

Causal Vector-valued Witsenhausen Counterexamples with Feedback

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Abstract—We study the continuous vector-valued Witsenhausen counterexample through the lens of empirical coordination coding. We characterize the region of achievable pairs of costs in three scenarios: (i) causal encoding and causal decoding, (ii) causal encoding and causal decoding with channel feedback, and (iii) causal encoding and noncausal decoding with channel feedback. In these vector-valued versions of the problem, the optimal coding schemes must rely on a time-sharing strategy, since the region of achievable pairs of costs might not be convex in the scalar version of the problem. We examine the role of the channel feedback when the encoder is causal and the decoder is either causal or non-causal, and we show that feedback improves the performance, only when the decoder is non-causal.

I. INTRODUCTION

In 1968, Witsenhausen introduced his famous counterexample demonstrating the suboptimality of affine strategies in the Linear Quadratic Gaussian (LQG) settings featuring non-classical information patterns [1]. Since then, it has become a cornerstone in the field of distributed decision-making [2]–[5] and information-theoretic control [6]–[13].

Due to the vector-valued formulation of the problem [14], many information-theoretic approaches have been adopted for analyzing this open problem [15]–[18]. The concept of empirical coordination, introduced in [19]–[21], plays an important role in establishing cooperative behavior among all agents in the network, providing single-letter solutions for problems such as characterizing the optimal cost, capacity region, and utility functions [22]–[24].

Feedback, as an intrinsic component of communication systems, offers the potential to enhance performance by enabling decision makers (DMs) to refine their actions based on previous outcomes. In many multi-terminal setups, feedback has proven beneficial for increasing the capacity region and assisting communication of the multiple-access channel [25, 26] as well as broadcast channel [27, 28]. Additionally, when considering the empirical coordination coding problem in a point-to-point scenario, [29, 30] showed that channel feedback could enable the DMs to directly coordinate their outputs with the system state, by simplifying the information constraint and by reducing the number of auxiliary random variables, and

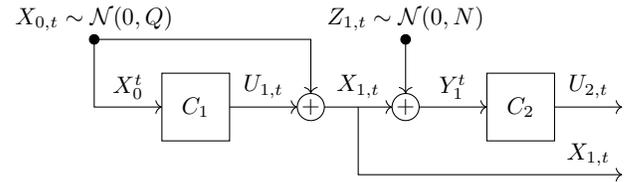


Figure 1. Vector-valued Witsenhausen counterexample with causal encoder and causal decoder.

therefore, enlarge the set of distributions. However, whether feedback facilitates coordination in the vector-valued Witsenhausen settings with causal controllers [31, 32] still remains an open question. Recently, it has been claimed in [33], that when constraint to Gaussian case, feedback does not contribute to performance improvements when the first DM is causal and the second DM is noncausal. In this article, we complete this study and we provide the necessary proof to the claimed information constraint.

In this paper, we aim to deepen the understanding of the problem by incorporating channel feedback for a comparative study, focusing on three novel setups: (i) causal encoding and causal decoding, (ii) causal encoding and causal decoding with channel feedback, and (iii) causal encoding and non-causal decoding with channel feedback. Our analysis shows that time-sharing is crucial in the first setup, in order to randomize between operating points and convexify the region of achievable pairs of costs. The minimum Gaussian cost identified in [33] can also be achieved by time-sharing between two affine policies. Time-sharing corresponds to randomized decision-making process for the original one-shot Witsenhausen counterexample which enables operating points to be achieved by randomization. The characterization of the region of achievable pairs of costs for the second model is derived using the genie-aided argument by focusing on its outer-bound. Comparing the results of the second and third models, we conclude that feedback can enlarge the region of achievable pairs of costs when the decoder is non-causal but not when it is causal.

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II. SYSTEM MODEL

The vector-valued Witsenhausen counterexample setup consists of source states and channel noises that are drawn independently according to the i.i.d. Gaussian distributions $X_0^n \sim \mathcal{N}(0, Q\mathbb{I})$ and $Z_1^n \sim \mathcal{N}(0, N\mathbb{I})$, for some block length $n > 0$, and $Q, N > 0$. We denote by X_1 the memoryless interim state and Y_1 and output of the memoryless additive channel, generated by

$$X_1 = X_0 + U_1 \quad \text{with } X_0 \sim \mathcal{N}(0, Q), \quad (1)$$

$$Y_1 = X_1 + Z_1 = X_0 + U_1 + Z_1 \quad \text{with } Z_1 \sim \mathcal{N}(0, N). \quad (2)$$

We denote by $\mathcal{P}_{X_0} = \mathcal{N}(0, Q)$ the generative Gaussian probability distribution of the state, and by $\mathcal{P}_{X_1, Y_1 | X_0, U_1}$ the channel probability distribution according to (1) and (2).

We now present the three models and their corresponding results. The proofs of these results are given in Appendix.

A. Causal encoding and Causal decoding

Let's first consider the model with two causal DMs, as illustrated in Fig. 1.

Definition II.1. For $n \in \mathbb{N}$, a ‘‘control design’’ with causal encoder and causal decoder is a tuple of stochastic functions $c = (\{f_{U_1, t | X_0^t}^{(t)}\}_{t=1}^n, \{g_{U_2, t | Y_1^t}^{(t)}\}_{t=1}^n)$ defined by

$$f_{U_1, t | X_0^t}^{(t)} : \mathcal{X}_0^t \longrightarrow \mathcal{U}_{1, t}, \quad g_{U_2, t | Y_1^t}^{(t)} : \mathcal{Y}_1^t \longrightarrow \mathcal{U}_{2, t}, \quad (3)$$

which induces a distribution over sequences of symbols:

$$\prod_{t=1}^n \mathcal{P}_{X_0, t} \prod_{t=1}^n f_{U_1, t | X_0^t}^{(t)} \prod_{t=1}^n \mathcal{P}_{X_1, t, Y_1, t | X_0, t, U_1, t} \prod_{t=1}^n g_{U_2, t | Y_1^t}^{(t)}. \quad (4)$$

We denote by $\mathcal{C}_{ed}(n)$ the set of control designs with causal encoder and causal decoder.

Definition II.2. We define the two long-run cost functions $c_P(u_1^n) = \frac{1}{n} \sum_{t=1}^n (u_{1, t})^2$ and $c_S(x_1^n, u_2^n) = \frac{1}{n} \sum_{t=1}^n (x_{1, t} - u_{2, t})^2$. The pair of costs $(P, S) \in \mathbb{R}^2$ is said to be achievable if for all $\varepsilon > 0$, there exists $\bar{n} \in \mathbb{N}$ such that for all $n \geq \bar{n}$, there exists a control design $c \in \mathcal{C}_{ed}(n)$ such that

$$\mathbb{E} \left[|P - c_P(U_1^n)| + |S - c_S(X_1^n, U_2^n)| \right] \leq \varepsilon. \quad (5)$$

We denote by \mathcal{R}_{ed} the set of achievable pairs of costs for control designs in \mathcal{C}_{ed} .

Next, we characterize the costs region \mathcal{R}_{ed} .

Theorem II.3. The pair of Witsenhausen costs (P, S) is achievable if and only if there exists a joint distribution over the random variables $(X_0, T, U_1, X_1, Y_1, U_2)$ that decomposes according to

$$\mathcal{P}_{X_0} \mathcal{P}_T \mathcal{P}_{U_1 | X_0, T} \mathcal{P}_{X_1, Y_1 | X_0, U_1} \mathcal{P}_{U_2 | T, Y_1}, \quad (6)$$

such that

$$P = \mathbb{E} \left[U_1^2 \right], \quad S = \mathbb{E} \left[(X_1 - U_2)^2 \right], \quad (7)$$

where \mathcal{P}_{X_0} and $\mathcal{P}_{X_1, Y_1 | X_0, U_1}$ are the given Gaussian distributions, and T is the time-sharing auxiliary random variable with cardinality bound $|\mathcal{T}| \leq 2$.

Remark II.4. The probability distribution (6) satisfies

$$\begin{cases} X_0 \text{ is independent of } T, \\ (X_1, Y_1) \text{---} \ominus (X_0, U_1) \text{---} \ominus T, \\ U_2 \text{---} \ominus (T, Y_1) \text{---} \ominus (X_0, U_1, X_1). \end{cases} \quad (8)$$

The first property comes from the fact that the time-sharing random variable is independent of the source. The second Markov chain is due to the memoryless property of the channel. The last Markov chain comes from the causal decoding and the symbol-wise reconstruction.

Remark II.5. The cost region

$$\mathcal{R}_{ed} = \{(P, S) : P = \mathbb{E}[U_1^2], S = \mathbb{E}[(X_1 - U_2)^2], \text{ for } \mathcal{P} \text{ of the form of (6)}\} \quad (9)$$

characterized in Theorem II.3 is convex. This is because time-sharing synthesizes DMs to agree on the operating points. Therefore, we can construct a control scheme that achieves any point on the cord between two pairs of Witsenhausen costs.

Let's look at the following example delving into the effect of convexification using time-sharing:

Example. We consider a binary source X_0 with $\mathbb{P}(X_0 = 0) = \mathbb{P}(X_0 = 1) = \frac{1}{2}$, binary symmetric stochastic encoder $\sim \text{Bern}(\alpha)$ and decoder $\sim \text{Bern}(\beta)$ and a perfect channel with the set of two symbols $\mathcal{X}_0 = \mathcal{U}_1 = \mathcal{U}_2 = \{0, 1\}$, as represented in Fig. 2.

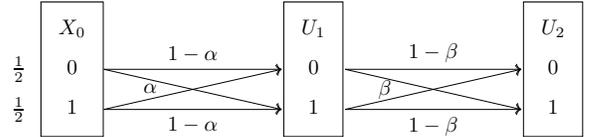


Figure 2. Binary information source and binary symmetric stochastic encoder and decoder.

The joint distribution of this system is presented in Fig. 3. The goal of the two DMs is to design their strategies through pairs (α, β) to achieve a desired empirical distribution.

	$U_2 = 0$	$U_2 = 1$		$U_2 = 0$	$U_2 = 1$
$U_1 = 0$	$\frac{1}{2}(1-\alpha)(1-\beta)$	$\frac{1}{2}(1-\alpha)\beta$	$U_1 = 0$	$\frac{1}{2}\alpha(1-\beta)$	$\frac{1}{2}\alpha\beta$
$U_1 = 1$	$\frac{1}{2}\alpha\beta$	$\frac{1}{2}\alpha(1-\beta)$	$U_1 = 1$	$\frac{1}{2}(1-\alpha)\beta$	$\frac{1}{2}(1-\alpha)(1-\beta)$
	$X_0 = 0$			$X_0 = 1$	

Figure 3. The joint distribution induced by a binary system depicted in Fig. 2

In this example, we initially don't apply the time-sharing technique. Fig. 4 shows an empirical distribution (third row) that is generated by directly combining the distributions given

by $(\alpha = 0, \beta = 0)$ (first row) and $(\alpha = 1, \beta = 1)$ (second row). However, one can easily show that this combined empirical distribution can not be achieved by any single choice of pair $(\alpha, \beta) \in [0, 1]^2$, thus it is achievable only through time-sharing.

		$U_2 = 0$	$U_2 = 1$			$U_2 = 0$	$U_2 = 1$
$U_1 = 0$	$U_1 = 1$	$\frac{1}{2}$	0	$U_1 = 0$	$U_1 = 1$	0	0
		$X_0 = 0$	$X_0 = 1$			$X_0 = 0$	$X_0 = 1$
		$U_2 = 0$	$U_2 = 1$			$U_2 = 0$	$U_2 = 1$
$U_1 = 0$	$U_1 = 1$	0	0	$U_1 = 0$	$U_1 = 1$	0	$\frac{1}{2}$
		$X_0 = 1$	$X_0 = 1$			$X_0 = 1$	$X_0 = 1$
		$U_2 = 0$	$U_2 = 1$			$U_2 = 0$	$U_2 = 1$
$U_1 = 0$	$U_1 = 1$	$\frac{1}{4}$	0	$U_1 = 0$	$U_1 = 1$	0	$\frac{1}{4}$
		$X_0 = 0$	$X_0 = 1$			$X_0 = 0$	$X_0 = 1$

Figure 4. First row: joint distribution given by $(\alpha = 0, \beta = 0)$, second row: joint distribution given by $(\alpha = 1, \beta = 1)$, third row: a convex combination of the above two cases, but is not achievable by any single pair of (α, β) .

B. Causal-causal with Channel Feedback

Now suppose the causal encoder has a delayed access to the sequence of channel output Y_1^{t-1} at each stage t to help its decision, see Fig. 5.

Definition II.6. For $n \in \mathbb{N}$, a “control design” with causal encoder and causal decoder with channel feedback is a tuple of stochastic functions $c = (\{f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)}\}_{t=1}^n, \{g_{U_2,t|Y_1^t}^{(t)}\}_{t=1}^n)$ defined by

$$f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)} : \mathcal{X}_0^t \times \mathcal{Y}_1^{t-1} \longrightarrow \mathcal{U}_{1,t}, \quad g_{U_2,t|Y_1^t}^{(t)} : \mathcal{Y}_1^t \longrightarrow \mathcal{U}_{2,t}, \quad (10)$$

which induces a distribution over sequences of symbols:

$$\prod_{t=1}^n \mathcal{P}_{X_{0,t}} \prod_{t=1}^n f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)} \prod_{t=1}^n \mathcal{P}_{X_{1,t}, Y_{1,t}|X_{0,t}, U_{1,t}} \prod_{t=1}^n g_{U_2,t|Y_1^t}^{(t)}, \quad (11)$$

where $Y_1^0 = \emptyset$. We denote by $\mathcal{C}_{ed,f}(n)$ the set of control designs with causal encoder and causal decoder with channel feedback.

We define the achievable pairs of cost (P, S) similarly as in Definition II.2 and we denote by $\mathcal{R}_{ed,f}$ the region of achievable pairs of costs.

Theorem II.7. $\mathcal{R}_{ed,f} = \mathcal{R}_{ed}$.

The proof of Theorem II.7 is stated in Appendix B, where a genie-aided method is used to prove the converse result.

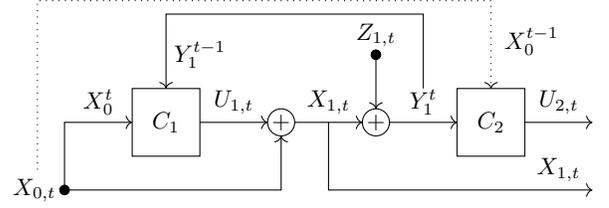


Figure 5. Witsenhausen counterexample for causal-encoding and causal-decoding with channel feedback Y_1^{t-1} to the encoder. The dotted line describes the source feed-forward X_0^{t-1} to the decoder, that is used for the genie-aided argument in the converse proof of Theorem II.7 in Appendix B.

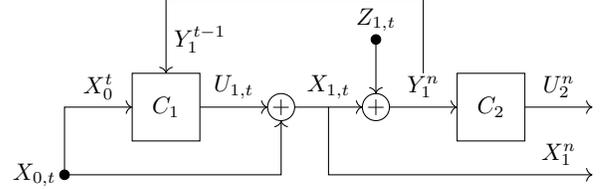


Figure 6. Witsenhausen counterexample for causal-encoding and noncausal-decoding with channel feedback Y_1^{t-1} to the encoder.

Adding channel feedback does not help for enlarging the region of achievable pairs of costs when both DMs are causal. This result is a consequence of the two causalities such that the two DMs only deal with new information: Even though at time $t \in [1 : n]$ the encoder has the opportunity to provide more insights for its previous decisions based on the past channel output Y_1^{t-1} , the causal decoder has already responded to its past actions and therefore does not allow further modifications for this time.

C. Causal-noncausal with Channel Feedback

In this section, we consider the model of causal-encoding and noncausal-decoding with channel feedback as illustrated in Fig. 6. The main result in this chapter provides the information constraint that completes the proof of Corollary III.6 in [33].

Definition II.8. For $n \in \mathbb{N}$, a “control design” with causal encoder and noncausal decoder with channel feedback is a tuple of stochastic functions $c = (\{f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)}\}_{t=1}^n, g_{U_2^n|Y_1^n})$ defined by

$$f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)} : \mathcal{X}_0^t \times \mathcal{Y}_1^{t-1} \longrightarrow \mathcal{U}_{1,t}, \quad g_{U_2^n|Y_1^n} : \mathcal{Y}_1^n \longrightarrow \mathcal{U}_2^n, \quad (12)$$

which induces a distribution over sequences of symbols:

$$\prod_{t=1}^n \mathcal{P}_{X_{0,t}} \prod_{t=1}^n f_{U_1,t|X_0^t, Y_1^{t-1}}^{(f,t)} \prod_{t=1}^n \mathcal{P}_{X_{1,t}, Y_{1,t}|X_{0,t}, U_{1,t}} g_{U_2^n|Y_1^n}, \quad (13)$$

where $Y_1^0 = \emptyset$. We denote by $\mathcal{C}_{e,f}(n)$ the set of control designs with causal encoder and noncausal decoder with channel feedback.

Again, we denote by $\mathcal{R}_{e,f}$ the region of achievable pairs of rates for control designs in $\mathcal{C}_{e,f}(n)$.

Theorem II.9. *The pair of Witsenhausen costs is achievable (P, S) if and only if there exists a joint distribution over the random variables $(X_0, W_1, U_1, X_1, Y_1, U_2)$ that decomposes according to*

$$\mathcal{P}_{X_0} \mathcal{P}_{W_1} \mathcal{P}_{U_1|X_0, W_1} \mathcal{P}_{X_1, Y_1|X_0, U_1} \mathcal{P}_{U_2|X_0, W_1, Y_1}, \quad (14)$$

such that

$$I(W_1; Y_1) - I(U_2; X_0|W_1, Y_1) \geq 0, \quad (15)$$

$$P = \mathbb{E}[U_1^2], \quad S = \mathbb{E}[(X_1 - U_2)^2], \quad (16)$$

where \mathcal{P}_{X_0} and $\mathcal{P}_{X_1, Y_1|X_0, U_1}$ are given Gaussian distributions, and W_1 is an auxiliary random variables.

Remark II.10. *Distribution (14) satisfies the follows*

$$\begin{cases} X_0 \text{ is independent of } W_1, \\ (X_1, Y_1) \text{ --- } (X_0, U_1) \text{ --- } W_1, \\ U_2 \text{ --- } (X_0, Y_1, W_1) \text{ --- } (U_1, X_1). \end{cases} \quad (17)$$

Remark II.11. *Compared to the single-letter result in [32, Theorem II.3], the presence of channel feedback enables the decoder to directly coordinate with the source state X_0 instead of its noisy representation (W_2 in the reference paper). Therefore, feedback enlarges the set of achievable pairs. A similar observation has been also pointed out in [29, 30, 34]*

Sketch of the proof for Theorem II.9. The converse proof for Theorem II.9 can be adapted from the converse proof of Theorem III.2 in [29], which dealt with a state-independent channel. We can modify the arguments from this reference to apply to the state-dependent channel in our setting, without affecting the outcome of the result.

Additionally, the coding scheme for the achievability proof for Theorem II.9 also extends the approach provided in [29] with changing the channel from state-independent to state-dependent. The cost analysis is directly derived from the arguments presented in [32]. \square

APPENDIX A: PROOF OF THEOREM II.3

Converse. For a pair (P, S) , assume that we have a control design $c \in \mathcal{C}_{ed}(n)$ of block length $n \in \mathbb{N}$ and small $\varepsilon > 0$ which induces a joint p.m.f. \mathcal{P}^n of the form (4) such that

$$|P - \mathbb{E}[c_P(U_1^n)]| + |S - \mathbb{E}[c_S(X_1^n, U_2^n)]| < \varepsilon. \quad (18)$$

This is implied by condition (5).

let Q be an independent time random variable uniformly distributed over $\{1, \dots, n\}$. Define auxiliary random variables $X_0 = X_{0,Q}, U_1 = U_{1,Q}, X_1 = X_{1,Q}, Y_1 = Y_{1,Q}, U_2 = U_{2,Q}$ with distribution

$$\mathbb{P}((X_0, U_1, X_1, Y_1, U_2) = (x_0, u_1, x_1, y_1, u_2)) \quad (19)$$

$$= \frac{1}{n} \sum_{q=1}^n \mathbb{P}((X_{0,q}, U_{1,q}, X_{1,q}, Y_{1,q}, U_{2,q}) = (x_0, u_1, x_1, y_1, u_2))$$

$$\forall (x_0, u_1, x_1, y_1, u_2) \in \mathcal{X}_0 \times \mathcal{U}_1 \times \mathcal{X}_1 \times \mathcal{Y}_1 \times \mathcal{U}_2.$$

Then, the expected long-run costs could be reformulated as

$$\mathbb{E}[c_P(U_1^n)] = \mathbb{E}\left[\frac{1}{n} \sum_{q=1}^n U_{1,q}^2\right] = \mathbb{E}[U_1^2], \quad (20)$$

$$\mathbb{E}[c_S(X_1^n, U_2^n)] = \mathbb{E}\left[\frac{1}{n} \sum_{q=1}^n (X_{1,q} - U_{2,q})^2\right] \quad (21)$$

$$= \mathbb{E}[(X_1 - U_2)^2]. \quad (22)$$

Therefore, given (18), we obtain that

$$|P - \mathbb{E}[U_1^2]| + |S - \mathbb{E}[(X_1 - U_2)^2]| < \varepsilon, \quad (23)$$

which is valid for all $\varepsilon \geq 0$. Hence, we have (7).

Now, we define the new auxiliary random variables $W_q = Y_1^{q-1}$ for $q = 1, \dots, n$ and $T = (W_Q, Q)$. These auxiliary random variables satisfy the following Markov chains

- $X_0 \perp\!\!\!\perp T$: This is because the source is i.i.d. generated, independent of the time stage Q as well as the past channel output sequence Y_1^{Q-1} due to causal encoding.
- $(X_1, Y_1) \text{ --- } (X_0, U_1) \text{ --- } T$: This comes from the discrete memoryless channel, the characterization of the output (X_1, Y_1) depends only on the input (X_0, U_1) .
- $U_2 \text{ --- } (T, Y_1) \text{ --- } (X_0, U_1, X_1)$: This is the consequence of the causal decoding that the reconstruction is fully characterized by sequence Y_1^Q up to stage $Q = 1, \dots, n$.

Therefore, the distribution of all the introduced auxiliary random variables decomposes as (6). \square

Achievability. Consider a joint distribution \mathcal{P} of the form of (6) with $\mathbb{E}[U_1^2] = P$ and $\mathbb{E}[(X_1 - U_2)^2] = S$. Fix a small $\varepsilon > 0$ and a blocklength $n \in \mathbb{N}$.

The encoder and decoder simply conduct symbol-by-symbol approaches based on the given distribution \mathcal{P} : Before the transmission, the encoder selects a typical sequence $T^n \in \mathcal{T}_\varepsilon(\mathcal{P}_T)$ and shares it to the decoder. Then, at each time $t \in \{1, \dots, n\}$, the encoder observes $X_{0,t}$, and outputs $U_{1,t} \sim \mathcal{P}_{U_1|X_0, T}(\cdot|X_{0,t}, T_t)$. The channel generates $X_{1,t}, Y_{1,t} \sim \mathcal{P}_{X_1, Y_1|X_0, U_1}(\cdot|X_{0,t}, U_{1,t})$. Then, the decoder draws $U_{2,t} \sim \mathcal{P}_{U_2|T, Y_1}(\cdot|T_t, Y_{1,t})$.

In such case, since each symbol is generated i.i.d. according to its distribution, from the law of large numbers (LLN), we have

$$c_P(U_1^n) = \frac{1}{n} \sum_{t=1}^n U_{1,t}^2 \xrightarrow{n \rightarrow \infty} P. \quad (24)$$

$$c_S(X_1^n, U_2^n) = \frac{1}{n} \sum_{t=1}^n (X_{1,t} - U_{2,t})^2 \xrightarrow{n \rightarrow \infty} S \quad (25)$$

Since convergence in probability implies convergence in \mathcal{L}^1 measure, given the existence of finite second moment, we have

$$\mathbb{E}[|c_P(U_1^n) - P|] < \frac{1}{2}\varepsilon, \quad (26)$$

$$\mathbb{E}[|c_S(X_1^n, U_2^n) - S|] < \frac{1}{2}\varepsilon. \quad (27)$$

for sufficiently large n . \square

Proof of the Cardinality Bound of T . Consider the set \mathcal{P} of all p.m.f.s on $\mathcal{U}_1 \times \mathcal{X}_1 \times \mathcal{U}_2$ satisfying the form of (6) and the following two continuous functions

$$f_1(\mathcal{P}, t) = \mathbb{E} \left[U_1^2 \mid T = t \right], \quad (28)$$

$$f_2(\mathcal{P}, t) = \mathbb{E} \left[(X_1 - U_2)^2 \mid T = t \right]. \quad (29)$$

From the support lemma in [35], [15, Appendix C], we can establish the time-sharing cardinality bound $|\mathcal{T}| \leq 2$ for the convex-hull operation. \square

APPENDIX B: PROOF OF THEOREM II.7

Converse. Consider a control design illustrated in Fig. 5 together with the dotted line. This induces a distribution of the following form

$$\prod_{t=1}^n \mathcal{P}_{X_0,t} \prod_{t=1}^n f_{U_1,t|X_0^t,Y_1^{t-1}}^{(f,t)} \prod_{t=1}^n \mathcal{P}_{X_1,t,Y_1,t|X_0,t,U_1,t} \prod_{t=1}^n g_{U_2,t|Y_1^t,X_0^{t-1}}^{(f,t)}. \quad (30)$$

This control design is clearly more powerful than that in Definition II.6 because the causal decoder also receives a past sequence of source information X_0^{t-1} at each instant t . Therefore, the achievable cost region induced by this superior system of adding the source feed-forward serves as an outerbound for the region without the source feed-forward. We denote this new region by $\mathcal{R}_{ed,f,f}$ and we have

$$\mathcal{R}_{ed,f,f} \supseteq \mathcal{R}_{ed,f} \supseteq \mathcal{R}_{ed} \quad (31)$$

Now, we show that $\mathcal{R}_{ed,f,f} = \mathcal{R}_{ed}$.

Similar to the converse proof of Theorem II.3 in Appendix A, we introduce a time-sharing random variable $Q \sim \text{Unif}[1, \dots, n]$ and define a sequence of new random variables $X_0 = X_{0,Q}, U_1 = U_{1,Q}, X_1 = X_{1,Q}, Y_1 = Y_{1,Q}, U_2 = U_{2,Q}$ with their joint distribution (19). In this way, the n -stage long-run costs for the control design of (30) also could be reformulated to (20) and (22).

Next, let $W_q = (X_0^{q-1}, Y_1^{q-1})$ for $q = 1, \dots, n$ and $T = (W_Q, Q)$ be new auxiliary random variables. Then, it holds that

- $X_0 \perp\!\!\!\perp T$. This follows from the i.i.d. source and causal encoding.
- $(X_1, Y_1) \ominus (X_0, U_1) \ominus T$: This comes from the discrete memoryless channel.
- $U_2 \ominus (T, Y_1) \ominus (X_0, U_1, X_1)$: This is the consequence of the causal decoding at the presence of Y_1^{Q-1}, X_0^{Q-1} at each time instant Q .

Given the above Markov chains, the single-letter joint distribution that characterizes a desired control scheme of adding a source feed-forward also decomposes as (6).

Therefore, we have shown that $\mathcal{R}_{ed,f,f} = \mathcal{R}_{ed}$. Namely, adding both the channel feedback AND the source feed-forward information does not enlarge the achievable Witsenhausen cost region of \mathcal{R}_{ed} . Since we also have (31), by the genie-aided argument, we conclude $\mathcal{R}_{ed,f} = \mathcal{R}_{ed}$. \square

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