CRITICAL SLOPE p-ADIC L-FUNCTIONS

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Abstract. Let g be an eigenform of weight $k+2$ on $\Gamma_0(p) \cap \Gamma_1(N)$ with $p \nmid N$. If g is non-critical (*i.e.* of slope less than $k+1$), using the methods of [1, 20], one can attach a p-adic L-function to g which is uniquely determined by its interpolation property together with a bound on its growth. However, in the critical slope case, the corresponding growth bound is too large to uniquely determine the p-adic L-function with its standard interpolation property.

In this paper, using the theory of overconvergent modular symbols, we give a natural definition of p-adic L-functions in this critical slope case. If, moreover, the modular form is not in the image of theta then the p-adic L-function satisfies the standard interpolation property.

1. Introduction

Let p be a prime number, and let $f = \sum_n a_n q^n$ denote a normalized cuspidal eigenform of weight $k+2$ on $\Gamma_1(N)$ with nebentype ε and with $p \nmid N$. If f is a p-ordinary form, then by [1, 20] we can attach a p-adic L-function to f which interpolates special values of its L-series. On the other hand, if f is non-ordinary at p, we have *two* p-adic L-functions attached to f, one for each root of $x^2 - a_p x + \varepsilon(p)p^{k+1}$. These two roots correspond to the two p-stabilizations of f to level $\Gamma_0 := \Gamma_1(N) \cap \Gamma_0(p)$, and, more precisely, we are attaching a p-adic L-function to each of these forms.

In the case when f is p-ordinary, one of these p-stabilizations is p-ordinary and the other has slope $k+1$ (critical slope). The methods of [1, 20] only apply to forms of slope strictly less than $k+1$, which is why in this case we only have one p-adic L-function. It is the goal of this paper to give a natural construction of p-adic L-functions of critical slope forms, and thus to construct the "missing" p-adic L-function in the ordinary case.

The basic starting point of our method is the theory of overconvergent modular symbols developed by the second author. Let $\mathcal{D}_k$ denote the space of locally $\mathbb{Q}_p$-analytic distributions on $\mathbb{Z}_p$ endowed with the weight k action. This distribution space $\mathcal{D}_k$ admits a surjective map to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$. Thus, we get an induced Hecke-equivariant map

$$H^1_c(\Gamma_0, \mathcal{D}_k) \longrightarrow H^1_c(\Gamma_0, \mathrm{Sym}^k(\mathbb{Q}_p^2))$$

which we refer to as the *specialization* map. The target of this map is finite-dimensional while the source is infinite-dimensional if it is non-zero. Nonetheless, we have the following comparison theorem of the second author (see [19] and also [15, Theorem 5.12]).

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Theorem 1.1. *We have*

$$H^1_c(\Gamma_0,\mathcal{D}_k)^{(<k+1)} \xrightarrow{\cong} H^1_c(\Gamma_0,\mathrm{Sym}^k(\mathbb{Q}_p^2))^{(<k+1)}.$$

That is, the specialization map is an isomorphism on the subspace where U_p acts with slope strictly less than $k+1$.

This comparison theorem should be viewed as the analogue of Coleman's theorem on small slope overconvergent modular forms being classical.

Let f now be a normalized cuspidal eigenform of level Γ_0, which we assume for simplicity has its Fourier coefficients in $\mathbb{Z}_p$. (Note that we are certainly allowing the possibility that f is old at p.) Consider the modular symbol $\phi_f \in H^1_c(\Gamma_0,\mathrm{Sym}^k(\mathbb{Q}_p^2))$ attached to f. If f is of non-critical slope, by Theorem 1.1, ϕ_f lifts uniquely to a Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma_0,\mathcal{D}_k)$. Moreover, if we "integrate" this symbol from ∞ to 0, the resulting distribution we get is exactly the p-adic L-function of f (see [19] and [15, Prop 6.3]).

To define critical slope p-adic L-functions, we repeat this analysis for the slope $k+1$ subspace. To this end, let $\theta^{k+1} : M^\dagger_{-k}(\Gamma,\overline{\mathbb{Q}}_p) \longrightarrow M^\dagger_{k+2}(\Gamma,\overline{\mathbb{Q}}_p)$ denote the p-adic θ-operator which acts on q-expansions by $(q\frac{d}{dq})^{k+1}$. Here $M^\dagger_r(\Gamma,\overline{\mathbb{Q}}_p)$ is the space of overconvergent modular forms of weight r.

The following is the main theorem of this paper (see Theorem 8.1).

Theorem 1.2. *Let f be an eigenform in $S_{k+2}(\Gamma_0,\overline{\mathbb{Q}}_p)$ with slope $k+1$. Then*

$$H^1_c(\Gamma_0,\mathcal{D}_k)_{(f)} \longrightarrow H^1_c(\Gamma_0,\mathrm{Sym}^k(\mathbb{Q}_p^2))_{(f)}$$

is an isomorphism if and only if $f \notin \mathrm{im}(\theta^{k+1})$.

Here, the subscript (f) denotes the generalized eigenspace on which the Hecke-algebra acts via the eigenvalues of f.

In particular, if f is not in the image of θ^{k+1}, using the same arguments as above, we can associate a unique Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma_0,\mathcal{D}_k)$ which specializes to ϕ_f. We then simply *define* the p-adic L-function of f to be the value of Φ_f when integrated from ∞ to 0.

Many examples of these critical slope p-adic L-functions are computed in [15]. Their zeroes appear to contain interesting patterns which encode the classical μ- and λ-invariants of the corresponding ordinary p-adic L-function.[1]

The case when f is in the image of θ^{k+1} remains an interesting one. In this situation, we know that there is some non-zero Hecke-eigensymbol $\Phi_f \in H^1_c(\Gamma_0,\mathcal{D}_k)$ in the kernel of specialization with the same eigenpacket[2] as f. In fact, there are two such symbols, one in each of the eigenspaces of complex conjugation. Numerical experiments from [15] suggest that there is a 1-dimensional space of such symbols in each eigenspace; if this were true, we could define a p-adic L-function, at least up to scaling. However, we have been unable to establish this claim even in a particular case.[3]

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[1]Since the writing of this paper, Loeffler and Zerbes (see [11]) proved that analogous formulas involving Iwasawa invariants hold for the critical slope p-adic L-function defined via Kato's Euler system (see Remark 9.5).

[2]Here "eigenpacket" is synonymous with "system of Hecke-eigenvalues" – see section 6.2.

[3]Since the writing of this paper, Bellaïche has proven that this eigenspace is 1-dimensional and has defined critical slope p-adic L-functions in this case as well (see [3]).

In the course of the paper, we actually first prove a weaker version of the above theorem (see Theorem 6.7). The conclusion of this weaker theorem is the same, but its hypothesis is stronger. Namely, we assume that f does not possess a mod p companion form. (It is straightforward to see that if $f \in \mathrm{im}(\theta^{k+1})$ then f has a mod p companion form – see Proposition 6.9). We chose to include both proofs of these theorems in this paper as the proof of the weaker theorem is simpler in some aspects and may be more easily generalized to a wider class of reductive groups.

We now sketch a proof of the non-critical slope comparison theorem, and then sketch the two proofs dealing with the critical slope subspace. Using the Riesz decomposition (reviewed in section 4), it follows that the specialization map restricted to the slope less than $k+1$ subspace

$$H^1_c(\Gamma_0, \mathcal{D}_k)^{(<k+1)} \overset{\rho_k^*}{\twoheadrightarrow} H^1_c(\Gamma_0, \mathrm{Sym}^k(\mathbb{Q}_p^2))^{(<k+1)}$$

is surjective. Moreover, the kernel of the specialization map can be identified with

$$H^1_c(\Gamma_0, \mathcal{D}_{-2-k})(k+1);$$

here, we are twisting by the $(k+1)^{\mathrm{st}}$ power of the determinant, and thus the Hecke operator T_n acts by $n^{k+1}T_n$ (see section 3.4). In particular, U_p acts with slope at least $k+1$ on this space. It follows that the specialization map restricted to the slope less than $k+1$ subspace is also injective, proving the comparison theorem.

To deal with the critical slope case, assume that f is an eigenform of slope $k+1$. We wish to show (under some hypotheses) that the f-isotypic subspace

$$\left(H^1_c(\Gamma_0, \mathcal{D}_{-2-k})(k+1)\right)_{(f)}$$

vanishes. Assume the contrary, and let Ψ denote some non-zero Hecke-eigensymbol in this subspace. Let Ψ_0 denote the untwisted symbol in $H^1_c(\Gamma_0, \mathcal{D}_{-2-k})$. Since Ψ has slope $k+1$, the symbol Ψ_0 has slope 0.

Approach 1: By scaling, we may assume that Ψ_0 takes values in $\mathcal{D}^0_{-2-k}$, the unit ball of $\mathcal{D}_{-2-k}$. In section 3.5, we introduce a descending filtration $\mathrm{Fil}^r \mathcal{D}^0_m$ for any *negative* weight m, such that any normalized eigensymbol taking values in $\mathrm{Fil}^r \mathcal{D}^0_m$ has slope bounded below by r. Specifically, for $r = 1$, the subspace $\mathrm{Fil}^1 \mathcal{D}^0_m$ consists of distributions whose total measure is divisible by p. Since Ψ_0 has slope zero, its image $\overline{\Psi}_0$ in

$$H^1_c(\Gamma_0, \mathcal{D}^0_{-2-k}/\mathrm{Fil}^1 \mathcal{D}^0_{-2-k})$$

is a non-zero eigensymbol. Moreover, the quotient $\mathcal{D}^0_{-2-k}/\mathrm{Fil}^1 \mathcal{D}^0_{-2-k}$ is simply $\mathbb{F}_p$ with a certain non-trivial matrix action. Thus, this non-zero eigensymbol is related to an ordinary weight 2 modular form with some nebentype at p.

Using Hida theory, we can then find a congruent form of small weight with no nebentype at p. Precisely, let j be the unique integer with $k+2 \equiv j \pmod{p-1}$ and $2 \le j \le p$. Then there exists an eigenform g of level Γ_0 and weight $p+1-j$ such that $\overline{\rho}_g \otimes \omega^{k+1} \cong \overline{\rho}_f$. Here $\overline{\rho}_f$ and $\overline{\rho}_g$ are the residual Galois representations attached to f and g, and ω is the mod p cyclotomic character. In particular, g is a mod p companion form for f. Thus we deduce that the specialization map restricted to the f-isotypic subspace is an isomorphism as long as f does not possess a mod p companion form.

Approach 2: By [18], eigenpackets in $H^1_c(\Gamma_0, \mathcal{D}_k)$ are in one-to-one correspondence with eigenpackets in $M^{\dagger}_{k+2}(\Gamma,\overline{\mathbb{Q}}_p)$ for any p-adic weight k (with one exception when $k=0$). See section 7 where a proof of this result is given. In particular, the eigensymbol Ψ_0 corresponds to some overconvergent modular form $g\in M^{\dagger}_{-k}(\Gamma,\overline{\mathbb{Q}}_p)$ such that the eigenpacket of $\theta^{k+1}g$ is the same as the eigenpacket of f. By looking at q-expansions, we see that $f=\theta^{k+1}g$, and deduce that $f\in\mathrm{im}(\theta^{k+1})$. Thus, the specialization map restricted to the f-isotypic subspace is an isomorphism as long as f is not in the image of θ^{k+1}.

We close this introduction by mentioning two other approaches to constructing critical slope p-adic L-functions. The first combines Perrin-Riou's dual exponential map with the existence of Kato's zeta element (see [13, 3.2.2] and [6, Remark 4.12]). The second uses Emerton's theory of $\widehat{H}^1$ (see [7, 8]). At the end of section 9, we recall these two methods and discuss the role played by the condition $f\notin\mathrm{im}(\theta^{k+1})$.

The format of the paper is as follows: in the following section we review the basic definitions of modular symbols. In the third section, we introduce the relevant spaces of distributions, and the filtration on them described above. In the fourth section, we review the Riesz decomposition. In the fifth section, we prove the non-critical comparison theorem. In the sixth section, we prove a critical slope comparison theorem by the first approach outlined above. In the seventh and eighth sections, we present a proof of the results of [18] and use these results to give a second proof of the critical slope comparison theorem. In the final section, we discuss p-adic L-functions.

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2. Modular Symbols

2.1. **Basic definitions.** Let p be a prime and N an integer prime to p. Set $\Gamma_0=\Gamma_1(N)\cap\Gamma_0(p)$ and

$$S_0(p)=\left\{\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in M_2(\mathbb{Z})\text{ such that } p\nmid a, p\mid c\text{ and } ad-bc\neq 0\right\}.$$

If M is a right $\mathbb{Z}[S_0(p)]$-module, let $\widetilde{M}$ denote the associated locally constant sheaf on the open modular curve Y_{Γ_0}, and let

$$H^1_c(\Gamma_0,M):=H^1_c(Y_{\Gamma_0},\widetilde{M})$$

denote the space of one-dimensional compactly supported cohomology with coefficients in $\widetilde{M}$.

When the order of each torsion element of Γ_0 acts invertibly on M, the space $H^1_c(\Gamma_0,M)$ admits a description in terms of modular symbols. Indeed, let $\Delta_0:=\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ denote the set of degree 0 divisors on $\mathbb{P}^1(\mathbb{Q})$ endowed with a left $\mathrm{GL}_2(\mathbb{Q})$-action via linear fractional transformations. The space $\mathrm{Hom}(\Delta_0,M)$ admits a right action of $S_0(p)$ by

$$(\phi\big|\gamma)(D)=\phi(\gamma D)\big|\gamma$$

where $D\in\Delta_0$ and $\gamma\in S_0(p)$. Then by [2, Proposition 4.2] there exists a canonical isomorphism

$$H^1_c(\Gamma_0,M)\cong\mathrm{Hom}_{\Gamma_0}(\Delta_0,M) \tag{1}$$

where the target of the map is the set of Γ_0-invariant homomorphisms.

If $\mathcal{H}$ denotes the free polynomial algebra over $\mathbb{Z}$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q|Np$, then both $H^1_c(\Gamma_0, M)$ and $\mathrm{Hom}_{\Gamma_0}(\Delta_0, M)$ are naturally $\mathcal{H}$-modules. For instance, the U_p-operator on $\mathrm{Hom}_{\Gamma_0}(\Delta_0, M)$ can be explicitly realized by

$$\phi|U_p = \sum_{a=0}^{p-1} \phi| \left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right).$$

The isomorphism (1) is Hecke-equivariant, and throughout the paper, we will identify these two spaces when the order of the torsion elements of Γ_0 act invertibly on M (for example, whenever $N \geq 4$). Note that this condition holds when M is a vector space over a field of characteristic 0, when Γ_0 is torsion-free, or when M is a $\mathbb{Z}_p$-module with $p > 3$.

Also, as the congruence subgroup Γ_0 is fixed throughout the paper, we simply write $H^1_c(M)$ for $H^1_c(\Gamma_0, M)$ and refer to it as the space of M-valued modular symbols (of level Γ_0).

2.2. **Miscellany.** For $r \in \mathbb{Z}^{\geq 0}$, let $M(r)$ denote the $S_0(p)$-module whose underlying set is M and whose $S_0(p)$-action is twisted by the r^{th} power of the determinant. For an $\mathcal{H}$-module X, set $X(r)$ to be the $\mathcal{H}$-module whose underlying set is X and whose Hecke action by T_ℓ (resp. U_q) is given by $\ell^r T_\ell$ (resp. $q^r U_q$). We then have the tautological isomorphism

$$H^1_c(M(r)) \cong H^1_c(M)(r).$$

We also mention that the matrix $\iota = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ normalizes Γ_0, and thus induces an involution on $H^1_c(M)$; if 2 acts invertibly on M, this involution gives a decomposition

$$H^1_c(M) = H^1_c(M)^+ \oplus H^1_c(M)^-$$

into ± 1-eigenspaces for ι.

Lastly, if M is a Banach space and if Γ_0 acts by unitary operators on M, then $H^1_c(M)$ is also a Banach space under the norm

$$||\Phi|| = \sup_{D \in \Delta_0} ||\Phi(D)||.$$

This supremum exists as $||\Phi(D)||$ is constant on each of the finitely many Γ_0-orbits of Δ_0. (Note here that we are implicitly using the identification in (1).)

3. Distributions

3.1. **Definitions.** For each $r \in |\mathbb{C}_p^\times|$, let

$$B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p \mid \text{there exists some } a \in \mathbb{Z}_p \text{ with } |z - a| \leq r\}.$$

Then $B[\mathbb{Z}_p, r]$ is the $\mathbb{C}_p$-points of a $\mathbb{Q}_p$-affinoid variety. For example, if $r \geq 1$ then $B[\mathbb{Z}_p, r]$ is the closed disc in $\mathbb{C}_p$ of radius r around 0. If $r = \frac{1}{p}$ then $B[\mathbb{Z}_p, r]$ is the disjoint union of the p discs of radius $\frac{1}{p}$ around the points $0, 1, \ldots, p-1$.

Let $\mathbf{A}[r]$ denote the $\mathbb{Q}_p$-Banach algebra of $\mathbb{Q}_p$-affinoid functions on $B[\mathbb{Z}_p, r]$. For example, if $r \geq 1$

$$\mathbf{A}[r] = \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathbb{Q}_p[[z]] \text{ such that } \{|a_n| \cdot r^n\} \to 0 \right\}.$$

The norm on $\mathbf{A}[r]$ is given by the supremum norm. That is, if $f \in \mathbf{A}[r]$ then

$$||f||_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|_p.$$

For $r_1 < r_2$, there is a natural restriction map $\mathbf{A}[r_2] \to \mathbf{A}[r_1]$ that is injective, completely continuous and has dense image. We define

$$\mathcal{A} = \varinjlim_{s>0} \mathbf{A}[s] \text{ and } \mathcal{A}^\dagger(\mathbb{Z}_p, r) = \varinjlim_{s>r} \mathbf{A}[s].$$

The first of these spaces is naturally identified with the space of locally analytic $\mathbb{Q}_p$-valued functions on $\mathbb{Z}_p$, while $\mathcal{A}^\dagger(\mathbb{Z}_p, r)$ is identified with the space of $\mathbb{Q}_p$-overconvergent functions on $B[\mathbb{Z}_p, r]$. The topology on each is given by the inductive limit topology. Note that there are natural continuous inclusions

$$\mathcal{A}^\dagger(\mathbb{Z}_p, r) \hookrightarrow \mathbf{A}[r] \hookrightarrow \mathcal{A};$$

the image of each of these maps is dense in its target space.

Set $\mathbf{D}[r]$ (resp. $\mathcal{D}$, $\mathcal{D}^\dagger(\mathbb{Z}_p, r)$) equal to the space of continuous $\mathbb{Q}_p$-linear functionals on $\mathbf{A}[r]$ (resp. $\mathcal{A}$, $\mathcal{A}^\dagger(\mathbb{Z}_p, r)$) endowed with the strong topology. Note that

$$\mathcal{D} = \varprojlim_{s>0} \mathbf{D}[s] \text{ and } \mathcal{D}^\dagger(\mathbb{Z}_p, r) = \varprojlim_{s>r} \mathbf{D}[s],$$

with the projective limit topology.

We have that $\mathbf{D}[r]$ is a Banach space under the norm

$$||\mu||_r = \sup_{\substack{f \in \mathbf{A}[r] \\ f \neq 0}} \frac{|\mu(f)|}{||f||_r}$$

while $\mathcal{D}$ (resp. $\mathcal{D}^\dagger(\mathbb{Z}_p, r)$) has its topology defined by the family of norms $\{||\cdot||_s\}$ for $s \in |\mathbb{C}_p^\times|$ with $s > 0$ (resp. $s > r$). By duality, we have continuous linear injective maps

$$\mathcal{D} \hookrightarrow \mathbf{D}[r] \hookrightarrow \mathcal{D}^\dagger(\mathbb{Z}_p, r).$$

When $r = 1$, we simply write $\mathbf{A}$, $\mathcal{A}^\dagger$, $\mathbf{D}$, $\mathcal{D}^\dagger$ for $\mathbf{A}[1]$, $\mathcal{A}^\dagger(\mathbb{Z}_p, 1)$, $\mathbf{D}[1]$, $\mathcal{D}^\dagger(\mathbb{Z}_p, 1)$ respectively.

3.2. **Difference operator.** For future reference we record here a simple result about finite differences. Namely, we define the difference operator $\Delta : \mathcal{A}^\dagger \longrightarrow \mathcal{A}^\dagger$ by

$$(\Delta f)(z) = f(z+1) - f(z),$$

and define $\Delta : \mathcal{D}^\dagger \longrightarrow \mathcal{D}^\dagger$ by duality.

Proposition 3.1. *We have an exact sequence*

$$0 \longrightarrow \mathcal{D}^\dagger \xrightarrow{\Delta} \mathcal{D}^\dagger \xrightarrow{\rho} \mathbb{Q}_p \longrightarrow 0$$

where $\rho : \mathcal{D}^\dagger \longrightarrow \mathbb{Q}_p$ is defined by $\rho(\mu) = \mu(1)$.

Proof. It follows immediately from the definitions that ρ is surjective and that $\rho \circ \Delta = 0$. To show $\Delta : \mathcal{D}^\dagger \longrightarrow \mathcal{D}^\dagger$ is injective it suffices to show $\Delta : \mathcal{A}^\dagger \to \mathcal{A}^\dagger$ is surjective. For this, we introduce the notation $z^{(0)} = 1$ and for each $n \geq 1$ let $z^{(n)} = z(z-1)\cdots(z-n+1)$. Then for any $s \in |\mathbb{C}_p^\times|$ with $s > 1$ and any $\lambda \in \mathbb{C}_p^\times$

with $|\lambda| = s$, one has that the sequence $\left\{ \dfrac{z^{(n)}}{\lambda^n} \right\}_n$ is an ON-basis for $\mathbf{A}[s]$. Indeed, $\left\{ \dfrac{z^n}{\lambda^n} \right\}_n$ is an ON-basis, and for any integer $n \geq 0$ we have

$$\left\| \frac{z^{(n)}}{\lambda^n} - \frac{z^n}{\lambda^n} \right\|_s \leq s^{-1} < 1$$

from which it follows that the sequences $\left\{ \dfrac{z^{(n)}}{\lambda^n} \right\}_n$ and $\left\{ \dfrac{z^n}{\lambda^n} \right\}_n$ are related by a change of coordinates matrix with p-integral entries that is congruent to the identity matrix modulo the maximal ideal in the ring of integers of $\mathbb{C}_p$.

It follows from this that a function f on $\mathbb{Z}_p$ is in $\mathcal{A}^\dagger$ if and only if it is represented by a sum of the form

$$f(z) = \sum_{n=0}^{\infty} a_n z^{(n)}$$

that converges for all z with $|z| \leq s$ for some $s \in |\mathbb{C}_p^\times|$ with $s > 1$. This happens if and only if $|a_n|s^n \to 0$ as $n \to \infty$. In particular, we then have $|a_n/n|r^n \to 0$ as $n \to \infty$ for every r with $1 < r < s$ and therefore

$$g(z) := \sum_{n=1}^{\infty} \frac{a_{n-1}}{n} z^{(n)}$$

converges to a function $g \in \mathcal{A}^\dagger$. But an easy calculation shows $\Delta(z^{(n)}) = nz^{(n-1)}$ for all $n \geq 1$ and it follows that $\Delta(g) = f$.

Finally, we show $\ker(\rho) \subseteq \mathrm{Image}(\Delta)$. So let $\mu \in \ker(\rho)$. Then $\mu(1) = 0$. We then define $\nu \in \mathcal{D}^\dagger$ by

$$\nu(z^{(n)}) = \frac{1}{n+1}\mu(z^{(n+1)})$$

and note that for any $f \in \mathcal{A}^\dagger$ given by $f(z) = \sum_{n=0}^{\infty} a_n z^{(n)}$ as before, we have

$$(\nu|\Delta)(f) = \nu(\Delta f) = \sum_{n=1}^{\infty} a_n \nu(nz^{(n-1)}) = \sum_{n=1}^{\infty} a_n \mu(z^{(n)}) = \mu(f)$$

where the last equality above follows since $\mu(1) = 0$. Thus $\nu|\Delta = \mu$ and the proof of the proposition is complete. □

3.3. **The action of $\Sigma_0(p)$.** Let

$$\Sigma_0(p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p) \text{ such that } p \nmid a,\ p \mid c \text{ and } ad - bc \neq 0 \right\}$$

be the p-adic version of $S_0(p)$. Fix an integer k, and let $\Sigma_0(p)$ act on $\mathbf{A}[r]$ on the left by

$$(\gamma \cdot_k f)(z) = (a + cz)^k \cdot f\left(\frac{b + dz}{a + cz}\right)$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Sigma_0(p)$ and $f \in \mathbf{A}[r]$. Then $\Sigma_0(p)$ acts on $\mathbf{D}[r]$ on the right by

$$(\mu\big|_k \gamma)(f) = \mu(\gamma \cdot_k f).$$

where $\mu \in \mathbf{D}[r]$. These two actions then induce actions on $\mathcal{A}$, $\mathcal{A}^\dagger(\mathbb{Z}_p, r)$, $\mathcal{D}$, and $\mathcal{D}^\dagger(\mathbb{Z}_p, r)$.

To emphasize the role of k in these actions, we will include it as a subscript (*e.g.* $\mathbf{A}_k[r]$ denotes $\mathbf{A}[r]$ endowed with the weight k action). When $r = 1$, we simplify the notation and write $\mathbf{A}_k$, $\mathcal{A}_k^\dagger$, $\mathbf{D}_k$, $\mathcal{D}_k^\dagger$ for $\mathbf{A}_k[1]$, $\mathcal{A}_k^\dagger(\mathbb{Z}_p, 1)$, $\mathbf{D}_k[1]$, $\mathcal{D}_k^\dagger(\mathbb{Z}_p, 1)$ respectively.

Remark 3.2. We note that the space $\mathcal{A}_k$ can be viewed as a locally analytic induction. Indeed, let N^{opp} (resp. T) denote the subgroup of lower triangular (resp. diagonal) matrices in $\mathrm{GL}_2(\mathbb{Q}_p)$. Let I denote the Iwahori subgroup of $\mathrm{GL}_2(\mathbb{Z}_p)$, and let X denote its image in $N^{\mathrm{opp}}\backslash \mathrm{GL}_2(\mathbb{Q}_p)$. Then X inherits a natural right action by $\Sigma_0(p)$. Also, $\mathbb{Z}_p$ injects into X by sending z to $\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right)$.

Let λ denote the character of T that maps $\left(\begin{smallmatrix} a & 0 \\ 0 & d \end{smallmatrix}\right)$ onto a^k, and set

$$\mathcal{A}_\lambda := \{f : X \to \mathbb{Q}_p : f \text{ is locally analytic and } f(tx) = \lambda(t)f(x) \text{ for } t \in T\}\,.$$

Here, a function on X is locally analytic if its restriction to the image of $\mathbb{Z}_p$ in X is locally analytic. One then verifies that restriction to $\mathbb{Z}_p$ induces a $\Sigma_0(p)$-isomorphism between $\mathcal{A}_\lambda$ and $\mathcal{A}_k$. Similarly, $\mathbf{A}[r]$ can be viewed as a "rigid analytic induction".

3.4. **Finite-dimensional quotients.** Assume for this section that k is a non-negative integer. Let $\mathcal{P}_k$ denote the space of polynomials of degree at most k with coefficients in $\mathbb{Q}_p$. Then $\mathcal{P}_k$ is naturally a subspace of $\mathcal{A}_k^\dagger$ and is preserved under the weight k action of $\Sigma_0(p)$. Set $\mathcal{P}_k^\vee$ equal to the $\mathbb{Q}_p$-dual of $\mathcal{P}_k$ which we endow with the structure of a right $\Sigma_0(p)$-module via the action

$$(\ell\big|_k\gamma)(P) = \ell(\gamma \cdot_k P)$$

where $\ell \in \mathcal{P}_k^\vee$, $P \in \mathcal{P}_k$ and $\gamma \in \Sigma_0(p)$. (We note that $\mathcal{P}_k^\vee$ is isomorphic as a representation space to $\mathrm{Sym}^k(\mathbb{Q}_p^2)$ where $\mathbb{Q}_p^2$ is viewed as a right $\mathrm{GL}_2(\mathbb{Q}_p)$-module.)

Dualizing the inclusion of $\mathcal{P}_k$ into $\mathcal{A}_k^\dagger$, yields a $\Sigma_0(p)$-equivariant surjection

$$\rho_k : \mathcal{D}_k^\dagger \longrightarrow \mathcal{P}_k^\vee.$$

We have the following proposition.

Proposition 3.3. *We have an exact sequence*

$$0 \longrightarrow \mathcal{D}_{-2-k}^\dagger(k+1) \xrightarrow{\delta} \mathcal{D}_k^\dagger \xrightarrow{\rho_k} \mathcal{P}_k^\vee \longrightarrow 0$$

where δ is defined by

$$(\delta\mu)(f) = \mu((d/dz)^{k+1}f)$$

with $\mu \in \mathcal{D}_k^\dagger$ and $f \in \mathcal{A}_k^\dagger$.

Proof. To explicitly describe the kernel of ρ_k, note that $(d/dz)^{k+1}$ yields a $\Sigma_0(p)$-equivariant map

$$\mathcal{A}_k^\dagger \longrightarrow \mathcal{A}_{-2-k}^\dagger(k+1).$$

Here the action on the target of the map is twisted by the $(k+1)^{\mathrm{st}}$ power of the determinant. This map is easily seen to be surjective with kernel $\mathcal{P}_k$. Thus we have an exact sequence

$$0 \longrightarrow \mathcal{P}_k \longrightarrow \mathcal{A}_k^\dagger \longrightarrow \mathcal{A}_{-2-k}^\dagger(k+1) \longrightarrow 0.$$

Dualizing this sequence then yields the proposition. $\square$

3.5. **A filtration on $\mathbf{D}_{-k}$.** Let k be a positive integer. Set

$$\mathbf{D}^0_{-k} = \left\{\mu \in \mathbf{D}_{-k} \text{ with } \mu(z^j) \in \mathbb{Z}_p \text{ for all } j \geq 0\right\}$$

which is the unit ball of $\mathbf{D}_{-k}$, and consider the decreasing filtration of $\mathbf{D}^0_{-k}$ given by

$$\mathrm{Fil}^r\, \mathbf{D}^0_{-k} = \left\{\mu \in \mathbf{D}^0_{-k} \text{ such that } \mu(z^j) \in p^{r-j}\mathbb{Z}_p\right\}.$$

Proposition 3.4. *The filtration*

$$\mathbf{D}^0_{-k} \supset \mathrm{Fil}^1\, \mathbf{D}^0_{-k} \supset \cdots \supset \mathrm{Fil}^r\, \mathbf{D}^0_{-k} \supset \cdots$$

is stable under the weight $-k$ action of $\Sigma_0(p)$.

Proof. For $\mu \in \mathrm{Fil}^r\, \mathbf{D}^0_{-k}$, we have

$$(\mu\big|\gamma)(z^j) = \mu\left((a+cz)^{-k-j}(b+dz)^j\right).$$

Note that

$$(a+cz)^{-k-j}(b+dz)^j = a^{-k-j}(1+a^{-1}cz)^{-k-j}(b+dz)^j =$$
$$a^{-k-j}\left(\sum_{s=0}^{\infty}\binom{-k-j}{s}a^{-s}c^s z^s\right)\left(\sum_{t=0}^{j}\binom{j}{t}b^{j-t}d^t z^t\right) = \sum_{i=0}^{\infty} a_i z^i,$$

since $-k-j<0$. Moreover, since $a \in \mathbb{Z}_p^\times$ and $c \in p\mathbb{Z}_p$, a direct computation shows that $\mathrm{ord}_p(a_i) \geq i-j$. Substituting back in yields

$$(\mu\big|\gamma)(z^j) = \sum_{i=0}^{\infty} a_i \mu(z^i).$$

Thus, to show that $\mu\big|\gamma \in \mathrm{Fil}^r\, \mathbf{D}^0_{-k}$, we need to show that $\sum_{i=0}^{\infty} a_i\mu(z^i)$ is divisible by p^{r-j} for $j<r$.

To see this, note that $\mu(z^i)$ is divisible by p^{r-i} as $\mu \in \mathrm{Fil}^r\, \mathbf{D}^0_{-k}$. For i between 0 and j, $\mu(z^i)$ is thus divisible by p^{r-j} . For i between j and r, $a_i\mu(z^i)$ is divisible by $p^{i-j}\cdot p^{r-i} = p^{r-j}$. Thus, $(\mu\big|\gamma)(z^j)$ is divisible by p^{r-j}, and $\mu\big|\gamma$ is in $\mathrm{Fil}^r\, \mathbf{D}^0_{-k}$. $\square$

Remark 3.5. Note that the subset

$$\left\{\mu \in \mathbf{D}^0_{-k} \text{ such that } \mu(z^j) = 0 \text{ for } j<r\right\}$$

is not preserved by the weight k action of $\Sigma_0(p)$.

Remark 3.6. In this paper, we only make use of the first step of this filtration. In [15], we make more extensive use of an analogous filtration on $\mathbf{D}_k$ for $k \geq 0$ in order to do explicit computations with overconvergent modular symbols.

The following lemma will be useful in our study of critical slope modular symbols. In what follows, $\mathbb{F}_p(a^j)$ denotes the $\Sigma_0(p)$-module $\mathbb{F}_p$ on which $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ acts by a^j.

Lemma 3.7. *We have*

$$\mathbf{D}^0_{-k}/\,\mathrm{Fil}^1\, \mathbf{D}^0_{-k} \cong \mathbb{F}_p(a^{-k})$$

as $\Sigma_0(p)$-modules.

Proof. As a group, $\mathbf{D}^0_{-k}/\operatorname{Fil}^1\mathbf{D}^0_{-k}$ is isomorphic to $\mathbb{F}_p$ and is generated by any μ such that $\mu(\mathbf{1}) \neq 0$. To see the $\Sigma_0(p)$-action, note that

$$\begin{aligned}(\mu|\gamma)(\mathbf{1}) &= \mu\left((a+cz)^{-k}\right) = a^{-k}\mu\left((1+a^{-1}cz)^{-k}\right)\\ &= a^{-k}\mu\left(\sum_{i=0}^{\infty}\binom{-k}{i}a^{-i}c^iz^i\right)\\ &\equiv a^{-k}\mu(\mathbf{1}) \pmod p\end{aligned}$$

which proves the claim. □

4. Slope decompositions

Let X be a Banach space over $\mathbb{Q}_p$ equipped with a completely continuous endomorphism $U = U_X$. If $h \in \mathbb{R}$, we define $X^{(<h)}$ to be the subspace on which U acts with slope less than h (see Definition 4.7). In this section, we observe that the association of X to $X^{(<h)}$ preserves exact sequences.

Let $\mathcal{C}$ denote the category of Banach spaces over $\mathbb{Q}_p$ which are equipped with a completely continuous operator U. If X and Y are in $\mathcal{C}$, we say $f : X \to Y$ is U-equivariant if $f \circ U_X = U_Y \circ f$.

Theorem 4.1 (Riesz decomposition)**.** *Let $X \in \mathcal{C}$. For each irreducible polynomial Q in $\mathbb{Q}_p[T]$ with $Q(0) \neq 0$, the space X decomposes into a direct sum of two closed subspaces preserved by U:*

$$X \cong X(Q) \oplus X'(Q)$$

such that $Q(U)$ is nilpotent on $X(Q)$ and invertible on $X'(Q)$. Moreover, $X(Q)$ is finite-dimensional over $\mathbb{Q}_p$.

Proof. See [16, pg. 82 - Remarques 3]. □

Lemma 4.2. *With X and Q as above, we have*
1. $X(Q) = \bigcup_n \ker(Q(U)^n)$*;*
2. $X'(Q) = \bigcap_n \operatorname{im}(Q(U)^n)$.

In particular, the Riesz decomposition is unique.

Proof. For the first part, $X(Q) \subseteq \bigcup_n \ker(Q(U)^n)$ as $Q(U)$ is nilpotent on $X(Q)$. Moreover, the projection of $\bigcup_n \ker(Q(U)^n)$ to $X'(Q)$ is zero since $Q(U)$ acts invertibly on $X'(Q)$. Thus, the above containment is an equality.

For the second part, since $Q(U)$ acts invertibly on $X'(Q)$, we have $X'(Q) \subseteq \bigcap_n \operatorname{im}(Q(U)^n)$. Since $X(Q)$ is finite-dimensional, $Q(U)^n$ annihilates $X(Q)$ for some n. Applying $Q(U)^n$ to the Riesz decomposition of X then yields $\operatorname{im}(Q(U)^n) = X'(Q)$, proving the lemma. □

Corollary 4.3. *For $X, Y \in \mathcal{C}$ with $f : X \to Y$ a U-equivariant linear map, we have $f(X(Q)) \subseteq Y(Q)$ and $f(X'(Q)) \subseteq Y'(Q)$.*

Proof. This follows immediately from the previous lemma as $f(\ker(Q(U_X)^n)) \subseteq \ker(Q(U_Y)^n)$ and $f(\operatorname{im}(Q(U_X)^n)) \subseteq \operatorname{im}(Q(U_Y)^n)$ by the U-equivariance. □

Corollary 4.4. *Let $X_1, X_2, X_3 \in \mathcal{C}$, and let*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

be an exact sequence (of vector spaces) with each map U-equivariant. Then for any irreducible polynomial $Q \in \mathbb{Q}_p[T]$ with $Q(0) \neq 0$,

$$0 \to X_1(Q) \to X_2(Q) \to X_3(Q) \to 0$$

is an exact sequence.

Proof. Exactness is clear except for the surjection $X_2(Q) \to X_3(Q)$. Take $x_3 \in X_3(Q)$, and let $x_2 + x_2'$ be some preimage with $x_2 \in X_2(Q)$ and $x_2' \in X_2'(Q)$. As the image of x_2' lands in $X_3'(Q)$, we have that x_2 maps to x_3. □

Definition 4.5. If $Q \in \mathbb{Q}_p[T]$ is a monic polynomial, we define $v(Q) := \sup_\alpha v(\alpha)$ where α runs over the roots of Q in $\overline{\mathbb{Q}}_p$. In particular, $Q(0) \neq 0$ if and only if $v(Q) < \infty$.

Lemma 4.6. *Let $X \in \mathcal{C}$, and let $Q \in \mathbb{Q}_p[T]$ be a monic polynomial with $Q(0) \neq 0$. Then*

$$v(Q) < -v(\|U\|) \Longrightarrow X(Q) = 0$$

where $\|U\|$ denotes the norm of U acting on X.

Proof. Since $Q(0) \neq 0$, we may write

$$\frac{Q(T)}{Q(0)} = \prod_i (1 - T/\alpha_i)$$

where all $\alpha_i \neq 0$. Now assume $v(Q) < v(\|U\|)$. Then we also have

$$\|U/\alpha_i\| < 1 \quad \text{for } i = 1, \dots, d.$$

In particular, substituting U for T, we can write $Q(U)/Q(0) = 1 - V$ where V is a continuous operator on X whose norm satisfies $\|V\| < 1$. From this we conclude that $1 - V$, hence also $Q(U)^n$, is an invertible operator on X for every $n \geq 0$. But by Lemma 4.2 we know that every element of $X(Q)$ is contained in the kernel of $Q(U)^n$ for some n. Hence $X(Q) = 0$ as claimed. □

Definition 4.7. For $X \in \mathcal{C}$ and $h \in \mathbb{R}$, set

$$X^{(<h)} := \bigoplus_{v(Q)<h} X(Q)$$

where Q runs over all monic irreducible polynomials of $\mathbb{Q}_p[T]$, and $v(Q)$ denotes the valuation of any root of Q. We define $X^{(=h)}$ and $X^{(\leq h)}$ similarly.

Proposition 4.8. *Let $X_1, X_2, X_3 \in \mathcal{C}$, and let*

$$0 \to X_1 \to X_2 \to X_3 \to 0$$

be an exact sequence (of vector spaces) with each map U-equivariant. Then

$$0 \to X_1^{(<h)} \to X_2^{(<h)} \to X_3^{(<h)} \to 0$$

is exact. The same assertion is true for $X_i^{(=h)}$ and $X_i^{(\leq h)}$.

Proof. This follows from Corollary 4.4 as $X_i^{(<h)}$ is a direct sum over various $X(Q)$. □

Lemma 4.9. *Let $X \in \mathcal{C}$ and $h \in \mathbb{R}$.*

1. *The vector spaces $X^{(<h)}$, $X^{(=h)}$ and $X^{(\leq h)}$ are all finite-dimensional.*
2. *If the operator $U : X \longrightarrow X$ satisfies $\|U\| \leq p^{-h}$, then $X^{(<h)} = 0$.*

Proof. Let P_U be the characteristic power series of U acting on X. Then $X(Q) \neq 0$ if and only if the reciprocal of a root of Q is a root of P_U. Since each $X(Q)$ is finite-dimensional and P_U has only finitely roots with slope less than or equal to h, (1) follows. Assertion (2) is an immediate consequence of Lemma 4.6. □

5. Comparison theorem

As the map $\rho_k : \mathcal{D}_k^\dagger \to \mathcal{P}_k^\vee$ defined in section 3.4 is $\Sigma_0(p)$-equivariant, it induces a map on cohomology

$$H_c^1(\mathcal{D}_k^\dagger) \xrightarrow{\rho_k^*} H_c^1(\mathcal{P}_k^\vee)$$

which we call the *specialization* map.

In this section, we sketch a proof of a theorem of second author which states that ρ_k^* is a isomorphism on the subspace of slope strictly less than $k+1$.

5.1. **Some lemmas.**

Lemma 5.1. *We have an exact sequence*

$$0 \to H_c^1(\mathcal{D}_{-2-k}^\dagger)(k+1) \to H_c^1(\mathcal{D}_k^\dagger) \xrightarrow{\rho_k^*} H_c^1(\mathcal{P}_k^\vee) \to 0.$$

Proof. The sequence

$$0 \to \mathcal{D}_{-2-k}^\dagger(k+1) \to \mathcal{D}_k^\dagger \xrightarrow{\rho_k} \mathcal{P}_k^\vee \to 0$$

induces a long exact sequence of compactly supported cohomology groups[4] and since $H_c^*(\mathcal{D}_{-2-k}^\dagger(k+1)) \cong H_c^*(\mathcal{D}_{-2-k}^\dagger)(k+1)$ for $* = 1, 2$, this long exact sequence takes the form

$$H_c^1(\mathcal{D}_{-2-k}^\dagger)(k+1) \to H_c^1(\mathcal{D}_k^\dagger) \to H_c^1(\mathcal{P}_k^\vee) \to H_c^2(\Gamma_0, \mathcal{D}_{-2-k}^\dagger)(k+1).$$

First note that the leftmost map is clearly injective as we may view these compactly supported cohomology groups as spaces of modular symbols via (1). To complete the proof, it therefore suffices to prove $H_c^2(\Gamma_0, \mathcal{D}_{-2-k}^\dagger) = 0$. This is an immediate consequence of the next lemma. □

Lemma 5.2. *For $k \in \mathbb{Z}$,*

$$H_c^2(\Gamma_0, \mathcal{D}_k^\dagger) = \begin{cases} \mathbb{Q}_p & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By Poincaré duality we have an isomorphism $H_c^2(\Gamma_0, \mathcal{D}_k^\dagger) \cong H_0(\Gamma_0, \mathcal{D}_k^\dagger)$, where the latter is just the group of Γ_0-coinvariants of $\mathcal{D}_k^\dagger$. So letting $I \subseteq \mathbb{Z}[\Gamma_0]$ be the augmentation ideal, we need to compute $\mathcal{D}_k^\dagger / I\mathcal{D}_k^\dagger$.

We first observe that for $\tau = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the element $\tau - 1 \in I$ acts on $\mathcal{D}_k^\dagger$ as the difference operator Δ studied at the end of section 3.1. In particular, Proposition 3.1 asserts that the sequence

$$0 \longrightarrow \mathcal{D}_k^\dagger \xrightarrow{\tau-1} \mathcal{D}_k^\dagger \xrightarrow{\rho} \mathbb{Q}_p \longrightarrow 0$$

is exact. When $k = 0$, one sees at once from the definitions that $I\mathcal{D}_k^\dagger \subseteq \ker(\rho)$, so in this case the above sequence implies $I\mathcal{D}_k^\dagger = \ker(\rho)$, proving the first equality.

[4]For a nice treatment of cohomology with supports, we recommend section II 2.5ff of [10].

Now suppose $k \neq 0$ and consider the distribution $\delta_0' \in \mathcal{D}_k^\dagger$ defined by $\delta_0'(f) = f'(0)$ for $f \in \mathcal{A}_k^\dagger$. Choose $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0$ to be any element for which $ac \neq 0$. Setting $\mu := \delta_0'|(\gamma - 1)$ we have $\mu \in I\mathcal{D}_k^\dagger$. But $\rho(\mu) = \mu(1) = \delta_0'(\gamma \cdot 1)$ and since $(\gamma \cdot 1)(z) = (a + cz)^k$ we have $\rho(\mu) = kca^{k-1} \neq 0$. Thus when $k \neq 0$ we have $\ker(\rho) \subsetneq I\mathcal{D}_k^\dagger$ and therefore $I\mathcal{D}_k^\dagger = \mathcal{D}_k^\dagger$, proving the final assertion of the lemma. $\square$

For $r \in |\mathbb{C}_p^\times|$, $\mathbf{D}_k[r]$ is a Banach space and therefore $H_c^1(\mathbf{D}_k[r])$ is naturally a Banach space as well. In particular, for $h \in \mathbb{R}$, we have a well-defined subspace $H_c^1(\mathbf{D}_k[r])^{(<h)}$. In contrast with this, $H_c^1(\mathcal{D}_k)$ is a Frechet space, and not a Banach space, and thus the arguments of the previous section do not apply. We sidestep this issue by directly defining $H_c^1(\mathcal{D}_k)^{(<h)} := H_c^1(\mathcal{D}_k) \cap H_c^1(\mathbf{D}_k)^{(<h)}$; here, to make sense of this intersection, we are implicitly using (1) and are identifying these spaces as subsets of $\mathrm{Hom}(\Delta_0, \mathbf{D}_k)$. Similarly, we define $H_c^1(\mathcal{D}_k^\dagger)^{(<h)} := H_c^1(\mathcal{D}_k^\dagger) \cap H_c^1(\mathbf{D}_k[r])^{(<h)}$, which we will see does not depend on the choice of r (see the next lemma).

Lemma 5.3. *For any $h \in \mathbb{R}$ and any $r > 1$ with $r \in |\mathbb{C}_p^\times|$, the natural maps*

$$H_c^1(\mathcal{D}_k)^{(<h)} \xrightarrow{\sim} H_c^1(\mathbf{D}_k)^{(<h)} \xrightarrow{\sim} H_c^1(\mathcal{D}_k^\dagger)^{(<h)} \xrightarrow{\sim} H_c^1(\mathbf{D}_k[r])^{(<h)}$$

are isomorphisms.

Proof. By (1), we may view each of these compactly supported cohomology spaces as spaces of modular symbols. From this optic, it is clear that all of the above maps are injective. To prove the lemma, it then suffices to show that any $\Phi \in H_c^1(\mathbf{D}_k[r])^{(<h)}$ actually takes values in $\mathcal{D}_k$. Since U_p acts invertibly on $H_c^1(\mathbf{D}_k[r])^{(<h)}$, for each $n \geq 1$, $\Phi = \Psi \big| U_p^n$ for some $\Psi \in H_c^1(\mathbf{D}_k)$. For $D \in \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ and $g \in \mathbf{A}_k[r]$, we have

$$\begin{aligned}\Phi(D)(g) = (\Psi|U_p^n)(D)(g) &= \sum_{a=0}^{p^n-1} \left(\Psi\big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)\right)(D)(g) \\ &= \sum_{a=0}^{p^n-1} \left(\Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) D\right) \big| \left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right)\right)(g) = \sum_{a=0}^{p^n-1} \Psi\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) D\right)\left(\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) g\right).\end{aligned}$$

Since $g \in \mathbf{A}_k[r]$, we have $\left(\begin{smallmatrix} 1 & a \\ 0 & p^n \end{smallmatrix}\right) g$ extends naturally to an element of $\mathbf{A}_k[rp^{-n}]$. Thus, the above computation shows that $\Phi(D)$ extends to $\mathbf{D}_k[rp^{-n}]$ for all n, and thus to $\mathcal{D}_k$. $\square$

5.2. **Proof of comparison theorem.**

Theorem 5.4 (Stevens)**.** *We have*

$$H_c^1(\mathcal{D}_k)^{(<k+1)} \xrightarrow{\sim} H_c^1(\mathcal{P}_k^\vee)^{(<k+1)}$$

is an isomorphism.

Proof. From Lemma 5.1, we have the following exact sequence of Hecke modules:

$$0 \to H_c^1(\mathcal{D}_{-2-k}^\dagger)(k+1) \to H_c^1(\mathcal{D}_k^\dagger) \xrightarrow{\rho_k^*} H_c^1(\mathcal{P}_k^\vee) \to 0.$$

By Proposition 4.8 and Lemma 5.3, passing to the slope less than $k+1$ subspaces gives an exact sequence

$$0 \to H_c^1(\mathbf{D}_{-2-k})(k+1)^{(<k+1)} \to H_c^1(\mathbf{D}_k)^{(<k+1)} \to H_c^1(\mathcal{P}_k^\vee)^{(<k+1)} \to 0.$$

As U_p has norm ≤ 1 on $H^1_c(\mathbf{D}_{-2-k})$, it follows that U_p induces operators on $H^1_c(\mathbf{D}_{-2-k})(k+1)$ of norm $\leq p^{-(k+1)}$. From Lemma 4.9.2, it follows that the first term of the above exact sequence vanishes. Thus $H^1_c(\mathbf{D}_k)^{(<k+1)} \cong H^1_c(\mathcal{P}_k^\vee)^{(<k+1)}$ and the theorem follows from Lemma 5.3. □

Remark 5.5. This theorem can be viewed as an overconvergent modular symbol version of Coleman's theorem that non-critical overconvergent modular forms are classical.

6. The critical slope subspace I

We now study the restriction of the specialization map to the subspace where U_p acts with slope equal to $k+1$. Throughout this section, we assume that there are no elements of order p in Γ_0.

6.1. **Some lemmas on filtrations.** Let k be a positive integer.

Lemma 6.1. *If $\mu \in \operatorname{Fil}^r \mathbf{D}^0_{-k}$, then $\mu\big|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right) \in p^r \mathbf{D}^0_{-k}$.*

Proof. We have that

$$(\mu\big|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right))(z^j) = \mu\left((a+pz)^j\right) = \sum_{i=0}^{j} \binom{j}{i} a^{j-i} p^i \mu(z^i)$$

which is divisible by p^r as $\mu(z^i) \in p^{r-i}\mathbb{Z}_p$. □

Lemma 6.2. *We have that*

$$H^1_c(\operatorname{Fil}^r \mathbf{D}^0_{-k})\big|U_p \subseteq p^r H^1_c(\mathbf{D}^0_{-k}).$$

Proof. For $\Phi \in H^1_c(\operatorname{Fil}^r \mathbf{D}^0_{-k})$, we have

$$(\Phi\big|U_p)(D) = \sum_{a=0}^{p-1} \Phi(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)D)|\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right),$$

which, by Lemma 6.1, is divisible by p^r as $\Phi(\left(\begin{smallmatrix} 1 & a \\ 0 & p \end{smallmatrix}\right)D) \in \operatorname{Fil}^r \mathbf{D}^0_{-k}$.

Note that we are implicitly using the identification in (1) here which is the reason for the assumption on the torsion in Γ_0 given at the start of the section. Indeed, we need this hypothesis as p does not act invertibly on $\operatorname{Fil}^r \mathbf{D}^0_{-k}$. □

Lemma 6.3. *If $\Phi \in H^1_c(\mathbf{D}^0_{-k})$ is a U_p-eigensymbol with slope h and $||\Phi|| = 1$, then the image of Φ in $H^1_c(\mathbf{D}^0_{-k}/\operatorname{Fil}^r \mathbf{D}^0_{-k})$ is non-zero for $r > h$.*

Proof. This lemma follows from Lemmas 6.1 and 6.2. □

6.2. **Some linear algebra.** Recall that $\mathcal{H}$ denotes the free polynomial algebra over $\mathbb{Z}$ generated by the Hecke operators T_ℓ for $\ell \nmid Np$ and U_q for $q|Np$. We define an *eigenpacket* of $\mathcal{H}$ over a ring R to be a homomorphism $\eta : \mathcal{H} \to R$. If M is a (right) $\mathcal{H}$-module, we say that an eigenpacket η occurs in M, if there is some non-zero $m \in M$ such that $m\big|T = \eta(T)m$ for all $T \in \mathcal{H}$.

Let V be a finite-dimensional vector space over $\mathbb{Q}_p$ with an action of $\mathcal{H}$. For $T \in \mathcal{H}$ and $\alpha \in \overline{\mathbb{Q}}_p$, let $V_{(\alpha,T)}$ denote the generalized eigenspace of T acting on $V \otimes \overline{\mathbb{Q}}_p$ with eigenvalue α. For η, an eigenpacket of $\mathcal{H}$ over $\overline{\mathbb{Q}}_p$, we define

$$V_{(\eta)} = \bigcap_{T \in \mathcal{H}} V_{(\eta(T),T)},$$

the η-isotypic subspace of V. Note that an eigenpacket η occurs in V if and only if $V_{(\eta)} \neq 0$.

Lemma 6.4. *Let V', V, and V'' be finite-dimensional $\mathbb{Q}_p$-vector spaces equipped with an action of $\mathcal{H}$. If*

$$0 \to V' \to V \to V'' \to 0$$

is an $\mathcal{H}$-equivariant exact sequence, then

$$0 \to V'_{(\eta)} \to V_{(\eta)} \to V''_{(\eta)} \to 0$$

is exact.

Proof. By basic linear algebra (i.e. Jordan canonical form), for a fixed $T \in \mathcal{H}$, passing to a generalized eigenspace of T preserves exact sequences. Since $\mathcal{H}$ is a commutative algebra, forming the η-isotypic subspace is done by repeatedly restricting to generalized eigenspaces of elements of $\mathcal{H}$. □

Remark 6.5. Passage to eigenspaces (as opposed to generalized eigenspaces) does not preserve exact sequences. For instance, let $V = \mathbb{Q}_p e_1 \oplus \mathbb{Q}_p e_2$, and let $V'' = V/\mathbb{Q}_p e_1$. Let T act on V by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and on V'' trivially. The natural map $V \to V''$ is then T-equivariant, and the image of e_2 in V'' is an eigenvector. However, no preimage of this vector in V is an eigenvector.

If $f = \sum_n a_n q^n$ is an eigenform in $S_k(\Gamma_0, \overline{\mathbb{Q}}_p)$, then $\eta_f : \mathcal{H} \to \overline{\mathbb{Q}}_p$ given by $\eta_f(T_\ell) = a_\ell$ and $\eta_f(U_q) = a_q$ is the eigenpacket attached to f. To simplify notation, we write $V_{(f)}$ for $V_{(\eta_f)}$.

6.3. **A lemma relating modular symbols and modular forms.** Let $\Gamma_1 = \Gamma_1(Np)$.

Lemma 6.6. *If η is an eigenpacket of $\mathcal{H}$ which occurs in $H^1_c(\Gamma_0, \overline{\mathbb{F}}_p(a^j))$, then there is some eigenform g in $M_2(\Gamma_1, \omega^j, \overline{\mathbb{Q}}_p)$ whose eigenpacket reduces to η.*

Proof. One checks that the natural map

$$H^1_c(\Gamma_0, \mathbb{F}_p(a^j)) \to H^1_c(\Gamma_1, \mathbb{F}_p)^{(\omega^j)}$$

is an isomorphism. (Here again we are using the the identification in (1)). We claim that the natural map

$$H^1_c(\Gamma_1, \mathbb{Z}_p) \to H^1_c(\Gamma_1, \mathbb{F}_p)$$

is surjective. To see this, note that the cokernel of this map equals the p-torsion in $H^2_c(\Gamma_1, \mathbb{Z}_p)$. By Poincaré duality, we have $H^2_c(\Gamma_1, \mathbb{Z}_p) \cong H_0(\Gamma_1, \mathbb{Z}_p) = \mathbb{Z}_p$, which is torsion-free. Thus, η lifts to some eigenpacket occurring in $H^1_c(\Gamma_1, \mathcal{O})^{(\omega^j)}$ where $\mathcal{O}$ is some finite extension of $\mathbb{Z}_p$. By the classical Eichler-Shimura isomorphism [17, Chapter 8], the lemma then follows[5]. □

[5] We note that in [17], Eichler-Shimura is stated in the form $S_{k+2} \oplus S_{k+2} \cong H^1_{par}(\Gamma, V_k)$ where $V_k := \mathrm{Sym}^k(\mathbb{C}^2)$ and the isomorphism is given by $(f,g) \mapsto \xi_f|\iota + \xi_g$ where ι is the operator induced by the matrix $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. To obtain the form used here, we note that H^1_{par} is defined by the exactness of the sequence

$$0 \longrightarrow H^1_{par}(\Gamma, V_k) \longrightarrow H^1(\Gamma, V_k) \xrightarrow{\rho} \bigoplus_{x \in \mathrm{cusps}(\Gamma)} H^1(\Gamma_x, V_k)$$

where, for $x \in \mathrm{cusps}(\Gamma)$, Γ_x is the stabilizer in Γ of some fixed representative of x in $\mathbb{P}^1(\mathbb{Q})$ and ρ is the direct sum of the restriction morphisms. But the composition of the canonical map

6.4. **Main theorem I.** Let f be a critical slope eigenform in $S_{k+2}(\Gamma_0,\overline{\mathbb{Q}}_p)$. Since f is necessarily the critical p-stabilization of a p-ordinary form of level $\Gamma_1(N)$, by Hida theory, f has the same residual Galois representation as some eigenform in $S_j(\Gamma_0,\overline{\mathbb{Q}}_p)$ with $1 \leq j \leq p-1$ and $j \equiv k+2 \pmod{p-1}$. We thus say that f possesses a *mod p companion form*, if there is an eigenform g in $M_{p+1-j}(\Gamma_0,\overline{\mathbb{Q}}_p)$ with $\overline{\rho}_f \cong \overline{\rho}_g \otimes \omega^{k+1} \cong \overline{\rho}_g \otimes \omega^{j-1}$.

Theorem 6.7. *Let f be an eigenform in $S_{k+2}(\Gamma_0,\overline{\mathbb{Q}}_p)$ with slope $k+1$. If Γ_0 has no elements of order p, and if f does not possess a mod p companion form, then*

$$H^1_c(\Gamma_0,\mathcal{D}_k)_{(f)} \xrightarrow{\sim} H^1_c(\Gamma_0,\mathcal{P}_k^\vee)_{(f)}$$

is an isomorphism.

Proof. By Proposition 4.8 and Lemma 5.1, we have an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)^{(=k+1)} \to H^1_c(\mathbf{D}_k)^{(=k+1)} \to H^1_c(\mathcal{P}_k^\vee)^{(=k+1)} \to 0$$

of finite-dimensional vector spaces. By Lemma 6.4, passing to f-isotypic subspaces gives an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)} \to H^1_c(\mathbf{D}_k)_{(f)} \to H^1_c(\mathcal{P}_k^\vee)_{(f)} \to 0.$$

Assume that $\Psi \in H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)}$ is some non-zero $\mathcal{H}$-eigenvector, and we will produce a mod p companion form for f.

Let Ψ_0 be the corresponding untwisted $\mathcal{H}$-eigensymbol in $H^1_c(\mathbf{D}_{-2-k})$. By scaling, we may assume that $||\Psi_0|| = 1$, and in particular, Ψ_0 takes values in $\mathbf{D}^0_{-2-k}$. Since Ψ has slope $k+1$, we have that Ψ_0 has slope 0. Thus, by Lemma 6.3, the reduction of Ψ_0 modulo $\operatorname{Fil}^1 \mathbf{D}^0_{-2-k}$ is non-zero.

By Lemma 3.7, we have

$$H^1_c(\mathbf{D}^0_{-2-k}/\operatorname{Fil}^1 \mathbf{D}^0_{-2-k}) \cong H^1_c(\mathbb{F}_p(a^{-2-k})).$$

Thus, by Lemma 6.6, there is some eigenform h in $M_2(\Gamma_1,\omega^{-2-k},\overline{\mathbb{Q}}_p)$ whose eigenpacket reduces to the eigenpacket attached to the image of Ψ_0. In particular, this means that $\overline{\rho}_h \otimes \omega^{k+1} \cong \overline{\rho}_f$.

Note that h is necessarily p-ordinary as Ψ_0 has slope 0. Thus, by Hida theory, there is a unique weight between 2 and p for which $\overline{\rho}_h$ occurs at level Γ_0 (as opposed to Γ_1). To find this weight, note that

$$\det \overline{\rho}_h = \omega^{-1-k} = \omega^{p-j}$$

where j is the unique integer satisfying $j \equiv k+2 \pmod{p-1}$ and $1 \leq j \leq p-1$. Thus, there is some p-ordinary eigenform g in $M_{p+1-j}(\Gamma_0,\overline{\mathbb{Q}}_p)$ with $\overline{\rho}_g \cong \overline{\rho}_h$, and f possesses a mod p companion form. $\square$

$\mathcal{E}_{k+2} \xrightarrow{\xi} H^1(\Gamma,V_k)$ with ρ maps $\mathcal{E}_{k+2}$ isomorphically to the image of ρ, so we have a canonical splitting $H^1_{par}(\Gamma,V_k) \oplus \mathcal{E}_{k+2} \xrightarrow{\sim} H^1(\Gamma,V_k)$ of Hecke modules. Combining this with [17] we obtain a Hecke-equivariant isomorphism

$$S_{k+2} \oplus M_{k+2} \cong H^1(\Gamma,V_k).$$

(Here and elsewhere we refer to this statement as the classical Eichler-Shimura isomorphism.) By Lefschetz duality, there is a Hecke-equivariant perfect pairing over $\mathbb{C}$ between $H^1(\Gamma,V_k)$ and $H^1_c(\Gamma,V_k)$. Thus a Hecke eigenpacket occurs in $H^1_c(\Gamma,V_k)$ if and only if it occurs in $H^1(\Gamma,V_k)$, which by the above version of Eichler-Shimura, is equivalent to its occurrence in M_{k+2}. For the application given here, we take $k=0$, but in chapter 7, we will need the general statement.

6.5. **On the condition of Theorem 6.7.** The hypothesis of Theorem 6.7 (on the non-existence of a mod p companion form) can be readily verified for a given modular form. In this section, we give examples of forms that satisfy and fail this condition.

Example 6.8. Let f_0 be the normalized newform on $\Gamma_0(11)$ that corresponds to the elliptic curve $X_0(11)$. Let $p = 3$, and let f correspond to the critical 3-stabilization of f_0 to level 33. (Note that $\Gamma_0(11)$ is torsion-free and so we are free to take $p = 3$ in this example.) We need to verify that f does not possess a mod 3 companion form. That is, we need to see that there is no form g in $M_2(\Gamma_0(33), \overline{\mathbb{Q}}_p)$ such that $\overline{\rho}_g \otimes \omega \cong \overline{\rho}_f$. To see this, note that $a_5(f) = 1$, and thus any such g would need to satisfy $a_5(g)$ reduces to 2 in $\mathbb{F}_3$. However, $M_2(\Gamma_0(33), \overline{\mathbb{Q}}_p)$ is six-dimensional with three dimensions coming Eisenstein series and the remaining three dimensions coming from the two 3-stabilizations of f_0 and from one newform on $\Gamma_0(33)$ corresponding to the unique elliptic curve over $\mathbb{Q}$ with conductor 33. All of the Eisenstein series have a_5 equal to 6 while all of cuspforms have a_5 congruent to 1. Thus f has no mod 3 companion form.

Let θ^{k+1} denote the θ-operator on overconvergent modular forms which takes $S^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$ to $S^\dagger_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ and which acts on q-expansions by $(qd/dq)^{k+1}$.

Proposition 6.9. *If $f \in S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ is a critical slope eigenform which is in the image of θ^{k+1}, then f possesses a mod p companion form.*

Proof. Write $f = \theta^{k+1}h$ with $h \in S^\dagger_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$. Since f has slope $k + 1$, we must have that h is a p-ordinary form as $a_p(f) = p^{k+1}a_p(h)$. Thus, by Hida theory (as in the proof of Theorem 6.7), there is a modular form g in $S_{p+1-j}(\Gamma_0, \overline{\mathbb{Q}}_p)$ with $j \equiv k+2 \pmod{p-1}$, $1 \leq j \leq p-1$ and with $\overline{\rho}_g \cong \overline{\rho}_h$. Since $\rho_h \otimes \chi^{k+1} \cong \rho_f$ where χ is the cyclotomic character, we immediately see that g is a mod p companion form for f. □

Example 6.10. Let $E/\mathbb{Q}$ be any elliptic curve with CM, and let f_0 be the corresponding normalized newform. Let p be a good ordinary prime for E, and let f be the criticial p-stabilization of f_0. By [4, Prop 7.1], the form f is in the image of θ, and in particular, the form f fails the conditions of Theorem 6.7.

In [15], computations were done for f corresponding to the CM elliptic curve $X_0(32)$ with $p = 5$. In these computations, an approximation to an overconvergent Hecke-eigensymbol was found with the same eigenvalues as f. However, this symbol was in the kernel of specialization, and no symbol was found which specialized to the classical modular symbol attached to f.

Example 6.11. Let f_0 again be the normalized newform on $\Gamma_0(11)$ that corresponds to $X_0(11)$, but now take $p = 5$. Let f correspond to the critical 5-stabilization of f_0 to level 55. In this case, f does possess a mod 5 companion form. Indeed, $\overline{\rho}_f \cong \mathbf{1} \oplus \omega$ where $\mathbf{1}$ is the trivial character. We thus need to find an eigenform g of weight 4 such that $\overline{\rho}_g \cong \omega^3 \oplus \mathbf{1}$. But then simply the Eisenstein series E_4 (stabilized to level 55) serves as a companion form for f.

We note that despite the fact that the hypotheses of Theorem 6.7 are not satisfied in the previous example, its conclusion still holds in this case. This will be established in the following two sections.

7. Eigenpackets in $H^1_c(\Gamma_0, \mathcal{D}_k)$ and $M^\dagger_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$

Let $\epsilon : \mathcal{H} \longrightarrow \mathbb{Q}_p$ be the eigenpacket attached to E_2^{crit}, the critical slope Eisenstein series of weight 2 on $\Gamma_0(p)$. Explicitly, we have $\epsilon(T_\ell) = \ell + 1$ for $\ell \nmid Np$ and $\epsilon(U_q) = q$ for $q|Np$. We will also let $\mathcal{E}_2^{\mathrm{crit}} := \mathbb{Q}_p E_2^{\mathrm{crit}}$.

The following theorem of the second author compares the eigenpackets which occur in spaces of overconvergent modular symbols and in spaces of overconvergent modular forms.

Theorem 7.1 (Stevens [18, 19])**.** *Let k be any integer. A finite slope eigenpacket for $\psi : \mathcal{H} \longrightarrow \overline{\mathbb{Q}}_p$ occurs in $H^1_c(\Gamma_0, \mathcal{D}_k)$ if and only if either ψ occurs in $M^\dagger_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ or $k = 0$ and $\psi = \epsilon$.*

The proof, which will occupy the rest of this section, comes down to comparing, for a fixed element $u \in U_p\mathcal{H}$, the characteristic Fredholm series of u acting on overconvergent modular forms with the characteristic Fredholm series for u acting on overconvergent modular symbols.

On the modular form side we have the work of Coleman-Mazur, Buzzard and others, which in particular gives us the following theorem. In the following, let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$ denote the space of Iwasawa functions on weight space.

Theorem 7.2. *Let $u \in U_p\mathcal{H}$. Then there is a (unique) power series $Q(T) \in \Lambda[[T]]$ such that for every non-negative integer $k \in \mathbb{Z}$ we have*

$$Q(k, T) = \det\left(1 - uT; S^\dagger_{k+2} \oplus M^\dagger_{k+2}\right).$$

Moreover, for such k we have the congruence

$$Q(k, T) \equiv \det\left(1 - uT; S_{k+2} \oplus M_{k+2}\right) \pmod{p^{k+1}}.$$

To compare this with corresponding constructions on the modular symbols side, we fix $\Omega \subseteq \mathcal{X}$ to be an affinoid disk (viewed as a $\mathbb{Q}_p$-affinoid) in weight space with $0 \in \Omega$. We form the Frechet $A(\Omega)$-module

$$\mathcal{D}_\Omega := \mathcal{D} \widehat{\otimes}_{\mathbb{Q}_p} A(\Omega)$$

and endow it with the canonical action of Σ_0, with respect to which specialization of the second factor to any point $k \in \Omega$ induces a Σ_0-morphism $\mathcal{D}_\Omega \longrightarrow \mathcal{D}_k$. It can be shown that $H^1_c(\mathcal{D}_\Omega)$ is a projective limit of ON-able $A(\Omega)$-modules on which U_p acts in a completely continuous manner. Hence our fixed element $u \in U_p\mathcal{H}$ also acts completely continuously and we can form the Fredholm determinant

$$P_\Omega(T) = \det(1 - uT; H^1_c(\mathcal{D}_\Omega)) \in A^0(\Omega)[[T]]$$

where $A^0(\Omega)$ denotes the ring of integers in $A(\Omega)$. We then have the following theorem.

Theorem 7.3. *For any affinoid disk $\Omega \subseteq \mathcal{X}$ as above, we have $Q|_\Omega = P_\Omega$.*

To prove this, we note that for any $k \in \Omega$ and any generator π_k of the maximal ideal associated to k in $A(\Omega)$, the space

$$H^1_c(\mathcal{D}_\Omega)(k) := H^1_c(\mathcal{D}_\Omega)/\pi_k H^1_c(\mathcal{D}_\Omega)$$

inherits a structure of $\mathbb{Q}_p$-Frechet space on which u is again completely continuous, and the characteristic Fredholm series of u on this space is equal to the specialization of P_Ω at k:

$$P_\Omega(k, T) = \det\left(1 - uT; H^1_c(\mathcal{D}_\Omega)(k)\right) \in \mathbb{Z}_p[[T]].$$

We also have an exact sequence

$$0 \longrightarrow \mathcal{D}_\Omega \xrightarrow{\pi_k} \mathcal{D}_\Omega \longrightarrow \mathcal{D}_k \longrightarrow 0$$

and consequently a long exact sequence in cohomology, from which we may extract the $\mathcal{H}$-equivariant exact sequence of $A(\Omega)$-modules

$$0 \longrightarrow H^1_c(\mathcal{D}_\Omega)(k) \longrightarrow H^1_c(\mathcal{D}_k) \longrightarrow H^2_c(\mathcal{D}_\Omega) \longrightarrow H^2_c(\mathcal{D}_\Omega).$$

On the other hand, it follows from Lemma 5.2 that the natural map $\mathcal{D}_\Omega \longrightarrow \mathbb{Q}_p$ given by taking the total measure and composing with evaluation at 0 induces an $\mathcal{H}$-equivariant isomorphism over $A(\Omega)$:

$$H^2_c(\mathcal{D}_\Omega) \cong H^2_c(\mathbb{Q}_p) \cong \mathcal{E}_2^{\mathrm{crit}}$$

where $A(\Omega)$ acts on $\mathcal{E}_2^{\mathrm{crit}}$ via specialization to 0. In particular, for any $k \neq 0$ in $\mathcal{X}$ we have an isomorphism

$$H^1_c(\mathcal{D}_\Omega)(k) \cong H^1_c(\mathcal{D}_k) \quad (k \neq 0)$$

while for $k = 0$ we have the exact sequence

$$0 \longrightarrow H^1_c(\mathcal{D}_\Omega)(0) \longrightarrow H^1_c(\mathcal{D}_0) \longrightarrow \mathcal{E}_2^{\mathrm{crit}} \longrightarrow 0.$$

But from the comparison theorem (Theorem 5.4) for modular symbols we have, for any integer $k \geq 0$,

$$\det\left(1 - uT; H^1_c(\mathcal{D}_k)\right) \equiv \det\left(1 - uT; H^1_c\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)\right) \pmod{p^{k+1}}.$$

On the other hand, since $H^1_c\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)$ is dual to $H^1\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)$ and the action of u on the first is the transpose of the action of u on the second we have

$$\det\left(1 - uT; H^1_c\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)\right) = \det\left(1 - uT; H^1\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)\right)$$

while the classical Eichler-Shimura[6] theorem tells us

$$\det\left(1 - uT; H^1\left(\mathrm{Sym}^k(\mathbb{Q}_p^2)\right)\right) = \det\left(1 - uT; S_{k+2} \oplus M_{k+2}\right).$$

Comparing this with Theorem 7.2 we obtain, for any non-negative integer $k \geq 0$, the congruence

$$P_\Omega(k, T) \equiv Q(k, T) \pmod{p^{k+1}}.$$

In fact, this congruence is easily improved to an equality. Indeed, for any positive integer n we may choose a positive integer $k' > n - 1$ sufficiently close to k in $\mathcal{X}$ so that we have the following congruences

$$P_\Omega(k, T) \equiv P_\Omega(k', T) \equiv Q(k', T) \equiv Q(k, T) \pmod{p^n}.$$

It follows that $P_\Omega(k, T) \equiv Q(k, T) \pmod{p^n}$ for any positive integer n and therefore $P_\Omega(k, T) = Q(k, T)$, as claimed. Since this is true for any non-negative integer $k \in \Omega$ and since Ω necessarily contains infinitely many such integers, it follows that $P_\Omega = Q|_\Omega$ and Theorem 7.3 is proved.

Let

$$\widetilde{M}_{k+2} := \begin{cases} M^\dagger_{k+2} & \text{if } k \neq 0; \\ M^\dagger_2 \oplus \mathcal{E}_2^{\mathrm{crit}} & \text{if } k = 0. \end{cases}$$

The following corollary is an easy consequence of the above discussion.

[6]Ibid

Corollary 7.4. *For any $u \in U_p\mathcal{H}$ and any integer k, the set of non-zero eigenvalues of u occurring in $\widetilde{M}_{k+2}$ is the same as the set of non-zero eigenvalues of u occurring in $H^1_c(\mathcal{D}_k)$.*

We now complete the proof of the main theorem of this section.

Proof of Theorem 7.1. Let $\psi : \mathcal{H} \longrightarrow K$ be a finite slope eigenpacket as in the statement of the theorem. Choose $h \in \mathbb{Q}$ with $h > \max\{1, \mathrm{ord}_p(\psi(U_p))\}$. It suffices to prove

$$\psi \text{ occurs in } H^1_c(\mathcal{D}_k)^{(<h)} \iff \psi \text{ occurs in } \widetilde{M}^{(<h)}_{k+2}.$$

Let $\mathcal{R}$ be the image of $\mathcal{H}$ in the endomorphism ring of $H^1_c(\mathcal{D}_k)^{(<h)}$ over $\mathbb{Q}_p$ and $\widetilde{\mathcal{R}}$ be the image of $\mathcal{H}$ in the endomorphism ring of $\widetilde{M}^{(<h)}_{k+2}$. Then both $\mathcal{R}$ and $\widetilde{\mathcal{R}}$ are finite-dimensional $\mathbb{Q}_p$-algebras and we have canonical surjective homomorphisms

$$\varphi : \mathcal{H} \longrightarrow \mathcal{R} \qquad \text{and} \qquad \widetilde{\varphi} : \mathcal{H} \longrightarrow \widetilde{\mathcal{R}}\,.$$

One checks easily that ψ occurs in $H^1_c(\mathcal{D}_k)^{(<h)}$ if and only if $\ker(\varphi) \subseteq \ker(\psi)$ and this in turn is equivalent to $\mathrm{Rad}(\ker(\varphi)) \subseteq \ker(\psi)$. Similarly, ψ occurs in $\widetilde{M}^{(<h)}_{k+2}$ if and only if $\mathrm{Rad}(\ker(\widetilde{\varphi})) \subseteq \ker(\psi)$. So it suffices to show

$$\mathrm{Rad}(\ker(\varphi)) = \mathrm{Rad}(\ker(\widetilde{\varphi})).$$

So let $T \in \mathrm{Rad}(\ker(\varphi))$. Then there is a non-negative integer n such that $\varphi(T)^n = 0$ in $\mathcal{R}$. We claim the element $t := \widetilde{\varphi}(T^n) \in \widetilde{\mathcal{R}}$ is a nilpotent endomorphism of $\widetilde{M}^{(<h)}_{k+2}$. For this it suffices to show that t has no non-zero eigenvalue.

Suppose to the contrary that $\lambda \in \overline{\mathbb{Q}}_p$ were a non-zero eigenvalue of t occurring in $\widetilde{M}^{(<h)}_{k+2}$. Since U_p commutes with T, there must be a λ-eigenvector $f \in \widetilde{M}^{(<h)}_{k+2}$ for t that is simultaneously an eigenvector for U_p. Thus $f|U_p = \alpha f$ for some $0 \neq \alpha \in \overline{\mathbb{Q}}_p$ with $\mathrm{ord}_p(\alpha) < h$. Now let m be an arbitrary positive integer and let $u = T^n U_p^m$. Clearly, $f|u = \lambda\alpha^m$ with $\lambda\alpha^m \neq 0$, so by Corollary 7.4 the eigenvalue $\lambda\alpha^m$ occurs as an eigenvalue for u in $H^1_c(\mathcal{D}_k)$. But then there must be a $\lambda\alpha^m$-eigenvector $x \in H^1_c(\mathcal{D}_k)$ for u that is simultaneously an eigenvector for U_p. Let β be the eigenvalue of U_p acting on x. Clearly β is not zero and therefore x is also an eigenvector for T^n and the eigenvalue of T^n on x is $\lambda \cdot (\alpha/\beta)^m$, which is not 0. But T^n annihilates $H^1_c(\mathcal{D}_k)^{(<h)}$, so we must have $\mathrm{ord}_p(\beta) \geq h$. On the other hand, T^n preserves the integral structure of $H^1_c(\mathcal{D}_k)$, hence any eigenvalue of T^n occurring in this space must be integral. Putting all of this together, we have $\mathrm{ord}_p(\lambda) \geq m(\mathrm{ord}_p(\beta) - \mathrm{ord}_p(\alpha) \geq m(h - \mathrm{ord}_p(\alpha))$. Since this is true for all positive integers m and since $h > \mathrm{ord}_p(\alpha)$ we must have $\lambda = 0$, which contradicts our initial assumption that $\lambda \neq 0$. It follows that t has no non-zero eigenvalue occurring in $\widetilde{M}^{(<h)}_{k+2}$ and consequently that t is a nilpotent endomorphism of this space. Thus $T \in \mathrm{Rad}(\ker(\widetilde{\varphi}))$ and we conclude that

$$\mathrm{Rad}(\ker(\varphi)) \subseteq \mathrm{Rad}(\ker(\widetilde{\varphi})).$$

The opposite inclusion is proved in precisely the same way. This completes the proof of Theorem 7.1. □

Remark 7.5. We note that we will only apply this theorem when k is negative and when the eigenpacket has slope 0 (and in particular the exceptional case of E_2^{crit} does not intervene).

Remark 7.6. Following Coleman and Mazur [5] one may also define the spaces $M^{\dagger}_{k+2}$ for any $k \in \mathcal{X}(K)$ with $K/\mathbb{Q}_p$ finite. With those conventions, all of the results in this section remain true and our proofs remain valid, also for non-integral weights k.

8. The critical slope subspace II

We now strengthen the results of Theorem 6.7 using the results of the previous section.

Theorem 8.1. *Let f be an eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ with slope $k+1$. Then*

$$H^1_c(\Gamma_0, \mathcal{D}_k)_{(f)} \longrightarrow H^1_c(\Gamma_0, \mathcal{P}^{\vee}_k)_{(f)}$$

is an isomorphism if and only if $f \notin \operatorname{im}(\theta^{k+1})$.

Proof. As in the proof of Theorem 6.7, we have an exact sequence

$$0 \to H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)} \to H^1_c(\mathbf{D}_k)_{(f)} \to H^1_c(\mathcal{P}^{\vee}_k)_{(f)} \to 0.$$

Assume there is a non-zero $\mathcal{H}$-eigensymbol $\Psi \in H^1_c(\mathbf{D}_{-2-k})(k+1)_{(f)}$, and let Ψ_0 be the untwisted eigensymbol in $H^1_c(\mathbf{D}_{-2-k})$. By Theorem 7.1, there is some overconvergent eigenform $g \in M^{\dagger}_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$ with the same eigenpacket as Ψ_0. The eigenpacket of $\theta^{k+1}g \in M^{\dagger}_{k+2}(\Gamma, \overline{\mathbb{Q}}_p)$ then equals the eigenpacket of f. Thus, looking at q-expansions, we deduce that $f = \theta^{k+1}g$.

Conversely, assume that $f = \theta^{k+1}g$ for some $g \in M^{\dagger}_{-k}(\Gamma, \overline{\mathbb{Q}}_p)$. By Theorem 7.1, there is some $\mathcal{H}$-eigensymbol Ψ_0 in $H^1_c(\mathbf{D}_{-2-k})$ with the same eigenpacket as g. Twisting this symbol then gives an $\mathcal{H}$-eigensymbol Ψ in $\big(H^1_c(\mathbf{D}_{-2-k})(k+1)\big)_{(f)}$. The image of Ψ in $H^1_c(\mathbf{D}_k)_{(f)} \cong H^1_c(\mathcal{D}_k)_{(f)}$ is then a non-zero symbol in the kernel of specialization. $\square$

Remark 8.2. When f is in the image of θ^{k+1}, the above theorem implies that there is an $\mathcal{H}$-eigensymbol $\Psi \in H^1_c(\mathcal{D}_k)^{\pm}_{(f)}$ which is in the kernel of specialization. However, from Theorem 7.1 alone, one cannot conclude that this symbol (up to scaling) is unique. Indeed, it is *a priori* possible that there are multiple $\mathcal{H}$-eigenvectors in $H^1_c(\mathcal{D}_k)^{\pm}_{(f)}$, while in $(M^{\dagger}_k)_{(f)}$, $\mathcal{H}$ acts non-semisimply and there is only a one-dimensional space of overconvergent eigenforms.

Remark 8.3. We note that by Proposition 6.9, we have that Theorem 8.1 does indeed imply Theorem 6.7.

9. p-adic L-functions

Let $f = \sum a_n q^n$ be a normalized eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ with slope $h < k+1$. In this case, there is a p-adic L-function $\mu_f \in \mathcal{D}$ which interpolates the special values of twists of the complex L-series of f. Specifically, if χ is a finite order character of $\mathbb{Z}_p^{\times}$ with conductor p^n and j is an integer between 0 and k, then

$$\mu_f(x^j \cdot \chi) = \frac{1}{a_p^n} \cdot \frac{p^{n(j+1)}}{(-2\pi i)^j} \cdot \frac{j!}{\tau(\chi^{-1})} \cdot \frac{L(f, \chi^{-1}, 1)}{\Omega_f^{\pm}} \tag{2}$$

where $\tau(\chi^{-1})$ is a Gauss sum and $\Omega^{\pm}_f$ are certain complex periods. We note that the p-adic L-function μ_f is uniquely determined by this interpolation property and by a bound on its growth (i.e. that it is h-admissible).

We now describe an alternative construction of this p-adic L-function via overconvergent modular symbols to motivate our definition of p-adic L-functions for critical slope forms.

Let K_f denote the finite extension of $\mathbb{Q}_p$ containing the Fourier coefficients of f. By Eichler-Shimura theory and multiplicity one, the f-isotypic subspace of $H^1_c(\Gamma_0, \mathcal{P}_k^\vee))^\pm \otimes K_f$ is one-dimensional. Let $\phi_f^\pm$ denote a non-zero element of this subspace, normalized to have size 1, and set $\phi_f = \phi_f^+ + \phi_f^-$.

Since we are assuming that f is non-critical, by Theorem 5.4, there is a unique overconvergent modular symbol $\Phi_f \in H^1_c(\mathcal{D}_k) \otimes K_f$ which specializes to ϕ_f. The following theorem relates Φ_f to the p-adic L-function of f.

Proposition 9.1. *With f, ϕ_f and Φ_f as above, we have*

$$\Phi_f(\{\infty\} - \{0\})\big|_{\mathbb{Z}_p^\times} = \mu_f,$$

the p-adic L-function of f.

Proof. See [19] or [15, Prop 6.3]. □

We now consider the case where f has slope equal to $k+1$. In light of Proposition 9.1, we make the following definition of the p-adic L-function of f.

Definition 9.2. Let f be an eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ of slope $k+1$ which is not in the image of θ^{k+1}. Let Φ_f be the unique overconvergent eigensymbol of Theorem 8.1 which specializes to ϕ_f. We define the p-adic L-function of f to be

$$\mu_f := \Phi_f(\{\infty\} - \{0\})\big|_{\mathbb{Z}_p^\times},$$

which is a locally analytic distribution on $\mathbb{Z}_p^\times$.

Proposition 9.3. *Let f be an eigenform in $S_{k+2}(\Gamma_0, \overline{\mathbb{Q}}_p)$ of slope $k+1$ which is not in the image of θ. Then μ_f is a $(k+1)$-admissible distribution. Further, μ_f satisfies the interpolation property in (2).*

Proof. The admissibility claim follows from [15, Lemma 6.2]. The interpolation property is a formal consequence of Φ_f being a U_p-eigensymbol lifting ϕ_f as in Proposition 9.1. □

Remark 9.4. Since μ_f is a $(k+1)$-admissible distribution, it is *not* uniquely determined by the above interpolation property. To uniquely determine this distribution by interpolation, one would also need to specify its values at the characters of the form $x^{k+1}\chi$ with χ of finite order. We point out here that our method of producing μ_f from overconvergent modular symbols does not directly give a way of understanding its values at such characters.

Remark 9.5. We now sketch an alternative construction of a critical slope p-adic L-functions given by combining Perrin-Riou's dual exponential map with Kato's zeta-element (see [13] and [6] for more details).

Let f now be an eigenform on $\Gamma_0(N)$ with $p \nmid N$, and let V_f denote the p-adic representation attached to f. Consider Perrin-Riou's dual-exponential map (see [14] and [12, 2.1]),

$$\varprojlim_n H^1(\mathbb{Q}_{n,p}, V_f) \xrightarrow{\exp^*} \mathcal{D}_k \otimes D_{\mathrm{cris}}(V_f)$$

where $\mathbb{Q}_{n,p}$ is the n-th layer of the local cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$. Kato's zeta-element $\mathbf{z}(f) = (z_n(f)) \in \varprojlim_n H^1(\mathbb{Q}_n, V_f)$ is a norm-coherent system of global cohomology classes; here, $\mathbb{Q}_n$ is the n-th layer of the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$.

Let $\mathbf{z}_p(f)$ denote the restriction to p of $\mathbf{z}(f)$, and let

$$L_p(f) = \exp^*(\mathbf{z}_p(f)) \in \mathcal{D}_k \otimes D_{\mathrm{cris}}(V_f).$$

We have that $D_{\mathrm{cris}}(V_f)$ decomposes under the φ-action into eigenspaces with eigenvalues α and β, the roots of $x^2 - a_p x + p^{k+1}$. Let $L_{p,\alpha}(f)$ and $L_{p,\beta}(f)$ denote the two projections of $L_p(f)$ onto these eigenspaces, which we can identify as locally analytic p-adic distributions.

If f is non-ordinary at p, then these two p-adic distributions are precisely the p-adic L-functions attached to f. If f is ordinary at p, and α is a p-unit, then $L_{p,\alpha}(f)$ is the ordinary p-adic L-function of f. In this case, one *defines* $L_{p,\beta}(f)$ to be the critical slope p-adic L-function attached to f.

As long as V_f is not locally split at p, this distribution satisfies the interpolation property of equation (2). However, if V_f is locally split at p, then $L_{p,\beta}(f)$ vanishes at all characters of the form $x^j\chi(x)$ where χ has finite order and $0 \leq j < k-2$, and in particular, it is not even clear that $L_{p,\beta}(f)$ is non-zero.

Comparing the above construction to the construction of this paper, we note that if f is in the image of θ^{k+1}, then V_f is locally split at p (see [9, Prop 1.2]), and further, the converse of this statement is known as well under some mild hypotheses. In this case, by Theorem 8.1, there is at least one $\mathcal{D}_k$-valued modular symbol Φ_f in the kernel of specialization with the same eigenpacket as f. For this symbol, we note that $\Phi_f(\{\infty\} - \{0\})$ also vanishes at all characters of the form $x^j\chi(x)$ with χ of finite order and $0 \leq j < k-2$.

Remark 9.6. In [7, Section 4.5], Emerton gives a construction of a two-variable p-adic L-function which specializes correctly to one-variable p-adic L-functions at classical points of non-critical slope. Moreover, under some mild assumptions on the residual representation (*e.g.* globally irreducible, p-distinguished), Emerton's local-global compatibility theorem [8] implies that this two-variable p-adic L-function extends to critical slope forms which are not in the image of θ^{k+1}.

Remark 9.7. We close with the remark that it is *a priori* unclear that the constructions of this paper match the constructions mentioned in Remarks 9.5 and 9.6 as these critical slope p-adic L-functions are not uniquely determined by their interpolation property.

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