A family of barely expansive polynomials

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Abstract

This paper examines expansive integer polynomials, i.e. polynomials with integer coefficients whose roots lie outside of the unit circle. Our question regards the so-called expansivity gap, i.e. how close the roots of such polynomials can be to the unit circle. We recall a previous result of the author which gave $1/(c(n)H^{n-1})$ as a lower bound on the expansivity gap, where n is the degree, H is the height of the polynomial, and $c(n)$ is some n-dependent value. This paper defines a family of expansive integer polynomials, and proves that their expansivity gap is approx. $1/(2H^{n-1})$. This indicates how much the lower bound might be improved, and in fact, if the degree is constant, the lower bound is asymptotically sharp. The construction of the polynomials uses the so-called Motzkin numbers, and some linear algebra tools for expansive polynomials, developed by the author for his previous lower bound results.

1 Introduction

In this paper, $f(x)$ denotes an integer polynomial of degree n and coefficients $a_i \in \mathbb{Z}$:

$$f(x) = a_n x^n + \ldots + a_1 x + a_0$$

with $n \geq 1$ and $a_n \neq 0$.

We are interested in the following type of polynomials:

Definition 1.1. *The polynomial f is* expansive *if all of its roots lie outside the unit circle, i.e. for all roots x_i (either real or complex) we have: $|x_i| > 1$.*

The goal is to estimate the following quantity:

Definition 1.2. *The* expansivity gap *of an expansive polynomial, whose roots are $x_1, x_2, \ldots, x_n$, is:*

$$\varepsilon := \min_{i=1}^{n} |x_i| - 1.$$

In the estimations, we use the degree n of the polynomial and the following property:

Definition 1.3. *Denote by $H(f)$ the height of $f(x)$:*

$$H(f) := \max_{i=0}^{n} |a_i|.$$

In [2], the author proved, among others, the following:

Theorem 1.4. *The expansivity gap of an expansive integer polynomial $f(x)$ has the following lower bound:*

$$\varepsilon \geq \frac{1}{\binom{n}{2} n! H^{n-1} + \binom{n}{2} + 1}, \tag{1.1}$$

where $H := H(f)$ is the height of $f(x)$, and $n \geq 3$.

In this paper we prove the following:

Theorem 1.5. *For each $n \geq 2$ and for each sufficiently large H, there exists an expansive integer polynomial $f(x)$ of degree n and height H whose expansivity gap is:*

$$\varepsilon = \frac{1}{2H^{n-1}} + O\left(\frac{1}{H^n}\right).$$

This latter result shows that the lower bound is quite close, and its dependence on H is asymptotically sharp.

To prove this result, we use the determinant-based tool developed by the author and used to prove Theorem 1.4 in [2]. This involves the following determinants. Here the polynomial coefficients can be any real numbers, not neccessarily integers.

Definition 1.6. *[2, Def. 3.1] For a polynomial f of degree n, define the determinant $D_k^{\pm}(f)$ for each $1 \le k \le n$ and both signs $+$ or $-$ as a function of the coefficients of f as follows. The size of $D_k^{\pm}(f)$ is $k \times k$, and the element in the ith row and jth column is the following:*

$$d_{ij} = a_{j-i} \pm a_{i+j+n-k-1} \quad (1 \le i, j \le k),$$

with the convention that indices outside the allowed range indicate zero values, i.e. $a_i = 0$ for $i < 0$ and $i > n$.

For example for $n = 7$ and $k = 6$:

$$D_6^-(f) = \begin{vmatrix} a_0 - a_2 & a_1 - a_3 & a_2 - a_4 & a_3 - a_5 & a_4 - a_6 & a_5 - a_7 \\ -a_3 & a_0 - a_4 & a_1 - a_5 & a_2 - a_6 & a_3 - a_7 & a_4 \\ -a_4 & -a_5 & a_0 - a_6 & a_1 - a_7 & a_2 & a_3 \\ -a_5 & -a_6 & -a_7 & a_0 & a_1 & a_2 \\ -a_6 & -a_7 & & & a_0 & a_1 \\ -a_7 & & & & & a_0 \end{vmatrix}.$$

These determinants can be used to characterize the expansivity of real polynomials:

Theorem 1.7. *[2, Theorem 3.2] Assume that f has real coefficients, i.e. all $a_k \in \mathbb{R}$, and $a_0 > 0$. Then f is expansive if and only if:*

1. *for all k between $1 \le k \le n-1$ and for both signs: $D_k^{\pm}(f) > 0$, and*
2. *$f(\pm 1) > 0$.*

This is proven in [2] using the well-known Schur–Cohn-test [1].

The rest of this paper is devoted to the proof of Theorem 1.5. In Section 2.1, we define the Motzkin-triangle and state some of its simple properties. In Section 2.2, we use the Motzkin-triangle to define the family of polynomials for the main theorem, and we calculate the quantities of Theorem 1.7 for these polynomials. In Section 2.3, we use these quantities to finish the proof of Theorem 1.5.

2 Proof of Theorem 1.5

2.1 Motzkin-triangle and its inverse

For the construction of the polynomials, we need a certain table of values, the so-called Motzkin triangle [3]. Let $M_{n,k}$ $(n, k \ge 0)$ be the number of possible paths from the origin $(0,0)$ to the point (n,k) using the steps $(1,1), (1,0), (1,-1)$ and never going below the x-axis. We can take the inverse of $(M_{n,k})$, considered as an infinite matrix, and denote the values by $N_{n,k}$. The first few values of $M_{n,k}$ and $N_{n,k}$ are:

$M_{n,k}$	$k=0$	1	2	3	4	5	6	7
$n=0$	1	0	0	0	0	0	0	0
1	1	1	0	0	0	0	0	0
2	2	2	1	0	0	0	0	0
3	4	5	3	1	0	0	0	0
4	9	12	9	4	1	0	0	0
5	21	30	25	14	5	1	0	0
6	51	76	69	44	20	6	1	0
7	127	196	189	133	70	27	7	1

$N_{n,k}$	$k=0$	1	2	3	4	5	6	7
$n=0$	1	0	0	0	0	0	0	0
1	-1	1	0	0	0	0	0	0
2	0	-2	1	0	0	0	0	0
3	1	1	-3	1	0	0	0	0
4	-1	2	3	-4	1	0	0	0
5	0	-4	2	6	-5	1	0	0
6	1	2	-9	0	10	-6	1	0
7	-1	3	9	-15	-5	15	-7	1

It is also convenient to set $M_{-1,-1} := 1$ and $M_{-1,k} = M_{n,-1} := 0$ for $k, n \geq 0$, and the same for N. The $M_{n,k}$-values have a simple recurrence relation:

$$M_{n,k} = M_{n-1,k-1} + M_{n-1,k} + M_{n-1,k+1} \quad (n, k \geq 0). \tag{2.1}$$

We show that the $M_{n,k}$- and $N_{n,k}$-values have the following properties:

Lemma 2.1.

$$\sum_{k=i}^{j} M_{j,k} N_{k,i} = \delta_{i,j} \qquad (i, j \geq -1), \tag{2.2}$$

$$\sum_{k=i}^{j-l} M_{j,l+k} N_{k,i} = M_{j-i-1,l-1} \qquad (l \geq 0, -1 \leq i \leq j), \tag{2.3}$$

$$\sum_{k=0}^{\min(i,j)} M_{i,k} M_{j,k} = M_{i+j,0} \qquad (i, j \geq 0), \tag{2.4}$$

where $\delta_{i,j}$ is the Kronecker delta function, i.e. it is 1 if $i = j$ and 0 otherwise.

Proof. (2.2) just means that N is the inverse of M.

(2.3) is essentially the generalization of (2.2), being the same for $l = 0$ (if $i \leq j$). If $l \geq 1$, it can be easily proven by an induction on j starting from $j = i$, using the recurrence relation (2.1):

$$\begin{aligned}
&\sum_{k=i}^{j-l+1} M_{j+1,l+k} N_{k,i} = \\
&\quad = \sum_{k=i}^{j-l+1} M_{j,l+k-1} N_{k,i} + \sum_{k=i}^{j-l} M_{j,l+k} N_{k,i} + \sum_{k=i}^{j-l-1} M_{j,l+k+1} N_{k,i} = \\
&\quad = M_{j-i-1,l-2} + M_{j-i-1,l-1} + M_{j-i-1,l} = M_{j-i,l-1}.
\end{aligned}$$

(2.4) is proved by induction on i with constant $i+j$, assuming $i \leq j$. The case $i = 0$ is trivial, and for $i \to i+1$ (assuming $i+1 \leq j-1$), it goes as follows:

$$\sum_{k=0}^{i+1} M_{i+1,k} M_{j-1,k} =$$

$$= \sum_{k=0}^{i+1} M_{i,k-1} M_{j-1,k} + \sum_{k=0}^{i+1} M_{i,k} M_{j-1,k} + \sum_{k=0}^{i+1} M_{i,k+1} M_{j-1,k} =$$

$$= \sum_{k=0}^{i} M_{i,k} M_{j-1,k+1} + \sum_{k=0}^{i} M_{i,k} M_{j-1,k} + \sum_{k=0}^{i} M_{i,k} M_{j-1,k-1} = \sum_{k=0}^{i} M_{i,k} M_{j,k}.$$

$\square$

2.2 The construction

Now we define the family of polynomials for which Theorem 1.5 will be proved. For each $n \geq 2$ and $H \geq 1$, define the coefficients of $f(x)$ as follows:

$$\begin{aligned} a_0 &:= H, \\ a_1 &:= H - (M_{n-3,0} + 1), \\ a_2 &:= H - M_{n-2,0}, \\ a_i &:= -M_{n-2,i-2} \qquad\qquad (3 \leq i \leq n). \end{aligned} \tag{2.5}$$

For the first few degrees, these polynomials are:

$$\begin{array}{ll} n = 2: & (H-1)x^2 + (H-1)x + H \\ n = 3: & -x^3 + (H-1)x^2 + (H-2)x + H \\ n = 4: & -x^4 - 2x^3 + (H-2)x^2 + (H-2)x + H \\ n = 5: & -x^5 - 3x^4 - 5x^3 + (H-4)x^2 + (H-3)x + H \\ n = 6: & -x^6 - 4x^5 - 9x^4 - 12x^3 + (H-9)x^2 + (H-5)x + H \\ n = 7: & -x^7 - 5x^6 - 14x^5 - 25x^4 - 30x^3 + (H-21)x^2 + (H-10)x + H \end{array}$$

An interesting side-note is that the constant 1 in the definition of a_1 can sometimes be replaced by -1, more precisely by an ω such that $\omega^{n-2} = 1$, i.e. $a_1 := H - (M_{n-3,0} + \omega)$. For odd n, it remains only $\omega = +1$, but for even n, it can be either $+1$ or -1, moreover if $n = 2$, it can be any $\omega \in \mathbb{Z}$.

First we need to ensure that these polynomials are indeed expansive (at least for sufficiently large H). For this, we use Theorem 1.7 on these polynomials, and consider the quantities $f(\pm 1)$ and $D_k^{\pm}(f)$ appearing in it. The crux in our proof is the following lemma about these quantities, especially the third part.

Lemma 2.2.

1. *$f(\pm 1) > 0$ for sufficiently large H parameters.*

2. *For any $1 \le k \le n-1$ and any sign in $\{+,-\}$, $D_k^{\pm}(f) > 0$ (as in Def. 1.6) for sufficiently large H.*

3. *$D_{n-1}^{-}(f) = 1$ for all $H \in \mathbb{N}^+$.*

Proof. The first statement of the lemma is trivial.

Denote the $k \times k$ components of $D_k^{\pm}(f)$ by $d_{i,j}$, which is, by Def. 1.6 and (2.5):

$$d_{i,j} = a_{j-i} \pm a_{i+j+n-k-1} =$$
$$= \mp M_{n-2,i+j+n-k-3} + \begin{cases} 0 & (j < i), \\ H \pm H & (j = i = 1 \wedge k = n-1), \\ H & (j = i,\ \text{otherwise}), \\ H - M_{n-3,0} - \omega & (j = i+1), \\ H - M_{n-2,0} & (j = i+2), \\ -M_{n-2,j-i-2} & (j \ge i+3). \end{cases} \tag{2.6}$$

For example for $n = 7$:

$$D_6^{-}(f) = \begin{vmatrix} 21 & H+20 & H+4 & -16 & -20 & -13 \\ 30 & H+25 & H+4 & H-16 & -29 & -25 \\ 25 & 14 & H+5 & H-9 & H-21 & -30 \\ 14 & 5 & 1 & H & H-10 & H-21 \\ 5 & 1 & & & H & H-10 \\ 1 & & & & & H \end{vmatrix}.$$

These determinants are polynomials in H of degree at most k. H occurs only in the main diagonal and the two diagonals above it. All occurences have 1 as coefficient except on the top left corner, where it is 1 if $k < n-1$, 2 if $k = n-1$ with the $(+)$ sign, and 0 if $k = n-1$ with the $(-)$ sign. Therefore, in the former two cases, the leading term of the polynomial is H^k or $2H^k$, which proves that the determinant is positive for sufficiently large H. The remaining of this proof will show that in the third case, the determinant is the constant polynomial 1.

From now on, let $n \ge 2$, and let D be the matrix of the determinant $D_{n-1}^{-}(f)$. The latter will be multiplied by matrices with determinant 1, bringing it to a simplified form where it is obvious that its determinant is also 1.

In the first step, we remove the off-diagonal H values from D. This can be achieved by row-operations from the bottom to the top, which can be equivalently described as a product $D = ED'$, where D' is the resulting matrix having only diagonal H values, and E is an appropriate upper triangular matrix representing the inverse transformation, more specifically a matrix containing 1's exactly in the three diagonals where D has H's, and otherwise 0. For example ($n = 6$):

$$\begin{bmatrix} 9 & H+7 & H & -8 & -8 \\ 12 & H+9 & H-1 & H-8 & -12 \\ 9 & 4 & H+1 & H-5 & H-9 \\ 4 & 1 & 0 & H & H-5 \\ 1 & 0 & 0 & 0 & H \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 0 & -1 \\ 4 & H+5 & -2 & -3 & -3 \\ 5 & 3 & H+1 & -5 & -4 \\ 3 & 1 & 0 & H & -5 \\ 1 & 0 & 0 & 0 & H \end{bmatrix}.$$

Using the recurrence relation (2.1) of $M_{n,k}$, it is straightforward to prove that, as the example suggests, the resulting D' matrix has a similar, but simplier structure than D:

$$d'_{i,j} = M_{n-3,i+j-3} + \begin{cases} 0 & (j < i), \\ 0 & (j = i = 1), \\ H & (j = i \geq 2), \\ -M_{n-3,j-i-1} - \omega N_{j-i-1,0} & (j > i). \end{cases} \tag{2.7}$$

In the second step of the determinant transformation, we calculate $D'' := M^{-1} D' M$, where M is a matrix from the transposed Motzkin triangle starting from -1, more precisely whose elements are $m_{i,j} = M_{j-2,i-2}$ and whose invert's elements are $n_{i,j} = N_{j-2,i-2}$. For example ($n = 6$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 & 0 & -1 \\ 4 & H+5 & -2 & -3 & -3 \\ 5 & 3 & H+1 & -5 & -4 \\ 3 & 1 & 0 & H & -5 \\ 1 & 0 & 0 & 0 & H \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & H+2 & -1 & 0 & 0 \\ 0 & 1 & H+2 & -1 & 0 \\ 0 & 1 & 1 & H+2 & -1 \\ 1 & 0 & 0 & 0 & H \end{bmatrix}.$$

To calculate this product, we split D' into $A - B + C$ where A has the top-left triangular pattern, B has the top-right triangular pattern of numbers, and C has all the H. More precisely (cf. (2.7)):

$$\begin{aligned} a_{i,j} &= M_{n-3,i+j-3}, \\ b_{i,j} &= M_{n-3,j-i-1} + \omega N_{j-i-1,0} \quad (j > i), \\ c_{i,j} &= \delta_{i,j} H \quad (i \geq 2). \end{aligned}$$

Now calculating termwise:

$$(M^{-1}AM)_{i,j} = \sum_{l=1}^{\min(j,n-i)} \sum_{k=i}^{n-l} N_{k-2,i-2} M_{n-3,l+k-3} M_{j-2,l-2} =$$

$$\overset{(2.3)}{=} \sum_{l=1}^{\min(j,n-i)} M_{n-i-2,l-2} M_{j-2,l-2} \overset{(2.4)}{=} \begin{cases} \delta_{i,n-1} & (j=1), \\ \delta_{j,1} & (i=n-1), \\ M_{n-i+j-4,0} & (\text{otherwise}). \end{cases}$$

$$\begin{aligned} (M^{-1}BM)_{i,j} &= \sum_{k=1}^{n-1} \sum_{l=k+1}^{n-1} N_{k-2,i-2}(M_{n-3,l-k-1} + \omega N_{l-k-1,0}) M_{j-2,l-2} = \\ &= \sum_{m=1}^{n-2} (M_{n-3,m-1} + \omega N_{m-1,0}) \sum_{k=1}^{n-m-1} M_{j-2,m+k-2} N_{k-2,i-2} = \\ &\overset{(2.3)}{=} \sum_{m=1}^{j-i} (M_{n-3,m-1} + \omega N_{m-1,0}) M_{j-i-1,m-1} = \\ &\overset{(2.4),(2.2)}{=} \begin{cases} M_{n-i+j-4,0} + \omega \delta_{i+1,j} & (j \geq i+1), \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

$$(M^{-1}CM)_{i,j} = \sum_{k=2}^{n-1} N_{k-2,i-2} H M_{j-2,k-2} \overset{(2.2)}{=} H \begin{cases} \delta_{i,j} & (i \geq 2), \\ 0 & (\text{otherwise}). \end{cases}$$

Putting these together:

$$d''_{i,j} = (M^{-1}D'M)_{i,j} = \begin{cases} \delta_{i,n-1} & (j=1), \\ H\delta_{j,n-1} & (i=n-1 \wedge j \geq 2), \\ M_{n-i+j-4,0} + H\delta_{i,j} & (2 \leq j \leq i \leq n-2), \\ -\omega \delta_{i+1,j} & (j \geq i+1). \end{cases}$$

It is now easy to see that $\det D'' = \omega^{n-2} = 1$, which finally proves Lemma 2.2. □

2.3 Finishing the proof

Theorem 1.5 will be proved for the family of polynomials $f(x)$ in (2.5). It is already clear that they are expansive for sufficiently large H, since Lemma 2.2 proved that all quantities of $f(x)$ required to be positive by Theorem 1.7 are indeed positive. To find the smallest root of $f(x)$, we need to examine the same quantities but for $f_\varepsilon(x) := f\left((1+\varepsilon)x\right)$, and the smallest ε for which one of these quantities becomes non-positive will give the size of the smallest root as $1+\varepsilon$.

Let $Q_0(H)$ be any of these quantities in Theorem 1.7 (i.e. $D_k^{\pm}(f)$ or $f(\pm 1)$), and let $Q(H,\varepsilon)$ be the same for $f_\varepsilon(x)$, i.e. replace the coefficients $a_0, a_1, \ldots, a_n$ by $a_0, a_1(1+\varepsilon), \ldots, a_n(1+\varepsilon)^n$, respectively. It is a bivariate polynomial in H and ε, and let $d := \deg_H Q$ and $N := \deg_\varepsilon Q$. It can be expanded by ε:

$$Q(H,\varepsilon) = Q_0(H) + Q_1(H)\varepsilon + Q_2(H)\varepsilon^2 + \ldots + Q_N(H)\varepsilon^N,$$

where all $\deg Q_i \le d$.

We need to find the smallest ε for each sufficiently large H such that $Q(H,\varepsilon) = 0$ for any of the examined Q polynomials. First we prove that such ε exists and $\varepsilon = O(1/H)$.

For any $\varepsilon = O(1/H)$, the expansion of $Q(H,\varepsilon)$ can be written as follows, using that $Q_k(H) = O(H^d)$:

$$Q(H,\varepsilon) = Q_0(H) + \varepsilon\left(Q_1(H) + O(H^{d-1})\right). \tag{2.8}$$

For those Q polynomials where $\deg Q_0 = d$, i.e. $Q_0(H) = cH^d + O(H^{d-1})$ with $c > 0$ (not negative because of Lemma 2.2), the expansion (2.8) becomes:

$$Q(H,\varepsilon) = cH^d + O(H^{d-1}),$$

which is positive for sufficiently large H. Using the results from the beginning of the proof of Lemma 2.2, this holds for:

1. $Q := f(\pm 1)$, where $d = 1$ and $c = 2 \pm 1$;
2. for all $Q := D_k^{\pm}(f)$ with $k < n-1$, where $d = k$ and $c = 1$;
3. and for $Q := D_{n-1}^{+}(f)$, where $d = n-1$ and $c = 2$.

The only remaining case is $Q := D_{n-1}^{-}(f)$, where $Q_0(H) = 1$. We prove that here the next polynomial has full degree, i.e. $\deg Q_1 = n-1$. Examine the coefficient of H^{n-1} in the expansion of $Q(H,\varepsilon)$ by H. The structure of $Q(H,\varepsilon)$ is similar to (2.6), but the a_k coefficients are replaced by $a_k(1+\varepsilon)^k$. It is still true that this is an $(n-1)\times(n-1)$ determinant where H appears only in the upper triangular elements, and only as linear terms, so the term H^{n-1} can come only from the product of the main diagonal entries. They are $q_{k,k} = a_0 - a_{2k}(1+\varepsilon)^{2k}$, which is $H + M_{n-2,2k-2}(1+\varepsilon)^{2k}$ for $k \ge 2$ and

$q_{1,1} = -H\varepsilon(\varepsilon+2) + M_{n-2,0}(1+\varepsilon)^2$, so the coefficient of H^{n-1} is $-\varepsilon(\varepsilon+2)$. Therefore, the leading term of Q_1 is $-2H^{n-1}$.

Substituting this and $Q_0(H) = 1$ into (2.8) gives:

$$Q(H,\varepsilon) = 1 + \varepsilon\left(-2H^{n-1} + O(H^{n-2})\right). \tag{2.9}$$

Now if $\varepsilon = 1/H$ exactly, then:

$$Q(H,\varepsilon) = 1 - 2H^{n-2} + O(H^{n-3}),$$

which is negative for sufficiently large H. Since $Q(H,0) > 0$, there must be a zero for some $\varepsilon = O(1/H)$. We can find its order by making (2.9) equal to zero and rearranging:

$$\varepsilon = \frac{1}{2H^{n-1} + O(H^{n-2})} = \frac{1}{2H^{n-1}} + O\left(\frac{1}{H^n}\right).$$

Since any other Q polynomials are positive for $\varepsilon = O(1/H)$, this is the first ε for which the conditions of Theorem 1.7 fail. This finishes the proof of Theorem 1.5.

References

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