

Multiple Quantum Hypothesis Testing Expressions and Classical-Quantum Channel Converse Bounds

Gonzalo Vazquez-Vilar
Universidad Carlos III de Madrid, Spain
Email: gvazquez@ieee.org

Abstract—Alternative exact expressions are derived for the minimum error probability of a hypothesis test discriminating among M quantum states. The first expression corresponds to the error probability of a binary hypothesis test with certain parameters; the second involves the optimization of a given information-spectrum measure. Particularized in the classical-quantum channel coding setting, this characterization implies the tightness of two existing converse bounds; one derived by Matthews and Wehner using hypothesis-testing, and one obtained by Hayashi and Nagaoka via an information-spectrum approach.

I. INTRODUCTION

Optimal discrimination among multiple quantum states—quantum hypothesis testing—is at the core of several information processing tasks involving quantum-mechanical systems. When the number of hypotheses is two, quantum hypothesis testing allows a simple formulation in terms of two kinds of pairwise errors. The quantum version of the Neyman-Pearson lemma establishes the optimum binary test in this setting. This problem was first studied by Helstrom in [1] (see also [2], [3]). When the number of hypotheses is larger than two, a (classical) prior distribution is usually placed over the hypotheses. While there exists no closed form for the optimal test in general, optimality conditions can be obtained [4], [5]. For historical notes on the subject see [6, Ch. IV].

In the context of reliable communication, hypothesis testing has been instrumental in the derivation of converse bounds to the error probability both in the classical and quantum settings (see, e.g., [7], [8]). Recently, hypothesis testing gained interest as a very general approach to obtain converse bounds in the finite block-length regime. In classical channel coding, Polyanskiy, Poor and Verdú derived the meta-converse bound based on an instance of binary hypothesis testing [9]. A similar approach was used by Wang and Renner to derive a finite block-length converse bound for classical-quantum channels [10], and by Matthews and Wehner to obtain a family of converse bounds for general quantum channels [11]. The results by Matthews and Wehner are general enough to recover the meta-converse bound in the classical setting and Wang-Renner converse bound in the classical-quantum setting.

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The information-spectrum method studies the asymptotics of a certain random variable, often referred to as information density or information random variable. Using a quantum analogue of this quantity, Hayashi and Nagaoka studied quantum hypothesis testing [12], and classical-quantum channel coding [13], obtaining general bounds for both problems.

In this paper, we derive two alternative exact expressions for the minimum error probability of multiple quantum hypothesis testing when a (classical) prior distribution is placed over the hypotheses. The expressions obtained illustrate connections among hypothesis testing, information-spectrum measures and converse bounds in classical-quantum channel coding. An application to classical-quantum channel coding shows that Matthews-Wehner converse bound [11, Th. 19] and Hayashi-Nagaoka lemma [13, Lemma 4] with certain parameters yield the exact error probability. This work thus generalizes several results derived in [14] in the classical setting.

II. BACKGROUND

A. Notation

In the general case, a quantum state is described by a density operator ρ acting on some finite dimensional complex Hilbert space \mathcal{H} . Density operators are self-adjoint, positive semidefinite, and have unit trace. A measurement on a quantum system is a mapping from the state of the system ρ to a classical outcome $m \in \{1, \dots, M\}$. A measurement is represented by a collection of positive self-adjoint operators $\{\Pi_1, \dots, \Pi_M\}$ such that $\sum \Pi_m = \mathbb{1}$, where $\mathbb{1}$ is the identity operator. These operators form a POVM (positive operator-valued measure). A measurement $\{\Pi_1, \dots, \Pi_M\}$ applied to ρ has outcome m with probability $\text{Tr}(\rho\Pi_m)$.

For two self-adjoint operators A, B , the notation $A \geq B$ means that $A - B$ is positive semidefinite. Similarly $A \leq B$, $A > B$, and $A < B$ means that $A - B$ is negative semidefinite, positive definite and negative definite, respectively. For a self-adjoint operator A with spectral decomposition $A = \sum_i \lambda_i E_i$, where $\{\lambda_i\}$ are the eigenvalues and $\{E_i\}$ are the orthogonal projections onto the corresponding eigenspaces, we define

$$\{A > 0\} \triangleq \sum_{i:\lambda_i > 0} E_i. \quad (1)$$

This corresponds to the projector associated to the positive eigenspace of A . We shall also use $\{A \geq 0\} \triangleq \sum_{i:\lambda_i \geq 0} E_i$, $\{A < 0\} \triangleq \sum_{i:\lambda_i < 0} E_i$ and $\{A \leq 0\} \triangleq \sum_{i:\lambda_i \leq 0} E_i$.

B. Binary Hypothesis Testing

Let us consider a binary hypothesis test (with simple hypotheses) discriminating between the density operators ρ_0 and ρ_1 acting on \mathcal{H} . In order to distinguish between the two hypotheses we perform a measurement. We define a test measurement $\{T, \bar{T}\}$, such that T and $\bar{T} \triangleq \mathbb{1} - T$ are positive semidefinite. The test decides ρ_0 (resp. ρ_1) when the measurement outcome corresponding to T (resp. \bar{T}) occurs.

Let $\epsilon_{j|i}$ denote the probability of deciding ρ_j when ρ_i is the true hypothesis, $i, j = 0, 1$, $i \neq j$. More precisely, we define

$$\epsilon_{1|0}(T) \triangleq 1 - \text{Tr}(\rho_0 T) = \text{Tr}(\rho_0 \bar{T}), \quad (2)$$

$$\epsilon_{0|1}(T) \triangleq \text{Tr}(\rho_1 T). \quad (3)$$

Let $\alpha_\beta(\rho_0 \|\rho_1)$ denote the minimum error probability $\epsilon_{1|0}$ among all tests with $\epsilon_{0|1}$ at most β , that is,

$$\alpha_\beta(\rho_0 \|\rho_1) \triangleq \inf_{T: \epsilon_{0|1}(T) \leq \beta} \epsilon_{1|0}(T). \quad (4)$$

The function $\alpha_\beta(\cdot \|\cdot)$ is the inverse of the function $\beta_\alpha(\cdot \|\cdot)$ appearing in [11], which is itself related to the hypothesis-testing relative entropy as $D_{\text{H}}^\alpha(\rho_0 \|\rho_1) = -\log \beta_\alpha(\rho_0 \|\rho_1)$ [10].

When ρ_0 and ρ_1 commute, the test T in (4) can be restricted to be diagonal in the (common) eigenbasis of ρ_0 and ρ_1 , then (4) reduces to the classical case [14].

The quantum version of the Neyman-Pearson lemma characterizes the form of the test minimizing (4). Let $t \geq 0$ and let P_t^+ , P_t^- , P_t^0 denote the projectors spanning the positive, negative and null eigenspaces of the matrix $\rho_0 - t\rho_1$, respectively, i. e.,

$$P_t^+ \triangleq \{\rho_0 - t\rho_1 > 0\}, \quad (5)$$

$$P_t^- \triangleq \{\rho_0 - t\rho_1 < 0\}, \quad (6)$$

$$P_t^0 \triangleq \mathbb{1} - P_t^+ - P_t^-. \quad (7)$$

Lemma 1 (Neyman-Pearson lemma): The operator T_{NP} is an optimal test between ρ_0 and ρ_1 if and only if

$$T_{\text{NP}} = P_t^+ + p_t^0, \quad (8)$$

where $0 \leq p_t^0 \leq P_t^0$.

Proof: A slightly different formulation of this result is usually given in the literature. The statement included here can be found in, e.g., [15, Lem. 3]. ■

Therefore, for any $t \geq 0$ and $0 \leq p_t^0 \leq P_t^0$ such that $\text{Tr}\{\rho_1 T_{\text{NP}}\} = \beta$, the resulting test T_{NP} minimizes (4). Moreover, the following lower bound holds.

Lemma 2: For any test discriminating between ρ_0 and ρ_1 , and for any $t' \geq 0$, it holds that

$$\alpha_\beta(\rho_0 \|\rho_1) \geq \text{Tr}\left(\rho_0(P_{t'}^- + P_{t'}^0)\right) - t'\beta. \quad (9)$$

Proof: For any operator $A \geq 0$ and $0 \leq T \leq \mathbb{1}$, it holds that $\text{Tr}(A\{A > 0\}) \geq \text{Tr}(AT)$ [12, Eq. 8]. For $A = \rho_0 - t'\rho_1$ and $T = T_{\text{NP}}$, this inequality becomes

$$\text{Tr}((\rho_0 - t'\rho_1)P_{t'}^+) \geq \text{Tr}((\rho_0 - t'\rho_1)T_{\text{NP}}), \quad (10)$$

which after some algebra yields

$$-\text{Tr}(\rho_0 T_{\text{NP}}) \geq -\text{Tr}(\rho_0 P_{t'}^+) + t' \text{Tr}(\rho_1(P_{t'}^+ - T_{\text{NP}})). \quad (11)$$

Summing one to both sides of (11) and noting that $\alpha_\beta(\rho_0 \|\rho_1) = 1 - \text{Tr}(\rho_0 T_{\text{NP}})$ and $\beta = \text{Tr}(\rho_1 T_{\text{NP}})$, we obtain

$$\alpha_\beta(\rho_0 \|\rho_1) \geq \text{Tr}(\rho_0(P_{t'}^- + P_{t'}^0)) + t' \text{Tr}(\rho_1 P_{t'}^+) - t'\beta. \quad (12)$$

The result thus follows by lower-bounding $\text{Tr}(\rho_1 P_{t'}^+) \geq 0$. ■

III. MULTIPLE QUANTUM HYPOTHESIS TESTING

We consider a hypothesis testing problem discriminating among M possible states acting on \mathcal{H} , where M is assumed to be finite. The M alternatives τ_1, \dots, τ_M are assumed to occur with (classical) probabilities p_1, \dots, p_M , respectively.

A M -ary hypothesis test is a POVM $\mathcal{P} \triangleq \{\Pi_1, \Pi_2, \dots, \Pi_M\}$, $\sum \Pi_i = \mathbb{1}$. The test decides the alternative τ_i when the measurement with respect to \mathcal{P} has outcome i . The probability that the test \mathcal{P} decides τ_j when τ_i is the true underlying state is thus $\text{Tr}(\tau_i \Pi_j)$ and the average error probability is

$$\epsilon(\mathcal{P}) \triangleq 1 - \sum_{i=1}^M p_i \text{Tr}(\tau_i \Pi_i). \quad (13)$$

We define the minimum average error probability as

$$\epsilon \triangleq \min_{\mathcal{P}} \epsilon(\mathcal{P}). \quad (14)$$

The test \mathcal{P} minimizing (14) has no simple form in general.

Lemma 3 (Holevo-Yuen-Kennedy-Lax conditions): A test $\mathcal{P}^* = \{\Pi_1^*, \dots, \Pi_M^*\}$ minimizes (14) if and only if, for each $m = 1, \dots, M$,

$$(\Lambda(\mathcal{P}^*) - p_m \tau_m) \Pi_m^* = \Pi_m^* (\Lambda(\mathcal{P}^*) - p_m \tau_m) = 0, \quad (15)$$

$$\Lambda(\mathcal{P}^*) - p_m \tau_m \geq 0, \quad (16)$$

where

$$\Lambda(\mathcal{P}^*) \triangleq \sum_{i=1}^M p_i \tau_i \Pi_i^* = \sum_{i=1}^M p_i \Pi_i^* \tau_i \quad (17)$$

is required to be self-adjoint¹.

Proof: The theorem follows from [4, Th. 4.1, Eq. (4.8)] or [5, Th. I] after simplifying the resulting optimality conditions. ■

We next show an alternative characterization of the minimum error probability ϵ as a function of a binary hypothesis test with certain parameters.

Let $\text{diag}(\rho_1, \dots, \rho_M)$ denote the block-diagonal matrix with diagonal blocks ρ_1, \dots, ρ_M . We define

$$\mathcal{T} \triangleq \text{diag}(p_1 \tau_1, \dots, p_M \tau_M), \quad (18)$$

$$\mathcal{D}(\mu_0) \triangleq \text{diag}\left(\frac{1}{M} \mu_0, \dots, \frac{1}{M} \mu_0\right), \quad (19)$$

where μ_0 is an arbitrary density operator acting on \mathcal{H} . Note that both \mathcal{T} and $\mathcal{D}(\mu_0)$ are density operators themselves, since they are self-adjoint, positive semidefinite and have unit trace.

¹The operator $\Lambda(\mathcal{P})$ takes a role of the Lagrange multiplier associated to the constraint $\sum \Pi_i = \mathbb{1}$, which, involving self-adjoint operators requires Λ to be self-adjoint.

Theorem 1: The minimum error probability of an M -ary test discriminating among states $\{\tau_1, \dots, \tau_M\}$ with prior classical probabilities $\{p_1, \dots, p_M\}$ satisfies

$$\epsilon = \max_{\mu_0} \alpha_{\frac{1}{M}}(\mathcal{T} \parallel \mathcal{D}(\mu_0)), \quad (20)$$

where \mathcal{T} and $\mathcal{D}(\cdot)$ are given in (18) and (19), respectively, and where the optimization is carried out over (unit-trace non-negative) density operators μ_0 .

Proof: For any $\mathcal{P} = \{\Pi_1, \Pi_2, \dots, \Pi_M\}$ let us define the binary test $T' \triangleq \text{diag}(\Pi_1, \dots, \Pi_M)$. For this test we obtain

$$\epsilon_{1|0}(T') = 1 - \sum_{i=1}^M p_i \text{Tr}(\tau_i \Pi_i) = \epsilon(\mathcal{P}), \quad (21)$$

$$\epsilon_{0|1}(T') = \frac{1}{M} \sum_{i=1}^M \text{Tr}(\mu_0 \Pi_i) \quad (22)$$

$$= \frac{1}{M} \text{Tr} \left(\mu_0 \left(\sum_{i=1}^M \Pi_i \right) \right) \quad (23)$$

$$= \frac{1}{M} \text{Tr}(\mu_0) = \frac{1}{M}. \quad (24)$$

The (possibly suboptimal) test T' has thus $\epsilon_{1|0}(T') = \epsilon(\mathcal{P})$ and $\epsilon_{0|1}(T') = \frac{1}{M}$. Therefore, using (4) and maximizing the resulting expression over μ_0 , we obtain

$$\epsilon(\mathcal{P}) \geq \max_{\mu_0} \alpha_{\frac{1}{M}}(\mathcal{T} \parallel \mathcal{D}(\mu_0)). \quad (25)$$

It remains to show that, for $\mathcal{P} = \mathcal{P}^*$ defined in Lemma 3, the lower bound (25) holds with equality. To this end, we next demonstrate that the optimality conditions for T_{NP} in Lemma 1 and for $\mathcal{P}^* = \{\Pi_1^*, \dots, \Pi_M^*\}$ in Lemma 3 are equivalent for a specific choice of μ_0 .

Let $\mathcal{P}^* = \{\Pi_1^*, \dots, \Pi_M^*\}$ satisfy (15)-(16) and define

$$\mu_0^* \triangleq \frac{1}{c_0^*} \sum_{i=1}^M p_i \tau_i \Pi_i^* = \frac{1}{c_0^*} \Lambda(\mathcal{P}^*), \quad (26)$$

where c_0^* is a normalizing constant such that μ_0^* is unit trace.

Lemma 1 shows that the test T_{NP} achieving (25) is associated to the non-negative eigenspace of the matrix $\mathcal{T} - t\mathcal{D}(\mu_0)$. Given the block-diagonal structure of the matrix $\mathcal{T} - t\mathcal{D}(\mu_0)$, it is enough to consider binary tests T_{NP} with block-diagonal structure. Then, we write $T_{\text{NP}} = \text{diag}(T_1^{\text{NP}}, \dots, T_M^{\text{NP}})$.

For the choice $\mu_0 = \mu_0^*$, and $t = M c_0^*$, the m -th block-diagonal term in $\mathcal{T} - t\mathcal{D}(\mu_0)$ is given by

$$p_m \tau_m - \frac{t}{M} \mu_0 = p_m \tau_m - \Lambda(\mathcal{P}^*). \quad (27)$$

The m -th block of the Neyman-Pearson test T_m^{NP} must lie in the non-negative eigenspace of the matrix (27). However, since (16) implies that (27) is negative semidefinite, each block T_m^{NP} can only lie in the null eigenspace of (27), $m = 1, \dots, M$.

According to (15), the operator Π_m^* belongs to the null eigenspace of (27), $m = 1, \dots, M$. As a result, the choice

$$T_{\text{NP}} = \text{diag}(\Pi_1^*, \dots, \Pi_M^*) \quad (28)$$

satisfies the optimality conditions in Lemma 1. Moreover, since $\epsilon_{1|0}(T_{\text{NP}}) = \epsilon(\mathcal{P}^*) = \epsilon$ and $\epsilon_{0|1}(T_{\text{NP}}) = \frac{1}{M}$, Lemma 1

implies that (20) holds with equality for $\mu_0 = \mu_0^*$. Given the bound in (25), other choices of μ_0 cannot improve the result, and Theorem 1 thus follows. \blacksquare

Combining Theorem 1 and Lemma 2, we obtain a characterization for ϵ based on information-spectrum measures.

Theorem 2: The minimum error probability of an M -ary test discriminating among states $\{\tau_1, \dots, \tau_M\}$ with prior classical probabilities $\{p_1, \dots, p_M\}$ satisfies

$$\epsilon = \max_{\mu_0, t \geq 0} \left\{ \sum_{i=1}^M p_i \text{Tr} \left(\tau_i \{ p_i \tau_i - t \mu_0 \leq 0 \} \right) - t \right\}. \quad (29)$$

where the optimization is carried out over (unit-trace non-negative) density operators μ_0 acting on \mathcal{H} , and over the scalar threshold $t \geq 0$.

Proof: Applying Lemma 2 to (20), and using the definitions of \mathcal{T} in (18) and $\mathcal{D}(\cdot)$ in (19), yields, for any $\mu_0, t' \geq 0$,

$$\epsilon \geq \sum_{i=1}^M p_i \text{Tr} \left(\tau_i \{ p_i \tau_i - \frac{t'}{M} \mu_0 \leq 0 \} \right) - \frac{t'}{M}. \quad (30)$$

It remains to show that there exist μ_0 and $t' \geq 0$ such that (30) holds with equality. In particular, let us choose $\mu_0 = \mu_0^*$ defined in (26), and $t' = M c_0^*$ where $c_0^* = \sum_{i=1}^M p_i \text{Tr}(\tau_i \Pi_i^*)$ is the normalizing constant from (26).

For this choice of μ_0 and t' , the projector spanning the negative semidefinite eigenspace of the operator $p_i \tau_i - \frac{t'}{M} \mu_0$ can be rewritten as

$$\left\{ p_i \tau_i - \frac{t'}{M} \mu_0 \leq 0 \right\} = \left\{ p_i \tau_i - \Lambda(\mathcal{P}^*) \leq 0 \right\} \quad (31)$$

$$= \mathbb{1}, \quad (32)$$

where the last identity follows from (16). The right-hand side of (30) thus becomes

$$\sum_{i=1}^M p_i \text{Tr}(\tau_i) - \frac{t'}{M} = 1 - \frac{t'}{M}. \quad (33)$$

The result follows since $\frac{t'}{M} = c_0^* = \sum_i p_i \text{Tr}(\tau_i \Pi_i^*) = 1 - \epsilon$. \blacksquare

The alternative expressions derived in Theorems 1 and 2 are not easier to compute than the original optimization in (14), all of them requiring to solve a semidefinite program. We recall from the proofs of the theorems that a density operator μ_0 maximizing (20) and (29) is

$$\mu_0^* = \frac{1}{c_0^*} \sum_{i=1}^M p_i \tau_i \Pi_i^*, \quad (34)$$

for some $\mathcal{P}^* = \{\Pi_1^*, \dots, \Pi_M^*\}$ satisfying the conditions in Lemma 3 and where c_0^* is a normalizing constant. Hence, the optimal M -ary hypothesis test \mathcal{P}^* characterizes the optimal μ_0 . Conversely, the optimal μ_0 is precisely the Lagrange multiplier associated to the minimization in (14), after an appropriate re-scaling.

The expressions derived in Theorems 1 and 2 can be used to determine the tightness of several converse bounds from the literature, as we show in the next section.

IV. APPLICATION TO CLASSICAL-QUANTUM CHANNELS

We consider the channel coding problem of transmitting M equiprobable messages over a one-shot classical-quantum channel $x \rightarrow W_x$, with $x \in \mathcal{X}$ and $W_x \in \mathcal{H}$.

A channel code is defined as a mapping from the message set $\{1, \dots, M\}$ into a set of M codewords $\mathcal{C} = \{x_1, \dots, x_M\}$. For a source message m , the decoder receives the associated density operator W_{x_m} and must decide on the transmitted message. The minimum error probability for a code \mathcal{C} is

$$P_e(\mathcal{C}) \triangleq \min_{\{\Pi_1, \dots, \Pi_M\}} \left\{ 1 - \frac{1}{M} \sum_{m=1}^M \text{Tr}(W_{x_m} \Pi_m) \right\}. \quad (35)$$

This problem corresponds precisely to the M -ary quantum hypothesis testing problem described in Section III. Then, direct application of Theorems 1 and 2 yields two alternative expressions for $P_e(\mathcal{C})$.

Let \mathbb{A} and \mathbb{B} denote the input and output of the system, respectively. The joint state induced by a codebook \mathcal{C} is

$$\rho_{\mathcal{C}}^{\mathbb{A}\mathbb{B}} = \frac{1}{M} \sum_{x \in \mathcal{C}} |x\rangle\langle x|^{\mathbb{A}} \otimes W_x^{\mathbb{B}}, \quad (36)$$

and $\rho_{\mathcal{C}}^{\mathbb{A}} = \frac{1}{M} \sum_{x \in \mathcal{C}} |x\rangle\langle x|^{\mathbb{A}}$ its input marginal.

According to (20) in Theorem 1 we obtain

$$P_e(\mathcal{C}) = \max_{\mu_0} \alpha_{\frac{1}{M}}(\rho_{\mathcal{C}}^{\mathbb{A}\mathbb{B}} \| \rho_{\mathcal{C}}^{\mathbb{A}} \otimes \mu_0^{\mathbb{B}}). \quad (37)$$

The expression (37) is precisely the finite block-length converse bound by Matthews and Wehner [11, Eq. (45)], particularized for a classical-quantum channel with an input state induced by the codebook \mathcal{C} . Therefore, Theorem 1 implies that the quantum generalization of the meta-converse bound proposed by Matthews and Wehner is tight for a fixed codebook \mathcal{C} .

Minimizing the right-hand side of (37) over all distributions P_X defined over the input alphabet \mathcal{X} , not necessarily induced by a codebook, yields a lower bound on $P_e(\mathcal{C})$ for any codebook \mathcal{C} . By fixing μ_0 to be the state induced at the system output, this lower bound recovers the converse bound by Wang and Renner [10, Th. 1]. This bound is not tight in general since (i) the minimizing P_X does not need to coincide with the input state induced by the best codebook, and (ii) the choice of μ_0 in [10, Th. 1] does not maximize the resulting bound in general.

Using the characterization in Theorem 2, the error probability $P_e(\mathcal{C})$ can be equivalently written as

$$P_e(\mathcal{C}) = \max_{\mu_0, t' \geq 0} \left\{ \frac{1}{M} \sum_{x \in \mathcal{C}} \text{Tr}(W_x \{W_x - t' \mu_0 \leq 0\}) - \frac{t'}{M} \right\}. \quad (38)$$

The objective of the maximization in (38) coincides with the information-spectrum bound [13, Lemma 4]. Then, (38) shows that the Hayashi-Nagaoka lemma yields the exact error probability for a fixed code, after optimization over the free parameters $\mu_0, t' \geq 0$.

V. CONCLUDING REMARKS

In Theorem 1, the minimum error probability of an M -ary quantum hypothesis test is expressed as an instance of a binary quantum hypothesis test with certain parameters. This expression implies the tightness of the converse bound [11, Th. 19] by Matthews and Wehner, and identifies the weakness of [10, Th. 1] by Wang and Renner in classical-quantum channel coding. For more general channels and entanglement-assisted codes, it is not clear whether the bounds in [11, Th. 18 and Th. 19] coincide with the exact error probability. To study this, a generalization of Theorem 1 imposing less structure over the test alternatives is needed. Theorem 2 shows that the minimum error probability can be written as an optimization problem involving information-spectrum measures. In particular, this expression shows that the Hayashi-Nagaoka lemma [13, Lemma 4] yields the exact error probability after optimization over its free parameters.

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