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Abstract Linear Algebra

What are scalars? What are vectors? What is a linear combination? And what is linear independence and dependence?

$\mathcal{F} = \mathbb{R}$ is the canonical example!!

Definition 2-1

A field consists of a set, denoted by $\mathscr{F}$, of elements called *scalars* and two operations called addition "+" and multiplication "·"; the two operations are defined over $\mathscr{F}$ such that they satisfy the following conditions:

1. To every pair of elements α and β in $\mathscr{F}$, there correspond an element $\alpha + \beta$ in $\mathscr{F}$ called the *sum* of α and β, and an element $\alpha \cdot \beta$ or $\alpha\beta$ in $\mathscr{F}$, called the *product* of α and β.
2. Addition and multiplication are respectively commutative: For any α, β in $\mathscr{F}$,

$$\alpha + \beta = \beta + \alpha \qquad \alpha \cdot \beta = \beta \cdot \alpha$$

3. Addition and multiplication are respectively associative: For any α, β, γ in $\mathscr{F}$,

$$(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma) \qquad (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma)$$

4. Multiplication is distributive with respect to addition: For any α, β, γ in $\mathscr{F}$,

$$\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$$

5. $\mathscr{F}$ contains an element, denoted by 0, and an element, denoted by 1, such that $\alpha + 0 = \alpha$, $1 \cdot \alpha = \alpha$ for every α in $\mathscr{F}$.
6. To every α in $\mathscr{F}$, there is an element β in $\mathscr{F}$ such that $\alpha + \beta = 0$. The element β is called the *additive inverse*.
7. To every α in $\mathscr{F}$ which is not the element 0, there is an element γ in $\mathscr{F}$ such that $\alpha \cdot \gamma = 1$. The element γ is called the *multiplicative inverse*. ∎

To show something is a field you must check all seven axioms (super boring!)

To show something is not a field, you only need to show that one of the axioms fails!

Examples

a) $\mathbb{R}$

b) $\mathbb{C}$

c) $\mathbb{Q}$ (rational numbers)

Non-examples

a) $\mathbb{Z}$ integers (Fails #7)

b) $\mathbb{N}$ (Fails #5, #6, #7)

c) A = 2x2 real matrix. (Fails #2 and others)

For us, $\mathbb{R}$ and $\mathbb{C}$ will be our main fields. We will do some examples using $\mathbb{Q}$ just to make us think!

Want to define vectors

Please have the following example in mind:

Let A and B be 2×3 real matrices.

$$[A+B]_{ij} \triangleq [A]_{ij} + [B]_{ij}$$

$\triangleq$ Definition

$x := y$ means y is already understood, and I am defining x.

$$[A+B]_{ij} := [A]_{ij} + [B]_{ij}$$

OR

$[A]_{ij} + [B]_{ij} =: [A+B]_{ij}$

$\alpha \in \mathbb{R}$, $[\alpha A]_{ij} := \alpha [A]_{ij}$

Another example to have in mind $\mathbb{R}^2 =$ column vectors with two entries

Vector space

Definition 2-2

A linear space over a field $\mathscr{F}$, denoted by $(\mathscr{X}, \mathscr{F})$, consists of a set, denoted by $\mathscr{X}$, of elements called *vectors*, a field $\mathscr{F}$, and two operations called *vector addition* and *scalar multiplication*. The two operations are defined over $\mathscr{X}$ and $\mathscr{F}$ such that they satisfy all the following conditions:

1. To every pair of vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ in $\mathscr{X}$, there corresponds a vector $\mathbf{x}_1 + \mathbf{x}_2$ in $\mathscr{X}$, called the sum of $\mathbf{x}_1$ and $\mathbf{x}_2$.
2. Addition is commutative: For any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{x}_2 + \mathbf{x}_1$.
3. Addition is associative: For any $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ in $\mathscr{X}$, $(\mathbf{x}_1 + \mathbf{x}_2) + \mathbf{x}_3 = \mathbf{x}_1 + (\mathbf{x}_2 + \mathbf{x}_3)$.
4. $\mathscr{X}$ contains a vector, denoted by $\mathbf{0}$, such that $\mathbf{0} + \mathbf{x} = \mathbf{x}$ for every $\mathbf{x}$ in $\mathscr{X}$. The vector $\mathbf{0}$ is called the zero vector or the origin.
5. To every $\mathbf{x}$ in $\mathscr{X}$, there is a vector $\bar{\mathbf{x}}$ in $\mathscr{X}$, such that $\mathbf{x} + \bar{\mathbf{x}} = \mathbf{0}$.
6. To every α in $\mathscr{F}$, and every $\mathbf{x}$ in $\mathscr{X}$, there corresponds a vector $\alpha\mathbf{x}$ in $\mathscr{X}$ called the *scalar product* of α and $\mathbf{x}$.
7. Scalar multiplication is associative: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.
8. Scalar multiplication is distributive with respect to vector addition: For any α in $\mathscr{F}$ and any $\mathbf{x}_1, \mathbf{x}_2$ in $\mathscr{X}$, $\alpha(\mathbf{x}_1 + \mathbf{x}_2) = \alpha\mathbf{x}_1 + \alpha\mathbf{x}_2$.
9. Scalar multiplication is distributive with respect to scalar addition: For any α, β in $\mathscr{F}$ and any $\mathbf{x}$ in $\mathscr{X}$, $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$.
10. For any $\mathbf{x}$ in $\mathscr{X}$, $1\mathbf{x} = \mathbf{x}$, where 1 is the element 1 in $\mathscr{F}$. ∎

Examples of Vector Spaces

$(\mathcal{X}, \mathcal{F})$

Vectors field

a) Every field forms a vector space over itself: $(\mathcal{F}, \mathcal{F})$

$(\mathbb{R}, \mathbb{R})$, $(\mathbb{C}, \mathbb{C})$, $(\mathbb{Q}, \mathbb{Q})$

b) $(\mathbb{C}, \mathbb{R})$

Non-example

a) $(\mathbb{R}, \mathbb{C})$ under the usual rules of multiplication of complex numbers.

Examples continued

c) $\mathcal{F} = \mathbb{R}$, $D \subset \mathbb{R}$ where

$D = [a, b]$, or $D = [a, \infty)$, etc.

$\mathcal{X} := \{ f : D \to \mathbb{R} \}$ real-valued functions from D to $\mathbb{R}$.

$[\forall t \in D, f(t) \in \mathbb{R}] \leftrightarrow \{f: D \to \mathbb{R}\}$

Define vector addition and scalar times vector multiplication by:

a) $\forall f, g \in \mathcal{X}$ define $f+g$ by

$\forall t \in D, (f+g)(t) := f(t) + g(t)$

b) $\forall f \in \mathcal{X}, \forall \alpha \in \mathcal{F}$ define $\alpha \cdot f$ by

$\forall t \in D, (\alpha \cdot f)(t) := \alpha \cdot f(t)$

Fact: $(\mathcal{X}, \mathcal{F})$ is a vector space.

To show this, you check all 10 of the axioms! (Boring)

We'll do #8 as an example

 $\forall \alpha \in \mathcal{F} = \mathbb{R}, \forall f, g \in \mathcal{X}$ $\alpha(f+g) = \alpha f + \alpha g$

Method: Show LHS = RHS
(left hand side = right hand side)

Let $t \in D$

$$[\alpha(f+g)](t) := \alpha \cdot [f+g](t)$$
$$= \alpha \cdot [f(t) + g(t)]$$
$$= \alpha f(t) + \alpha g(t) \quad \text{LHS}$$

$$[\alpha f + \alpha g](t) := [\alpha f](t) + [\alpha g](t)$$
$$= \alpha f(t) + \alpha g(t) \quad \text{RHS}$$

RHS = LHS ✓ $\square$

More Vector Spaces

Let $\mathbb{F}$ be a field. Define

$\mathbb{F}^n :=$ set of n-tuples of elements of $\mathbb{F}$

$$:= \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} \;\middle|\; 1 \le i \le n,\ \alpha_i \in \mathbb{F} \right\}$$

Define vector addition by

$$\begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} + \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} := \begin{bmatrix} \alpha_1+\beta_1 \\ \vdots \\ \alpha_n+\beta_n \end{bmatrix}$$

and scalar times vector multiplication by

$$\alpha \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} := \begin{bmatrix} \alpha\beta_1 \\ \vdots \\ \alpha\beta_n \end{bmatrix}$$

$\therefore (\mathbb{F}^n, \mathbb{F})$ is a vector space.

"Matrices"

$\mathbb{F}^{n \times m} = \{ A = n \times m$ matrix with entries in $\mathbb{F} \}$

$(\mathbb{F}^{n\times m}, \mathbb{F})$ is a vector space with the obvious rules of

Vector addition and scalar times vector multiplication.

Notation $A \subset B$ means for A and B two sets ???

$A \subset B \Leftrightarrow (\forall a \in A \Rightarrow a \in B)$

$A = B \Leftrightarrow (A \subset B \text{ and } B \subset A)$

$\boxed{\text{We have no notation like } A \subseteq B \text{ in ROB 501}}$

Subspaces

Def. Let $(\mathcal{X}, \mathcal{F})$ be a vector space, and let $\mathcal{Y} \subset \mathcal{X}$.
~~Then $(\mathcal{Y}, \mathcal{F})$ is a subs~~
Then $\mathcal{Y}$ is a subspace of $\mathcal{X}$ if $(\mathcal{Y}, \mathcal{F})$ is a vector space when you use the rules of

vector addition and scalar times vector multiplication inherited from $(\mathcal{X}, \mathcal{F})$.