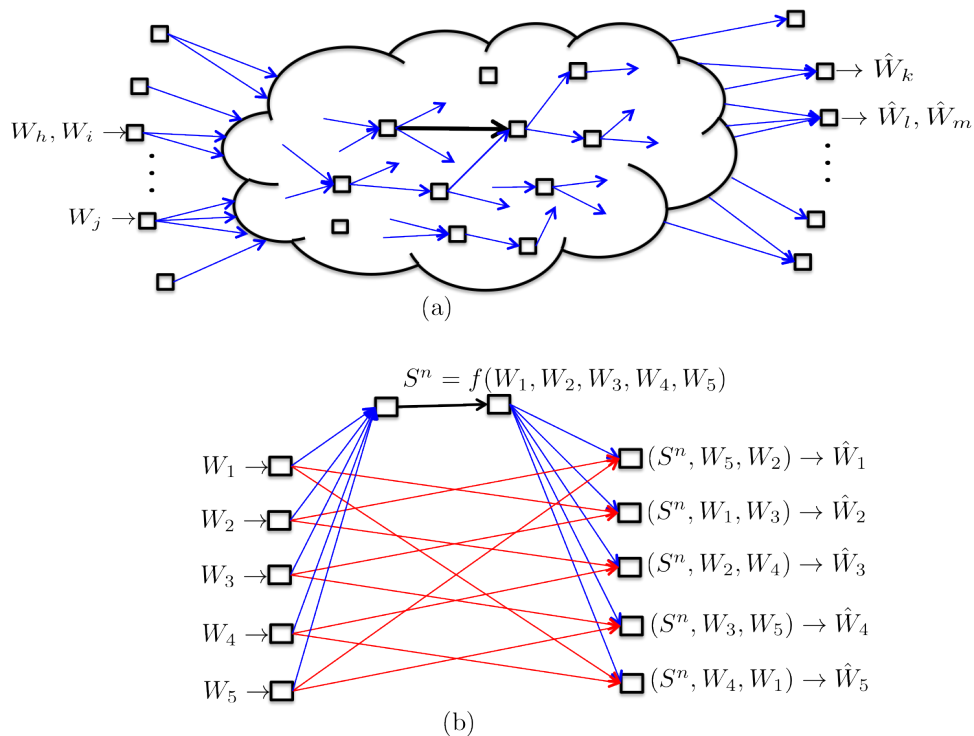
Figure 1: (a) General network coding problem: If only one link (shown in black) in the intermediate network has finite (unit) capacity, and all the other links have infinite capacity, then the remaining problem is the index coding problem. (b) Example of an index coding setting[3].

be able to decode its desired message based on the sequence of symbols sent over the bottleneck link and the set of undesired messages available to it as antidotes.

Index coding can also be seen as "source coding with side information", or as a broadcast channel with cognitive receivers, i.e., where certain receivers have full knowledge of certain messages a-priori.

1.2 Single Bottleneck Wireless Networks – Wireless Index Coding

Consider a wireless network shown in Figure 2(a) comprised of the source nodes shown on the left, which communicate with destination nodes shown on the right, through an intermediate network of relay nodes. Depending on propagation path loss different pairs of nodes may be connected or disconnected. Because this is a wireless setting, signals emerging from the same transmitter are broadcast, and signals arriving at the same receiver interfere. All transmitters are subject to power constraint P, and generally the receivers experience additive white Gaussian noise (AWGN) in addition to the superposition of fading signals from connected transmitters. As an analogue to the index coding problem defined by a single bottleneck link, let us assume only one of the receivers in the intermediate network experiences AWGN, e.g., of unit variance, while all other receivers experience no noise. It means all the other receivers have infinite resolution of the complex valued signals, essentially providing them infinite capacity links to their respective connected transmitters. Eliminating messages that have infinite capacity paths between their sources and all their desired destinations, what remains is the wireless index coding problem, introduced in [5]. Depending on the wireless network topology, the resulting wireless index coding problem can in general be quite challenging. The problem can be quite involved, for example, if the wireless network contains cycles that provide feedback from the output of the bottleneck receiver to the distributed or partially cooperating nodes transmitting to the bottleneck receiver. Fig. 2(b) shows a relatively simple example of the wireless index coding problem that corresponds to the index coding problem of Fig. 1(b), in the sense that the capacity of the index coding
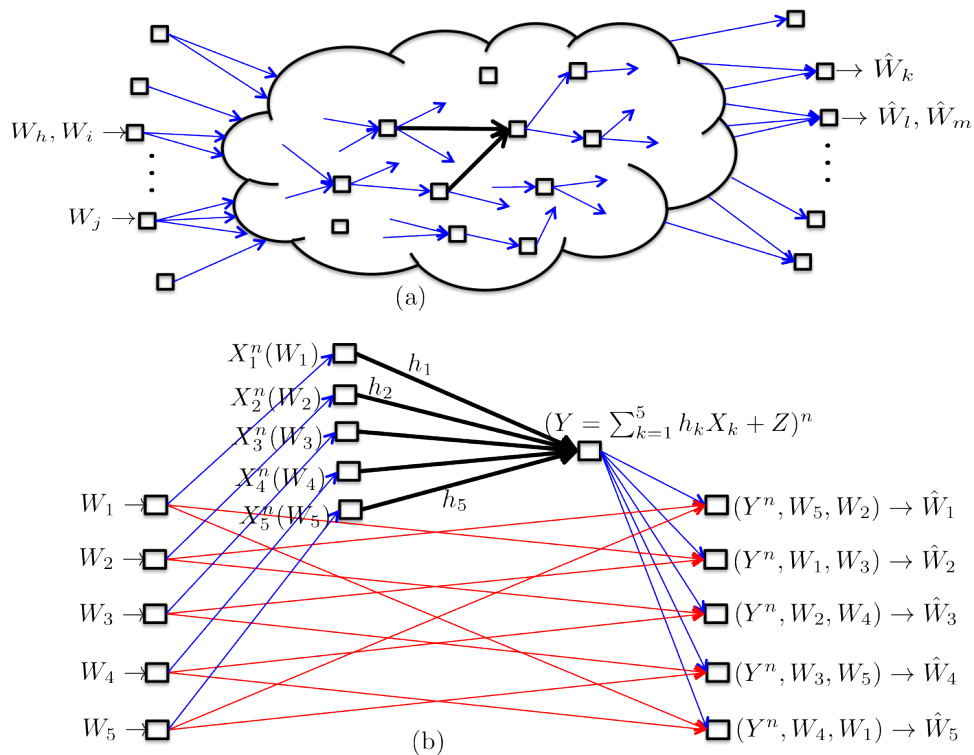
Figure 2: (a) Wireless network: If only one receiver (shown with incoming signals in black) in the intermediate network has non-zero (unit) AWGN variance, and all the other receivers have zero noise (infinite capacity), then the remaining problem is the wireless index coding problem. (b) Example of a wireless index coding setting.

problem maps directly to the degrees of freedom (DoF) of the wireless index coding problem. As we show later in the paper (Section 4.3, Theorem 5,) the capacity per message of the index coding problem in Figure 1(b) is 2/5, as is the DoF value per message for the wireless index coding problem in Fig. 2(b), and in both cases the "unit" for measurement is the capacity/DoF of the bottleneck link/receiver. The index coding problem normalizes the bottleneck link capacity to unity, so that the rates of all the messages are measured as multiples of the bottleneck link capacity. Similarly, the wireless index coding problem normalizes the number of signal dimensions (DoF) available to the bottleneck receiver to unity, and the DoF of all the messages are measured as multiples of the bottleneck DoF. As explained in [5], the relationship between the index coding problem and the wireless index coding problem goes much further, and much more can be said about their similarities and differences. For instance, if full cooperation is allowed between all sources directly transmitting to the bottleneck receiver in a wireless index coding problem, the DoF of the resulting network is the same as the capacity of the corresponding index coding problem (in their respective units). The DoF of the wireless index coding problem are, in general, bounded above by the capacity of the index coding problem. It also highlights the main difference between the index coding problem and the wireless index coding problem — all sources are necessarily allowed to fully cooperate in the former because the bottleneck transmitter has full knowledge of all messages, but not necessarily in the latter (depending on the topology of the original network in Figure 2(a)). However, if the index coding problem has a capacity optimal vector linear coding solution that can be translated to the complex field, then the same solution may be applied in the wireless index coding problem as well. This is because vector linear solutions are comprised of a superposition of separately encoded messages, and a superposition over complex field is naturally provided by the wireless medium [5]. Somewhat surprisingly, this is a very common situation, e.g., all the instances of the index coding problems studied in this paper have capacity optimal vector linear coding solutions that translate to the complex field, thereby simultaneously providing the DoF characterization for

the corresponding wireless index coding problem.

In the wireless index coding problem discussed above, the bottleneck is concentrated at one receiver, lending the bottleneck a multiple access character. Another formulation of the wireless index coding problem is also conceivable where the bottleneck may be concentrated at one transmitter, e.g., all receivers experience additive noise and there is only one transmitter with finite power (all other transmitters have infinite power), which would lend the bottleneck a broadcast character, and which could be a similarly interesting and promising research avenue.

The motivation for studying single bottleneck networks in both wired and wireless settings is evident from an information-theoretical perspective as a stepping stone to a broader understanding of communication networks. What is surprising is that the index coding problem, in spite of its simple formulation, not only already captures much of the complexity of the full-fledged network capacity problem, but also contains a class of problems known as topological interference management [5] problems (TIM) which are of immediate practical interest for cellular wireless networks. Indeed, it is the TIM setting that motivates most of the instances of index coding that we solve in this work.

1.3   Topological Interference Management Problem

Consider, as an example, the cellular downlink setting shown in Figure 3(a) comprised of 5 partially overlapping cells depicted as circles. Each cell (circle) contains a transmitter (base station) near its center, shown as a black square, and 3 receivers (users), shown as white squares. Propagation path loss is modeled by the assumption that each base station transmitter (black square) is only heard within the circular region defining its own cell. This gives rise to the connectivity pattern where each transmitter can be heard by three receivers and each receiver can hear three transmitters. The resulting wireless network connectivity is shown in Figure 3(b) where the links show the non-zero channel coefficients. The knowledge of non-zero channel coefficient values, which are assumed to be drawn from identical distributions, is not available to the (blind) transmitters. Depending upon the message sets, e.g., whether each transmitter sends a message to only one user in its cell, or whether each transmitter sends 3 independent messages to the 3 users in its cell, we have the partially connected interference channel or X channel setting, respectively (More details are provided in Sections 3.2 and 3.3). Since both settings will benefit significantly from interference alignment and no knowledge of non-zero channel coefficient values is assumed, this is known as the topological interference management problem [5].
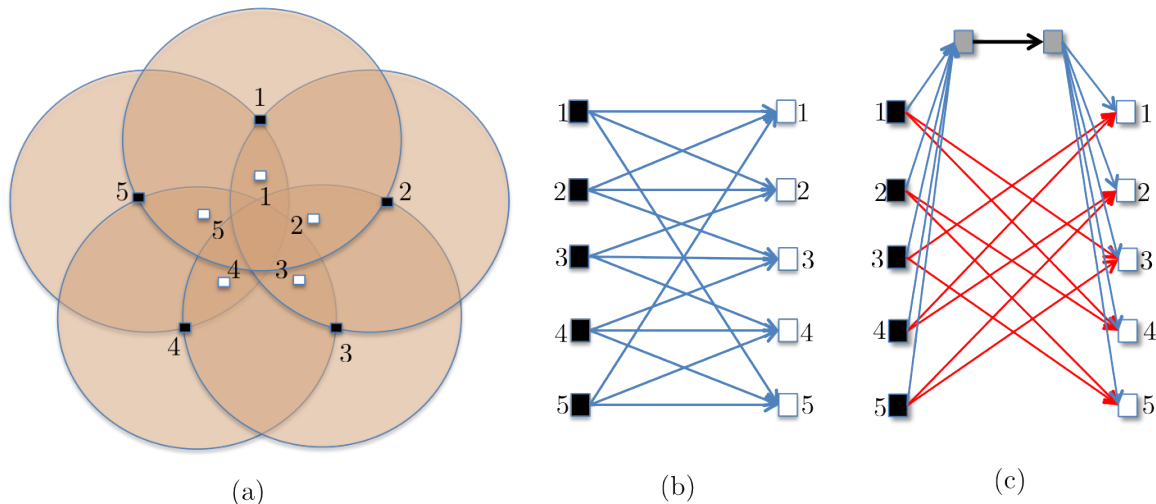

Figure 3: Topological Interference Management problem: (a) Cellular layout governing the connectivity pattern, (b) Locally connected wireless network representation, (c) Corresponding index coding problem

The key to the TIM problem lies in its close relationship to a corresponding index coding problem.

For the TIM problem of Figure 3(a) and Figure 3(b), the corresponding index coding problem is shown in Figure 3(c). In the corresponding index coding problem, the graph connecting black (transmitter) and white (receiver) nodes in Figure 3(c) is complementary to that in Figure 3(b), i.e., a black and white node pair is connected in the index coding problem of Figure 3(c) by an antidote link if and only if it is not connected in the locally connected network representation of Figure 3(b). Thus, antidote links which allow a receiver to subtract from its received signal the contribution from the corresponding undesired messages, especially in a superposition based coding scheme as must be the case in the TIM problem due to the distributed nature of the source nodes and the additive nature of the wireless medium, play the same role as a zero channel coefficient value in the local connectivity pattern. The main difference in the index coding problem is that full cooperation between sources is allowed, which makes the index coding capacity an outer bound on the DoF of the TIM setting. However, as we will see in this paper, optimal solutions to the TIM based index coding problems that we study are based on vector linear coding and robust to the choice of the underlying field as the real or complex field. An important implication is that, for the scenarios that we study, our optimal solutions of the index coding problem automatically yield optimal solutions to the corresponding TIM problem as well (See [5] for more details).

Finally, we note that the TIM problem has a natural counterpart in the network coding setting, which is a *blind linear network coding* problem[4]. Consider, for example, a wired network where the intermediate nodes perform random linear network coding, creating a linear channel matrix comprised of polynomials in the network coding coefficients of the intermediate nodes. Recent work [12, 20, 13, 21] has investigated how to exploit the knowledge of these effective channel matrices at the source nodes to achieve interference alignment by linear precoding, in a manner that mimics the wireless interference channel. However, suppose that the source nodes only know the end-to-end connectivity but do not know the channel coefficient values, e.g., because they do not keep track of all the network coding coefficients. Aside from the significant distinction of working over finite fields, the resulting *blind network coding* problem is virtually identical to the TIM problem, and is similarly related to a corresponding index coding problem. For instance, if we take the graph of Figure 3(b) to represent the resulting connectivity of a blind network coding problem, with no knowledge of the channel realizations at the transmitters, then the corresponding index coding problem is shown in Figure 3(c). Solving this index coding problem will solve both the corresponding TIM problem as well as the corresponding blind network coding problem.

The preceding discussion of closely related problems sheds light on the significance of the index coding problem. In spite of having only one finite capacity link, the richness of the index coding problem is evident. The complexity of index coding problem is further underscored in the result by Rouayheb et al. in [3] and its generalization by Effros et. al. in [22], where an equivalence is established between the general network coding problem restricted to linear codes, and the index coding problem. On the other hand, the relative simplicity of the index coding problem setting does make this setting more tractable in certain respects. As an example, we note that the equivalence of ϵ-error capacity and zero-error capacity, which remains open in the general network coding problem, has been established for the index coding problem [16].

1.4 Organization of this paper

The rest of this paper is organized as follows. In Section 2, we formally describe the index coding problem. In Section 3, we show that the index coding problem can, in fact, be viewed as problem in interference alignment. We will use instances of index coding problems motivated by the TIM setting as examples to establish this connection. In Section 4, we present the main results of our paper. The interference alignment perspective of Section 3 serves as a common thread that ties together all the results of Section 4. We conclude the main part of this paper with some interesting open questions in Section 5. We present proofs of the results of Section 4 in Appendices A-E.

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[4]It is important to distinguish our *blind* network coding setting from the *non-coherent* network coding setting (See, for example, [18, 19].) In the former setting, we assume that the transmitters are unaware of the network coding co-efficients, whereas the receivers have perfect knowledge of the coding co-efficient matrix. In the latter setting, neither transmitters nor receivers have knowledge of the network coding co-efficients.

2 Index Coding – Problem Formulation

The index coding problem consists of a set of M independent messages

$$\mathcal{W} = \{W_1, W_2, \ldots, W_M\},$$

and a set of K destination nodes

$$\mathcal{D} = \{D_1, D_2, \cdots, D_K\},$$

with the kth destination node D_k identified as

$$D_k = (\mathcal{W}_k, \mathcal{A}_k),$$

where $\mathcal{W}_k \subseteq \mathcal{W}$ is the set of messages desired by D_k, the set $\mathcal{A}_k \subset \mathcal{W}$ is comprised of the messages available to destination D_k as side information (antidotes). The set of antidotes $\mathcal{A}_k$ satisfies $\mathcal{W}_k \cap \mathcal{A}_k = \phi$, i.e., a destination node does not desire a message that is already available to it.

An $(\mathcal{S}, n, \mathcal{R})$ index coding scheme corresponds to the choice of a finite alphabet $\mathcal{S}$ of cardinality $|\mathcal{S}| > 1$, a coding function, f, and a decoding function $g_{k,i}$, for each desired message W_i at each destination D_k. The coding function f maps all the messages to the sequence of transmitted symbols

$$f(W_1, W_2, \cdots, W_M) = S^n$$

where $S^n \in \mathcal{S}^n$ is the sequence of symbols transmitted over n channel uses. Here, $\forall m \in \{1, 2, \cdots, M\}$, message W_m is a random variable uniformly distributed over the set

$$W_m \in \{1, 2, \cdots, |\mathcal{S}|^{nR_m}\},$$

and $\mathcal{R} \in \mathbb{R}_+^M$ is simply a rate vector

$$\mathcal{R} = (R_1, R_2, \cdots, R_M)$$

that satisfies the condition that $|\mathcal{S}|^{nR_m}$ an integer for every $m \in \{1, 2, \ldots, M\}$. At each destination, D_k, there is a decoding function for each desired message

$$g_{k,i}(S^n, \mathcal{A}_k) = \hat{W}_{k,i}, \quad \forall i \text{ such that } W_i \in \mathcal{W}_k.$$

The decoding is said to be in error if any desired message is decoded incorrectly. The probability of error is

$$P_e = 1 - \text{Prob}[\hat{W}_{k,i} = W_i, \quad \forall i, k \text{ such that } W_i \in \mathcal{W}_k].$$

A rate tuple $\mathcal{R} = (R_1, R_2, \ldots, R_M) \in \mathbb{R}_+^M$ is said to be *achievable* if for every $\epsilon, \delta > 0$ there exists a $(\mathcal{S}, n, (\overline{R}_1, \overline{R}_2, \cdots, \overline{R}_M))$ coding scheme, for some $\mathcal{S}, n$, such that $\forall m \in \{1, 2, \ldots, M\}$, $\overline{R}_m \geq R_m - \delta$, and the probability of error $P_e \leq \epsilon$. The capacity region of the index coding problem is defined as the set of all achievable rate tuples $(R_1, R_2, \ldots, R_M)$ and is denoted by $\mathcal{C}$.

As an example of the notation, in Figure 1(b), we have $M = K = 5$, $\mathcal{W} = \{W_1, W_2, \cdots, W_5\}$, $D_1 = (\{W_1\}, \{W_5, W_2\})$, $D_2 = (\{W_2\}, \{W_1, W_3\})$, $D_3 = (\{W_3\}, \{W_2, W_4\})$, $D_4 = (\{W_4\}, \{W_3, W_5\})$, $D_5 = (\{W_5\}, \{W_4, W_1\})$.

The definition of capacity region presented above is in the classical sense of asymptotically vanishing probability of error, also known as ϵ-error capacity, which can in general be larger than the zero-error capacity where only achievable schemes with $P_e = 0$ are allowed. Even for networks comprised of noiseless links, e.g., in the network coding problem, the equivalence of the two is not known. Remarkably, for index coding, it has been shown by Langberg and Effros in [16] that the ϵ-error capacity is the same as the zero-error capacity.

For a $(n, \mathcal{S}, \mathcal{R})$ coding scheme, the parameter n may be interpreted as the blocklength and $\mathcal{S}$ to be the alphabet of coding. It is noteworthy that we could have chosen $\mathcal{S}$ to be an arbitrary fixed set above without

changing the capacity region. For instance, we can equivalently[5] define the capacity region to be the set of all rates $(R_1, R_2, \ldots, R_M)$ such that, for every $\epsilon, \delta > 0$, there exists a $(\{0,1\}, n, (\overline{R}_1, \overline{R}_2, \ldots, \overline{R}_M))$ coding scheme, where $\overline{R}_m \geq R_m - \delta$, whose probability of error smaller than ϵ. Nonetheless, we explicitly identify the parameter $\mathcal{S}$ in our definitions above mainly because this parameter helps distinguish between linear and non-linear coding schemes, or scalar and vector coding (See Definitions 2 and 3).

We refer to the general index coding problem statement presented above as the **groupcast** setting, where each message may be desired by multiple destination nodes. This general term includes within its scope both the multiple unicast setting where each message is desired by exactly one destination node, and the multisource multicast (or simply, multicast) setting where each message is desired by all destination nodes.

Notation: For any subset of messages $\overline{\mathcal{W}} \subset \mathcal{W}$ we define the compact notation,

$$\overline{\mathcal{W}}^c \triangleq \mathcal{W} - \overline{\mathcal{W}}$$

as the set of messages in $\mathcal{W}$ that are not in $\overline{\mathcal{W}}$. Further, we use the compact notation $W_{i,j} = \{W_i, W_j\}, W_{i,j,k} = \{W_i, W_j, W_k\}$, etc. Also we define the compact notation, $R_{i_{1:L}} = R_{i_1} + R_{i_2} + \ldots + R_{i_L}$, $\mathcal{K} = \{1, 2, \ldots, K\}$ and $\mathcal{M} = \{1, 2, \ldots, M\}$

A few important classes of the index coding problem are formalized next.

Definition 1 —Multiple Unicast Index Coding
The index coding problem is called a multiple unicast index coding problem if and only if

$$\forall k_1, k_2 \in \mathcal{K}, k_1 \neq k_2, \qquad \mathcal{W}_{k_1} \cap \mathcal{W}_{k_2} = \phi \tag{1}$$

In other words, a multiple unicast index coding problem is one where no message is desired by more than one destination.

Definition 2 —Scalar Index Coding Scheme
An $(\mathcal{S}, n, \mathcal{R})$ index coding scheme is called a scalar coding scheme if and only if

$$\mathcal{R} \;=\; \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n} \right) \tag{2}$$

In other words, a scalar index coding scheme sends one symbol for each message over n channel uses.

Definition 3 — Linear Index Coding Scheme
A linear $(\mathcal{S}, n, \mathcal{R})$ index coding scheme, achieving the rate vector $\mathcal{R} = \left(\frac{P_1}{n}, \frac{P_2}{n}, \cdots, \frac{P_M}{n} \right)$ over n channel uses, corresponds to a choice of

1. *a finite field $\mathbb{F} = \mathcal{S}$ as the alphabet*

2. *$\mathbf{V}_m \in \mathbb{F}^{n \times P_m}, \forall m \in \mathcal{M}$ as precoding matrices*

3. *$\mathbf{U}_{m,k} \in \mathbb{F}^{P_m \times n}, \forall m \in \mathcal{M}, \forall k$ such that $W_m \in \mathcal{W}_k$, as receiver combining matrices*

such that the following properties are satisfied

> *Property 1:* $\mathbf{U}_{m,k}\mathbf{V}_i = 0, \quad \forall m, i \in \mathcal{M}, k \in \mathcal{K}$ *such that $m \neq i, W_m \in \mathcal{W}_k, W_i \notin \mathcal{A}_k$*
>
> *Property 2:* $det(\mathbf{U}_{m,k}\mathbf{V}_m) \neq 0, \quad \forall m \in \mathcal{M}, k \in \mathcal{K}$ *such that $W_m \in \mathcal{W}_k$*

where all operations are over $\mathbb{F}$.

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[5]This equivalence can be shown by "simulating" a different alphabet $\mathcal{S}$ over $\{0,1\}$ by accumulating several symbols together and coding over them. Specifically a coding scheme of blocklength m over $\mathcal{S}$ can be simulated over $\{0,1\}$ over blocklength of $\lceil m \log_2 |\mathcal{S}| \rceil$ through a one-to-one mapping between $|\mathcal{S}|^m$ and $\{0,1\}^n$. The rate of such a coding scheme over $\{0, 1, \}$ could be smaller than the rate over $\mathcal{S}$ because of an integrality gap, but this gap will vanish as $m \to \infty$. We omit a formal proof of this equivalence here for brevity. We note that a linear coding scheme over one alphabet, when mapped in this manner to another alphabet, may however lead to a non-linear coding scheme over the latter alphabet.

The transmitted symbol sequence $S^n \in \mathbb{F}^{n \times 1}$ in a linear index coding scheme is

$$S^n = \sum_{m=1}^{M} \mathbf{V}_m \mathbf{X}_m \tag{3}$$

where $\mathbf{X}_m = (x_{m,1}, x_{m,2}, \cdots, x_{m,P_m})^T \in \mathbb{F}^{P_m \times 1}$ is a $P_m \times 1$ vector representing W_m. In other words, message W_m is split into P_m independent scalar streams, each of which carries one symbol from $\mathbb{F}$, and is transmitted along the corresponding column vectors (the "beamforming" vectors) of the precoding matrix for $\mathbf{V}_m$. The decoding operation for message $W_m \in \mathcal{W}_k$, desired at destination D_k, is

$$\hat{\mathbf{X}}_m = (\mathbf{U}_{m,k}\mathbf{V}_m)^{-1} \mathbf{U}_{m,k} \left(S^n - \sum_{\mathbf{X}_i : W_i \in \mathcal{A}_k} \mathbf{V}_i \mathbf{X}_i \right). \tag{4}$$

The above decoding operation can be interpreted as occurring in three steps. In the first step, the contribution from undesired messages available as antidotes, $W_i \in \mathcal{A}_k$, is eliminated from S^n, then in the second step, the remaining undesired symbols are zero-forced by Property 1, and in the final step, the desired symbols $\mathbf{X}_m$ are recovered by the invertibility of $\mathbf{U}_{m,k}\mathbf{V}_m$, which is guaranteed by Property 2.

Note that a linear encoding scheme, as explained above, is a zero-error encoding scheme. The linear coding scheme described above is also called *vector* linear coding schemes. This includes the special case where $P_i = 1, \forall i \in \{1, 2, \cdots, M\}$, which is called a *scalar* linear encoding scheme.

2.1 Discussion on Index Coding Nomenclature

Our definitions and choice of nomenclature are motivated by commonly used of terminology in classical network coding and information theory. However, the quantities defined here are sometimes referred to by different names in some index coding literature. We discuss some of these differences here.

Symmetric Capacity

Consider an achievable rate point $(R_1, R_2, \ldots, R_M)$ and let $R = \min_{m \in \{1,2,\ldots,M\}} R_m$. The quantity R is often referred to in information theory literature as an achievable *symmetric* rate. The maximum achievable symmetric rate is known as the *symmetric capacity* of the problem. Note that the quantity $\frac{1}{R}$ can be interpreted as the average number of time slots required for *all* the destinations to get a message from $\mathcal{S}$. In much of the index coding literature, the minimum value of $\frac{1}{R}$ over the set of all achievable rates - or equivalently, the reciprocal of the symmetric capacity - is the primary object of study and is often denoted as β (e.g. [17, 23, 24, 25, 26]). Indeed, in these works, the index coding problem is formulated as one where the goal is to minimize β, which is equivalent to determining the symmetric capacity of the problem.

Groupcast and Multiple Unicast Index Coding Problem

For the multiple unicast index coding problem, note that there is no loss of generality in assuming that $\mathcal{W}_k = \{W_k\}$, i.e., the kth destination intends to decode the kth message. Under this assumption, the directed graph, whose vertex set is $\mathcal{K}$ and edge set is $\{(k, j) : j \in \mathcal{A}_k, k \in \mathcal{K}\}$, is known as the *side information graph* of the index coding problem. Thus, a multiple unicast index coding problem can be equivalently represented using its side information graph. In fact, as we shall discuss later in Section 4.1, references [2, 27] describe an achievable rate for the index coding problem in terms of a certain functional of this side information graph. To distinguish from the multiple unicast setting, the groupcast index coding problem is sometimes referred to as *index coding over hypergraphs.* The motivation for this nomenclature comes from [26], where the groupcast index coding problem was modeled via hypergraphs. We refer the interested reader to the mentioned reference for details. In reference, [28], which is concurrent to our paper, the groupcast setting is also referred to as *bipartite index coding* since the problem is modeled in the reference using directed bipartite graphs. Finally, it is worth noting that the groupcast setting is also sometimes referred

to as *multiple multicast* in some network coding literature (See, for example, [29]). Our nomenclature of *groupcast*, which describes a particular structure of the messages that generalizes the multiple unicast setting, follows conventions in information theoretic literature where the message demand structure is often separated from the physical communication scenario. Thus the term groupcast can be used in the context of various network communication scenarios, such as wireless index coding. We use the term *groupcast* instead of the term "multiple multicast" to avoid ambiguity, especially since the term *multisource multicast* (or simply, multicast) is sometimes used to refer to the scenario where there are multiple sources, and every destination intends to decode every one of the sources (see, for example, [30]).

3 Index Coding as an Interference Alignment Problem

In wireless communications, interference is a natural phenomenon because of the broadcast nature of the medium. In wireline network communications, interference occurs because of multiple data streams contending for a common link. Indeed this scenario is best exemplified by the index coding problem where a single bottleneck link is shared by all data streams. In wireless systems, interference alignment provides surprising gains by exploiting the inherent diversity in distributed linear systems - i.e., the notion that every receiver sees a different alignment of signal dimensions, and therefore signals can be designed to align at one receiver and stay separable at a different receiver. In the index coding problem too, we exploit the inherent diversity that exists among the receivers because the set of antidotes is different for different receivers. This diversity ensures that even if two signals align along the bottleneck link, they can be decoded at the desired receivers if they have the appropriate set of antidotes. This alignment frees up the available dimensions (on the bottleneck link) for other messages and hence makes the system more efficient.

To better understand the role of interference alignment in the index coding problem, let us examine a $(n, \mathcal{S}, \mathcal{R})$ linear index coding scheme, where $\mathcal{R} = (\frac{P_1}{n}, \frac{P_2}{n}, \ldots, \frac{P_M}{n})$. For simplicity of exposition, let us consider a symmetric rate setting, i.e., $P_m = P, \forall m \in \mathcal{M}$. For any subset of the set of messages $\mathcal{B} \subset \mathcal{W}$, we use the following notation.

$$\mathcal{V}_\mathcal{B} = \{\mathbf{v} \in \mathbb{F}^n : \mathbf{v} \text{ is a column vector of } \mathbf{V}_i, i \in \mathcal{B}\},$$

i.e., $\mathcal{V}_\mathcal{B}$ is the set of all column vectors which belong to at least one matrix $\mathbf{V}_i : i \in \mathcal{B}$.

Receiver k receives a n dimensional vector S^n which is a linear combination of the MP column vectors of $\mathcal{V}_\mathcal{W}$. If all the MP column vectors are linearly independent, then, clearly the desired signal is resolvable and a symmetric rate of $1/M$ per message is achievable. Note that this is the rate achieved by "routing", where each of the M messages occupies P time-slots that do not overlap with the time-slots occupied by any of the remaining $M - 1$ messages[6]. The routing solution is in fact one such realization of $\mathcal{V}_\mathcal{W}$. In general, however, the messages may be resolvable even if the MP column vectors of $\mathcal{V}_\mathcal{W}$ are linearly *dependent*, because of the presence of antidotes. In fact, a rate greater than $1/M$ is possible only by making the column vectors of $\mathcal{V}_\mathcal{W}$ linearly *dependent*. Consider, for instance a receiver, say destination D_k, that wants to decode messages $\mathcal{W}_k$ and has antidotes for messages $\mathcal{A}_k$. It receives the linear combination of MP vectors in an n dimensional space, of which it can cancel the impact of $|\mathcal{A}_k|P$ vectors, $\mathcal{V}_{\mathcal{A}_k}$, using the antidotes. Therefore, from the perspective of destination D_k, it observes (after cancellation), the linear combination of $|\mathcal{W}_k|P$ desired vectors along $\mathcal{V}_{\mathcal{W}_k}$, and $(M - |\mathcal{A}_k| - |\mathcal{W}_k|)P$ interfering vectors along the columns of $\mathcal{V}_{\mathcal{W}-(\mathcal{A}_k \cup \mathcal{W}_k)}$. A necessary condition for the resolvability of messages $\mathcal{W}_k$ at destination D_k can be expressed as

$$\text{span}(\mathcal{V}_{\mathcal{W}_k}) \cap \text{span}(\mathcal{V}_{\mathcal{W}-(\mathcal{A}_k \cup \mathcal{W}_k)}) = \{0\}. \tag{5}$$

This means that a necessary condition for resolvability at destination D_k, is that the dimension of interference $\text{span}(\mathcal{V}_{\mathcal{W}-(\mathcal{A}_k \cup \mathcal{W}_k)})$ should be smaller than $n - |\mathcal{W}_k|P$ (because the vectors are all observed in an n dimensional space). Clearly, if $n - |\mathcal{W}_k|P < |\mathcal{W} - (\mathcal{A}_k \cup \mathcal{W}_k)|P$, then, the interfering vectors need to *align* in an $n - |\mathcal{W}_k|P$ dimensional space.

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[6]More formally, routing involves partitioning the column vectors of the $MP \times MP$ identity matrix into M partitions of P vectors each, and forming a linear index coding scheme by respectively setting the precoding matrix of each message to the set of column vectors of one of these M partitions.

Next we illustrate the role of interference alignment in index coding, with a series of examples presented in increasing order of complexity.

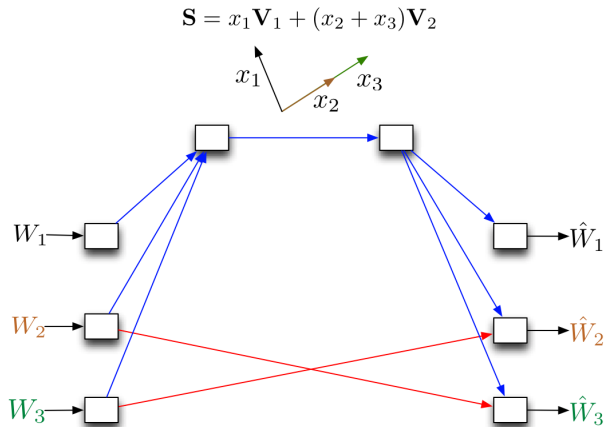## 3.1   Example 1: Scalar linear index coding with One-to-One Alignment

Figure 4: A simple example where interference alignment is useful in the index coding problem. In the example, the alignment is realized with $\mathbf{V}_2 = \mathbf{V}_3$. The alignment enables transmission of 3 scalars to the 3 corresponding users in a 2 dimensional vector space, ensuring that a rate of $1/2$ is achievable. The field $\mathbb{F}$ can be chosen arbitrarily.

A simple interference alignment solution for an index coding problem is demonstrated in Figure 4, where $K = 3$ and each user sends $P = 1$ vector. Because of alignment of $\mathbf{V}_2$ and $\mathbf{V}_3$, user 1 is able to resolve x_1. For instance, one may choose $\mathbf{V}_1 = [0,1]^T$, $\mathbf{V}_2 = \mathbf{V}_3 = [1,0]^T$, so that the two transmitted symbols on the bottleneck link are $S^2 = (S_1, S_2) = (x_2 + x_3, x_1)$, from which each destination is able to recover its desired message. Since only one symbol is sent per message, this is an example of a scalar linear index coding solution. Furthermore, since the alignment of vectors takes place in a one-to-one fashion, i.e., $\mathbf{V}_2$ aligns with $\mathbf{V}_3$, we refer to this as a one-to-one alignment solution, to be distinguished from the subspace alignment solutions to be presented soon.

3.2   Example 2: Vector linear index coding with One-to-One Alignment

The next index coding example comes from the TIM setting shown in Figure 3(a), (b), (c). Depending upon whether each base station transmitter has a message only for one corresponding receiver, or an independent message for each of the receivers that are within-range, we have the interference channel or the X channel setting, respectively. Here we consider the interference channel setting.

The interference channel setting means that base station (source) i, $1 \leq i \leq 5$, has only one message, W_i, for its corresponding receiver (destination) i. In the corresponding index coding problem, as in the interference channel setting, there are $M = 5$ distinct messages and $K = 5$ destinations. The goal is to achieve a symmetric rate of $\frac{2}{5}$ per message, which is also the capacity optimal solution. The problem is essentially identical to the setting considered in [26] to show that vector linear index coding outperforms scalar linear index coding. The achievable scheme is a vector linear scheme operating over 5 channel uses. The precoding vectors $\mathbf{V}_i, 1 \leq i \leq 5$ are 5×2 matrices and $\mathbf{X}_i$ is a 2×1 vector $\mathbf{X}_i = [x_{i,1}, x_{i,2}]^T$ representing W_i. Therefore, the transmitted symbol sequence S^5 is

$$S^5 = \sum_{i=1}^{5} \mathbf{V}_i \mathbf{X}_i. \tag{6}$$

$$S_1 = x_{5,1} + x_{2,2}$$
$$S_2 = x_{3,1} + x_{5,2}$$
$$S_3 = x_{1,1} + x_{3,2}$$
$$S_4 = x_{4,1} + x_{1,2}$$
$$S_5 = x_{2,1} + x_{4,2}$$

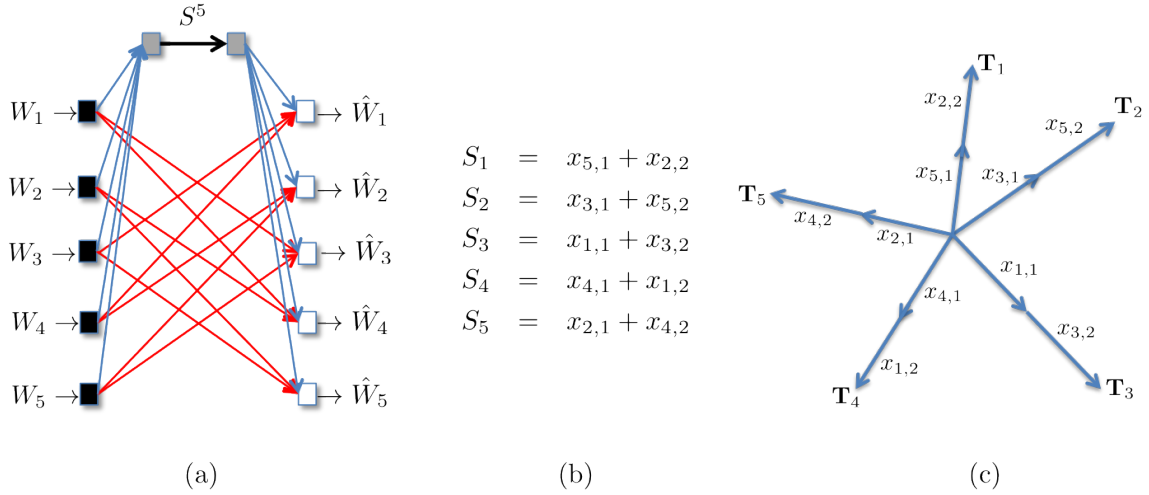(a)   (b)   (c)

Figure 5: Index coding problem corresponding to a TIM interference channel setting, (a) Desired message and antidote sets, (b) Capacity optimal solution shows the 5 dimensional S^5 space, along which 2 symbols per message are sent with one-to-one pairwise alignments along the 5 orthogonal basis vectors. The field $\mathbb{F}$ can be chosen arbitrarily.

To see why interference alignment is necessary, note that each destination has access to the 5-dimensional symbol S^5, and it also observes 2 messages as cognitive information. After removing the known streams from S^5 the receiver is left with 6 remaining unknown symbols in a 5-dimensional space. Since each destination is interested in two desired streams which span a 2-dimensional space, the remaining 4 streams that constitute interference must align in way that they occupy at most a 3-dimensional space.

Suppose $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, \mathbf{T}_5$ are 5 linearly independent vectors over the 5-dimensional space. These 5 vectors can be chosen to be the columns of 5×5 identity matrix. At destination D_1, messages W_2 and W_5, each composed of two independent scalar streams, should align such that they together occupy a 3-dimensional space. One way to do so is to perfectly align the precoding vector of one of two streams of W_2 with one of two streams of W_5, e.g., $\mathbf{V}_{2,2} = \mathbf{V}_{5,1} = \mathbf{T}_1$. Similarly, in order to satisfy the alignment constraint at all the destinations, the precoding vectors are chosen to be

$$\mathbf{V}_{2,2} = \mathbf{V}_{5,1} = \mathbf{T}_1, \ \mathbf{V}_{5,2} = \mathbf{V}_{3,1} = \mathbf{T}_2, \ \mathbf{V}_{3,2} = \mathbf{V}_{1,1} = \mathbf{T}_3, \ \mathbf{V}_{1,2} = \mathbf{V}_{4,1} = \mathbf{T}_4, \ \mathbf{V}_{4,2} = \mathbf{V}_{2,1} = \mathbf{T}_5. \quad (7)$$

Thus, the 4 undesired signal vectors are aligned at each destination such that they occupy only 3 dimensions. Now what we need to prove is the resolvability of desired messages at the corresponding destinations. They are resolvable because at each destination D_i, the desired messages are received in a space $\mathcal{V}_{\mathcal{W}_i}$ that is

linearly independent from the space of interfering messages denoted as $\mathcal{I}_i$.

$$
\mathcal{V}_{\mathcal{W}_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{I}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathcal{V}_{\mathcal{W}_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_3} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathcal{V}_{\mathcal{W}_4} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \mathcal{I}_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_5} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{I}_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{8}
$$

Finally, if we choose the receiver combining matrices as follows, we can easily verify that both property 1 and property 2 are satisfied

$$
\text{At destination } D_1: \quad \mathbf{U}_{1,1} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad \text{At destination } D_2: \quad \mathbf{U}_{2,2} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
\text{At destination } D_3: \quad \mathbf{U}_{3,3} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \text{At destination } D_4: \quad \mathbf{U}_{4,4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\text{At destination } D_5: \quad \mathbf{U}_{5,5} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{9}
$$

Note that the field can be chosen arbitrarily, i.e., any choice of $\mathbb{F}$ works for the linear solution presented above. The solution also translates to the real or complex fields, thereby establishing the DoF for the corresponding TIM problem as 2/5 per message (See [5] for more details).

3.3   Example 3: Scalar linear index coding with Subspace Alignment

Like the previous example, this example also corresponds to the TIM problem in Fig. 3(a),(b),(c). The difference is in the message sets. While in the previous example, each base station served only one receiver, here we assume that each base station has 3 independent messages, one for each of the 3 receivers that are within receiving range of the base station. In the parlance of wireless networks, while the previous setting is an interference network, the current setting is an X network. In an X network, there is an independent message to be communicated between each transmitter-receiver pair that are within range of each other, i.e., have a non-zero channel coefficient between them.

The index coding problem for the X network setting is also well defined. There is an independent message between each source and destination pair that are not connected via an antidote link. The specific index coding problem that we solve in this example is shown in Figure 6. Notice that an X network setting is a mulitple unicast setting, since each message has a unique source and a unique destination. It is also sometimes referred to as the *all unicast* setting, since here all possible non-trivial unicast flows are simultaneously active (the unicast flows between source destination pairs connected by infinite capacity antidote links are ignored because they trivially have infinite rate). Also note that as always it is possible, without loss of generality, to represent this index coding problem with only one message per source and only one message per destination by increasing the number of sources and destinations, but we prefer the compact representation shown in Figure 6 which directly reflects the X channel setting. X networks often lead to interesting interference alignment problems. Indeed, that is the case with this example as well, where one-to-one alignment does not

suffice and the optimal index coding scheme is a scalar linear subspace alignment scheme. Next we proceed to describe the alignment solution.

$$S_1 = x_3 + x_{10} + x_{13} + x_{14} + x_{15}$$
$$S_2 = x_5 + x_{10} + x_{11} + x_{14} + x_{15}$$
$$S_3 = x_7 + x_{10} + x_{11} + x_{12} + x_{15}$$
$$S_4 = x_2 + x_8 - x_{10} - x_{15}$$
$$S_5 = x_4 + x_9 + x_8 - x_{10} - x_{11} - x_{15}$$
$$S_6 = x_1 + x_6 + x_8 + x_{14}$$

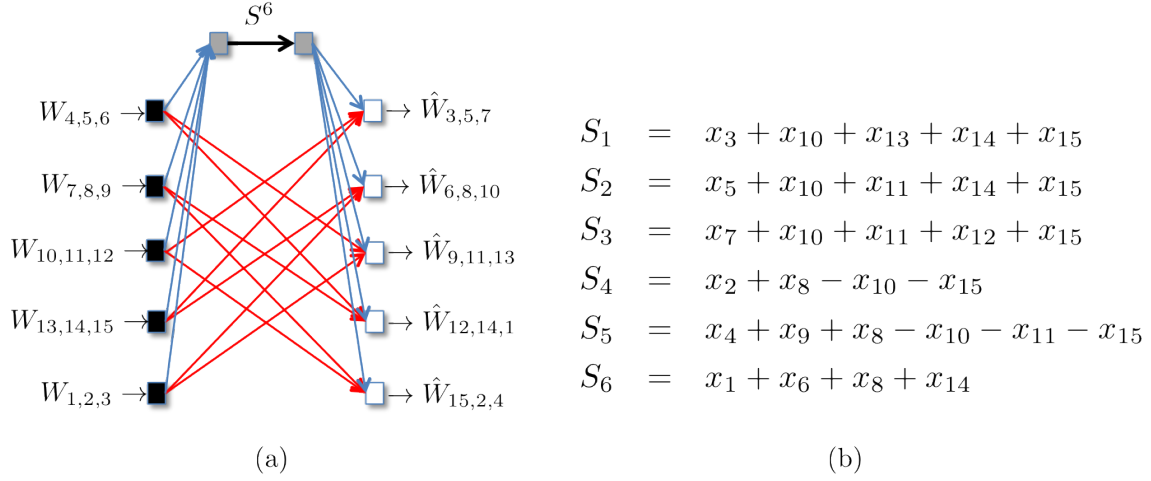(a)   (b)

Figure 6: Index coding problem corresponding to a TIM X channel setting, (a) Desired message and antidote sets, (b) Capacity optimal solution shows the 6 dimensional S^6 space, along which 15 symbols are sent. The field $\mathbb{F}$ can be chosen arbitrarily.

There are $M = 15$ distinct messages and $K = 5$ destinations. Each black square on the left is a base station (source) sending 3 distinct messages. Base station (source) $i, 1 \leq i \leq 4$ sends messages $W_{3i+1:3i+3}$ and the 5-th base station sends $W_{1:3}$. We have

$$\mathcal{W}_1 = \{W_{3,5,7}\}, \quad \mathcal{A}_1 = \{W_{1,2,4,6,8,9} \cup \mathcal{W}_1\}^c$$
$$\mathcal{W}_2 = \{W_{6,8,10}\}, \quad \mathcal{A}_2 = \{W_{4,5,7,9,11,12} \cup \mathcal{W}_2\}^c$$
$$\mathcal{W}_3 = \{W_{9,11,13}\}, \quad \mathcal{A}_3 = \{W_{7,8,10,12,14,15} \cup \mathcal{W}_3\}^c$$
$$\mathcal{W}_4 = \{W_{12,14,1}\}, \quad \mathcal{A}_4 = \{W_{10,11,13,15,2,3} \cup \mathcal{W}_4\}^c$$
$$\mathcal{W}_5 = \{W_{15,2,4}\}, \quad \mathcal{A}_5 = \{W_{13,14,1,3,5,6} \cup \mathcal{W}_5\}^c$$

Our goal is to achieve the symmetric rate of $\frac{1}{6}$ per message, which is also the capacity of this network (the outer bound follows from a subsequent solution of a broader class of X networks, presented in Section 4.3.3). The achievable scheme is a scalar linear achievable scheme over 6 channel uses and hence the precoding vectors $\mathbf{V}_i, 1 \leq i \leq 15$, are 6×1 vectors. Therefore, the transmitted symbol sequence S^6 is

$$S^6 = \sum_{i=1}^{15} \mathbf{V}_i x_i \tag{10}$$

where the scalar symbol x_i represents message W_i. To see why interference alignment is necessary, note that each destination has access to the 6-dimensional symbol S^6, and it also observes 6 messages as cognitive information. After removing the known symbols from S^6 the receiver is left with 9 remaining unknown symbols in a 6-dimensional space. Since each destination is interested in 3 desired symbols, they must occupy a 3-dimensional space, leaving only 3 dimensions within which the remaining 6 undesired symbols that constitute interference, must align.

Suppose $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6$ are 6 linearly independent vectors over 6-dimensional space. These 6 vectors can be chosen to be the columns of 6×6 identity matrix. Since $W_{3,5,7}$ are desired at destination D_1, they should span a 3-dimensional space and therefore are sent over 3 linearly independent vectors $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$. On the other hand, $W_{2,4}$ are the interfering messages at destination D_1 and both are desired at destination D_5. So these two messages should be sent over vectors that are linearly independent of each other and

linearly independent from $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$. Therefore, we send W_2, W_4 over $\mathbf{T}_4, \mathbf{T}_5$, respectively. Also W_1 is desired at destination 4 and is considered as an interfering message at both destination D_1 and destination D_5. So W_1 should be sent over a vector that is linearly independent from $\mathbf{T}_1, \ldots, \mathbf{T}_5$ and is chosen to be sent along $\mathbf{T}_6$.

Interfering messages W_1, W_2, W_4 occupy a 3-dimensional space at destination D_1. The remaining interfering messages at destination D_1, i.e., $W_{6,8,9}$, should be sent along precoding vectors chosen such that they remain in the same span as span$(\mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6)$, i.e.,

$$\text{span}(\mathbf{V}_6, \mathbf{V}_8, \mathbf{V}_9) \in \text{span}(\mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6) \tag{11}$$

One way to satisfy (11) is by choosing $\mathbf{V}_6 = \mathbf{T}_6$, $\mathbf{V}_9 = \mathbf{T}_5$. The remaining precoding vector is designed later to satisfy the following

$$\text{At destination } D_1 : \text{span}(\mathbf{V}_8) \in \text{span}(\mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6) \tag{12}$$

Similarly, to satisfy the requirement of aligning interfering messages in 3-dimensional space at different destinations, we design the precoding vectors as follows:

$$\text{At destination } D_2 : \quad \text{span}(\mathbf{V}_{11}) \in \text{span}(\mathbf{T}_2, \mathbf{T}_3, \mathbf{T}_5) \tag{13}$$
$$\text{At destination } D_3 : \quad \text{span}(\mathbf{V}_{14}) \in \text{span}(\mathbf{V}_8, \mathbf{T}_3, \mathbf{V}_{10}) \tag{14}$$
$$\text{At destination } D_4 : \quad \text{span}(\mathbf{V}_2) = \text{span}(\mathbf{T}_4) \in \text{span}(\mathbf{V}_{11}, \mathbf{V}_{10}, \mathbf{T}_1) \tag{15}$$
$$\text{At destination } D_5 : \quad \text{span}(\mathbf{V}_5) = \text{span}(\mathbf{T}_2) \in \text{span}(\mathbf{V}_{14}, \mathbf{T}_1, \mathbf{T}_6) \tag{16}$$
$$\mathbf{V}_{12} = \mathbf{T}_3, \ \mathbf{V}_{15} = \mathbf{V}_{10}, \ \mathbf{V}_{13} = \mathbf{T}_1 \tag{17}$$

leading to the formulation

$$\mathbf{V}_8 = a_1\mathbf{T}_4 + a_2\mathbf{T}_5 + a_3\mathbf{T}_6 \tag{18}$$
$$\mathbf{V}_{11} = b_1\mathbf{T}_2 + b_2\mathbf{T}_3 + b_3\mathbf{T}_5 \tag{19}$$
$$\mathbf{V}_{10} = c_1\mathbf{V}_{14} + c_2\mathbf{T}_3 + c_3\mathbf{V}_8 \tag{20}$$
$$\mathbf{V}_{10} = d_1\mathbf{V}_{11} + d_2\mathbf{T}_1 + d_3\mathbf{T}_4 \tag{21}$$
$$\mathbf{V}_{14} = e_1\mathbf{T}_1 + e_2\mathbf{T}_2 + e_3\mathbf{T}_6 \tag{22}$$

where $a_1, b_1, \ldots, d_3, e_3$ are linear combination coefficients. In order to satisfy (18)-(22), the only thing that restricts us from choosing the linear combination coefficients to be random is that $\mathbf{V}_{10}$ should satisfy both (20) and (21). If we substitute from (18),(19) and (22) into (20) and (21), $\mathbf{V}_{10}$ should satisfy the following two equations

$$\mathbf{V}_{10} = c_1e_1\mathbf{T}_1 + c_1e_2\mathbf{T}_2 + c_2\mathbf{T}_3 + c_3a_1\mathbf{T}_4 + c_3a_2\mathbf{T}_5 + (c_3a_3 + c_1e_3)\mathbf{T}_6 \tag{23}$$
$$\mathbf{V}_{10} = d_2\mathbf{T}_1 + d_1b_1\mathbf{T}_2 + d_1b_2\mathbf{T}_3 + d_3\mathbf{T}_4 + d_1b_3\mathbf{T}_5 \tag{24}$$

To satisfy (23) and (24), we have the following

$$d_2 = c_1e_1 \tag{25}$$
$$d_1b_1 = c_1e_2 \tag{26}$$
$$d_1b_2 = c_2 \tag{27}$$
$$d_3 = c_3a_1 \tag{28}$$
$$d_1b_3 = c_3a_2 \tag{29}$$
$$c_3a_3 + c_1e_3 = 0 \tag{30}$$

Clearly, there are many solutions. One of the solutions for these system of nonlinear polynomial equations is $a_1 = a_2 = a_3 = b_1 = b_2 = c_1 = c_2 = d_1 = d_2 = e_1 = e_2 = e_3 = 1$, $b_3 = c_3 = d_3 = -1$, according to which,

the precoding vectors are chosen as follows

$$
\begin{aligned}
\mathbf{V}_8 &= \mathbf{T}_4 + \mathbf{T}_5 + \mathbf{T}_6 \\
\mathbf{V}_{11} &= \mathbf{T}_2 + \mathbf{T}_3 - \mathbf{T}_5 \\
\mathbf{V}_{14} &= \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_6 \\
\mathbf{V}_{10} &= \mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 - \mathbf{T}_4 - \mathbf{T}_5
\end{aligned}
$$

Note that this is not a one-to-one alignment solution, e.g., $\mathbf{V}_8$ does not align with $\mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6$ individually. In fact it is pairwise linearly independent of all three. $\mathbf{V}_8$ aligns only within the subspace spanned by $\mathbf{T}_4, \mathbf{T}_5, \mathbf{T}_6$. This is referred to as subspace alignment.

After satisfying all the alignment constraints, we need to prove the resolvability of desired messages at the corresponding destinations. They are resolvable because at each destination D_i, the desired messages are received in a space denoted as $\mathcal{V}_{\mathcal{W}_i}$ which is linearly independent from the space of interfering messages, $\mathcal{I}_i$.

$$
\mathcal{V}_{\mathcal{W}_1} = [\mathbf{V}_3\ \mathbf{V}_5\ \mathbf{V}_7] =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
\mathcal{I}_1 =
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_2} = [\mathbf{V}_6\ \mathbf{V}_8\ \mathbf{V}_{10}] =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & -1 \\
0 & 1 & -1 \\
1 & 1 & 0
\end{bmatrix}, \quad
\mathcal{I}_2 =
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_3} = [\mathbf{V}_9\ \mathbf{V}_{11}\ \mathbf{V}_{13}] =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
1 & -1 & 0 \\
0 & 0 & 0
\end{bmatrix}, \quad
\mathcal{I}_3 =
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 1
\end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_4} = [\mathbf{V}_{12}\ \mathbf{V}_{14}\ \mathbf{V}_1] =
\begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 1 & 1
\end{bmatrix}, \quad
\mathcal{I}_4 =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & -1 \\
0 & 0 & 0
\end{bmatrix}
$$

$$
\mathcal{V}_{\mathcal{W}_5} = [\mathbf{V}_{15}\ \mathbf{V}_2\ \mathbf{V}_4] =
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0 \\
-1 & 1 & 0 \\
-1 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad
\mathcal{I}_5 =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

Finally, the receiver combining matrices are chosen as follows, so that both property 1 and property 2 are satisfied

At destination D_1: $\mathbf{U}_{3,1} = [1\ 0\ 0\ 0\ 0\ 0], \quad \mathbf{U}_{5,1} = [0\ 1\ 0\ 0\ 0\ 0], \quad \mathbf{U}_{7,1} = [0\ 0\ 1\ 0\ 0\ 0]$

At destination D_2: $\mathbf{U}_{6,2} = [-1\ 0\ 0\ -1\ 0\ 1], \quad \mathbf{U}_{8,2} = [1\ 0\ 0\ 1\ 0\ 0], \quad \mathbf{U}_{10,2} = [1\ 0\ 0\ 0\ 0\ 0]$

At destination D_3: $\mathbf{U}_{9,3} = [0\ 1\ 0\ 0\ 1\ -1]$, $\mathbf{U}_{11,3} = [0\ 1\ 0\ 1\ 0\ -1]$, $\mathbf{U}_{13,3} = [1\ 0\ 0\ 1\ 0\ -1]$

At destination D_4: $\mathbf{U}_{12,4} = [0\ 0\ 1\ 0\ 1\ 0]$, $\mathbf{U}_{14,4} = [0\ 1\ 0\ 0\ 1\ 0]$, $\mathbf{U}_{1,4} = [0\ -1\ 0\ 0\ -1\ 1]$

At destination D_5: $\mathbf{U}_{15,5} = [0\ 0\ 1\ 0\ 0\ 0]$, $\mathbf{U}_{2,5} = [0\ 0\ 1\ 1\ 0\ 0]$, $\mathbf{U}_{4,5} = [0\ 0\ 1\ 0\ 1\ 0]$.

Note that the field can be chosen arbitrarily, i.e., any choice of $\mathbb{F}$ works for the linear solution presented above. The solution also translates to the real or complex fields, thereby establishing the DoF for the corresponding TIM problem as $1/6$ per message.

4 Background and Results

In this section we present the results of this work along with the relevant background comprised of related prior work. In particular, we demonstrate the power of the interference alignment alignment perspective, which was described in the previous section, through the following results.

1. Our first result settles an open question related to the sufficiency of linear coding for multiple unicast index coding problems. We show in Theorem 1 in Section 4.1 that linear codes are insufficient for the multiple unicast index coding problem.

2. In our second main result presented in Theorem 3 in Section 4.2, we explicitly characterize the index coding scenarios where a rate of $\frac{1}{L+1}$ per message is achievable in index coding scenarios where each destination desires to decode at least L messages. Our result generalizes a result of Blasiak et. al. [17], where the acheivability of a rate of $1/2$ was characterized.

3. In Theorems 5 and 6 presented in Section 4.3, we solve the index coding problem for certain symmetric antidote structures, some of which are motivated by applications to topological interference management.

4.1 Insufficiency of Linear Codes for Multiple Unicast Index Coding

The index coding problem was introduced in the multiple *unicast* setting by Birk and Kol in [2] and subsequently studied by Bar-Yossef et al. in [31], where the rate achievable by the optimal scalar linear code was related to a graph functional that they called the minrank of the graph. For a fixed finite field, the minrank of a graph $G = (V, E)$ is defined as follows: An $|E| \times |E|$ matrix $\mathcal{A} = (a_{i,j})$ over the finite field is said to *fit* graph $G = (V, E)$ if $a_{i,i} \neq 0$, and $a_{i,j} = 0$ whenever (i, j) is not an edge of the side information graph. It was recognized in [2, 31] that the reciprocal of the symmetric rate achievable by the optimal scalar linear code over field $\mathbb{F}$ is equal to the minrank of the side information graph evaluated over $\mathbb{F}$[7]. Algorithms for finding or approximating the minrank function are proposed in [32] and [23]. Haviv and Langberg in [24] investigated the minranks of random digraphs. Within the class of scalar index coding schemes, the insufficiency of linear codes was established by Lubetzky and Stav in [25]. Alon et al. further established the sub-optimality of scalar linear solutions in [26] by highlighting the benefits of vector linear coding. Since vector linear coding is a more powerful form of linear coding than scalar linear coding, its optimality relative to non-linear coding schemes has been a topic of great interest. This line of work has produced a series of results in general contexts that have progressively closed in on the index coding problem. The series of breakthroughs started with the work of Dougherty et al. in [33] who used the connections between the network coding problem and representability of matroids to establish a gap between vector linear coding and non-linear coding for the multiple unicast *network coding* problem. However, the gap result of Dougherty et al. did not apply to the index coding problem which is a special case of the network coding problem. Building upon the work of [33], Rouayheb et al. established connections between the representability of matroids and the problem of groupcast index coding [3]. Most recently, Blasiak et al. further developed this relationship in [34], and used the construction of lexicographic products of graphs to show that for the problem of groupcast index

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[7] The connection of the achievable rate to the minrank of the graph was first shown in the context of coding over the binary field in [2]. An extension of this concept to an arbitrary finite field was described in [31].

coding, vector linear codes are strictly out-performed by non-linear codes. However, the gap result for the problem of groupcast index coding, does not apply to the multiple *unicast* index coding problem. Hence, in the original setting of multiple *unicast* index coding, which is also the most commonly studied form of index coding, the optimality of linear codes remains open.

Our first result, presented in Theorem 1, settles this issue.

Theorem 1 *Linear coding is insufficient to achieve the capacity region of the multiple* **unicast** *index coding problem.*

The proof of Theorem 1 is presented in Appendix A. The key to this result is the construction of an equivalent multiple unicast problem for an arbitrary groupcast setting. Loosely speaking, this equivalent multiple unicast setting has the following two properties:

- Any rate that is achievable with linear coding in the groupcast setting is also achievable with linear coding in the multiple-unicast setting and *vice-versa*.

- Any rate achievable in the groupcast setting (using any achievable scheme including possibly non-linear strategies) is achievable in the multiple unicast setting[8].

Since [34] has a construction of a problem of groupcast index coding where linear coding schemes are insufficient, the above conditions imply that linear coding is insufficient in the equivalent multiple unicast setting as well. Our equivalent multiple unicast construction is described formally next (See Construction 1). The above two properties are stated formally in Theorem 2. In the sequel, the overline notation is used to describe entities in the multiple unicast index coding setting (for example, a destination in the multiple unicast setting is denoted as $\overline{D}_{i,j}$.).

The basic idea of our construction of the equivalent multiple unicast index coding problem from a groupcast index coding problem is to replace each groupcast message that is desired by multiple destinations in the groupcast setting, with a new set of independent messages in the multiple unicast setting. Specifically, in the multiple unicast construction, there is one independent message for each destination that originally desires the groupcast message, implying that, as required for a multiple unicast setting, no message is required by more than one destination. The equivalent multiple unicast construction has, in addition to the destinations in the original groupcast setting, one auxiliary destination corresponding to each groupcast message. Thus, the number of destinations in the multiple unicast construction is equal to the number of destinations in the original groupcast setting plus the number of messages in the original groupcast setting. We next present a formal description of our equivalent multiple unicast construction (Construction 1). Later, we state Theorem 2 and provide an intuitive overview of the proof, which uses the interference alignment perspective.

Construction 1 *Consider an arbitrary problem of groupcast index coding where there are M messages and K destination nodes. Without loss of generality[9], we assume that*

- *each message is desired by L destinations, and*

- *each destination desires exactly one message, so that $K = LM$.*

In this index coding problem, the set of messages, the set of destinations, and the set of antidotes are respectively denoted by $\mathcal{W}, \mathcal{D}, \{\mathcal{A}_k : k = 1, 2, \ldots, K\}$, where

$$D_k = (\{W_m\}, \mathcal{A}_k), m = \lceil k/L \rceil$$

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[8]Note that this property goes only one way, i.e., we show that an arbitrary achievable scheme in the groupcast setting can be translated to an achievable scheme in the multiple unicast setting. The question of whether a rate achievable via an arbitrary coding scheme in the multiple unicast setting is also achievable in the groupcast setting is in fact an open problem. However the property that we show suffices to establish the insufficiency of linear codes in the multiple unicast setting.

[9]There is no loss of generality in the first assumption because if there is a message desired by $L' < L$ destinations, then we can add $L - L'$ virtual destinations with antidote sets identical to any of the L' (original) destinations, and which desire the appropriate subset of the messages desired by the original destinations. Similarly, there is no loss of generality in the second assumption because if a destination desires multiple messages, it can be equivalently replaced by multiple copies of itself, each interested in only one of the originally desired messages.

We now construct an equivalent multiple unicast index coding problem as follows. The multiple unicast setting has $LM+M$ messages and $K+M$ destinations. In this multiple unicast index coding problem, denote the messages as

$$\overline{\mathcal{W}} = \{\overline{W}_{1,0}, \overline{W}_{1,1}, \overline{W}_{1,2}, \ldots, \overline{W}_{1,L}, \overline{W}_{2,0}, \overline{W}_{2,1}, \ldots, \overline{W}_{2,L}, \ldots, \overline{W}_{M,0}, \overline{W}_{M,1}, \overline{W}_{M,2}, \ldots, \overline{W}_{M,L}\}.$$

The destinations in the multiple unicast system are denoted as

$$\overline{D}_{i,j} : i = 1, 2, \ldots, M, j = 0, 1, 2, \ldots, L$$

and the antidotes are denoted as

$$\overline{\mathcal{A}}_{i,j}, i = 1, 2, \ldots, M, j = 0, 1, 2, \ldots, L$$

so that destination $\overline{D}_{i,j} = (\overline{W}_{i,j}, \overline{\mathcal{A}}_{i,j})$. The set of antidotes is

$$\overline{\mathcal{A}}_{i,j} = \left\{ \begin{array}{ll} \mathcal{A}_{(i-1)L+j,*} \cup \overline{\mathcal{W}}_{*,0} \cup \{\overline{W}_{i,l} : l \neq j\}, & j \neq 0 \\ \overline{\mathcal{W}} - \overline{\mathcal{W}}_{i,*}, & j = 0 \end{array} \right\}, \tag{31}$$

where

$$\mathcal{A}_{k,*} = \{\overline{W}_{m,l} : W_m \in \mathcal{A}_k, l = 0, 1, 2, \ldots, L\} \tag{32}$$
$$\overline{\mathcal{W}}_{*,0} = \{\overline{W}_{1,0}, \overline{W}_{2,0}, \cdots, \overline{W}_{M,0}\} \tag{33}$$
$$\overline{\mathcal{W}}_{k,*} = \{\overline{W}_{k,0}, \overline{W}_{k,1}, \cdots, \overline{W}_{k,L}\}. \tag{34}$$

In this construction, $\overline{\mathcal{W}}_{*,0}$ is the set of auxiliary messages and $\overline{\mathcal{W}}_{k,*}$ is the set of independent messages that has replaced message W_k. We provide here an informal description of the construction. We will use an instance of the construction depicted in Fig. 7 to aid our description.

In the groupcast setting, a message W_m is desired to be decoded by L destinations, $D_{(m-1)L+1}, D_{(m-1)L+2}, \ldots, D_{mL}$. The equivalent multiple unicast construction described above can be intuitively viewed as one obtained by replacing the message W_m by L independent messages $\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}$ — each of these L messages desired uniquely by one destination as required for a multiple unicast index coding problem. In the example of Fig. 7, messages $\overline{W}_{1,1}$ and $\overline{W}_{1,2}$ are the independent messages that replace the groupcast message W_1. Similarly messages $\overline{W}_{2,1}$ and $\overline{W}_{2,2}$ are the independent messages that replace the groupcast message W_2.

There is a correspondence between destination $D_{(m-1)L+j}$ in the groupcast setting, and destination $\overline{D}_{m,j}$ in the equivalent multiple unicast setting. This correspondence is maintained in the antidote structure which is carried over from the groupcast setting to the multiple unicast setting. In particular, if a message W_l is present as an antidote at destination $D_{(m-1)L+j}$ in the groupcast setting, then, destination $\overline{D}_{m,j}$ has $\overline{W}_{l,1}, \overline{W}_{l,2}, \ldots, \overline{W}_{l,L}$ as antidotes in the multiple unicast setting (and vice-versa). Notice this in the example of Fig. 7, where the destination $\overline{D}_{2,2}$ has both $\overline{W}_{1,1}$ and $\overline{W}_{1,2}$ as antidotes.

Destination $\overline{D}_{m,l}$ has as antidotes, all the messages $\overline{W}_{m,l'}, l' \neq l$. For instance, in the example of Fig. 7, $\overline{D}_{1,1}$ has $\overline{W}_{1,2}$ as an antidote, and $\overline{D}_{1,2}$ has $\overline{W}_{1,1}$ as an antidote. Similarly, $\overline{D}_{2,1}$ has $\overline{W}_{2,2}$ as an antidote, and $\overline{D}_{2,2}$ has $\overline{W}_{2,1}$ as an antidote.

Finally, there is one auxiliary destination in the multiple unicast setting corresponding to each groupcast message. The mth auxiliary destination, denoted as $\overline{D}_{m,0}$, intends to decode an independent message $\overline{W}_{m,0}$, which is available as an antidote to all other destinations (including other auxiliary destinations) except those that desire to decode $\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}$. The mth auxiliary destination itself has all the messages as antidotes except $\overline{W}_{m,0}, \overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}$. For instance, in Fig. 7, destination $\overline{D}_{1,0}$ has $\overline{\mathcal{W}}_{2,*}$ as antidotes, and similarly, destination $\overline{D}_{2,0}$ has $\overline{\mathcal{W}}_{1,*}$ as antidotes. Auxilliary message $\overline{W}_{1,0}$ is available as an antidote for all the destinations $\overline{D}_{2,j}, j = 0, 1, 2$. A similar observation can be made for the availability of message $\overline{W}_{2,0}$.

To summarize the construction, it is instructive to observe the set of "interferers" seen by each destination in the multiple unicast setting. The set of interfering messages (i.e., those messages for which the antidotes are unavailable) seen by destination $\overline{D}_{m,j}$ for $j \neq 0$ are
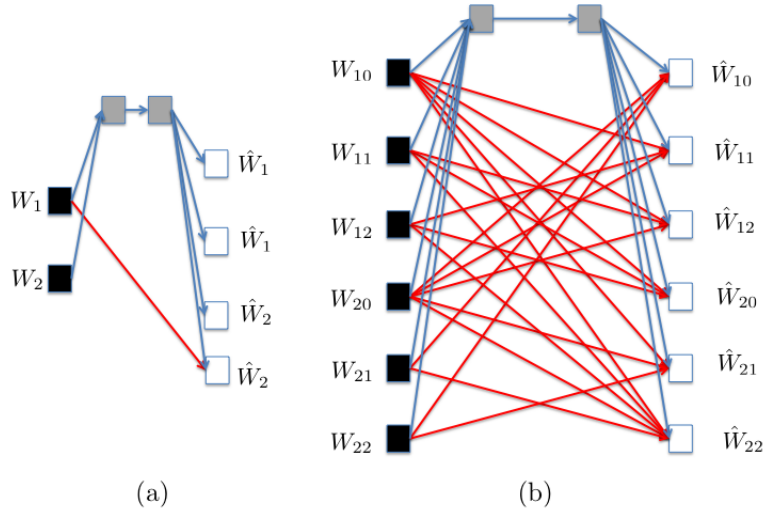
Figure 7: (a) A groupcast index coding problem where $M = 2$, $L = 2$ and each message is desired by two destinations (b) Equivalent multiple unicast index coding problem

- $\overline{W}_{l,1}, \overline{W}_{l,2}, \ldots, \overline{W}_{l,M}$ for every l such that W_l interferes with $D_{(m-1)L+j}$ in the groupcast setting, and

- the auxiliary message $W_{m,0}$.

The set of interferers seen by auxiliary destination $\overline{D}_{m,0}$ is $\{\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}\}$. The next step that we take, loosely speaking, is to take an index coding scheme that is able to resolve interference in the groupcast setting, and show that an analogous scheme can resolve the interference in the equivalent multiple unicast setting , and vice-versa. More specifically, we describe in Theorem 2, which is stated next, a relation between the rates achievable in the groupcast setting and the equivalent multiple unicast construction.

Theorem 2 *For every groupcast index coding problem, the equivalent multiple unicast index coding problem specified by Construction 1, satisfies the following properties:*

Property 1: *Rate tuple $\mathcal{R} = (R_1, R_2, \ldots R_M)$ is achievable in the groupcast setting only if the rate tuple*

$$R_{i,j} = \left\{ \begin{array}{cc} R_i, & j \neq 0 \\ 1 - R_i, & j = 0 \end{array} \right\} \tag{35}$$

is achievable in the equivalent multiple unicast problem, for all $i \in \{1, 2, \ldots, M\}, j \in \{0, 1, \ldots, L\}$.

Property 2: *Rate tuple $\mathcal{R} = (R_1, R_2, \ldots R_M)$ is achievable via **linear** coding in the groupcast setting **if and only if** the rate tuple*

$$R_{i,j} = \left\{ \begin{array}{cc} R_i, & j \neq 0 \\ 1 - R_i, & j = 0 \end{array} \right\} \tag{36}$$

is achievable via linear coding in the multiple unicast problem, for all $i \in \{1, 2, \ldots, M\}, j \in \{0, 1, \ldots, L\}$.

We present a formal proof of Theorem 2 in Appendix A, where we also show Theorem 1. Here, we summarize the intuition behind the proof.

Consider a linear coding achievable scheme in the groupcast setting. We can use this scheme to build a scheme over the multiple unicast setting as follows. All messages in the set $\{\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}\}$, simultaneously use (i.e., align in) $\mathbf{V}_m$ - the nR_m dimensional space used by W_m in the linear coding scheme in the groupcast setting. The auxiliary message $\overline{W}_{m,0}$ occupies a $n(1 - R_m)$ dimensional space that is

linearly independent of $\mathbf{V}_m$. This strategy leads to an achievable linear coding scheme in the multiple unicast setting. To see this, observe that the multiple unicast setting essentially inherits the antidote structure from the groupcast setting allowing destination $\overline{D}_{m,j}$ to cancel any interference from $\overline{W}_{l,j'}$ for $l \neq m$. This is because this interference arrives along $\mathbf{V}_l$ and achievability in the groupcast setting implies that the vectors $\mathbf{V}_m$ are linearly independent of all the vectors of $\mathbf{V}_l$. Also, notice that in our multiple unicast construction, a destination that desires one of the messages $\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}$ has the remaining $L-1$ messages from this set as antidotes, hence allowing these messages to align in $\mathbf{V}_m$ without interfering with each other. Therefore, the only other interference that a destination $\overline{D}_{m,j}$ faces is from the auxiliary message $\overline{W}_{m,0}$. This interference can be cancelled because by design of our scheme, the auxiliary messages chose precoding vectors that are linearly independent of the precoding vectors $\mathbf{V}_m$ used by message $\overline{W}_{m,j}$. Using a similar argument, it can be noted that all the auxiliary destinations can resolve their interference as well. This means that any achievable scheme in the groupcast setting naturally translates to the multiple unicast setting as per the rates prescribed by Theorem 2.

For the converse establishing equivalence (in Property 2), we need to show that a linear scheme achieving rate R_m for messages $\overline{W}_{m,1}, \ldots, \overline{W}_{m,L}$ and rate $1 - R_m$ for message $\overline{W}_{m,0}$ in the multiple unicast setting can be translated to a linear scheme achieving rate R_m for message W_m in the groupcast setting. The auxiliary destinations $\overline{D}_{m,0}$ play a key role ensuring that the achievable scheme can be translated from the multiple unicast to the groupcast setting. The message $\overline{W}_{m,0}$ desired by auxiliary destination $\overline{D}_{m,0}$ is provided as an antidote to all the other destinations and therefore does not affect achievability of rate R_l for any other destination $l \neq m$. Destination $\overline{D}_{m,0}$ faces interference from the messages $\overline{W}_{m,0}, \overline{W}_{m,1}, \ldots, \overline{W}_{m,L}$. This means that the space occupied by $\overline{W}_{m,0}$ has to be linearly independent of $\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots \overline{W}_{m,L}$. Now, if this message $\overline{W}_{m,0}$ has a rate of $1 - R_m$, then the interfering messages faced by this destination – $\overline{W}_{m,1}, \overline{W}_{m,2}, \ldots, \overline{W}_{m,L}$ – have to *together* occupy a space of dimension nR_m. This implies that if each of these interfering messages have to achieve a rate R_m, they have to align (nearly) perfectly. This aligned space can be used in the groupcast setting to encode W_m at rate R_m. Since, the groupcast setting and the multiple unicast setting have essentially similar antidote structures (with the exception of auxiliary messages), the use of this aligned space leads to an achievable scheme in the groupcast setting where the interference can be resolved, hence establishing the desired equivalence. The equivalence for non-linear schemes (Property 1) is based on random coding arguments and superposition coding for auxiliary messages, according to the detailed proof presented in Appendix A.

Remark 1 *It is worth noting that our result only shows the insufficiency of linear coding in the multiple unicast setting for achieving the entire capacity region. The question of whether linear coding suffices for the achievability of the symmetric capacity, a quantity that is commonly studied in previous index coding literature, for the multiple unicast setting remains open. This is because in Theorem 2, the rate of an auxiliary message is not equal to the rate of the other corresponding messages.*

4.2 Feasibility of Symmetric Rate $\frac{1}{L+1}$ when $\forall k, |\mathcal{W}_k| \geq L$

Recall that $\mathcal{W}_k$ denotes the set of messages that are intended to be decoded at destination D_k. Suppose that $\forall k, |\mathcal{W}_k| \geq L$, then, without loss of generality we can assume that $|\mathcal{W}_k| = L, \forall k \in \mathcal{K}$, i.e., each destination is interested in decoding exactly L distinct messages, i.e., $\mathcal{W}_k = \{W_{k_1}, W_{k_2}, \ldots, W_{k_L}\}, \forall k_i \in \mathcal{M}$. This is because destinations that wish to decode more than L messages can be split into multiple destinations with the same set of antidotes, that each wish to decode a subset of size L of the original messages, such that the union of these subsets is the original set of desired messages.

Note that the assumption $|\mathcal{W}_k| \geq L$ implicitly means that $M \geq L$. If $M = L$ or $M = L+1$, it is easy to achieve rate bigger than or equal to $\frac{1}{L+1}$ per message by sending each message separately at each time and each destination can achieve rate $\frac{1}{L}$ or $\frac{1}{L+1}$, respectively. If $M > L+1$, intuitively, since all the desired messages must pass through the bottleneck link of capacity 1 and all messages must simultaneously achieve rate $\frac{1}{L+1}$ each, then some overlap of signal dimensions within the bottleneck symbol S^n is unavoidable. The interfering messages that are available through antidote links can be subtracted. The desired signals consume a fraction $\frac{L}{L+1}$ of the capacity of the bottleneck link, which must be free from interference. This

leaves only the remaining $\frac{1}{L+1}$ of the signal space for interference within which all interfering messages, each of which carries rate $\frac{1}{L+1}$, should overlap nearly perfectly. The intuitive explanation is formalized for all possible coding schemes through a Shannon theoretic framework.

The following terminology is introduced specifically for the setting where each message wants to achieve rate $\frac{1}{L+1}$.

- **Alignment Relation:** We define a relation $W_i \overset{k}{\leftrightarrow} W_j$ as follows. $W_i \overset{k}{\leftrightarrow} W_j$ iff $W_i \notin \mathcal{A}_k$, $W_i \notin \mathcal{W}_k$, $W_j \notin \mathcal{A}_k$ and $W_j \notin \mathcal{W}_k$ for $k \in \mathcal{K}$ and distinct indices $i, j \in \mathcal{M}$. In the $\frac{1}{L+1}$ rate feasibility problem, the relation $W_i \overset{k}{\leftrightarrow} W_j$ represents the understanding that W_i and W_j must align (into $\frac{1}{L+1}$ of the signal space within the bottleneck link in order to leave the remaining $\frac{L}{L+1}$ of the signal space for $\mathcal{W}_k$).

- **Alignment Subsets:** The set of messages $\mathcal{W}$ is partitioned into alignment subsets, created as follows. We use the notation $W_i \leftrightarrow W_j$ to mean that $W_i \overset{k}{\leftrightarrow} W_j$ for some $k \in \mathcal{K}$. If $W_i \leftrightarrow W_j$, then both W_i, W_j belong to the same alignment subset. Further, if $W_i \leftrightarrow W_j$ and $W_j \leftrightarrow W_m$ then W_i, W_j, W_m all belong to the same alignment subset. For the $\frac{1}{L+1}$ rate feasibility problem, we expect that the messages within an alignment subset will need to align almost perfectly within the bottleneck link signal space.

As an example, consider an index coding problem shown in Fig. 8, where $\mathcal{W}_1 = \{W_1, W_2\}, \mathcal{W}_2 = \{W_1, W_3\}, \mathcal{W}_3 = \{W_2, W_4\}, \mathcal{A}_1 = \emptyset, \mathcal{A}_2 = \{W_4\}, \mathcal{A}_3 = \{W_3\}$. In this example, the alignment relations are $W_3 \overset{1}{\leftrightarrow} W_4$ and the alignment subsets are $\{W_1\}, \{W_2\}, \{W_3, W_4\}$ and the rate $\frac{1}{3}$ per message is feasible. Note that since the alignment subsets form a partition of the messages, they can be interpreted as dividing the messages into equivalence classes.
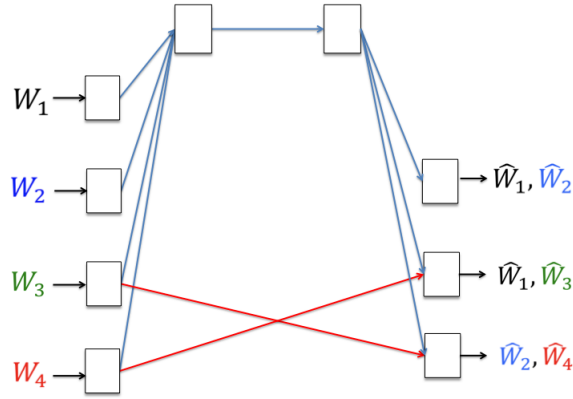

Figure 8: A groupcast index coding problem with $M = 4, K = 3$ and $L = 2$ where the rate $\frac{1}{3}$ per message is feasible

Based on this definition for the alignment subset, we have the following theorem for the $\frac{1}{L+1}$ rate feasibility.

Theorem 3 *The rate tuple $\mathcal{R}$ with $R_1 = R_2 = ... = R_M = \frac{1}{L+1}$ is not achievable in the groupcast index coding problem where $|\mathcal{W}_k| = L, \forall~ k \in \mathcal{K}$, if and only if there exist distinct indices $i, j \in \mathcal{M}$ such that W_i, W_j belong to the same alignment subset and $W_j \in \mathcal{W}_k$ and $W_i \notin \mathcal{A}_k$ for $k \in \mathcal{K}$*

The statement of Theorem 3 is intuitively interpreted as follows. If messages W_i, W_j belong to the same alignment set then they should overlap almost perfectly. Because of their overlap, it is not possible to recover one of them unless the other message is available as an antidote. If such an antidote is not available then the achievability of rate $\frac{1}{L+1}$ for every message becomes infeasible. Note that our feasibility condition refers

to all possible coding schemes and not just linear coding schemes. The theorem also implies the converse, specifically, the following: if for every pair of messages W_i, W_j that lie in the same alignment subset, a destination that desires one of these messages has the other as an antidote, then a rate of $\frac{1}{L+1}$ is achievable. In addition, since an alignment subset can have more than one member only if $M > L+1$, it is automatically implied by the theorem that if $M = L$ or if $M = L + 1$, then the rate tuple $R_1 = R_2 = \ldots R_M = \frac{1}{L+1}$ is achievable. A proof of Theorem 3 is presented in Appendix B.

As a special case of Theorem 3, for $L = 1$, we recover the feasibility condition for achievability of rate half per message, previously obtained by Blasiak et al. in [17].

Corollary 1 *The rate tuple $\mathcal{R}$ with $R_1 = R_2 = \ldots = R_M = \frac{1}{2}$ is not achievable in the groupcast index coding problem where each destination desires only one message, if and only if there exist distinct indices $i, j \in \mathcal{K}$ such that W_i, W_j belong to the same alignment subset and $\{W_j\} = \mathcal{W}_k$ and $W_i \notin \mathcal{A}_k$ for some $k \in \mathcal{K}$.*

The rate-half feasibility condition of Corollary 1 has previously been presented by Blasiak et al. in [17] using graph theoretic terminology which involves notions such as almost-alternating cycles and graph-compatible functions. While the two results are essentially identical, viewing the problem through the lens of interference alignment allows a more intuitively transparent statement in terms of alignment subsets. It is also worth comparing the feasibility condition of Corollary 1 to the graph coloring based achievable rate of [17]. The authors of [17] note that the reciprocal of the *fractional clique cover* of the side information graph also yields an achievable symmetric rate for the multiple unicast index coding problem. However, as also noted in the reference, this achieved rate is not tight. Here, we simply remark that the fractional clique cover based approach does not suffice for achievability, even for feasibility of a rate of $1/2$. For example, in an index coding problem with three users and messages where the kth destination intends to decode W_k with antidotes $\mathcal{A}_1 = \{W_1\}, \mathcal{A}_2 = \{W_3\}, \mathcal{A}_3 = \{W_1\}$, the fractional clique cover based solution yields a rate of $1/3$, whereas a rate of $1/2$ is achievable as per Corollary 1.

While the feasibility conditions are not limited to linear schemes, remarkably, linear coding is sufficient for achievability for feasible settings. The following theorem further elaborates on the linear coding scheme and the required field size.

Theorem 4 *Whenever rate half is feasible according to Corollary 1, it can be achieved through:*

1. *scalar linear coding over 2 channel uses if the finite field is large enough, specifically, $|\mathbb{F}| \geq Z$, or*

2. *vector linear coding over a given finite field $\mathbb{F}$ if the number of channel uses n is large enough, specifically, $n/2 \geq \log_{|\mathbb{F}|} Z$ and n is even,*

where Z is the number of alignment subsets.

Proof: From the detailed proof of achievability of Theorem 3 presented in Appendix B.1, it is evident that what is needed for half-rate achievability is simply a one-to-one mapping, from each alignment subset, to an $n/2$-dimensional subspace of an n-dimensional vector space, such that the subspaces assigned to any two alignment subsets are non-intersecting. The number of pairwise non-intersecting M_t-dimensional subspaces of an m-dimensional vector space over a field $\mathbb{F}$ is shown by [35] to be $\frac{|\mathbb{F}|^m - 1}{|\mathbb{F}|^{M_t} - 1}$ whenever M_t divides m. For rate-half achievability, we have $m = n$, $M_t = n/2$, and the result of Theorem 4 follows.

Note that while it clearly takes only polynomial complexity to identify the alignment subsets (also pointed out by Blasiak et al. [17]), finding the *minimum* possible number of alignment subsets is much more challenging. This is because it is generally possible to further consolidate alignment subsets as long as alignment constraints are not violated. In other words, two alignment subsets may be combined if there does not exist a message in either subset that cannot be aligned with a message in the other subset. Recall that two messages cannot be aligned if one of them is desired at any destination that does not have the other message as an antidote. Finding the minimum number of alignment subsets for a feasible problem is, however, NP-hard. As a consequence, deciding the feasibility of e.g., rate half, when both the field $\mathbb{F}$ and the number of channel uses n is fixed, is NP-complete. This is noted specifically in the context of rate-half feasibility over a binary field and 2 channel uses, by Dau et al. in [36]. On the other hand, as noted above,

if either the field size or the number of channel uses can be chosen to be large enough, then there is no need to consolidate the number of alignment sets, and determining the feasibility of rate half involves only polynomial complexity.

4.3 Symmetric Instances of the Multiple Unicast Index Coding Problem

As with several multiuser capacity problems, part of the difficulty of the index coding problem lies in the potentially unlimited number of parameters in the number of users and the desired and antidote message sets for each user. In order to limit the number of parameters while still covering broad classes of index coding problems, in this section we study symmetric instances of the multiple unicast index coding problem, e.g., where relative to its own position, each destination is associated with the same set of desired and antidote messages.

Solutions to the multiple unicast index coding problem have been found for a variety of symmetric settings. Since here $\mathcal{M} = \mathcal{K}$ and destination D_k desires only message W_k, the problem can be represented by a directed graph G on the vertex set $\mathcal{M}$, in which a vertex i is connected to a vertex j if and only if the destination D_i knows W_j. Bar-Yossef et al. in [27] found the optimal symmetric rate for directed acyclic graphs, perfect graphs, odd holes (undirected odd-length cycles of length at least 5) and odd anti-holes (complements of odd holes). Also, Blasiak et al. in [17] found the capacity per message of the following symmetric index coding instances (the parameters will become clear later on in this section).

- Neighboring antidotes where $D = U$, showing that the sum capacity is $\frac{U+1}{K}$ per message.

- $\mathcal{A}_k = \{W_{k+1}, W_{k+K/2}\}$ for even K, showing that the sum capacity is $\frac{2}{K}$ per message.

- Neighboring interference for arbitrary K where $U = D = 1$ and showing that the sum capacity is $\frac{\lfloor K/2 \rfloor}{K}$ per message.

Berliner and Langberg in [37] characterized the solution of index coding problems with outerplanar side information graphs in terms of the clique cover size of the information graph where the encoding functions are (scalar) linear. Ong et.al in [38] defined *uniprior* index coding problems as the case where $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ for $i \neq j$ and *single uniprior* as the case where $| \mathcal{A}_i |= 1$, $\forall i \in \mathcal{K}$. They derived the optimal symmetric rate for all single uniprior index coding problems.

In the following subsections, we present our capacity results for various symmetric classes of the index coding problem, mostly inspired by the natural settings for the TIM problem.

4.3.1 Neighboring antidotes

Consider a symmetric multiple unicast index coding problem where each destination has a total of $U + D = A < K$ antidotes, corresponding to the U messages before ("up" from) and D messages after ("down" from) its desired message. For this setting, we state the index coding capacity in the following theorem.

Theorem 5 *The capacity of the index coding problem with $M = K < \infty$, (all subscripts modulo K)*

$$D_k = (\{W_k\}, \{W_{k-U}, W_{k-U+1}, \cdots, W_{k-1}\} \cup \{W_{k+1}, W_{k+2}, \cdots, W_{k+D}\}) \tag{37}$$

and

$$U, D \in \mathbb{Z} \tag{38}$$
$$0 \leq U \leq D \tag{39}$$
$$U + D = A < K \tag{40}$$

is

$$C = \begin{cases} 1, & A = K - 1 \\ \frac{U+1}{K-A+2U}, & A \leq K - 2 \end{cases} \tag{41}$$

per message.

As an example, consider the $K = 5$ user setting with $A = 2$ antidotes, $U = D = 1$, where the capacity is $2/5$ per message. Incidentally, example 2 in Section 3.2 also has $K = 5$ users, $A = 2$ antidotes, and has the same capacity of $2/5$ per message. In general, however, for a fixed number of users, K, and a fixed number of total antidotes, A, the capacity per message depends on the relative position of antidotes. Notably, for fixed K, A values and neighboring antidotes as considered in Theorem 5, the capacity improves as the number of antidotes on either side becomes more evenly distributed, i.e., as $D - U$ becomes smaller. The best case setting, i.e., the setting with the highest capacity, is when the antidotes are symmetrically distributed on both sides, e.g., $A = 2U = 2D$, (for even A) which leads to a capacity of $\frac{A+2}{2K}$ per message. The worst case setting is when the antidotes are all on the same side, i.e., $A = D, U = 0$, which leads to a capacity of $\frac{1}{K-A}$ per message. No interference alignment is needed in the latter case.

The achievability scheme in general is a vector linear coding scheme with one-to-one alignments, where each message is sent through $U + 1$ scalar symbols over a $U + 1$ dimensional signal space. Adjacent messages overlap in U dimensions. Because of this interference alignment, at any receiver, the total number of signal dimensions occupied by the $K - A - 1$ interfering messages is equal to $U + (K - A - 1)$. The $U + 1$ dimensional desired signal space is chosen to not have an intersection with the interference space, so that the dimension of the total space, i.e., the number of channel uses n equals $U + (K - A - 1) + U + 1 = K - A + 2U$ and the capacity (normalized by the number of channel uses) is $(U + 1)/(K - A + 2U)$ per message. The details of the achievability proof and the converse are presented in Appendix C. See Figure 9 for an example.
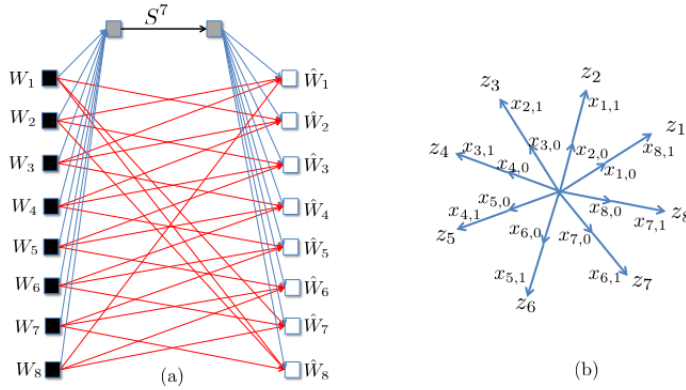

Figure 9: Multiple unicast with neighboring antidotes where $(D, U, K) = (2, 1, 8)$. (a) Desired message and antidote sets, (b) Capacity optimal solution – 2 symbols per message are sent with one-to-one pairwise alignments and a rate of $\frac{2}{7}$ per message is achieved using the scheme pictorially depicted. $z_1, \ldots, z_8$ may be chosen to be the 7 columns of the 7×7 identity matrix and the all 1's vector.

4.3.2 Neighboring interference

Consider the following TIM setting. We have a locally connected network with $M = K = \infty$ where each receiver k has only one desired message W_k from its corresponding (base station) transmitter k. There are totally $U + D + 1$ transmitters with non-zero channel coefficients to receiver k. One of them is the desired transmitter k, U of them are the transmitters before (up from) and D of them are the transmitters after (down from) the desired transmitter, as illustrated in Fig. 10. Basically, what is sent over the $U + D$ neighboring links constitutes interference for receiver k. The index coding problem for this locally connected network has the antidote graph that is the complement of the connectivity graph shown in Fig. 10.

For such a network, we have the following:

Theorem 6 *The capacity of the index coding problem associated with Fig. 10, where $M = K = \infty$,*

$$D_k = \left(\{W_k\}, \{W^c_{k-U, k-U+1, \cdots, k-1, k, k+1, k+2, \cdots, k+D}\}\right) \qquad (42)$$
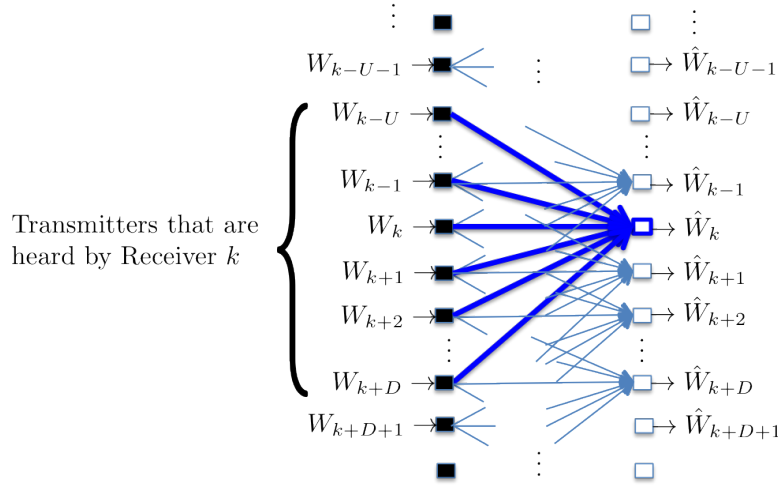
Figure 10: TIM setting: Locally connected network representation for multiple unicast with neighboring interference. The corresponding index coding problem has the complementary antidote graph, i.e., there is an antidote link for each transmitter receiver pair that are not connected above, and there is no antidote link for each transmitter receiver pair that are connected in the picture shown above.

and

$$U, D \in \mathbb{Z} \tag{43}$$
$$0 \leq U \leq D \tag{44}$$
$$\tag{45}$$

is

$$C = \frac{1}{D+1} \tag{46}$$

per message.

Evidently, if each destination has totally $L = U + D$ missing antidotes corresponding to the U messages before (up from) and D messages after (down from) its desired message, the best case setting, i.e., the setting with the highest capacity, is again when the missing antidotes are symmetrically distributed on both sides, i.e., $U = \lfloor \frac{L}{2} \rfloor$ and $D = \lceil \frac{L}{2} \rceil$, which leads to a capacity of $1/(\lceil \frac{L}{2} \rceil + 1)$ per message. The worst case setting is when the missing antidotes are all on the same side, i.e., $D = L$, $U = 0$, which leads to a capacity of $\frac{1}{L+1}$ per message. No interference alignment is needed in the latter case.

The achievability scheme in general is a scalar linear coding scheme with one-to-one alignments, where each message is sent through one scalar symbol over a $D + 1$ dimensional signal space. U messages before (up from) each desired message are aligned with the last U messages among D messages after (down from) that desired message, respectively. Because of this interference alignment, at any receiver, the total number of signal dimensions occupied by the $D + U$ interfering messages is equal to D. The one dimensional desired signal is chosen to not have intersection with the interfering signal space and over $D + 1$ channel uses, it is resolvable. The details of the achievability proof and the converse are presented in Appendix D.

4.3.3   X network setting with local connectivity

Consider the following TIM setting. We have a locally connected network where each destination is connected to L consecutive base stations. Suppose it is the X network setting where each base station has a distinct message for each connected destination. Without loss of generality, we can rename the destinations such that destination D_i is connected to sources $i, i+1, \ldots, i+L-1$ as depicted in Figure 11. The index coding
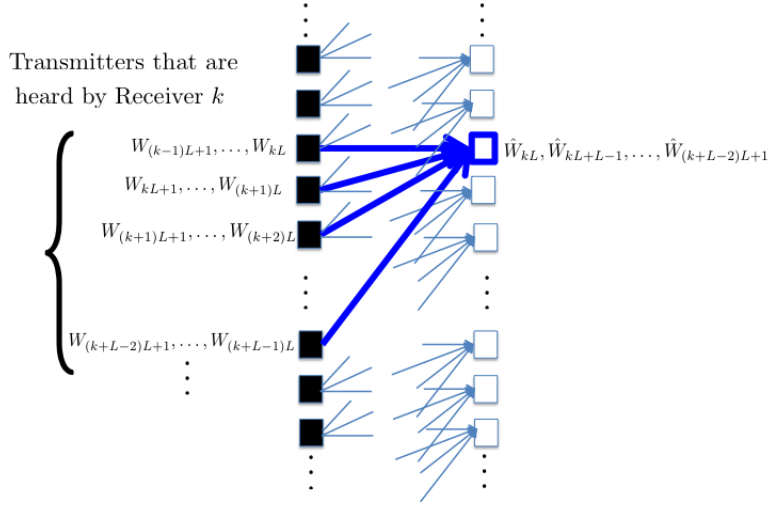
Figure 11: TIM setting: X network setting with local connectivity representation. The corresponding index coding problem has the complementary antidote graph, i.e., there is an antidote link for each transmitter receiver pair that are not connected above, and there is no antidote link for each transmitter receiver pair that are connected in the picture shown above.

problem for this locally connected network has the antidote graph that is the complement of the connectivity graph shown in Fig. 11. For such a network, we have the following:

Theorem 7 *The capacity of the symmetric index coding problem with $M = KL$ and $M, K \to \infty$, where*

$$
\begin{aligned}
\mathcal{W}_k &= \{W_{kL, kL+L-1, (k+1)L+L-2, \cdots, (k+i)L+L-i-1, \cdots, (k+L-2)L+1}\}, \\
\mathcal{A}_k &= \{W_{(k-1)L+1:(k-1)L+L-1, \cdots, (k+i)L+1:(k+i)L+L-i-2, (k+i)L+L-i:(k+i)L+L, \cdots, (k+L-2)L+2:(k+L-2)L+L} \cup \mathcal{W}_k\}^c
\end{aligned}
$$

is

$$ C = \frac{2}{L(L+1)} \tag{47} $$

per message.

The achievability scheme in general is a scalar linear coding scheme with one-to-one alignments, where each message is sent through one scalar symbol over a $\frac{L(L+1)}{2}$ dimensional signal space. $L^2 - L$ interfering messages are aligned into $\frac{L(L+1)}{2} - L$ dimensions using the pattern shown in (190). In particular, this pattern is repeated for each group of $L + 1$ consecutive transmitters for the shown TIM setting in Figure 11. The L desired symbols are chosen to not have intersection with the interfering signal space and over $\frac{L(L+1)}{2}$ channel uses, they are resolvable. The details of the achievability proof and the converse are presented in Section E.

5 Conclusion

As evident from this work, interference alignment is integral to the index coding problem. The interference alignment perspective allows us to not only solve fairly complex index coding problems, but also it makes the intuition behind the capacity optimal solutions quite transparent. As with wireless networks, while much of the initial intuition from interference alignment schemes is based on dimension counting based on linear codes, with few exceptions the dimension counting bounds are readily translated into tight information theoretic bounds. It is also remarkable that the interference alignment perspective allows us to prove the insufficiency of linear codes for multiple unicast index coding. As a side remark, we note that the capacity

results of the index coding settings explored in this work, since they rely only on vector linear achievable schemes that readily translate into the field of complex numbers, directly establish corresponding DoF results for the topological interference management settings as well. We end this paper with a couple of intriguing questions.

First, it is not clear that auxiliary messages and destinations, while convenient for our purpose, are *necessary* in the equivalent multiple unicast problem. Since the purpose of auxiliary messages and destinations is only to force the expanded messages into alignment, the natural question is if auxiliary messages and destinations are not included, would it be possible to achieve a higher min rate for the expanded messages in the multiple unicast setting, presumably through a non-aligned solution? While this possibility seems unlikely, we do not yet have a proof that it is impossible. Such a proof would be desirable because it would make the equivalence between groupcast and unicast settings more direct.

The second question pertains to the capacity of the X channel setting with finite number of users. We showed in Section 4.3.3 that the capacity solution for the case where the number of users is infinity, is $\frac{2}{L(L+1)}$ per message, where L is the number of messages per transmitter or per receiver. However, note that the finite user setting for the X channel, studied in Example 3 presented in Section 3.3 is also consistent with this result. In that example, we have 3 messages per source/destination, i.e., $L = 3$ and we achieve rate $\frac{2}{12}$ per message, even though the number of users is finite. A similar observation can be made for the $L = 2$ setting considered in [5]. This suggests an interesting possibility – is it always possible to achieve rate $\frac{2}{L(L+1)}$ per message with finite number of users K? Note that the outer bound applies to finite K settings as well. On the other hand, even if the optimal rate per message is the same for finite K as well as infinite K, evidently the alignment solution can be much more complex for finite K settings. Note that a sophisticated subspace alignment solution is needed in Section 3.3, but much simpler orthogonal solutions suffice when K is infinity, as shown in Section E.

Finally, we conclude with the observation that the index coding problem remains still an open problem of great interest, and we expect that the insights from interference alignment will continue to be useful not only in solving smaller networks or symmetric versions of extended networks as shown here, but perhaps also in other directions not explored in this work — e.g., designing interference alignment inspired algorithms for arbitrary index coding settings, and studying order optimality of interference alignment techniques in index coding settings modeled as random graphs.

A Proof of Theorem 1: Insufficiency of Linear Codes for Multiple Unicast Index Coding

This section contains a proof of Theorem 1. We first prove Properties 1 and 2 of Theorem 2 in Sections A.1 and A.2 respectively. In Section A.3, we show that Theorem 2 implies Theorem 1.

Notation: We use the overline notation to describe quantities in the equivalent multiple unicast setting. For example, for a given achievable scheme over the multiple unicast setting $\overline{\mathcal{S}}, \overline{n}, \overline{\mathcal{R}}$ represent the alphabet, the number of channel uses, and the rate vector respectively. The codeword and the encoding function for this achievable scheme are denoted as $\overline{S}^{\overline{n}}$ and $\overline{f}(\overline{W}_{1,0}, \overline{W}_{1,1}, \ldots, \overline{W}_{M,L})$ respectively. Also, we use the notation $\langle \mathcal{V} \rangle$ to denote a full rank matrix whose column-span is equal to the column-span of a set of column vectors $\mathcal{V}$.

A.1 Proof of Theorem 2 - Property 1

The rate $(R_1, R_2, \ldots, R_M)$ is achievable in the groupcast setting. This means that, for every given $\delta > 0$ there exists n and an encoding function $f(W_1, W_2, \ldots, W_M)$ such that over n channel uses, destination D_m can decode message $W_{\lceil m/L \rceil}$ of rate $R_{\lceil m/L \rceil} - \delta_{\lceil m/L \rceil}$ with a probability of error *equal*[10] to 0 for some $\delta_{\lceil m/L \rceil} \leq \delta$. Without loss of generality, assume that the alphabet for the achievable scheme is $\mathcal{S} = \{0, 1\}$. Now, we turn to the equivalent multiple unicast setting. In this setting, we will show that for any $\epsilon > 0$ there exists an achievable encoding scheme (over the same alphabet $\overline{\mathcal{S}} = \mathcal{S} = \{0, 1\}$) with rates as in (35)

---

[10]This is because, as noted in Section 2 the ϵ-error capacity is the same as the zero-error capacity in the index coding problem.

such that the probability of error for each message is smaller than ϵ. The encoding scheme for achievability can be described in the following three steps.

Step 1: Re-use f to encode $\overline{W}_{i,j}, j \neq 0, i = 1, 2, \ldots, M$, and

Step 2: form a random codebook for messages $\overline{W}_{i,0}, i = 1, 2, \ldots, M$, and then,

Step 3: superpose the $M + 1$ codewords formed in Step 1 and Step 2.

We will show that this approach achieves the desired rates in the multiple unicast setting. For the achievable scheme, we set the block length to be $\overline{n} = nn_0$ where n_0 is chosen to be sufficiently large for purposes that will be described. For this blocklength, we have $|\overline{\mathcal{W}}_{i,j}| = 2^{nn_0(R_i - \delta_i)}$. For Step 1 above, we first represent $\overline{W}_{i,j}$ as $nn_0(R_i - \delta_i)$ bits, and then split it into n_0 blocks of length $n(R_i - \delta_i)$. Thus, we have

$$\overline{W}_{i,j} = (\overline{W}_{i,j}^{[1]}, \overline{W}_{i,j}^{[2]}, \ldots, \overline{W}_{i,j}^{[n_0]}) \tag{48}$$

where $\overline{W}_{i,j}^{[k]}$ is a bit vector of length $n(R_i - \delta_i)$. Now, we set

$$\overline{W}_i^{[m]} = \sum_{j=1}^{L} \overline{W}_{i,j}^{[m]}$$

where the sum above is a bit-wise XOR of all the messages. Note that $\overline{W}_i^{[m]}$ is a bit-vector of length $n(R_i - \delta_i)$. Now, we are ready to form the codeword of Step 1, denoted as $\overline{S}_1^{nn_0} \in \{0, 1\}^{nn_0}$.

$$\overline{S}_1^{nn_0} = \begin{bmatrix} f(\overline{W}_1^{[1]}, \overline{W}_2^{[1]}, \ldots, \overline{W}_M^{[1]}) \\ f(\overline{W}_1^{[2]}, \overline{W}_2^{[2]}, \ldots, \overline{W}_M^{[2]}) \\ \vdots \\ f(\overline{W}_1^{[n_0]}, \overline{W}_2^{[n_0]}, \ldots, \overline{W}_M^{[n_0]}) \end{bmatrix} \triangleq \begin{bmatrix} \overline{S}_1^{[1]} \\ \overline{S}_1^{[2]} \\ \vdots \\ \overline{S}_1^{[n_0]} \end{bmatrix} \tag{49}$$

where $\overline{S}_1^{[m]}$ is an n bit column vector.

Now, we proceed to Step 2. Message $\overline{W}_{k,0}$, is encoded in $\{0, 1\}^{nn_0}$ by treating the codeword as n_0 block-symbols, each of size n, and using i.i.d. random coding for each block. In particular, we generate a random codebook of length n_0 over alphabet $\{0, 1\}^n$, where each of the n_0 entries are chosen uniformly over all possible n-bit vectors, and independently of each other. We denote by $\overline{S}_{i,0}^{nn_0}$, the codeword corresponding to $\overline{W}_{i,0}$, and set

$$\overline{S}_0^{nn_0} = \sum_{i=1}^{M} \overline{S}_{i,0}^{nn_0}, \tag{50}$$

where $\overline{S}_0, \overline{S}_{i,0} \in \{0, 1\}$.

We are now ready for Step 3. The codeword sent on the multiple unicast channel is set as

$$\overline{S}^{nn_0} = \overline{S}_1^{nn_0} + \overline{S}_0^{nn_0},$$

where $\overline{S}_1^{nn_0}$ and $\overline{S}_0^{nn_0}$ are chosen from (49) and (50), and the addition is bit-wise XOR of the corresponding entries of the vector.

Now we need to show that each message can be decoded with a probability of error smaller than ϵ. First, consider message $\overline{W}_{i,j}, j \neq 0$ to be decoded at destination $\overline{D}_{i,j}$. Note that this destination has, as antidotes, $\mathcal{W}_{*,0}$ because of (31). Therefore, it can compute $\overline{S}_0^{nn_0}$ and subtract it from the received codeword to obtain $f(\overline{W}_1^{[m]}, \overline{W}_2^{[m]}, \ldots, \overline{W}_M^{[m]})$ for $m = 1, 2, \ldots, n_0$. For all l such that $W_l \in \mathcal{A}_{(i-1)L+j}$, this destination has, as antidote, $\{\overline{W}_{l,j} : j = 0, 1, 2, \ldots, L\}$ and can therefore compute $\overline{W}_l$. Now, destination $\overline{D}_{i,j}$ resembles destination $D_{(i-1)L+j}$ in the groupcast setting. In particular, the decoding strategy used by $D_{(i-1)L+j}$ in

the groupcast setting can be used by $\overline{D}_{i,j}$ in the equivalent multiple unicast setting to decode $\overline{W}_i^{[m]}$, free of error. Also, note that the destination has $\{\overline{\mathcal{W}}_{i,l} : l \neq j\}$ as antidotes. Therefore, it can decode $\overline{W}_{i,j}^{[m]}$ from $\overline{W}_i^{[m]}$.

Now, consider destination $\overline{D}_{i,0}$. This destination has $\overline{\mathcal{W}} - \overline{\mathcal{W}}_{i,*}$ as antidotes. Therefore, it can compute $\overline{S}_{k,0}^{nn_0}, k \neq i$ and cancel the effect of this to obtain $\overline{S}_1^{nn_0} + \overline{S}_{i,0}^{nn_0}$. Intuitively, the signal obtained after this cancellation can be interpreted as an additive noise channel over alphabet $\{0,1\}^n$, where the noise vector faced by the mth channel use is $\overline{S}_1^{[m]}, m = 1, 2, \ldots, n_0$. Using the random coding argument, for sufficiently large n_0, $\overline{W}_{i,0}$ can be decoded by this destination with probability of error smaller than ϵ, for any $\epsilon > 0$ for any rate $R_{i,0}$ up to the mutual information

$$
\begin{aligned}
\frac{1}{nn_o} I\left(\overline{S}_1^{nn_0} + \overline{S}_{i,0}^{nn_0}, \overline{\mathcal{A}}_{i,0}; \overline{W}_{i,0}\right) &= \frac{1}{nn_o} I\left(\overline{S}_1^{nn_0} + \overline{S}_{i,0}^{nn_0}; \overline{W}_{i,0} | \overline{\mathcal{A}}_{i,0}\right) && (51) \\
&= \frac{1}{nn_o} H(\overline{S}_1^{nn_0} + \overline{S}_{i,0}^{nn_0} | \overline{\mathcal{A}}_{i,0}) - \frac{1}{nn_o} H(\overline{S}_1^{nn_0} + \overline{S}_{i,0}^{nn_0} | \overline{\mathcal{A}}_{i,0}, \overline{W}_{i,0}) && (52) \\
&= 1 - \frac{1}{nn_o} H(\overline{S}_1^{nn_0} | \overline{\mathcal{A}}_{i,0}, \overline{W}_{i,0}) && (53) \\
&= 1 - (R_i - \delta_i) && (54)
\end{aligned}
$$

where, (53) comes from the fact that $\overline{S}_{i,0}^{nn_0}$ is uniformly distributed over $\{0,1\}^{nn_0}$ through the random coding construction (and is independent of $\overline{\mathcal{A}}_{i,0}$). The final bound comes from the fact that given $(\overline{\mathcal{A}}_{i,0}, \overline{W}_{i,0})$, $S_1^{nn_o}$ is an invertible function of $\overline{W}_i$ and $H(\overline{W}_i) = nn_o(R_i - \delta_i)$.

A.2   Proof of Theorem 2 - Property 2

Here, we first consider the case where rate $(R_1, R_2, \ldots, R_K)$ is achievable in the groupcast setting via a *linear* coding based achievable scheme. This means that for any $\delta > 0$, there exists a set of parameters $n, \mathbf{U}_l, \mathbf{V}_k, l = 1, 2, \ldots, ML, k = 1, 2, \ldots, M$, such that

$$
\begin{aligned}
\mathbf{U}_l \mathbf{V}_k &= 0, \quad \forall l = 1, 2, \ldots, ML, W_k \notin \mathcal{A}_l, k \neq \lceil l/L \rceil && (55) \\
\det\left(\mathbf{U}_l \mathbf{V}_{\lceil l/L \rceil}\right) &\neq 0, \quad \forall l \in \{1, 2, \ldots, ML\}, && (56)
\end{aligned}
$$

where $\mathbf{V}_k$ is a $n \times n(R_k - \delta_k)$ dimensional precoding matrix, and $\mathbf{U}_l$ is a $n(R_{\lceil l/L \rceil} - \delta_{\lceil l/L \rceil}) \times n$ receive combining matrix, where $\delta_k \leq \delta$ for every k. The encoding scheme for this setting is

$$
S^n = \sum_{k=1}^{M} \mathbf{V}_k X_k,
$$

where $S \in \mathbb{F}$ and X_k is a $n(R_k - \delta_k) \times 1$ vector carrying message W_k. Now, we turn to the equivalent multiple unicast setting, where we provide linear encoding achievable scheme for rates

$$
\overline{R}_{i,j} = \left\{ \begin{array}{ll} R_i - \delta_i, & j \neq 0 \\ 1 - R_i, & j = 0 \end{array} \right\}. \tag{57}
$$

Providing such a linear encoding scheme automatically implies achievability of rate (36). The encoding scheme for the multiple unicast setting is formed by setting, for $\overline{W}_{i,j}, i \in \{1, 2, \ldots, M\} j \in \{1, 2, \ldots, L\}$, the precoding matrix $\overline{\mathbf{V}}_{i,j}$ for message $\overline{W}_{i,j}$ to be equal to the precoding matrix $\mathbf{V}_i$ used for the message W_i in the groupcast setting. That is, we set $\overline{\mathbf{V}}_{i,j} = \mathbf{V}_i$, for $j \neq 0$ implying that

$$
\overline{S}^n = \sum_{i,j} \overline{\mathbf{V}}_{i,j} \overline{X}_{i,j} = \sum_{i=1}^{M} \mathbf{V}_i \left(\overline{X}_{i,1} + \overline{X}_{i,2} + \ldots + \overline{X}_{i,L}\right) + \sum_{i=1}^{M} \overline{\mathbf{V}}_{i,0} \overline{X}_{i,0}
$$

where $\overline{S} \in \mathbb{F}$, $\overline{X}_{i,j}$ is a $n(R_i - \delta_i) \times 1$ vector for $j \neq 0$ and $n(1 - R_i) \times 1$ vector for $j = 0$. The precoding matrix $\overline{\mathbf{V}}_{i,0}$ for the auxiliary message $\overline{W}_{i,0}$ is chosen to be any $n \times n(1 - R_i)$ full rank matrix satisfying

$$\text{colspan}(\mathbf{V}_i) \cap \text{colspan}(\overline{\mathbf{V}}_{i,0}) = \{0\}.$$

Since $\text{rank}(\mathbf{V}_i) \leq n(R_i - \delta_i)$ there exists a matrix $\overline{\mathbf{V}}_{i,0}$ satisfying the above condition. This condition automatically implies that there exists receive combining matrix $\overline{\mathbf{U}}_{i,0}$ such that

$$\overline{\mathbf{U}}_{i,0}\mathbf{V}_i = 0,$$

$$\det(\overline{\mathbf{U}}_{i,0}\overline{\mathbf{V}}_{i,0}) \neq 0.$$

Since $\{\overline{W}_{i,j} : j = 1, 2, \ldots, L\}$ are the only messages not present at $\overline{D}_{i,0}$ as antidotes, and all these messages are encoded using $\mathbf{V}_i$, the above equations imply that the destination can decode message $\overline{W}_{i,0}$. Now, consider destination $\overline{D}_{i,j}, j \neq 0$. We can set the receiver combining matrices in the multiple unicast setting as $\overline{\mathbf{U}}_{i,j} = \mathbf{U}_{(i-1)L+j}$. Then, we have for $j \neq 0$,

$$\begin{aligned} \mathbf{U}_{(i-1)L+j}\mathbf{V}_k &= 0, \forall k \notin \mathcal{A}_{(i-1)L+j}, k \neq i \\ \Rightarrow \overline{\mathbf{U}}_{i,j}\overline{\mathbf{V}}_{k,l} &= 0, \forall \overline{W}_{k,l} \notin \overline{\mathcal{A}}_{i,j}, k \neq i \end{aligned}$$

and, because of (56), we have

$$\det\left(\overline{\mathbf{U}}_{i,j}\overline{\mathbf{V}}_{i,j}\right) = \det\left(\mathbf{U}_{(i-1)L+j}\mathbf{V}_i\right) \neq 0$$

as required. This ensures achievability of rates (36) via linear coding in the equivalent multiple unicast setting.

Now, we consider the case where rates in (36) are achievable in the multiple unicast problem via linear coding. This means that for any $\delta > 0$ there exist precoding and receive combining matrices $\overline{\mathbf{V}}_{i,j}, \overline{\mathbf{U}}_{i,j}, i = 1, 2, \ldots, M$ $j = 0, 1, 2, \ldots, L$ in the multiple unicast construction, such that

$$\overline{\mathbf{U}}_{i,j}\overline{\mathbf{V}}_{k,l} = 0, \forall k, l \text{ s.t. } \overline{W}_{k,l} \notin \overline{\mathcal{A}}_{i,j}, (k,l) \neq (i,j) \tag{58}$$

$$\det(\overline{\mathbf{U}}_{i,j}\overline{\mathbf{V}}_{i,j}) \neq 0$$

where $\overline{\mathbf{V}}_{i,j}$ is a $n \times n(R_i - \delta_{i,j})$ matrix if $j \neq 0$ and $n \times n(1 - R_i - \delta_{i,0})$ matrix if $j = 0$, for some $\delta_{i,j} \leq \delta, j = 0, 1, 2, \ldots, L, i = 1, 2, \ldots, M$. Similarly, $\overline{\mathbf{U}}_{i,j}$ is a $n(R_i - \delta_{i,j}) \times n$ matrix if $j \neq 0$ and it is a $n(1 - R_i - \delta_{i,0}) \times n$ matrix if $j = 0$. Now, note that the above implies that

$$\overline{\mathbf{U}}_{i,0}\overline{\mathbf{V}}_{i,j} = 0, j = 1, 2, \ldots, L.$$

We now intend to show that the rate tuple $(R_1, R_2, \ldots, R_M)$ is achievable in the groupcast setting via linear coding. Informally speaking, we set the precoding matrix $\mathbf{V}_i$ corresponding to message W_i in the groupcast setting to be the intersection of all the precoding matrices $\overline{\mathbf{V}}_{i,j}$ of the corresponding messages $\overline{W}_{i,j}, j = 1, 2, \ldots, L$. Note that in general, since the dimension of each encoding matrix $\overline{\mathbf{V}}_{i,j}$ is no bigger than nR_i, the dimension of this intersection is no bigger than nR_i. We show that the dimension of this intersection is sufficiently close to nR_i so as to lead to the achievability of rate tuple $(R_1, R_2, \ldots, R_M)$.

The achievable scheme in the groupcast setting sets precoding matrices $\mathbf{V}_i, i = 1, 2, \ldots, M$ as

$$\mathbf{V}_i = \langle \bigcap_{j=1}^{L} \text{colspan}(\overline{\mathbf{V}}_{i,j}) \rangle. \tag{59}$$

We intend to show that this strategy leads to a linear coding achievable scheme over the groupcast setting with an appropriate rate. Denoting $R'_i = \frac{\text{rank}(\mathbf{V}_i)}{n}$, we prove the following.

(C1) We need to show that there exist $nR_i' \times n$ decoding matrices $\mathbf{U}_i, i = 1, 2, \ldots, ML$ that satisfy the desired criteria, i.e., that

$$\mathbf{U}_l \mathbf{V}_m = 0, W_m \notin \mathcal{A}_l, m \neq \lceil l/L \rceil,$$
$$\det\left(\mathbf{U}_l \mathbf{V}_{\lceil l/L \rceil}\right) \neq 0.$$

(C2) We need to show that the above has an appropriate rate, i.e., we need to show that R_i' can be made arbitrarily close to R_i.

Instead of (C1) above, we show equivalently that there exist matrices $\mathbf{U}_l'$ for $l = 1, 2, \ldots, L$ such that

$$\mathbf{U}_l' \mathbf{V}_m = 0, \forall\, W_m \notin \mathcal{A}_l \cup \{W_{\lceil l/L \rceil}\}$$
$$\mathrm{rank}(\mathbf{U}_l' \mathbf{V}_{\lceil l/L \rceil}) = \mathrm{rank}(\mathbf{V}_{\lceil l/L \rceil}), \forall\, l \in \{1, 2, \ldots, ML\}.$$

For this, we set

$$\mathbf{U}_{(i-1)L+j}' = \overline{\mathbf{U}}_{i,j}. \tag{60}$$

Let $W_m \notin \mathcal{A}_l, m \neq \lceil l/L \rceil$. Because of (31), we infer that

$$\overline{W}_{m,k} \notin \overline{\mathcal{A}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)} \cup \{\overline{W}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)}\}, \forall k = 1, 2, \ldots, L \tag{61}$$
$$\Rightarrow \overline{\mathbf{U}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)} \overline{\mathbf{V}}_{m,k} = 0, \forall k = 1, 2, \ldots, L \tag{62}$$
$$\Rightarrow \mathbf{U}_l' \mathbf{V}_m = 0 \tag{63}$$

where the final equation comes from (59) and (60). In particular, the final equation comes from noting that $\mathrm{colspan}(\mathbf{V}_m) \subseteq \mathrm{colspan}(\mathbf{V}_{m,l}), l \neq 0$. Similarly, we have

$$\det\left(\overline{\mathbf{U}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)} \overline{\mathbf{V}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)}\right) \neq 0 \tag{64}$$
$$\Rightarrow \det\left(\mathbf{U}_l' \overline{\mathbf{V}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)}\right) \neq 0 \tag{65}$$
$$\Rightarrow \mathrm{rank}\left(\mathbf{U}_l' \mathbf{V}_{\lceil l/L \rceil}\right) = \mathrm{rank}\left(\mathbf{V}_{\lceil l/L \rceil}\right), \tag{66}$$

where, the final equation follows because $\mathrm{colspan}(\mathbf{V}_{\lceil l/L \rceil}) \subseteq \mathrm{colspan}(\overline{\mathbf{V}}_{\lceil l/L \rceil, l - L(\lceil l/L \rceil - 1)})$. This completes the proof of (C1). We now need to show (C2). To show this, we use the following Lemma.

Lemma 1 *Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three matrices respectively of sizes $n \times D_A, n \times D_B$ and $n \times D_C$. If $\mathrm{colspan}(\mathbf{A}) \cap \mathrm{colspan}(\mathbf{C}) = \{0\}$ and $\mathrm{colspan}(\mathbf{B}) \cap \mathrm{colspan}(\mathbf{C}) = \{0\}$, then,*

$$dim\left(colspan(\mathbf{A}) \cap colspan(\mathbf{B})\right) \geq rank(\mathbf{A}) + rank(\mathbf{B}) + rank(\mathbf{C}) - n.$$

Proof: Let the rank of $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ respectively be R_A, R_B and R_C. Also, let

$$R_{A \cup B} \overset{\triangle}{=} \dim\left(\mathrm{colspan}(\langle \mathrm{colspan}(\mathbf{A}) \cup \mathrm{colspan}(\mathbf{B}) \rangle)\right)$$

$$R_{A \cap B} \overset{\triangle}{=} \dim\left(\mathrm{colspan}(\mathbf{A}) \cap \mathrm{colspan}(\mathbf{B})\right).$$

We have

$$\mathrm{colspan}\left(\langle \mathrm{colspan}(\mathbf{A}) \cup \mathrm{colspan}(\mathbf{B}) \rangle\right) \cap \mathrm{colspan}(\mathbf{C}) = \{0\}$$
$$\Rightarrow R_{A \cup B} + \dim(\mathrm{colspan}(\mathbf{C})) = \dim\left(\mathrm{colspan}\left(\langle \mathrm{colspan}(\mathbf{A}) \cup \mathrm{colspan}(\mathbf{B}) \cup \mathrm{colspan}(\mathbf{C}) \rangle\right)\right)$$
$$\Rightarrow R_{A \cup B} + R_C \leq n$$
$$\Rightarrow R_A + R_B - R_{A \cap B} + R_C \leq n$$
$$\Rightarrow \dim\left(\mathrm{colspan}(\mathbf{A}) \cap \mathrm{colspan}(\mathbf{B})\right) = R_{A \cap B} \geq R_A + R_B + R_C - n$$

Using the fact that $\operatorname{colspan}(\overline{\mathbf{V}}_{i,j}) \cap \operatorname{colspan}(\overline{\mathbf{V}}_{i,0}) = \{0\}$ and the above lemma, we lower bound the rank of $\mathbf{V}_i = \langle \bigcap_{l=1}^{L} \operatorname{colspan}(\overline{\mathbf{V}}_{i,l}) \rangle$ as follows.

$$nR_i^{'} = \operatorname{rank}(\mathbf{V}_i) = \dim\left(\bigcap_{l=1}^{L} \operatorname{colspan}(\overline{\mathbf{V}}_{i,l})\right) \tag{67}$$

$$\geq \dim\left(\bigcap_{l=1}^{L-1} \operatorname{colspan}(\overline{\mathbf{V}}_{i,l})\right) + \operatorname{rank}(\overline{\mathbf{V}}_{i,L}) - (n - \operatorname{rank}(\overline{\mathbf{V}}_{i,0})) \tag{68}$$

$$\geq \dim\left(\bigcap_{l=1}^{L-2} \operatorname{colspan}(\overline{\mathbf{V}}_{i,l})\right) + \sum_{l=L-1}^{L} \operatorname{rank}(\overline{\mathbf{V}}_{i,l}) - 2(n - \operatorname{rank}(\overline{\mathbf{V}}_{i,0})) \tag{69}$$

$$\vdots \tag{70}$$

$$\geq \dim\left(\operatorname{colspan}(\overline{\mathbf{V}}_{i,1}) \cap \operatorname{colspan}(\overline{\mathbf{V}}_{i,2})\right) + \sum_{l=3}^{L} \operatorname{rank}(\overline{\mathbf{V}}_{i,l}) - (L-2)(n - \operatorname{rank}(\overline{\mathbf{V}}_{i,0})) \tag{71}$$

$$\geq \sum_{l=1}^{L} \operatorname{rank}(\overline{\mathbf{V}}_{i,l}) - (L-1)(n - \operatorname{rank}(\overline{\mathbf{V}}_{i,0})) \tag{72}$$

$$= \sum_{l=1}^{L} n(R_i - \delta_{i,l}) - (L-1)(n - n(1 - R_i - \delta_{i,0})) \tag{73}$$

$$\geq n\left(R_i - \sum_{i=1}^{L} \delta_{i,l} - (L-1)\delta_{i,0}\right) \tag{74}$$

where, the final equation above comes from the achievability of rates (35) on the multiple unicast setting. By choosing $\delta_{i,j}$ to be arbitrarily small, the rate R_i' can be made arbitrarily close to R_i. This completes the proof of (C2), and hence the proof of the Theorem 2.

A.3   Proof of Theorem 1

We show Theorem 1 using the results of Theorem 2 and [34]. [34] implies the following. There exists a groupcast setting where, there exists a rate vector $(R_1, R_2, \ldots, R_K)$ in its capacity region that not achievable via linear coding. Now, we consider the equivalent multiple unicast setting based on Construction 1. Because of Property 2 of Theorem 2, this implies that rate-tuple

$$R_{i,j} = \left\{ \begin{array}{ll} R_i & j \neq 0 \\ 1 - R_i & j = 0 \end{array} \right\}$$

is not achievable via *linear coding* in this multiple unicast setting. Further, because of Property 1 of Theorem 2, the achievability of $(R_1, R_2, \ldots, R_K)$ in the groupcast setting (using a non-linear scheme) implies that the above rate-tuple is achievable in the multiple unicast setting. Therefore, linear coding does not suffice for achievability in the multiple unicast setting.

B   Proof of Theorem 3: Feasibility of rate $\frac{1}{L+1}$ per message

In this section, we show Theorem 3. In Section B.1, we describe an alignment based achievable scheme that achieves a rate of $\frac{1}{L+1}$ for all the messages as long as the conditions of Theorem 3 are not satisfied. In Section B.2, we show a converse result which implies that a rate of $\frac{1}{L+1}$ is not achievable if the conditions of Theorem 3 are satisfied. In the process of showing the converse in Section B.2, we derive outer bounds for the capacity of the index coding problem that may be of independent interest (See Theorems 8 and 9).

B.1 Achievability

We partition $\mathcal{W} = \{W_1, W_2, \ldots, W_M\}$ into alignment subsets $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_T$, and define the mapping $P(m) : \mathcal{M} \to \{1, 2, \cdots, T\}$ so that $W_m \in \mathcal{P}_{P(m)}, \forall m \in \mathcal{M}$.

We use a scalar linear achievable scheme. In particular, we choose $n = L + 1$ and $\mathcal{S} = \mathbb{F}_q$, where q is chosen to be sufficiently large so that there exist T vectors $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_T$, in the $L + 1$ dimensional vector space over $\mathbb{F}_q$, such that every $L + 1$ of them are linearly independent. These are the linear coding vectors along which the aligned messages from each partition will be sent.

Destination D_r, which desires messages $\mathcal{W}_r = \{W_{r_1}, W_{r_2}, \cdots, W_{r_L}\}$, receives

$$
\begin{aligned}
S^n &= \sum_{i=1}^{M} x_i \mathbf{V}_{P(i)} \\
&= \sum_{i=1}^{L} x_{r_i} \mathbf{V}_{P(r_i)} + \underbrace{\sum_{i:W_i \in \mathcal{A}_r} x_i \mathbf{V}_i}_{\substack{\text{Side information} \\ \text{at destination } D_r}} + \left(\sum_{i:W_i \notin \mathcal{A}_r \cup \mathcal{W}_r} x_i \right) \mathbf{V}_{\mathcal{P}_t}
\end{aligned}
$$

where the last equation follows because all elements of $\{i : W_i \notin \mathcal{A}_r \cup \mathcal{W}_r\}$ belong to the same alignment subset by definition; this subset is denoted by $\mathcal{P}_t$. After cancelling the second term above, destination D_r obtains a linear combination of $L + 1$ vectors, $\mathbf{V}_{P(r_1)}, \mathbf{V}_{P(r_2)}, \ldots, \mathbf{V}_{P(r_L)}$ and $\mathbf{V}_{\mathcal{P}_t}$. We need to show that these $L + 1$ vectors are linearly independent if the feasibility condition is satisfied. The feasibility condition implies that for a feasible system, if two messages belong to the same alignment subset, then whenever one of them is desired, the other must be available as an antidote. Since this is not true for any two of the $L + 1$ vectors $\mathbf{V}_{P(r_1)}, \mathbf{V}_{P(r_2)}, \ldots, \mathbf{V}_{P(r_L)}$ and $\mathbf{V}_{\mathcal{P}_t}$, they must each belong to distinct partitions. Since every $L + 1$ coding vectors are linearly independent, we conclude that $x_{r_1}, x_{r_2}, \ldots, x_{r_L}$ are resolvable at destination D_r as required.

B.2 Outer Bounds

The main technical contribution of this section is an information theoretic outer bound to the capacity region of the index coding problem in Theorem 9. Since the proof of Theorem 9 is involved, we first begin with a simpler outer bound in Theorem 8. We then present Theorem 9. Following the theorem statement, we state and prove the converse of Theorem 3 as a corollary to Theorem 9 (Corollary 2). We present a proof of Theorem 9 in Section B.2.1.

Theorem 8 The achievable rates $(R_1, R_2, ..., R_M)$ for the index coding problem where $|\mathcal{W}_k| = L, \forall k \in \{1, 2, \ldots, K\}$, satisfy the following inequalities:

$$
\sum_{\{i:W_i \in \mathcal{W}_k\}} R_i + \sum_{\{i:W_i \in \mathcal{W}_j \cap (\mathcal{W}_k \cup \mathcal{A}_k)^c\}} R_i \leq 1, \quad \forall k \in \mathcal{K} \tag{75}
$$

In other words, the sum of the rates of the messages desired at a destination D_k and the rates of the interfering messages (i.e., undesired messages that are not available as antidotes to D_k) that are intended for any other destination D_j cannot exceed 1.

Proof: Consider an achievable coding scheme for the network. Assume all the messages *except* $\mathcal{W}_k$ and $\mathcal{W}_j \cap (\mathcal{W}_k \cup \mathcal{A}_k)^c$ are given by a genie to destination D_k and destination D_j. Further, assume that the genie provides $\mathcal{W}_k$ to destination D_j as well. Now, using the achievable scheme, destination D_k can decode $\mathcal{W}_k$. Having decoded $\mathcal{W}_k$, in this genie aided index coding problem, destination D_k has all the information that destination D_j has. Therefore, destination D_k can decode $\mathcal{W}_j \cap (\mathcal{W}_k \cup \mathcal{A}_k)^c$ as well. Since destination D_i can decode messages $\mathcal{W}_k$ and $\mathcal{W}_j \cap (\mathcal{W}_k \cup \mathcal{A}_k)^c$, the sum of $H(\mathcal{W}_k)$ and $H(\mathcal{W}_j \cap (\mathcal{W}_k \cup \mathcal{A}_k)^c)$ must be bounded above by the entropy of the bottleneck link $H(S^n)$.

Remark: A reader familiar with Carlieal's outer bound for the interference channel [39] may notice similarities with the approach in the above proof. Also note that the bound, like all the other bounds in this section, is information theoretic, so it applies to both linear and non-linear coding schemes.

As an example, suppose we have an index coding problem with $M = 4, K = 2$ and $L = 2$, where $\mathcal{W}_1 = \{W_1, W_2\}, \mathcal{W}_2 = \{W_3, W_4\}, \mathcal{A}_1 = \emptyset, \mathcal{A}_2 = \{W_1, W_2\}$ as shown in Fig. 12. Using the above outer bound, we have $R_1 + R_2 + R_3 + R_4 \leq 1$, which is tight because it is also achievable by sending each message separately in 4 consecutive time slots.
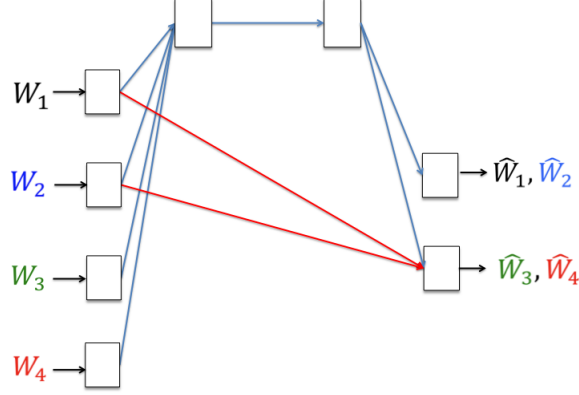

Figure 12: Index coding problem with $M = 4, K = 2$ and $L = 2$ where $R_1 + R_2 + R_3 + R_4 \leq 1$

Next we derive outer bounds which are valid for general index coding problems and are tight for the $\frac{1}{L+1}$ rate feasibility problem. The main result of this section is the following.

Theorem 9 For any $N + 1$ distinct indices $i_0, i_1, ..., i_N \in \mathcal{M}$ and N indices $j^1, ..., j^N \in \mathcal{K}$, if

$$W_{i_0} \overset{j^1}{\leftrightarrow} W_{i_1} \overset{j^2}{\leftrightarrow} W_{i_2} \overset{j^3}{\leftrightarrow} ... \overset{j^{N-1}}{\leftrightarrow} W_{i_{N-1}} \overset{j^N}{\leftrightarrow} W_{i_N} \quad and \quad W_{i_0} \notin \mathcal{A}_k \quad where \quad W_{i_N} \in \mathcal{W}_k \quad for \quad k \in \mathcal{K}, \tag{76}$$

then,

$$R_{i_0} + R_{j^1_{1:L}} + R_{i_1} + ... + R_{j^N_{1:L}} + R_{i_N} \leq N, \tag{77}$$

where $\mathcal{W}_{j^m} = \{W_{j^m_1}, W_{j^m_2}, \ldots, W_{j^m_L}\}$ *and* $R_{j^m_{1:L}} = \sum_{l=1}^{L} R_{j^m_l}$ *for* $m = 1, 2, \ldots, N$.

The converse for Theorem 3 follows from the above theorem. This converse is expressed next as a corollary to the above theorem.

Corollary 2 *The rate tuple $\mathcal{R}$ with $R_1 = R_2 = ... = R_M = \frac{1}{L+1}$ is not achievable in a single bottleneck network where $|\mathcal{W}_k| = L, \forall k \in \{1, 2, \ldots, K\}$ if and only if there exist distinct indices $i, j \in \mathcal{M}$ such that W_i, W_j belong to the same alignment subset and $W_j \in \mathcal{W}_k$ and $W_i \notin \mathcal{A}_k$ for $k \in \mathcal{K}$*

Proof of Corollary 2: If W_i, W_j belong to the same alignment subset, there must exist a chain of alignment relations connecting W_i, W_j as

$$W_i \overset{j^1}{\leftrightarrow} W_{i_1} \overset{j^2}{\leftrightarrow} W_{i_2} \overset{j^3}{\leftrightarrow} ... \overset{j^{N-1}}{\leftrightarrow} W_{i_{N-1}} \overset{j^N}{\leftrightarrow} W_j \tag{78}$$

where N is the length of the chain. If $W_j \in \mathcal{W}_k$ and $W_i \notin \mathcal{A}_k$ for $k \in \mathcal{K}$, from the result of Theorem 5 shown later in this paper we have an explicit rate bound

$$R_i + R_{j^1_{1:L}} + R_{i_1} + R_{j^2_{1:L}} + ... + R_{i_{N-1}} + R_{j^N_{1:L}} + R_j \leq N \tag{79}$$

Clearly, $R_i = R_{i_1} = R_{i_2} = \ldots = R_{i_{N-1}} = R_j = R_{j^1_1} = R_{j^1_2} = \ldots = R_{j^N_L} = \frac{1}{L+1}$ does not satisfy the above bound. This completes the proof of the corollary.

B.2.1 Proof of Theorem 9

Theorem 9 affords a simple proof for $N = 1$. We therefore begin with this case.

If $N = 1$, we have $W_i \overset{k}{\leftrightarrow} W_j, W_j \in \mathcal{W}_r, W_i \notin \mathcal{A}_r$ for $r \in \mathcal{K}$, then we intend to show that $R_i + R_j + R_{k_{1:L}} \leq 1$.

Consider any achievable index coding scheme. Now, we form a genie-enhanced index coding problem (where the scheme continues to remain achievable) as follows. Let $W_i \in \mathcal{W}_m$. In this enhanced problem, assume that destinations D_k, D_r, D_m are given all the messages except messages $W_{i,j,k_1,k_2,\dots,k_L}$ by a genie. We also assume that $W_{k_1,k_2,\dots,k_L}$ are provided via the genie to destinations D_r, D_m. Since $W_j \in \mathcal{W}_r$, for a reliable scheme, destination D_r can decode W_j. In this enhanced problem, either $m = r$ or destination D_m is a degraded version of destination D_r. In both cases, destination D_r can decode W_i as well. Now consider destination D_k. Achievability implies that destination D_k can decode $W_{k_1,k_2,\dots,k_L}$. Now, destination D_k has all the information available at destination D_r and can therefore decode W_i, W_j as well. Since all messages $W_{i,j,k_1,k_2,\dots,k_L}$ are decoded at a single destination, we can follow steps similar to the proof of Theorem 8, to conclude that $R_i + R_j + R_{k_{1:L}} \leq 1$.

As an example, suppose we have an index coding problem with $M = 4, K = 3$ and $L = 2$ where $\mathcal{W}_1 = \{W_1, W_3\}, \mathcal{W}_2 = \{W_2, W_3\}, \mathcal{W}_3 = \{W_3, W_4\}, \mathcal{A}_1 = \{W_2, W_4\}, \mathcal{A}_2 = \varnothing, \mathcal{A}_3 = W_2$ as shown in Fig. 13. We have the following alignment chain $W_1 \overset{2}{\leftrightarrow} W_4, W_4 \in \mathcal{W}_3, W_1 \notin \mathcal{A}_3$. Using the above outer bound, we have $R_1 + R_2 + R_3 + R_4 \leq 1$, which is tight because it is also achievable by sending each message separately in 4 consecutive time slots.
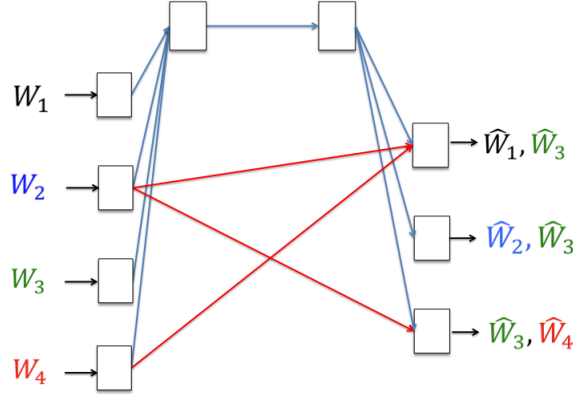

Figure 13: Index coding problem with $M = 4, K = 3$ and $L = 2$ where $R_1 + R_2 + R_3 + R_4 \leq 1$

Now, we consider the case where $N > 1$. Without loss of generality, we assume that $\mathcal{S} = \{0, 1\}$ in the proof. We first present the proof explicitly for $N = 2$ and $N = 3$. These cases do not afford a simple explanation as above, and capture all the ideas required for proving the theorem for an arbitrary value of N. We begin with $N = 2$. In what follows we show that for any 3 distinct indices $i, j, k \in \mathcal{M}$ and indices $l, m \in \mathcal{K}$ if $W_i \overset{l}{\leftrightarrow} W_j \overset{m}{\leftrightarrow} W_k, W_k \in \mathcal{W}_p, W_i \notin \mathcal{A}_p$ for $p \in \mathcal{K}$ then $R_i + R_j + R_k + R_{l_{1:L}} + R_{m_{1:L}} \leq 2$.

$$
\begin{aligned}
H(W_{l_1,l_2,\dots,l_L}) = nR_{l_{1:L}} &= I(W_{l_1,l_2,\dots,l_L}; S^n, \mathcal{W}_{\mathcal{A}_l}) + o(n) &(80)\\
&\leq I(W_{l_1,l_2,\dots,l_L}; S^n, W_{i,j,l_1,l_2,\dots,l_L}^c) + o(n) &(81)\\
&= I(W_{l_1,l_2,\dots,l_L}; S^n \mid W_{i,j,l_1,l_2,\dots,l_L}^c) + o(n) &(82)\\
&= H(S^n \mid W_{i,j,l_1,l_2,\dots,l_L}^c) \\
&\quad -H(S^n \mid W_{i,j}^c) + o(n) &(83)\\
&\leq n - H(S^n \mid W_{i,j}^c) + o(n) &(84)
\end{aligned}
$$

Similarly,

$$H(W_{m_1,m_2,\dots,m_L}) = nR_{m_{1:L}} \quad \leq \quad n - H(S^n \mid W_{j,k}^c) + o(n) \tag{85}$$

For destination D_p that is interested in W_k, we have

$$
\begin{aligned}
nR_k \quad &= \quad I(W_k; S^n, \mathcal{A}_p) + o(n) && (86) \\
&\leq \quad I(W_k; S^n, W_{i,k}^c) + o(n) && (87) \\
&= \quad I(W_k; S^n \mid W_{i,k}^c) + o(n) && (88) \\
&\leq \quad H(S^n \mid W_{i,k}^c) - H(S^n \mid W_i^c) + o(n) && (89) \\
&\leq \quad H(S^n \mid W_{i,k}^c) - nR_i + o(n) && (90) \\
&\leq \quad H(S^n \mid W_{i,j}^c) + H(S^n \mid W_{j,k}^c) \\
&\qquad -H(S^n \mid W_j^c) - nR_i + o(n) && (91) \\
&= \quad H(S^n \mid W_{i,j}^c) + H(S^n \mid W_{j,k}^c) \\
&\qquad -nR_j - nR_i + o(n) && (92) \\
&\leq \quad n(1 - R_{l_{1:L}}) + n(1 - R_{m_{1:L}}) - nR_i - nR_j + o(n) && (93) \\
\Rightarrow R_k \quad &\leq \quad 2 - R_i - R_j - R_{l_{1:L}} - R_{m_{1:L}}, && (94)
\end{aligned}
$$

where (91) follows from Lemma 2 (proved later in this section). Inequality (93) follows from substituting from (84) and (85) into (92), and (94) is obtained by dividing by n and taking the limit as $n \to \infty$.

As an example, suppose we have an index coding problem with $M = 5, K = 5$ and $L = 2$ where $\mathcal{W}_1 = \{W_1, W_5\}, \mathcal{W}_2 = \{W_1, W_2\}, \mathcal{W}_3 = \{W_2, W_5\}, \mathcal{W}_4 = \{W_2, W_4\}, \mathcal{W}_5 = \{W_2, W_3\}, \mathcal{A}_1 = \{W_2\}, \mathcal{A}_2 = \{W_3\}, \mathcal{A}_3 = \{W_1, W_4\}, \mathcal{A}_4 = \{W_1, W_3, W_5\}, \mathcal{A}_5 = \{W_1, W_4, W_5\}$ as shown in Fig. 14. We have the following alignment chain $W_3 \overset{1}{\leftrightarrow} W_4 \overset{2}{\leftrightarrow} W_5$, $W_5 \in \mathcal{W}_3, W_3 \notin \mathcal{A}_3$. Using the above outerbound, we have $R_3 + R_1 + R_5 + R_4 + R_1 + R_2 + R_5 \leq 2$.
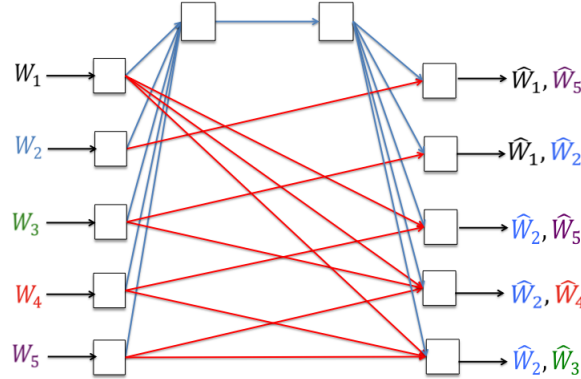

Figure 14: Index coding problem with $M = 5, K = 5$ and $L = 2$ where $R_3 + R_1 + R_5 + R_4 + R_1 + R_2 + R_5 \leq 2$

We next consider the case where the length of the alignment chain is $N = 3$. Suppose we have an alignment chain of length 3, i.e., $W_i \overset{p}{\leftrightarrow} W_j \overset{q}{\leftrightarrow} W_k \overset{r}{\leftrightarrow} W_l$ and the ends of the chain interfere, i.e., $W_l \in \mathcal{W}_m, W_i \notin \mathcal{A}_m$, then our goal is to show that $R_i + R_{p_{1:L}} + R_j + R_{q_{1:L}} + R_k + R_{r_{1:L}} + R_l \leq 3$.
We start with the inequalities at the destinations D_p, D_q, D_r, each of which is shown exactly through the steps followed in (80)-(84).

$$H(W_{p_1,p_2,\dots,p_L}) = nR_{p_{1:L}} \leq n - H(S^n \mid W_{i,j}^c) + o(n) \tag{95}$$
$$H(W_{q_1,q_2,\dots,q_L}) = nR_{q_{1:L}} \leq n - H(S^n \mid W_{j,k}^c) + o(n) \tag{96}$$

$$H(W_{r_1,r_2,\ldots,r_L}) = nR_{r_{1:L}} \le n - H(S^n \mid W_{k,l}^c) + o(n) \tag{97}$$

For destination D_m that is interested in W_l, we have

$$
\begin{aligned}
nR_l \;&=\; I(W_l; S^n, \mathcal{A}_m) + o(n) \tag{98}\\
&\le\; I(W_l; S^n, W_{i,l}^c) + o(n) \tag{99}\\
&=\; I(W_l; S^n \mid W_{i,l}^c) + o(n) \tag{100}\\
&=\; H(S^n \mid W_{i,l}^c) - H(S^n \mid W_i^c) + o(n) \tag{101}\\
&\le\; H(S^n \mid W_{i,l}^c) - nR_i + o(n) \tag{102}\\
&\le\; H(S^n \mid W_{i,j}^c) + H(S^n \mid W_{j,l}^c)\\
&\qquad -H(S^n \mid W_j^c) - nR_i + o(n) \tag{103}\\
&\le\; n(1 - R_{p_{1:L}}) + H(S^n \mid W_{j,l}^c) - nR_j - nR_i + o(n) \tag{104}\\
&\le\; n(1 - R_{p_{1:L}}) + H(S^n \mid W_{j,k}^c)\\
&\qquad + H(S^n \mid W_{k,l}^c) - H(S^n \mid W_k^c)\\
&\qquad - nR_j - nR_i + o(n) \tag{105}\\
&\le\; n(1 - R_{p_{1:L}}) + n(1 - R_{q_{1:L}}) + n(1 - R_{r_{1:L}}) - nR_k - nR_j\\
&\qquad - nR_i + o(n) \tag{106}\\
\Rightarrow R_l \;&\le\; 3 - R_{p_{1:L}} - R_{q_{1:L}} - R_{r_{1:L}} - R_k - R_j - R_i \tag{107}
\end{aligned}
$$

Note that Lemma 2 is applied twice, first in arriving at (103) and then to obtain (105).

Now we prove Theorem 9 for an arbitrary $N > 2$. Suppose we have an alignment chain of length N, $W_{i_0} \overset{j^1}{\leftrightarrow} W_{i_1} \overset{j^2}{\leftrightarrow} W_{i_2} \overset{j^3}{\leftrightarrow} \ldots \overset{j^{N-1}}{\leftrightarrow} W_{i_{N-1}} \overset{j^N}{\leftrightarrow} W_{i_N}$ and $W_{i_0} \notin \mathcal{A}_k$ where $W_{i_N} \in \mathcal{W}_k$ for $k \in \mathcal{K}$, then our goal is to show that $R_{i_0} + R_{j_{1:L}^1} + R_{i_1} + \ldots + R_{j_{1:L}^N} + R_{i_N} \le N$. We start with the inequalities at the destinations $j^1, \ldots, j^N$, each of which is shown exactly through the steps followed in (80)-(84).

$$H(W_{j_1^1, j_2^1, \ldots, j_L^1}) \;=\; nR_{j_{1:L}^1} \le n - H(S^n \mid W_{i_0,i_1}^c) + o(n) \tag{108}$$

$$H(W_{j_1^2, j_2^2, \ldots, j_L^2}) \;=\; nR_{j_{1:L}^2} \le n - H(S^n \mid W_{i_1,i_2}^c) + o(n) \tag{109}$$

$$\vdots \tag{110}$$

$$H(W_{j_1^N, j_2^N, \ldots, j_L^N}) \;=\; nR_{j_{1:L}^N} \le n - H(S^n \mid W_{i_{N-1},i_N}^c) + o(n) \tag{111}$$

For destination D_k that is interested in W_{i_N}, we have

$$
\begin{aligned}
nR_{i_N} \;&=\; I(W_{i_N}; S^n, \mathcal{A}_k) + o(n) \tag{112}\\
&\le\; I(W_{i_N}; S^n, W_{i_0,i_N}^c) + o(n) \tag{113}\\
&=\; I(W_{i_N}; S^n \mid W_{i_0,i_N}^c) + o(n) \tag{114}\\
&=\; H(S^n \mid W_{i_0,i_N}^c) - H(S^n \mid W_{i_0}^c) + o(n) \tag{115}\\
&\le\; H(S^n \mid W_{i_0,i_N}^c) - nR_{i_0} + o(n) \tag{116}\\
&\le\; H(S^n \mid W_{i_0,i_1}^c) + H(S^n \mid W_{i_1,i_N}^c)\\
&\qquad - H(S^n \mid W_{i_1}^c) - nR_{i_0} + o(n) \tag{117}\\
&\le\; n(1 - R_{j_{1:L}^1}) + H(S^n \mid W_{i_1,i_N}^c) - nR_{i_1} - nR_{i_0} + o(n) \tag{118}\\
&\le\; n(1 - R_{j_{1:L}^1}) + H(S^n \mid W_{i_1,i_2}^c) + H(S^n \mid W_{i_2,i_N}^c) - H(S^n \mid W_{i_2}^c)\\
&\qquad - nR_{i_1} - nR_{i_0} + o(n) \tag{119}\\
&\le\; n(1 - R_{j_{1:L}^1}) + n(1 - R_{j_{1:L}^2}) + H(S^n \mid W_{i_2,i_N}^c) - nR_{i_2} - nR_{i_1}\\
&\qquad - nR_{i_0} + o(n) \tag{120}\\
&\qquad \vdots
\end{aligned}
$$

$$\leq \quad n(1 - R_{j_{1:L}^1}) + n(1 - R_{j_{1:L}^2}) + \ldots + n(1 - R_{j_{1:L}^{N-2}}) + H(S^n \mid W_{i_{N-2},i_{N-1}}^c) + H(S^n \mid W_{i_{N-1},i_N}^c)$$

$$-H(S^n \mid W_{i_{N-1}}^c) - nR_{i_{N-2}} + \ldots - nR_{i_1} - nR_{i_0} \tag{121}$$

$$\leq \quad n(1 - R_{j_{1:L}^1}) + n(1 - R_{j_{1:L}^2}) + \ldots + n(1 - R_{j_{1:L}^{N-2}}) + n(1 - R_{j_{1:L}^{N-1}}) + n(1 - R_{j_{1:L}^N})$$

$$-nR_{i_{N-1}} - nR_{i_{N-2}} + \ldots - nR_{i_1} - nR_{i_0} \tag{122}$$

$$\Rightarrow \quad R_{i_N} \leq N - R_{j_{1:L}^1} - R_{j_{1:L}^2} - \ldots - R_{j_{1:L}^N} - R_{i_{N-1}} - R_{i_{N-2}} - \ldots - R_{i_0} \tag{123}$$

Note that Lemma 2 is applied in (117), (119) and (121). This proves Theorem 9.

Finally, we prove Lemma 2 which is used repeatedly in the proof of outer bounds.

Lemma 2

$$H(S^n \mid W_{j,k}^c) \leq H(S^n \mid W_{i,j}^c) + H(S^n \mid W_{i,k}^c) - H(S^n \mid W_i^c) \tag{124}$$

Proof: Since entropy function is a submodular function, for any two subsets of random variables $\mathcal{C}, \mathcal{D}$, we have the following

$$H(\mathcal{C}) + H(\mathcal{D}) \geq H(\mathcal{C} \cup \mathcal{D}) + H(\mathcal{C} \cap \mathcal{D}) \tag{125}$$

If we choose subsets $\mathcal{C}, \mathcal{D}$ as $\mathcal{C} = \{S^n, W_{i,j}^c\}$, $\mathcal{D} = \{S^n, W_{i,k}^c\}$ and use the submodular property of entropy function, we have,

$$H(S^n, W_{i,j}^c) + H(S^n, W_{i,k}^c) \geq H(S^n, W_i^c) + H(S^n, W_{i,j,k}^c) \tag{126}$$

which is equivalent to

$$H(S^n \mid W_{i,j}^c) + H(S^n \mid W_{i,k}^c) \quad \geq \quad H(S^n \mid W_i^c) + H(S^n \mid W_{i,j,k}^c)$$

$$\geq \quad H(S^n \mid W_i^c) + H(S^n \mid W_{j,k}^c) \tag{127}$$

where (127) is true because conditioning reduces entropy. This completes the proof

C   Proof of Theorem 5

C.1   Achievability

First, note that if $A = U + D = K - 1$, then it is obvious that each source can send 1 symbol per time slot and achieve a rate of 1. If $A = K - 2$, then, it is easy to verify that a rate of $\frac{1}{2}$ is achievable using Theorem 3. Here we show achievability for $A \leq K - 3$. Using linear coding, we show that each user can send $L = (U + 1)$ symbols in a $n = K - A + 2U$ dimensional space $\mathbb{F}_q^n$, when q is sufficiently large. Note that because $U \leq D$, we have $A - 2U = D - U \geq 0$, and therefore, $n \leq K$. To begin the construction of our achievable scheme, we pick K vectors over $\mathbb{F}_q^n$ such that any n of them are linearly independent. We denote the vectors by $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_K$. Note that over a sufficiently large field, the vectors $\mathbf{z}_i, i = 1, 2, \ldots, K$ can be chosen to satisfy this property[11]. Our construction for the $n \times (U + 1)$ matrix $\mathbf{V}_i$ — whose columns are the beamforming vectors for the $U + 1$ symbols comprising message W_i — is as follows.

$$\mathbf{V}_i = [\mathbf{z}_i \quad \mathbf{z}_{i+1} \quad \mathbf{z}_{i+2} \quad \ldots \mathbf{z}_{i+U}],$$

where all subscripts are interpreted modulo K. Note that any two adjacent messages overlap in U dimensions. For example, the signal spaces spanned by $\mathbf{V}_i$ and $\mathbf{V}_{i+1}$ overlap in dimensions $\mathbf{z}_{i+1}, \mathbf{z}_{i+2}, \cdots, \mathbf{z}_{i+U}$. This is the basis for interference alignment. With this construction, we intend to show that (5) is satisfied.

---

[11]One approach to such a construction is to use Vandermonde matrices.

First, note that the $U+1$ columns of $\mathbf{V}_i$ are linearly independent since $U+1 < n$. In particular, denoting $\mathbf{X}_i = [x_{i,0} \ \ x_{i,1} \ \ \ldots, x_{i,U}]$, we have

$$S^n = \sum_{i=1}^{K} \sum_{j=0}^{U} x_{i,j} \mathbf{z}_{i+j}.$$

Now, because of the symmetric nature of the problem, we only need to show that W_1 is linearly resolvable at D_1. This ensures resolvability at all other destination nodes. Note that destination D_1 has side information of $x_{i,j}$ for all $i \in \{K-U+1, K-U+2, \ldots, K, 2, 3, \ldots, D+1\}$. The precoding vectors are known to everyone apriori. Cancelling the effect of the known symbols, destination D_1 obtains

$$S_1^n = \underbrace{\sum_{j=0}^{U} x_{1,j} \mathbf{z}_{j+1}}_{\text{Desired Signal}} + \underbrace{\sum_{i=D+2}^{K-U} \sum_{j=0}^{U} x_{i,j} \mathbf{z}_{i+j}}_{\substack{\text{Aligned} \\ \text{Interference}}}$$

Thus, destination D_1 sees $(U+1) + (K - (D+1)) = n$ signal vectors $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_{U+1}, \mathbf{z}_{D+2}, \mathbf{z}_{D+3}, \ldots, \mathbf{z}_K$. Since these n vectors are linearly independent by design, the desired scalars $x_{1,j}, j = 0, 1, \ldots, U$ are resolvable at destination D_1. By symmetry, the same conclusion is applicable for all the K destinations.

To understand the role of alignment, note that there are $(K - A - 1)$ undesired messages that interfere at destination D_1, each occupying a $(U+1)$ dimensional signal space. However, because any two adjacent interferers overlap in U dimensions, the $(K - A - 1)$ interferers collectively occupy only $U + 1 + (K - A - 1) - 1 = K - D - 1$ dimensions. Since the desired signal occupies $U + 1$ dimensions, and $(U+1) + (K - D - 1) = n$, the observed signal space is big enough to resolve desired signals from the interference. See Figure 9 for an example.

C.2   Outer bound

The trivial bound that a message can not be transmitted at more than rate 1 (the capacity of the bottleneck link) is tight when $A = K - 1$, i.e., all undesired messages are available as antidotes. If $A = K - 2$, i.e., only one message is missing from the antidote set, then the rate $\frac{1}{2}$ per user is an outerbound, as shown in the proof of Theorem 8. Therefore we focus on the setting $A \leq K - 3$.

We first present an outer bound on the rates achievable via linear achievable schemes to gain intuition. Later, we will use this intuition to get information theoretic outer bounds for *any* achievable scheme.

C.2.1   Dimension Counting Outerbound

Our goal is to show that the sum-capacity per message is bounded above by $\frac{U+1}{K-A+2U}$. To show this, we first prove that if we want to achieve a rate d per message, then the total dimension of interference at each destination satisfies

$$\dim(\text{interference}) \geq \left\lceil \frac{K - A + 2U}{U + 1} - 1 \right\rceil d = \frac{K - A - 1 + U}{U + 1} d \tag{128}$$

$K - A - 1$ is the total number of interferers at each destination. Suppose $\mathcal{V}_i$ denotes the subspace assigned to user i. Hence, our goal is bounding $\sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+K-A-2})$, i.e., the total number of interfering dimensions at all the destinations. We define α_j as follows

$$\alpha_j \triangleq \sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-1}) \tag{129}$$

So our goal is bounding the total number of interference at all destinations, i.e., α_{K-A-1}.

Lemma 3 *For the following m and j*

$$\begin{cases} m = \lfloor \frac{K-A-1}{U+1} \rfloor, \ j = (K-A-1) \mod (U+1) & \textit{if } K-A-1 \mod (U+1) \neq 0 \\ m = \frac{K-A-1}{U+1} - 1, \ j = U+1 & \textit{if } K-A-1 \mod (U+1) = 0 \end{cases} \tag{130}$$

we have

$$\alpha_{K-A-1} \geq m\alpha_1 + \alpha_j \tag{131}$$

Proof:

$$
\begin{aligned}
\alpha_{K-A-1} \ &= \ \alpha_{m(U+1)+j} \\
&= \ \sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{m(U+1)+j-1}) \tag{132} \\
&\geq \ \sum_{i=1}^{K} \{ \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-1} \cup \mathcal{V}_{i+j-1+(U+1)} \cup \mathcal{V}_{i+j-1+2(U+1)} \cup \ldots \cup \mathcal{V}_{i+j-1+m(U+1)}) \tag{133} \\
&= \ \sum_{i=1}^{K} \{ \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-1}) + \dim(\mathcal{V}_{i+j-1+(U+1)}) + \dim(\mathcal{V}_{i+j-1+2(U+1)}) + \ldots \\
&\qquad + \dim(\mathcal{V}_{i+j-1+m(U+1)}) \} \tag{134} \\
&= \ m\alpha_1 + \alpha_j \tag{135}
\end{aligned}
$$

where (134) is true because $W_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+l(U+1)} \notin \mathcal{A}_{i+j-1+l(U+1)}$ for $l = 0, 1, \ldots, m$. This proves Lemma 3.

Lemma 4 *For $j = 2, 3, \ldots, U+1$, we have*

$$\alpha_j \ \geq \ \alpha_{j-1} + \frac{\alpha_1}{U+1} \tag{136}$$

$$\Rightarrow \quad \alpha_j \ \geq \ \frac{U+j}{U+1}\alpha_1 \tag{137}$$

Proof:

$$\sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2} \cup \mathcal{V}_{i+j+U-1}) \tag{138}$$

$$
\begin{aligned}
\leq \ &\sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2} \cup \mathcal{V}_{i+j-1}) + \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1} \cup \mathcal{V}_{i+j+U-1}) \\
&- \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1}) \tag{139} \\
\leq \ &\sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2} \cup \mathcal{V}_{i+j-1}) + \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1} \cup \mathcal{V}_{i+j}) \\
&+ \sum_{i=1}^{K} \dim(\mathcal{V}_{i+2} \cup \mathcal{V}_{i+3} \cup \ldots \cup \mathcal{V}_{i+j} \cup \mathcal{V}_{i+j+U-1}) - \sum_{i=1}^{K} \dim(\mathcal{V}_{i+2} \cup \mathcal{V}_{i+3} \cup \ldots \cup \mathcal{V}_{i+j}) \\
&- \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1}) \tag{140}
\end{aligned}
$$

$$\vdots$$

$$\leq \sum_{i=1}^{K} \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2} \cup \mathcal{V}_{i+j-1}) + \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1} \cup \mathcal{V}_{i+j})$$

$$+ \sum_{i=1}^{K} \dim(\mathcal{V}_{i+2} \cup \mathcal{V}_{i+3} \cup \ldots \cup \mathcal{V}_{i+j} \cup \mathcal{V}_{i+j+1}) + \ldots + \sum_{i=1}^{K} \dim(\mathcal{V}_{i+U} \cup \mathcal{V}_{i+U+1} \cup \ldots \cup \mathcal{V}_{i+j+U-2} \cup \mathcal{V}_{i+j+U-1})$$

$$- \sum_{i=1}^{K} \dim(\mathcal{V}_{i+U} \cup \mathcal{V}_{i+U+1} \cup \ldots \cup \mathcal{V}_{i+j+U-2}) - \ldots - \sum_{i=1}^{K} \dim(\mathcal{V}_{i+2} \cup \mathcal{V}_{i+3} \cup \ldots \cup \mathcal{V}_{i+j})$$

$$- \sum_{i=1}^{K} \dim(\mathcal{V}_{i+1} \cup \mathcal{V}_{i+2} \cup \ldots \cup \mathcal{V}_{i+j-1}) \tag{141}$$

where at each step, we use the submodular property of dim function. On the other hand, since $\{W_i, W_{i+1}, \ldots, W_{i+j-2}\} \notin \mathcal{A}_{i+j+U-1}$, we have

$$\dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2} \cup \mathcal{V}_{i+j+U-1}) = \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+j-2}) + \dim(\mathcal{V}_{i+j+U-1}) \tag{142}$$

By combining (141) and (142), we have

$$\alpha_{j-1} + \alpha_1 \leq \alpha_j + \alpha_j + \ldots + \alpha_j - \alpha_{j-1} - \ldots - \alpha_{j-1} \tag{143}$$

$$\alpha_j \geq \frac{1}{U+1}\{(U+1)\alpha_{j-1} + \alpha_1\} \tag{144}$$

This proves (136). Solving this recursive equation, we get (137). This proves Lemma 4.

Combining Lemma 3 and Lemma 4, we have the following

$$\alpha_{K-A-1} \geq m\alpha_1 + \frac{U+j}{U+1}\alpha_1 = \frac{m(U+1)+j+U}{U+1}\alpha_1 = \frac{K-A-1+U}{U+1}\alpha_1 \tag{145}$$

Then the dimension of desired message plus the dimension of interference at all destinations should be less than or equal to the total available dimensions, i.e., K

$$\alpha_1 + \frac{K-A-1+U}{U+1}\alpha_1 \leq \sum_{i=1}^{K}\{\dim(\mathcal{V}_{i-D-1}) + \dim(\mathcal{V}_i \cup \mathcal{V}_{i+1} \cup \ldots \cup \mathcal{V}_{i+K-A-2})\} \leq K \tag{146}$$

Hence the optimal symmetric rate per message is $d = \frac{1}{K}\alpha_1 \leq \frac{U+1}{K-A+2U}$

C.2.2 Information Theoretic Outerbound

We define α_j as follows

$$\alpha_j \triangleq \sum_{i=1}^{K} H(S^n | W_{i,i+1,\ldots,i+j-1}^c) \tag{147}$$

Our first goal is to bound α_{K-A-1}. We proceed as follows.

Lemma 5 *For the following m and j*

$$\begin{cases} m = \lfloor \frac{K-A-1}{U+1} \rfloor, \ j = (K-A-1) \mod (U+1) & \text{if } K-A-1 \mod (U+1) \neq 0 \\ m = \frac{K-A-1}{U+1} - 1, \ j = U+1 & \text{if } K-A-1 \mod (U+1) = 0 \end{cases} \tag{148}$$

we have

$$\alpha_{K-A-1} \geq m\alpha_1 + \alpha_j + o(n) \tag{149}$$

Proof: Note that destination $D_{i+j-1+(U+1)}$ can decode $W_{i+j-1+(U+1)}$ from $(S^n, W^c_{i,i+1,\ldots,i+j-1,i+j-1+(U+1)})$ with $P_e \to 0$ as $n \to \infty$. Therefore, we can write

$$
\begin{aligned}
nR_{i+j-1+(U+1)} &= H(W_{i+j-1+(U+1)}) \tag{150}\\
&= I(W_{i+j-1+(U+1)}; S^n | W^c_{i,i+1,\ldots,i+j-1,i+j-1+(U+1)}) \\
&\quad + H(W_{i+j-1+(U+1)} | S^n, W^c_{i,i+1,\ldots,i+j-1,i+j-1+(U+1)}) \tag{151}\\
&= H(S^n | W^c_{i,i+1,\ldots,i+j-1,i+j-1+(U+1)}) \\
&\quad - H(S^n | W^c_{i,i+1,\ldots,i+j-1}) + o(n) \tag{152}
\end{aligned}
$$

which gives us

$$
H(S^n | W^c_{i,i+1,\ldots,i+j-1,i+j-1+(U+1)}) = nR_{i+j-1+(U+1)} + H(S^n | W^c_{i,i+1,\ldots,i+j-1}) + o(n) \tag{153}
$$

Next, note that destination $D_{i+j-1+2(U+1)}$ does not have $W_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}$ as antidote. So, given $(S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)})$ it must be able to reliably decode $W_{i+j-1+2(U+1)}$.

$$
\begin{aligned}
nR_{i+j-1+2(U+1)} &= H(W_{i+j-1+2(U+1)}) \tag{154}\\
&= I(W_{i+j-1+2(U+1)}; S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}) \\
&\quad + H(W_{i+j-1+2(U+1)} | S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}) \tag{155}\\
&= I(W_{i+j-1+2(U+1)}; S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}) + o(n) \\
&= H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}) \\
&\quad - H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1)}) + o(n) \tag{156}
\end{aligned}
$$

which along with (153) gives us

$$
\begin{aligned}
H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1)}) &= nR_{i+j-1+2(U+1)} + nR_{i+j-1+(U+1)} \\
&\quad + H(S^n | W^c_{i,i+1,\ldots,i+j-1}) + o(n) \tag{157}
\end{aligned}
$$

Similarly, we note that destination $D_{i+j-1+l(U+1)}$ for $3 \leq l \leq m$ does not have $W_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+l(U+1)}$ as antidotes, so it must be able to decode $W_{i+j-1+l(U+1)}$ from $(S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+l(U+1)})$.

$$
\begin{aligned}
nR_{i+j-1+m(U+1)} &= H(W_{i+j-1+m(U+1)}) \tag{158}\\
&= I(W_{i+j-1+m(U+1)}; S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+m(U+1)}) + \\
&\quad H(W_{i+j-1+m(U+1)} | S^n, W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+m(U+1)}) \tag{159}\\
&= I(W_{i+j-1+m(U+1)}; S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+m(U+1)}) + o(n) \tag{160}\\
&= H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+m(U+1)}) - \\
&\quad H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+(m-1)(U+1)}) + o(n) \tag{161}
\end{aligned}
$$

which gives us

$$
\begin{aligned}
H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),\ldots,i+j-1+m(U+1)}) &= nR_{i+j-1+m(U+1)} + \ldots + nR_{i+j-1+2(U+1)} \\
&\quad + nR_{i+j-1+(U+1)} + H(S^n | W^c_{i,i+1,\ldots,i+j-1}) + o(n) \\
& \tag{162}
\end{aligned}
$$

Our goal is to bound α_{K-A-1}. We have

$$
\begin{aligned}
\alpha_{K-A-1} &= \alpha_{m(U+1)+j} \tag{163}\\
&= \sum_{i=1}^{K} H(S^n | W^c_{i,i+1,\ldots,i+K-A-2}) \tag{164}
\end{aligned}
$$

$$\geq \sum_{i=1}^{K} H(S^n | W^c_{i,\ldots,i+j-1,i+j-1+(U+1),i+j-1+2(U+1),\ldots,i+j-1+m(U+1)}) \tag{165}$$

$$= \sum_{i=1}^{K} \{nR_{i+j-1+(U+1)} + \ldots + nR_{i+j-1+m(U+1)} + H(S^n | W^c_{i,i+1,\ldots,i+j-1}) + o(n)\} \tag{166}$$

$$= m\alpha_1 + \alpha_j + o(n) \tag{167}$$

where (165) is true because conditioning reduces the entropy. (166) is derived by replacing (162) into (165). This proves Lemma 5.

Lemma 6 *For $j = 2, 3, \ldots, U+1$, we have*

$$\alpha_j \geq \alpha_{j-1} + \frac{\alpha_1}{U+1} + o(n) \tag{168}$$

$$\Rightarrow \quad \alpha_j \geq \frac{U+j}{U+1}\alpha_1 + o(n) \tag{169}$$

Proof: We use Lemma 7 (proved later in this section) at each step. So we have

$$\sum_{i=1}^{K} H(S^n, W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) \leq \sum_{i=1}^{K} \{H(S^n, W^c_{i,i+1,\ldots,i+j-1})$$
$$+ H(S^n, W^c_{i+1,i+2,\ldots,i+j-1,i+j+U-1})$$
$$- H(S^n, W^c_{i+1,i+2,\ldots,i+j-1})\} \tag{170}$$

$$\leq \sum_{i=1}^{K} \{H(S^n, W^c_{i,i+1,\ldots,i+j-1})$$
$$+ H(S^n, W^c_{i+1,i+2,\ldots,i+j-1,i+j})$$
$$+ H(S^n, W^c_{i+2,i+3,\ldots,i+j-1,i+j,i+j+U-1})$$
$$- H(S^n, W^c_{i+2,i+3,\ldots,i+j})$$
$$- H(S^n, W^c_{i+1,i+2,\ldots,i+j-1})\} \tag{171}$$

$$\vdots$$

$$\leq \sum_{i=1}^{K} \{H(S^n, W^c_{i,i+1,\ldots,i+j-1})$$
$$+ H(S^n, W^c_{i+1,i+2,\ldots,i+j-1,i+j})$$
$$+ \ldots + H(S^n, W^c_{i+U,i+U+1,\ldots,i+j+U-1})$$
$$- H(S^n, W^c_{i+U,i+U+1,\ldots,i+j+U-2})$$
$$- \ldots - H(S^n, W^c_{i+2,i+3,\ldots,i+j}) -$$
$$H(S^n, W^c_{i+1,i+2,\ldots,i+j-1})\} \tag{172}$$

We note that destination $D_{i+j+U-1}$ does not have $W_{i,i+1,\ldots,i+j-2,i+j+U-1}$ as antidotes, so it must be able to decode $W_{i+j+U-1}$ from $(S^n, W^c_{i,i+1,\ldots,i+j-2,i+j+U-1})$.

$$nR_{i+j+U-1} = H(W_{i+j+U-1}) \tag{173}$$
$$= I(W_{i+j+U-1}; S^n, W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) +$$
$$H(W_{i+j+U-1} | S^n, W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) \tag{174}$$
$$= I(W_i; S^n | W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) + o(n) \tag{175}$$
$$= H(S^n | W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) - H(S^n | W^c_{i,i+1,\ldots,i+j-2}) + o(n) \tag{176}$$

which gives us

$$H(S^n|W^c_{i,i+1,\ldots,i+j-2,i+j+U-1}) = nR_{i+j+U-1} + H(S^n|W^c_{i,i+1,\ldots,i+j-2}) + o(n) \tag{177}$$

Replacing (177) into (172), we have

$$\alpha_{j-1} + \alpha_1 + o(n) \le \alpha_j + \alpha_j + \ldots + \alpha_j - \alpha_{j-1} - \ldots - \alpha_{j-1} \tag{178}$$

$$\alpha_j \ge \frac{1}{U+1}\{(U+1)\alpha_{j-1} + \alpha_1\} + o(n) \tag{179}$$

Solving this recursive equation, we get

$$\alpha_j \ge \frac{U+j}{U+1}\alpha_1 + o(n) \tag{180}$$

This proves Lemma 6.

Combining Lemma 5 and Lemma 6, we have the following

$$\alpha_{K-A-1} \ge m\alpha_1 + \frac{U+j}{U+1}\alpha_1 + o(n) = \frac{m(U+1)+j+U}{U+1}\alpha_1 + o(n) = \frac{K-A-1+U}{U+1}\alpha_1 + o(n) \tag{181}$$

Finally, we note that destination D_{i-D-1} does not have any of the messages $W_{i,i+1,\ldots,i+K-A-2}$ as antidotes. So it must be able to decode W_{i-D-1} from $(S^n, W_{i-D-1,i,i+1,\ldots,i+K-A-2})$.

$$\sum_{i=1}^{K} nR_{i-D-1} \le \sum_{i=1}^{K} I\left(W_{i-D-1}; S^n|W^c_{i-D-1,i,i+1,\ldots,i+K-A-2}\right) + o(n) \tag{182}$$

$$= \sum_{i=1}^{K} \left\{H\left(S^n|W^c_{i-D-1,i,i+1,\ldots,i+K-A-2}\right) - H(S^n|W^c_{i,i+1,\ldots,i+K-A-2})\right\} + o(n)$$

$$\le \sum_{i=1}^{K}\{n - \frac{K-A-1+U}{U+1}R_i\} + o(n) \tag{183}$$

Rearranging terms and applying the limit $n \to \infty$ we have

$$\sum_{i=1}^{K} R_i \le \frac{U+1}{K-A+2U}K \tag{184}$$

so that we have the information theoretic capacity outer bound of $\frac{U+1}{K-A+2U}$ per message.

Lemma 7 *For $l = 0, 1, \ldots, U-1$, we have*

$$H(S^n|W^c_{i+l,i+l+1,\ldots,i+l+j-2,i+j+U-1}) \le H(S^n|W^c_{i+l,i+l+1,\ldots,i+l+j-1})$$
$$+H(S^n|W^c_{i+l+1,i+l+2,\ldots,i+l+j-1,i+j+U-1})$$
$$-H(S^n|W^c_{i+l+1,i+l+2,\ldots,i+l+j-1}) \tag{185}$$

Proof: Setting

$$\mathcal{C} = \{S^n, W^c_{i+l,i+l+1,\ldots,i+l+j-1}\},$$

$$\mathcal{D} = \{S^n, W^c_{i+l+1,i+l+2,\ldots,i+l+j-1,i+j+U-1}\}$$

and using the submodular property of entropy function

$$H(\mathcal{C}) + H(\mathcal{D}) \ge H(\mathcal{C} \cup \mathcal{D}) + H(\mathcal{C} \cap \mathcal{D}) \tag{186}$$

we have

$$H(S^n, W^c_{i+l,...,i+l+j-1}) + H(S^n, W^c_{i+l+1,...,i+l+j-1,i+j+U-1}) \geq H(S^n, W^c_{i+l+1,i+l+2,...,i+l+j-1})$$
$$+ H(S^n, W^c_{i+l,i+l+2,...,i+l+j-1,i+j+U-1})$$

Equivalently,

$$H(S^n | W^c_{i+l,...,i+l+j-1}) + H(S^n | W^c_{i+l+1,...,i+l+j-1,i+j+U-1}) \geq H(S^n | W^c_{i+l+1,i+l+2,...,i+l+j-1})$$
$$+ H(S^n | W^c_{i+l,i+l+2,...,i+l+j-1,i+j+U-1})$$
$$\geq H(S^n | W^c_{i+l+1,i+l+2,...,i+l+j-1})$$
$$+ H(S^n | W^c_{i+l,i+l+2,...,i+l+j-2,i+j+U-1})$$
$$(187)$$

where (187) is true because conditioning reduces the entropy. Equivalently,

$$H(S^n | W^c_{i+l,i+l+1,...,i+l+j-2,i+j+U-1}) \leq H(S^n | W^c_{i+l,i+l+1,...,i+l+j-1})$$
$$+ H(S^n | W^c_{i+l+1,i+l+2,...,i+l+j-1,i+j+U-1})$$
$$- H(S^n | W^c_{i+l+1,i+l+2,...,i+l+j-1})$$
$$(188)$$

This proves Lemma 7.

D   Proof of Theorem 6: Neighboring interference

D.1   Achievability

We use a scalar linear achievable scheme. In particular, we choose $\mathcal{S} = \mathbb{F}_q$, where q is a prime power q, $n = D + 1$. Suppose we define $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{D+1}$ as $D + 1$ linearly independent vectors in a $D + 1$ dimensional space. Then, starting from any arbitrarily chosen message, we assign these vectors to $D + 1$ consecutive messages respectively and repeat this assignment periodically. So if we assign $\mathbf{V}_m$ to message W_r, then message W_i will be encoded using $\mathbf{V}_{(i-r+m) \mod (D+1)}$. We now need to show resolvability of message W_r at destination D_r. Suppose $\mathbf{V}_m$ is the vector assigned to message W_r. Notice that there are $U + D$ interfering messages at this destination: $W_{r-U,r-U+1,...,r-1,r+1,r+2,...,r+D}$. The remaining messages are available as antidotes and can be cancelled. The U interfering messages $W_{r-U}, \ldots, W_{r-1}$ are encoded over $\mathbf{V}_{(m-U) \mod (D+1)}, \ldots, \mathbf{V}_{(m-1) \mod (D+1)}$, respectively. Since $\{i+D+1 \mod D+1\} = \{i \mod D+1\}$, these interfering vectors are the same as $\mathbf{V}_{(m+D+1-U) \mod (D+1)}, \ldots, \mathbf{V}_{(m+D) \mod (D+1)}$. Also the remaining D interfering messages of destination D_r, i.e., $W_{r+1}, \ldots, W_{r+D}$ are sent over $\mathbf{V}_{(m+1) \mod (D+1)}, \ldots, \mathbf{V}_{(m+D) \mod (D+1)}$, respectively. As we can see all the interfering messages align in the D-dimensional space spanned by vectors $\mathbf{V}_{(m+1) \mod (D+1)}, \ldots, \mathbf{V}_{(m+D) \mod (D+1)}$. This interference space is linearly independent of $\mathbf{V}_m$ because of our construction. Since we are operating in a $D + 1$ dimensional space, message W_r is linearly resolvable at destination D_r as required. This proves achievability.

Remark: $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{D+1}$ can be chosen to be columns of the $D + 1$ dimensional identity matrix and the field can be chosen to be $\mathbb{F}_2$. In other words, a $D + 1$-symbol long achievable scheme is obtained by simply sending the bit $\ldots \oplus x_{r-D-1} \oplus x_r \oplus x_{r+D+1} \oplus \ldots$, in the rth channel use, where $\oplus$ denotes the XOR.

D.2   Outerbound

For the converse, note that we can set $U = 0$. This is because, if $U > 0$, a genie can provide messages $W_{r-U,r-U+1,...,r-1}$ to destination D_r as antidotes to make an enhanced index coding problem where each user is only missing $D + 1$ antidotes after the desired message. In this enhanced index coding problem, we show that the capacity is $\frac{1}{D+1}$ per message. In this setting where $U = 0$, we intend to show that

$$R_i + R_{i+1} + \ldots + R_{i+D} \leq 1.$$

To do so, we give destinations $D_i, D_{i+1}, \ldots, D_{i+D}$ messages $W^c_{i,i+1,\ldots,i+D}$ through a genie. Now, in this genie-aided system, note that destination D_i, which is missing antidotes $W_{i+1,i+2,\ldots,i+D}$ can decode W_i (because of achievability in the original index coding problem). Having decoded W_i, this destination has all the messages present at destination D_{i+1}, and can therefore decode W_{i+1}. Having decoded $W_{i,i+1}$, destination D_i is now equipped with all messages present as antidotes at destination D_{i+2} and can therefore decode W_{i+2}. Continuing this argument, it can be shown that destination D_i can decode $W_{i,i+1,\ldots,i+D}$, and therefore we have, as required,

$$R_i + R_{i+1} + \ldots + R_{i+D} \leq 1.$$

E   Proof of Theorem 7: X network setting with local connectivity

E.1   Achievability

We use a scalar linear achievable scheme. In particular, we choose $\mathcal{S} = \mathbb{F}_q$, where q is a prime power, and $n = \frac{L(L+1)}{2}$. We choose $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{\frac{L(L+1)}{2}}$ to be $\frac{L(L+1)}{2}$ linearly independent vectors in a $n = \frac{L(L+1)}{2}$ dimensional space. Each message is sent over one of these vectors.

Consider an arbitrary destination, say destination D_k. After removing antidotes, this destination receives a linear combination of L^2 messages shown in (189) above the line, and it is interested in L messages

$$\mathcal{W}_k = \{W_{kL,kL+L-1,(k+1)L+L-2,\cdots,(k+i)L+L-i-1,\cdots,(k+L-2)L+1}\}$$

shown in black in (189). The destination faces $(L^2 - L)$ interfering messages shown in (189) with colors other than black (and above the line). The "black" messages are desired by destination D_k and are encoded using L linearly independent vectors, $\mathbf{V}_1, \ldots, \mathbf{V}_L$. Because the total number of dimensions is $\frac{L(L+1)}{2}$, the $L^2 - L$ interfering messages should align such that they span at most $\frac{L(L+1)}{2} - L$ dimensions. Among these interfering messages, the $L - 1$ "red" messages $W_{(k+1)L,(k+1)L+L-1),\ldots,(k+L-2)+2}$ are desired by destination D_{k+1} and should be linearly independent. These messages in red are encoded using $\mathbf{V}_{L+1}, \ldots, \mathbf{V}_{2L-1}$. Also all the $L-2$ "green" messages that are below the red messages and above the line, i.e., $W_{(k+2)L,(k+2)L+L-2,\ldots,(k+L-2)+3}$ are not available as antidotes at either destination D_k or destination D_{k+1} and therefore seen as interference at both these destinations. This implies that the red messages and black messages can not align with these green messages. So we assign $\mathbf{V}_{2L}, \ldots, \mathbf{V}_{3L-2}$ as encoding vectors respectively to green messages. Proceeding thus, we assign $V_{L(L+1)/2}$ to $W_{(k+L-1)L}$. So far, we have assigned $L(L+1)/2 - L$ linearly independent vectors to $L(L+1)/2 - L$ interferers at destination D_k.

$$
\begin{array}{ccccc}
W_{(k-1)L+1} & W_{(k-1)L+2} & \cdots & W_{(k-1)L+L-1} & W_{kL} \\
W_{kL+1} & W_{kL+2} & \cdots & W_{kL+L-1} & \color{red}{W_{(k+1)L}} \\
W_{(k+1)L+1} & W_{(k+1)L+2} & \cdots & \color{red}{W_{(k+1)L+L-1}} & \color{green}{W_{(k+2)L}} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
W_{(k+L-3)L+1} & W_{(k+L-3)L+2} & \cdots & \color{orange}{W_{(k+L-3)L+L-1}} & W_{(k+L-2)L} \\
W_{(k+L-2)L+1} & \color{purple}{W_{(k+L-2)L+2}} & \cdots & \color{orange}{W_{(k+L-2)L+L-1}} & \color{yellow}{W_{(k+L-1)L}} \\
\hline
W_{(k+L-1)L+1} & W_{(k+L-1)L+2} & \cdots & W_{(k+L-1)L+L-1} & W_{(k+L)L}
\end{array}
\tag{189}
$$

So the remaining interfering messages that are shown above the black messages at (189) should be sent over the vectors such that they stay in the same span of interfering messages below the black messages. The way that we satisfy this constraint is by sending the messages shown in (189) respectively over the following vectors

$$
\begin{array}{ccccc}
\mathbf{V}_{L+1} & \mathbf{V}_{L+2} & \cdots & \mathbf{V}_{2L-1} & \mathbf{V}_1 \\
\mathbf{V}_{2L} & \mathbf{V}_{2L+1} & \cdots & \mathbf{V}_2 & \mathbf{V}_{L+1} \\
\mathbf{V}_{3L-1} & \mathbf{V}_{3L} & \cdots & \mathbf{V}_{L+2} & \mathbf{V}_{2L} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\mathbf{V}_{\frac{L(L+1)}{2}} & \mathbf{V}_{L-1} & \cdots & \mathbf{V}_{\frac{L(L+1)}{2}-4} & \mathbf{V}_{\frac{L(L+1)}{2}-2} \\
\mathbf{V}_L & \mathbf{V}_{2L-1} & \cdots & \mathbf{V}_{\frac{L(L+1)}{2}-1} & \mathbf{V}_{\frac{L(L+1)}{2}} \\
\hline
\mathbf{V}_1 & \mathbf{V}_2 & \cdots & \mathbf{V}_{L-1} & \mathbf{V}_L
\end{array}
\tag{190}
$$

Evidently, at destination D_k, all the desired messages are seen over $\mathbf{V}_1, \ldots, \mathbf{V}_L$ and all the interfering messages are seen over $\mathbf{V}_{L+1}, \ldots, \mathbf{V}_{\frac{L(L+1)}{2}}$. Therefore, the desired messages are resolvable at destination D_k. Showing achievability for the remaining messages uses the same argument because of the inherent symmetry in the problem. Specifically, if we assign $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{L-1}, \mathbf{V}_L$ to $W_{(k+L-1)L+1}, W_{(k+L-1)L+2}, \ldots, W_{(k+L-1)L+L-1}, W_{(k+L)L}$ respectively and repeat the pattern shown in (190), assigning vectors periodically for the remaining messages, we can show resolvability at every destination. This follows because, if we choose any L consecutive rows (circularly) of (190), the vectors assigned to anti-diagonal messages which are the desired messages are linearly independent from each other and from the vectors assigned to interfering messages. This proves achievability.

Remark: $\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_{\frac{L(L+1)}{2}}$ can be chosen to be columns of $\frac{L(L+1)}{2}$ dimensional identity matrix, e.g., over $\mathbb{F}_2$ and therefore can be an orthogonal scheme. For the corresponding TIM problem this means that channel coherence is not required [5].

E.2 Outerbound

To prove the outerbound, consider a set of $\frac{L(L+1)}{2}$ messages

$$
\mathcal{W}_{\mathcal{O}} = \left\{ W_{kL, kL+L-1:kL+L, \cdots, (k+i)L-i-1:(k+i)L+L, \cdots, (k+L-2)L+1:(k+L-1)L} \right\},
$$

i.e., the set of "colored" messages shown (above the line) in (189). Among the messages in $\mathcal{W}_{\mathcal{O}}$, the number of messages intended for destination D_{k+i} is $L - i$, where $i \in \{0, 1, 2, \ldots, L - 1.\}$ Our goal is to argue that destination D_k can decode all these $\frac{L(L+1)}{2}$ messages and hence the symmetric rate per message is bounded as $C \leq \frac{2}{L(L+1)}$.

Consider any achievable index coding scheme. Assume that a genie provides all the messages except $\mathcal{W}_{\mathcal{O}}$ to all the destinations $D_k, D_{k+1}, \cdots, D_{k+L-1}$. With the considered index coding scheme, destination D_k, which has $\mathcal{W}_{\mathcal{O}}{}^c$ as antidotes, can decode the "black messages" in (189), i.e.,

$$
\left\{ W_{kL, kL+L-1, (k+1)L+L-2, \cdots, (k+i)L+L-i-1, \cdots, (k+L-2)L+1} \right\}.
$$

Destination D_{k+1} can decode its desired $L-1$ "red" messages - $\left\{ W_{kL+L, (k+1)L+L-1, \cdots, (k+i)L+L-i, \cdots, (k+L-2)L+2} \right\}$ - using antidotes $\mathcal{W}_{\mathcal{O}}{}^c \cup \{W_{kL}\}$. This automatically implies that destination D_k, having decoded all the black messages including W_{kL} can decode the red messages as well. Now, having decoded all the black messages and red messages, the set of messages known to destination D_k includes $\mathcal{W}_{\mathcal{O}}{}^c \cup W_{kL, kL+L-1, (k+1)L}$ - the antidote at destination D_{k+2}. Therefore all the green messages can be decoded at destination D_k. Proceeding further similarly we can argue that destination D_k can decode all the messages in $\mathcal{W}_{\mathcal{O}}$. This completes the proof.

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