Math 6450: Final Report

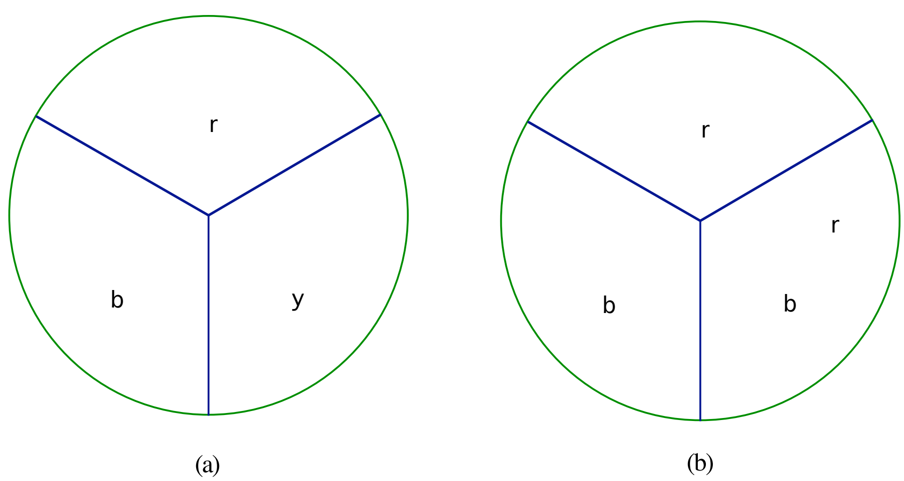
Group #2 Study Project
Nicole Fonger, Josh Goss, Ben Phillips, Chris Segroves
Wednesday June 24, 2009

1 Introduction

The following problem provided the original impetus for the study project that this author team pursued.

Problem: The three regions of the map in Figure 1(a) can be distinguished using three colors. These three regions can also be distinguished using the two colors shown in Figure 1(b). The idea here is that we might be able to color a region R with two colors (say alternating the colors in stripes) so that any region adjacent to R is not colored with these two colors. Here we understand neighboring regions to mean two regions with a boundary *line* in common, not simply a single point in common. Thus every map corresponds to a planar graph.

Figure 1: Two possible colorings



The idea of what came to be called a stripe-coloring was a natural consequence of the statement of this problem. We began by considering some potential practical applications.

First, the idea of a stripe-coloring could, perhaps, replace a more conventional coloring. Doing this may have the advantage of saving ink, making for a more economical situation. Figure 1 demonstrates this ink-saving idea because in map (a) three colors are used where in map (b), only two colors are used. Second, if a planar graph is considered to represent countries on a map, imagine a scenario where two countries are at war over a shared middle country. For example, in Figure 1(b), countries r and b are in dispute over country $\{r, b\}$. In this case, the stripe-colored country gives information about the colors of its neighbors.

The ink-saving and countries at war scenarios, along with our emerging ideas led us to consider several possible definitions and corresponding consequences.

2 Results

The results section is divided into 3 subsections: Stripe Colorings, 2-tone Stripe Colorings, and t-tone Colorings. The purpose for this division is to demonstrate the progression of our mathematical findings that developed over the course of this study project.

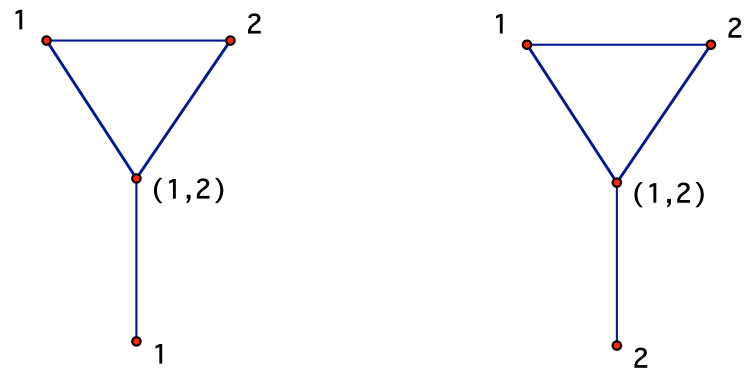
2.1 Stripe Colorings

Definition 1 *Let G be a graph of order n and let $\overline{n} = \{1, 2, ..., n\}$. A* ***proper stripe coloring*** *of G is a function $f : V(G) \to \mathcal{P}(\overline{n})$ such that $f(v) \neq f(u)$ if $uv \in E(G)$ and if $|f(v)| > 1$ then $f(v) = \bigcup f(u)$, where $uv \in E(G)$. If $|f(v)| > 1$ for then v is called a* ***striped vertex*** *and its label $f(v)$ is referred to as a* ***stripe****.*

As an example of two proper stripe colorings is given in figure 2.

Definition 2 *For a graph G of order n a proper stripe coloring f is called* ***minimum*** *if $|\bigcup f(v)| \leq |\bigcup g(v)|$ where g is any stripe coloring of G. The minimum such cardinality over all proper stripe colorings of G is called the* ***stripe chromatic number****, and will be denoted $\sigma(G)$.*

Figure 2: Two proper stripe-colorings of the Alavi graph



Definition 3 *If f is proper stripe coloring of a graph G with $t = max\{|f(v)| : v \in V(G)\}$, then f is called an **t-tone stripe coloring**.*

These definitions seemed most consistent with our initial ideas about the given problem, but quickly lead to a somewhat undesirable result.

Lemma 4 *Given a proper stripe coloring f of a graph G, no two adjacent vertices of G can both be colored with a stripe.*

Proof: Let u and v be adjacent vertices of a graph G. Suppose to the contrary that $f(u)$ and $f(v)$ are both stripes. By definition, $f(v) = \bigcup f(w)$ for all vertices w adjacent to v in G. So $f(u) \subseteq f(v)$. Likewise, $f(u) = \bigcup f(x)$ for all vertices x adjacent to u in G so $f(v) \subseteq f(u)$. Now we have $f(v) = f(u)$, a contradiction. $\square$

This idea, of not allowing adjacent stripes, seemed far too restrictive. It actually forced $\sigma(G)$ to almost always be an extremely trivial linear function of $\chi(G)$. However, for completeness we present some limited results.

Proposition 5 $\sigma(K_n) = n - 1$.

Proof: Let $G = K_n$. If $V(G) = \{v_1, v_2, \cdots, v_n\}$ we shall define a proper $(n-1)$-tone stripe coloring as follows:

$$f(v_i) = \begin{cases} i & \text{if } 1 \leq i \leq n-1 \\ \{1, 2, \cdots, n-1\} & \text{if } i = n \end{cases}.$$

Thus $\sigma(G) \leq n-1$. As no two adjacent vertices can be striped, it follows that any proper stripe coloring of G has at most one stripe, and thus must have at least $n-1$ different labels of cardinality 1. Thus $\sigma(G) \geq n-1$, so the result holds. $\square$

Proposition 6 $\chi(G) \geq \sigma(G)$.

Proof: Note that any proper coloring is also a proper stripe coloring. So $\sigma(G)$ certainly isn't larger than $\chi(G)$. $\square$

Note that when G is a bipartite graph, $\chi(G)$=$\sigma(G)$, thus this bound is sharp.

Proposition 7 $\sigma(C_n) = 2$ *for every* $n \geq 3$.

Proof: For n even, C_n is bipartite. For n odd, $\chi(C_n) = 3$, and can be colored such that one color class, C, has cardinality 1. Replacing the color used for C with a stripe yields a proper 2-tone stripe coloring, thus $\sigma(C_n) = 2$. $\square$

2.2 2-tone Stripe Colorings

As it quickly became apparent that our initial definition was undesirable, a less restrictive definition was suggested by Chartrand and Zhang. This definition too turned out to be somewhat undesirable.

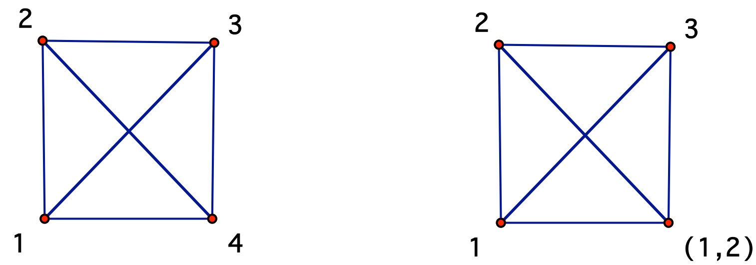
Definition 8 *Let* G *be a graph of order* n *and let* $\overline{n} = \{1, 2, ..., n\}$. *A* ***2-tone stripe coloring*** *of* G *is a function* $f : V(G) \to \mathcal{P}(\overline{n})$ *such that* $1 \leq |f(v)| \leq 2$ *and* $f(u) \neq f(v)$ *if* $uv \in E(G)$. *Denote the minimum* ***2-tone chromatic number*** *of* G *by* $\sigma_2(G)$.

Observe that $\sigma_2 \leq \chi(G)$. To illustrate this idea, consider the following example that shows strict inequality for this observation.

EXAMPLE $G = K_4$. As illustrated by the colorings in Figure 3, $\sigma_2(G) = 3$ and $\chi(G) = 4$.

This second definition was actually far too close to the standard idea of coloring to be very interesting. Once again, we were able to obtain a result which expressed $\sigma_2(G)$ as a function of $\chi(G)$.

Figure 3: A 1-tone stripe-coloring of K_4 and a 2-tone stripe-coloring of K_4



Theorem 9 *For every graph G with $\chi(G) = k$, $\sigma_2(G) = \left\lceil \frac{\sqrt{8k+1}-1}{2} \right\rceil$.*

Proof: Let G be a graph with $\chi(G) = k$. Let $\sigma_2(G) = s$. Then we have that s is the smallest positive integer such that $\binom{s+1}{2} \geq k$. This inequality is logically equivalent to the quadratic inequality $s^2 + s - 2k \geq 0$. From the quadratic formula, we see that the non-negative root of the governing quadratic is

$$\frac{\sqrt{8k+1}-1}{2}.$$

By the minimality of the integer $s = \sigma_2(G)$, we obtain the desired result. □

To illustrate the preceding result, consider the following example:

Example Let $G = K_5$. Since $\chi(G) = 5$, we conclude that $s = 3$ is the minimal integer such that $\binom{s+1}{2} \geq 5$. Thus $\sigma_2(K_5) = 3$.

2.3 t-tone Colorings

After two undesirable definitions, a third and final definition was considered, this time yielding quite interesting consequences. This definition was originally suggested by Chartrand.

Definition 10 *Let G be a graph, and let $\overline{k} = \{1, 2, \cdots, k\}$, where k is a positive integer. A function $f : V(G) \to \mathcal{P}(\overline{k})$ is called a **proper t-tone k-coloring** of G if*

1. *$|f(v)| = t$ for each vertex v of G*

2. *if* $d(u,v) = i$ *then* $|f(u) \cap f(v)| \le i - 1$ *for* $1 \le i \le \mathrm{Diam}(G)$

The t***-tone chromatic number*** *of* G*, denoted by* $\tau_t(G)$*, is the smallest positive integer* k *for which* G *has a proper* t*-tone* k*-coloring.*

First, we present a helpful, but unsurprising result.

Theorem 11 *If* H *is a subgraph of* G*, then* $\tau_t(H) \le \tau_t(G)$*.*

Proof: Let f be a proper t-tone $\tau_t(G)$-coloring of G. The restriction of f to H must then be a proper t-tone coloring of H. Thus $\tau_t(H)$ is no more than $\tau_t(G)$. □

Much of the work to be discussed is for the special case $t = 2$, that is, a 2-tone k-coloring.

Theorem 12 *Let* $G = K_{a_1,a_2,\cdots,a_k}$ *denote the complete* k*-partite graph, with partite sets of orders* $a_1, a_2, \cdots a_k$*. Then*

$$\tau_2(G) = \sum_{i=1}^{k} \left\lceil \frac{1+\sqrt{8a_i+1}}{2} \right\rceil .$$

Proof: Let $G = K_{a_1,\ldots,a_k}$ and let $\chi(G) = k$. Let $C_1, \ldots, C_k$ denote the color classes of G. Let $|C_i| = a_i$, for $1 \le i \le k$. Now let f be a proper 2-tone $\tau_2(G)$-coloring of G. Notice that for any pair of distinct vertices $u, v \in C_i$, We have $d(u,v) = 2$. Hence $|f(v) \cap f(u)| \le 1$. So the number of colors required for the i^{th} partite set is s_i such that $\binom{s_i}{2} \ge a_i$. By algebraic computations similar in nature to those demonstrated in proof of Theorem 9, and the fact that s_i is positive, we get that $s_i = \left\lceil \frac{\sqrt{8a_i+1}+1}{2} \right\rceil$. Furthermore, since every element in C_i is adjacent to every element in C_j for all $i \ne j$ with $1 \le i, j \le k$, they cannot have any colors in common. Therefore the 2-tone chromatic number of G is the sum of all such s_i, as desired. □

Corollary 13 *Let* $G = K_{k(a)}$ *be the complete multi-partite graph with* k *partite sets, each of order* a*. Then*

$$\tau_2(G) = k \left\lceil \frac{\sqrt{8a+1}+1}{2} \right\rceil .$$

Proof: This follows easily from the previous theorem by setting $a_i = a$ for each relevant i. $\square$

This theorem and its corollary can be used to give an upper bound on $\tau_2(G)$.

Corollary 14 *Let G be a connected graph with $\chi(G) = k$. Let f be a traditional k-coloring of G, with color classes $C_1, \cdots, C_k$. Set*

$$a = \max_f \{|C_i|\}.$$

That is, a is the largest size of any color class over all proper k-colorings of G. Then

$$\tau_2(G) \leq k \left\lceil \frac{\sqrt{8a+1}+1}{2} \right\rceil.$$

Proof: It is easily seen that G is a subgraph of $K_{k(a)}$ and so the result follows from the previous corollary. $\square$

Even though the chromatic number is often difficult to calculate, many upper bounds are well known. As such, we present the following upper bound. While the bound is generally worse than that given above, it is easier to calculate the bound, and so may occasionally be of use.

Corollary 15 *If G is a graph, then*

$$\tau_2(G) \leq \left\lfloor \frac{n+\omega(G)}{2} \right\rfloor \left\lceil \frac{\sqrt{8\alpha(G)+1}+1}{2} \right\rceil.$$

Corollary 16 *If G is a star, $K_{1,a}$, then*

$$\tau_2(G) = \left\lceil \frac{\sqrt{8a+1}+5}{2} \right\rceil.$$

Notice that the above corollary shows that even for planar graphs, $\tau_2(G)$ can become arbitrarily large. It also gives us a somewhat useful lower bound on $\tau_2(G)$, which can be stated in terms of the maximum degree of G, denoted $\Delta(G)$.

Theorem 17 *Let G be a graph and let $\Delta(G) = d$. Then*

$$\tau_2(G) \geq \left\lceil \frac{\sqrt{8d+1}+5}{2} \right\rceil.$$

Proof: If $\Delta(G) = d$ then G has a $K_{1,d}$ subgraph. Any proper 2-tone coloring of G must also be a proper 2-tone coloring of this star subgraph, and so the bound is obtained. □

This bound is sharp, as the specified value is obtained when G is a tree.

Theorem 18 *If G is a tree with $\Delta(G) = d$ then*

$$\tau_2(G) = \left\lceil \frac{\sqrt{8d+1}+5}{2} \right\rceil.$$

Proof: We shall first prove the result for trees T with degree set $D = \{1, d\}$, where $d \geq 2$ is a positive integer such that $\frac{\sqrt{8d+1}+5}{2}$ is an integer. Such integers d clearly exist, and can be shown to always be triangular numbers. Let $\mathcal{F}$ be the family of all such trees. We shall proceed by induction on l, the number of vertices of degree d in T. If $l = 1$ then $T = K_{1,d}$ and so the result holds by corollary 16. Then, assume the result holds for all trees with degree set D and $l < k$, where $k \geq 2$. Let T be a tree with $l = k$ and degree set D. By the fundamental theorem of graph theory, there exists a vertex $v \in V(T)$ with $\deg(v) = d$ and which is adjacent to $d-1$ end vertices. Let S be the set of end vertices which are adjacent to v. Then the tree $T - S$ satisfies the inductive hypothesis, and so there exists a proper 2-tone coloring of $T - S$ which uses $C = \frac{\sqrt{8d+1}+5}{2}$ colors. Now, the vertices of S must all receive different labels, and each pair of labels may share at most one color. Furthermore, none of the vertices of S can be colored with either of the colors used for v, nor can they be the same label as the neighbor of v in $T - S$. Thus, to color S we need $\binom{C-2}{2} - 1$ different labels. But, we have that

$$\begin{aligned}
\binom{C-2}{2} - 1 &= \left(\frac{1}{2}\right)\left(\frac{1+\sqrt{8d+1}}{2}\right)\left(\frac{-1+\sqrt{8d+1}}{2}\right) - 1 \\
&= \frac{(8d+1)-1}{8} - 1 \\
&= d-1
\end{aligned}$$

and so we can color every vertex of S with a label using the C colors used to color $T-S$. Thus we find that the result hods for trees with degree sequence D, for all such integers d.

Finally, note that any tree can be found as a subgraph of a tree in $\mathcal{F}$, by simply adding end vertices as needed. Thus the result holds for trees in general as well. □

The bound in theorem 17 can be generalized to t-tone colorings as well. For $t \geq 2$, let $\Delta_t(G)$ denote the maximum number of vertices in G which are a positive distance of at most $\lfloor \frac{t}{2} \rfloor$ away from any particular vertex of G. $\Delta_t(G)$ can be viewed as a sort of generalization of the maximum degree, so for completeness we set $\Delta_1(G) = \Delta(G)$. The proof of the following theorem is similar in idea to that of theorem 17.

Theorem 19 *For a graph G, $\tau_t(G)$ must be large enough to ensure that*

$$\binom{\tau_t(G)}{n} \geq \Delta_t(G).$$

A similar lower bound can be established if the diameter of G is known.

Theorem 20 *Let G be a graph with diameter d and order n. Then*

$$\binom{\tau_d(G)}{t} \geq n.$$

Proof: If the diameter of G is d, then any proper d-tone coloring must assign a different label to every vertex of G. Thus, we must at least have at least n different labels, as each vertex is within distance d of each other vertex. □

Just as with traditional coloring, complete graphs are easily dealt with.

Proposition 21 *If $G = K_n$ then $\tau_t(G) = tn$.*

Proof: Notice that if $G = K_n$ we have that every pair of vertices of G are adjacent. Thus every vertex must have a different t-element label, and none of these labels may share an element. Hence we need a minimum of $2t$ total elements for a proper 2-tone coloring of G. If $V(G) = \{v_1, v_2, \cdots, v_n\}$ then

$f(v_i) = \{ti-(t-1), ti-(t-2), \cdots, ti\}$ is a proper t-tone coloring of G using tn colors, and thus the result follows. □

As with regular chromatic numbers, complete graphs are extremal cases, providing A reasonably good lower bound, as well as a worst case upper bound.

Theorem 22 *Let G be a graph of order n. Then*

$$t\omega(G) \leq \tau_t(G) \leq tn.$$

These bounds are sharp when $G = K_n$.

Proof: Clearly, G is a subgraph of K_n, and thus $\tau_t(G) \leq \tau_t(K_n) = tn$. Similarly, if the largest complete subgraph G contains has order $\omega(G)$, then $t\omega(G) = \tau_t(K_{\omega(G)}) \leq \tau_t(G)$. These bounds are easily seen to be sharp for completes graphs. □

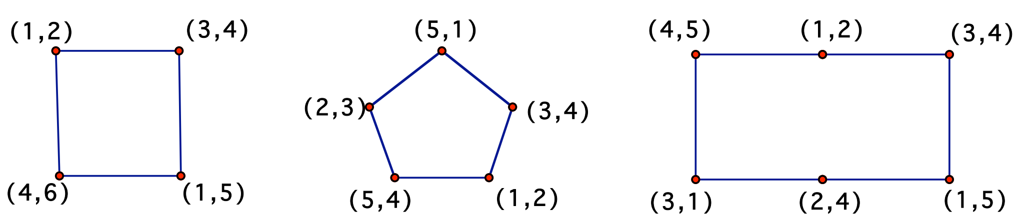
Some insight into t-tone colorings in general can be obtained by looking at specific families of graphs.

Proposition 23

$$\tau_2(C_n) = \begin{cases} 6 & \text{if } n = 3, 4, 7 \\ 5 & \text{otherwise} \end{cases}$$

An illustration of cycle colorings can be seen in figure 4, while illustrations of proposition 23 can be seen in figures 5, 6, and 7.

Figure 4: A 2-tone 6-coloring of C_4, a 2-tone 5-coloring of C_5, and a 2-tone 5-coloring of C_6



Proposition 24 *Let $G = C_n \times K_2$, $n \geq 3$. Then $\tau_2(G) = 6$.*

Figure 5: A 2-tone 5-coloring of $K_2 \times C_5$

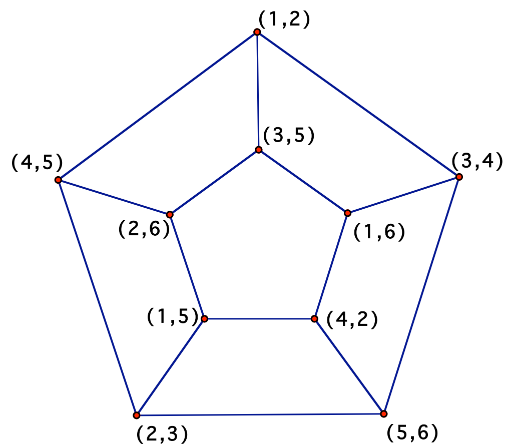
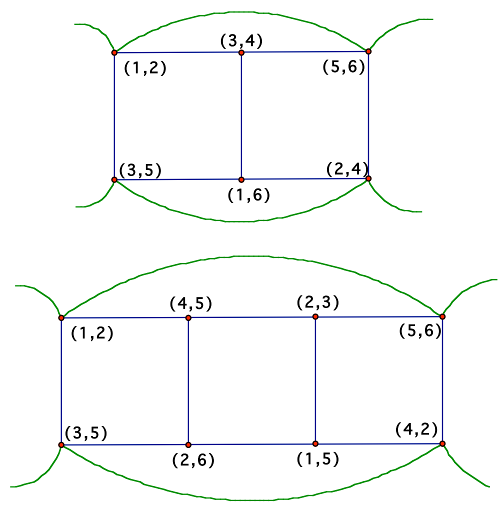
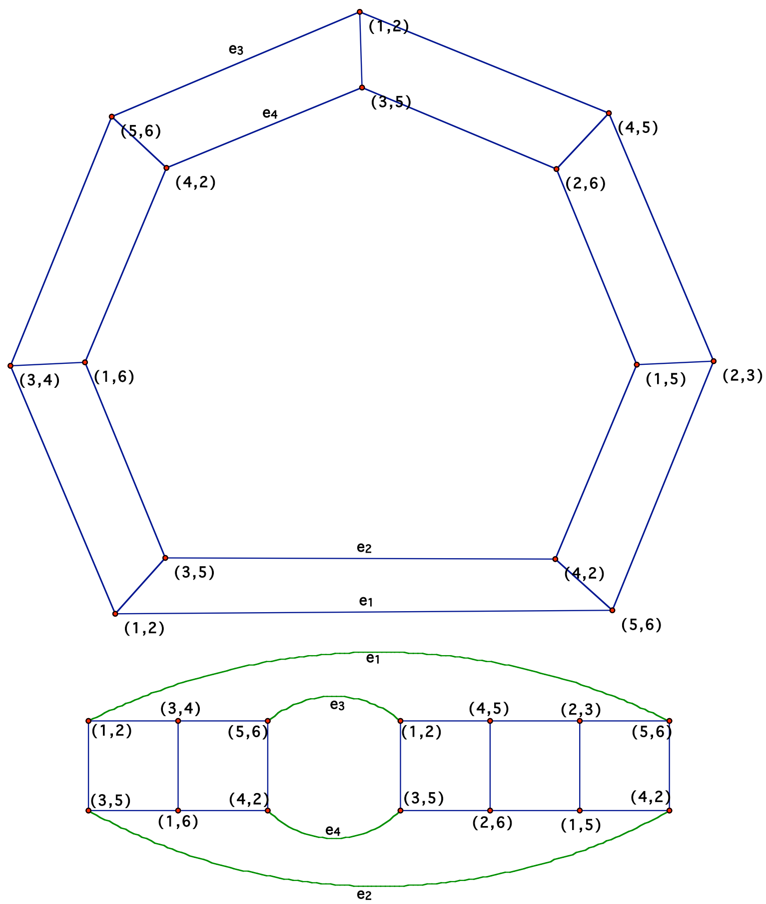


Figure 6: 2-tone colorings for $K_2 \times C_{3t}$ and $K_2 \times C_{4t}$



Recall that the kth power of a graph G, denoted G^k, is the graph with $V(G^k) = V(G)$ and $uv \in E(G^k)$ if and only if $d_G(u,v) \leq k$. It would seem as though $\chi(G^2)$ is in some way related to $\tau_2(G)$. This relationship is somewhat mysterious. For example, for some graphs, $\chi(G^2)$ is a lower bound on $\tau_2(G)$, while for other graphs, such as the Petersen graph P, we have that $\chi(P^2) \geq \tau_2(P)$, owning to the fact that P has diameter 2. However, we were able to find one clear relationship.

Figure 7: A 2-tone coloring for $K_2 \times C_7$, built from smaller cycles



Theorem 25 *Let G be a graph. Then*

$$\tau_2(G) \leq \chi(G) + \chi(G^2).$$

Proof: Let f_1 be a chromatic coloring of G using the colors $1, 2, \cdots, \chi(G)$, and f_2 a chromatic coloring of G^2 using the colors $\chi(G)+1, \cdots, \chi(G)+\chi(G^2)$. Then we define a 2-tone coloring f of G by setting $f(v) = \{f_1(v), f_2(v)\}$. Then if u, v are adjacent in G we have that $f_1(v), f_2(v), f_1(u)$ and $f_2(u)$ are four different colors. Similarly, if $d(u,v) = 2$ for vertices u and v of G, then $f_2(u) \neq f_2(v)$ and so the labels on u and v assigned by f are not the same. Thus f is a proper 2-tone coloring of G using $\chi(G) + \chi(G^2)$ colors. $\square$

A similar result holds for t-tone colorings: than an upper bound on $\tau_t(G)$ is

$$\tau_t(G) \leq \sum_{i=1}^{t} \chi(G^i).$$

The proof of this is nearly identical to that of theorem 25, and as such has been omitted. A similar, and generally better bound can be obtained recursively.

Theorem 26 *Let G be a graph and $t > 1$ integer. Then $\tau_t(G) \leq \tau_{t-1}(G) + \chi(G^t)$.*

Proof: Let f be a minimum proper $(t-1)$-tone coloring of G using the colors $1, 2, \cdots, \tau_{t-1}(G)$ and let g be a chromatic coloring of G^t, using the colors $\tau_{t-1}(G) + 1, \cdots \tau_{t-1}(G) + \chi(G^t)$. Then we form a proper t-tone coloring, c of G by setting $c(v) = f(v) \cup \{g(v)\}$. A similar argument to the one employed in theorem 25 shows this to be a proper t-tone coloring of G, using $\tau_{t-1} + \chi(G^t)$ colors. Thus the result holds. $\square$

It should be noted that finding the chromatic number of the tth power of a graph is in general much harder than finding $\chi(G)$. However, the two previous results can give a nice lower bound on $\chi(G^t)$ by first calculating $\tau_t(G)$ and either $\chi(G^i)$ for all $1 \leq i < t$ or $\tau_{t-1}(G)$. This could be potentially useful when dealing with certain classes of graphs, such as bipartite graphs. However, even for bipartite graphs, such as paths, the t-tone chromatic number is difficult to determine.

Conjecture 27 *Let P_n be the path on n vertices, and assume that $n \geq t+1$. Then*

$$\tau_t(P_n) = \frac{(j+2)(6t - j(j+1))}{6}$$

where $j = max\{l \ s.t. \ 1 + ... + l \leq n\}$

The conjecture has been verified for the cases of $t = 1, 2, 3$ and 4. However, a proof of this has not yet been obtained.

3 2-tone Colorings for Hypercubes

Recall the n-cube graph, which can be defined recursively:

$$Q_1 = K_2, \quad Q_n = Q_{n-1} \times K_2 \text{ for } n \geq 2.$$

While $\tau_2(Q_n)$ is known for various cases, in general it is not. It is known, for example, that $\tau_2(Q_1) = 2$, $\tau_2(Q_2) = \tau_2(Q_3) = 6$ and that $7 \leq \tau_2(Q_4) \leq 8$. However, by using the language of binary codes, we were able to establish an upper bound.

Proposition 28 *Let $G = Q_n$. Then $\tau_2(G) \leq 2^{n-1} + 2$.*

Proof: Describe the vertices of Q_n as binary code words of length n, where the distance between vertices can be tracked by the hamming distance between their corresponding code words. We shall assign a label to each vertex by utilizing the digit sum. We must assign 2 colors to each code word. If the code word has even digit sum, let the first of these colors be 1, and 2 otherwise. For the second color, list the first 2^{n-1} code words according to their standard total ordering. If a code word falls in position j of this ordering, assign it the color $j+2$, For the remaining code words, assign them the same second color as the word of maximum hamming distance away. This will yield a proper 2-tone $(2^{n-1}+2)$-coloring of Q_n, as distance 1 vertices must have different first colors and different second colors. Similarly, distance 2 vertices may share the same first color, but do not share the same second color. There are 2 possible choices for the first color and 2^{n-1} possible choices for the second color, thus we have used a total of $2^{n-1} + 2$ colors. □

The upper bound above may not be particularly close to the actual value of $\tau_2(Q_n)$. In fact, we conjecture the following:

Conjecture 29

$$\tau_2(Q_n) \leq \left\lceil \frac{\sqrt{8n+1}+5}{2} \right\rceil + \left\lceil \frac{\binom{n}{2}}{n} \right\rceil.$$

This would provide a better upper bound then above, but is still not likely to be the best possible. We have found that Q_6 can be colored using 8 colors, whereas the conjectured upper bound yields a value of 9. The pair of numbers in the parentheses is the 2-tone coloring of that vertex, the number outside of the vertex is a count of how many times that specific number pair occurs, except in the case of Q_4, where the numbers are included to show a possible relation between the coloring of Q_4 and Q_5.

For any given vertex in the hypercube, the number of vertices at a given distance compose various rows of Pascal's triangle. Additionally, the language of binary codes reveals a great deal of information about the structure of the hyper-cubes. However, as yet, this remains an open question. The t-tone chromatic number of Q_n seems to be closely related to $\chi(Q_p^t)$, which has been studied by various people, including Östergård, Ngo, Du and Graham. Patric Östergård, for example, has shown that as n tends to infinity, $\chi(Q_n^2)$ asymptotically approaches n and $\chi(Q_n^3)$ asymptotically approaches $2n$. This would seem to be a rich, but difficult area of further research.

Figures 8-10 show proper 2-tone colorings of the first several hypercubes. These were obtained by considering the two tone color of each vertex as a pair, then constructing permutations on the pairs to get from one hypercube to the next larger hypercube. These permutations must all be products of disjoint 2-cycles, with the conditions that if a pair of colors is used on a vertex, that pair can not be part of any 2-cycle. Also, for any two adjacent vertices, the four colors used to color them can not be contained in two 2-cycles; for example, if two adjacent vertices are colored (1,2) and (3,4), then the 2-cycles (1 3) and (2 4) can not both be in the same permutation, and neither can (1 4) and (2 3). This method may work in general, but we have yet to verify this.

Figure 8: A 2-tone coloring of Q_4

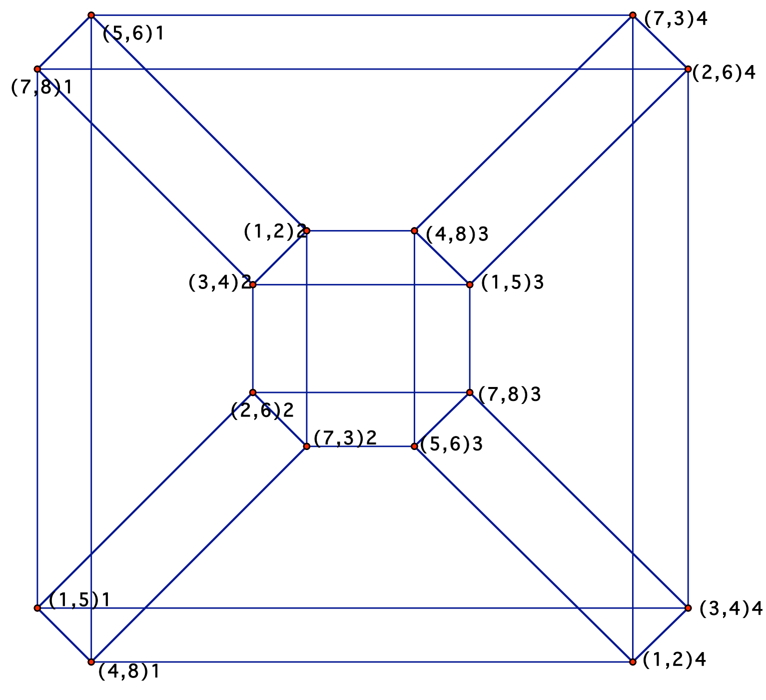


Figure 9: A 2-tone coloring of Q_5

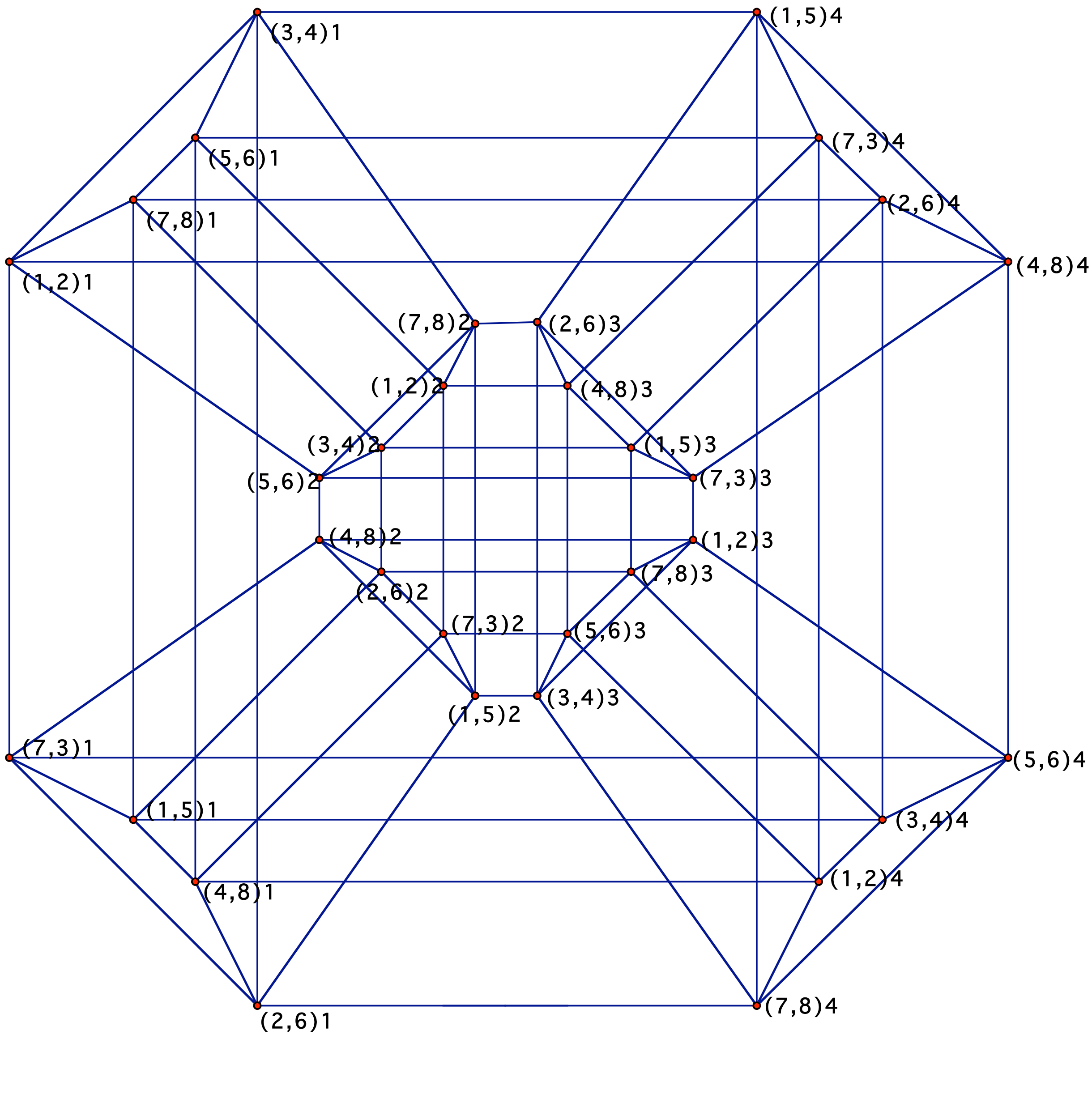


Figure 10: A 2-tone coloring of Q_6