TILING PROOFS OF SOME FORMULAS FOR THE PELL NUMBERS OF ODD INDEX

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Abstract

We provide tiling proofs of several algebraic formulas for the Pell numbers of odd index, all of which involve alternating sums of binomial coefficients, as well as consider polynomial generalizations of these formulas. In addition, we provide a combinatorial interpretation for a Diophantine equation satisfied by the Pell numbers of odd index.

1. Introduction

Combinatorial proofs which use tilings have recently been given to explain and extend a variety of algebraic identities, including ones involving Fibonacci numbers [2], determinants [1], and binomial coefficients [4, 8]. Here, we use tilings to explain (and generalize) several formulas for the Pell numbers of odd index, all of which involve alternating sums of binomial coefficients. We also look at an example of a Diophantine equation whose solution has a tiling interpretation.

Let p_n, $n \geqslant 0$, denote the sequence of Pell numbers given by the recurrence

$$p_n = 2p_{n-1} + p_{n-2}, \qquad n \geqslant 2, \tag{1}$$

with initial conditions $p_0 = 1$, $p_1 = 2$. (See A000129 in [10] for more information on these numbers.) From (1), one sees that p_n counts tilings of a board of length n with cells labelled $1, 2, ..., n$ using squares and dominos, where squares are painted black or white (which we'll term *Pell n-tilings*). For example, $p_0 = 1$ counts the empty tiling and $p_2 = 5$ since a board of length 2 may be covered (exactly) by either a domino or by two squares, each painted black or white. Benjamin, Plott, and Sellers [5] have provided combinatorial proofs of several recent Pell number identities which were q-generalized by Briggs, Little, and Sellers [6].

Let $a_n := p_{2n-1}$, $n \geqslant 1$, denote the n^{th} Pell number of odd index with $a_0 := 0$. (See A001542 in [10].) The a_n satisfy the recurrence [7]

$$a_n = 6a_{n-1} - a_{n-2}, \qquad n \geqslant 2, \tag{2}$$

with initial conditions $a_0 = 0$, $a_1 = 2$. From (2), it is obvious that the a_n are actually all even, which may also be realized by toggling the color of the first square in a Pell tiling of odd length. For a combinatorial explanation of (2), first perform one of the following six operations on a Pell $(2n-3)$-tiling, where $n \geqslant 2$: (i) Add two squares painted black or white to the end in one of four ways, (ii) add a domino to the end, or (iii) insert a domino directly prior to the final piece. Note that all Pell $(2n-1)$-tilings arise once from performing these six operations on the Pell $(2n-3)$-tilings except those ending in at least two dominos, which arise twice, of which there are a_{n-2}.

In this note, we provide combinatorial proofs of several formulas [7] for the a_n and generalizations, thereby avoiding the use of such algebraic techniques as induction, generating functions, and Binet formulas. In addition, we provide a bijective proof that the squares of the numbers $\frac{a_n}{2}$, $n \geqslant 0$, are all triangular numbers, which supplies a combinatorial insight into why the Diophantine equation

$$T^2 = \binom{Y+1}{2} \tag{3}$$

has $T = \frac{a_n}{2}$ as its solution.

If $n \geqslant 1$ and $0 \leqslant k \leqslant \lfloor n/2 \rfloor$, recall that there are $\binom{n-k}{k}$ ways to tile a board of length n with k dominos and $n-2k$ squares, such tilings being equivalent to sequential arrangements of k dominos and $n-2k$ squares. Similarly, there are $2^{n-2k}\binom{n-k}{k}$ Pell n-tilings containing k dominos, since each of the $n-2k$ squares may be painted either black or white. Summing over k gives the well-known explicit formulas

$$f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}, \qquad n \geqslant 0, \tag{4}$$

and

$$p_n = \sum_{k=0}^{\lfloor n/2 \rfloor} 2^{n-2k} \binom{n-k}{k}, \qquad n \geqslant 0, \tag{5}$$

for the Fibonacci and Pell numbers, respectively. Throughout, we represent Pell tilings as sequences in b, w and d, standing for black square, white square, and domino, respectively. If n is a positive integer, then let $[n]$ denote the set $\{1, 2, ..., n\}$ and P_n denote the set of all Pell n-tilings. By convention, we let $[0]$ be the empty set and P_0 be the singleton set consisting of the empty tiling.

2. Combinatorial Proofs

In this section, we provide combinatorial interpretations of six formulas for the odd Pell number $a_n := p_{2n-1}$, $n \geqslant 1$. The proofs are divided into four parts as follows:

(1) Describe a set of tilings C_m whose signed sum is given by the right-hand side;

(2) Set aside an exceptional subset $C'_m \subseteq C_m$, all of whose members have positive sign;

(3) Define a sign-changing involution of $C_m - C'_m$;

(4) Provide an argument for the cardinality of C'_m.

In all the proofs of this section (except for Identity 3), the sign of a tiling $\lambda \in C_m$ will be given by $(-1)^{v(\lambda)}$, where $v(\lambda)$ denotes the number of dominos in λ. The tilings themselves will consist of dominos and squares, circled and marked in various ways.

Identity 1.

$$a_{n+1} = 2\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k 6^{n-2k}\binom{n-k}{k}, \qquad n \geqslant 0.$$

Proof. We start with descriptions of C_n and C'_n.

`Description of the set` $\boldsymbol{C_n}$`:` Let C_n denote the set of tilings of $[n]$ in which each square is marked with a member of $[6]$. Then half the right-hand side gives the signed sum over all the members of C_n according to the number of dominos k.

`Description of the set` $\boldsymbol{C'_n}$`:` Let $C'_n \subseteq C_n$ consist of those tilings λ which satisfy the following three conditions:

(i) λ contains no dominos;

(ii) There are no two consecutive squares which cover the numbers $2i-1$ and $2i$ for some i, $1 \leqslant i \leqslant \lfloor \frac{n}{2} \rfloor$, and are marked with 1 and 2, respectively;

(iii) There are no two consecutive squares which cover the numbers $2i$ and $2i+1$ for some i, $1 \leqslant i \leqslant \lfloor \frac{n-1}{2} \rfloor$, and are marked with 3 and 4, respectively.

We proceed with an involution.

`Sign-changing involution of` $\boldsymbol{C_n - C'_n}$`:` Given $\lambda \in C_n - C'_n$, let i_0 be the smallest index $i \geqslant 1$ such that one of the following occurs:

(i) The numbers $2i-1$ and $2i$ are covered by a domino or by squares marked with 1 and 2, respectively;

(ii) The numbers $2i$ and $2i+1$ are covered by a domino or by squares marked with 3 and 4, respectively.

If (i) occurs, then either replace a domino covering the numbers $2i_0-1$ and $2i_0$ with two squares marked with 1 and 2, respectively, or vice-versa. Similarly, exchange the two cases within (ii) if (ii) occurs, which results in a sign-changing involution of $C_n - C'_n$ and implies $|C'_n|$ equals half the right side of Identity 1.

`Cardinality of the set` $\boldsymbol{C'_n}$`:` To complete the proof, we show $a_{n+1} = 2|C'_n|$, $n \geqslant 0$. Given a square painted black or white and $c = c_1c_2\cdots c_n \in C'_n$, where c_i denotes the member of $[6]$ assigned to the i^{th} square, we'll construct a member of P_{2n+1}, denoted c', in n steps according to the iterative procedure below. (If $1 \leqslant i \leqslant$

$c_i, 1 \leqslant i \leqslant n$	Addition to λ_{i-1} when i is odd	Addition to λ_{i-1} when i is even
1	d at end	wb
2	wb	d prior to last square
3	bw	d at end
4	d prior to last square	bw
5	ww	ww
6	bb	bb

n, then λ_i denotes the resulting member of P_{2i+1} obtained after making i additions as described below, with λ_0 denoting the painted square you start with.)

For example, if $n = 5$, $c = 41323 \in C'_5$, and $\lambda_0 = w$, then $\lambda_1 = dw$, $\lambda_2 = dw^2b$, $\lambda_3 = dw^2b^2w$, $\lambda_4 = dw^2b^2dw$, and $\lambda_5 = dw^2b^2dwbw$, which implies $c' = \lambda_5 \in P_{11}$. Note that $c \in C'_n$ implies that there is always a square ending λ_{i-1} whenever $c_i = 2$ and i is even or whenever $c_i = 4$ and $i \geqslant 3$ is odd. The mapping $c \mapsto c'$ is reversible, upon starting with the last piece of a member of P_{2n+1} and considering the parity of n. $\square$

Identity 2.

$$a_{n+1} = 2\sum_{k=0}^{n}(-1)^k 8^{n-k}\binom{2n+1-k}{k}, \qquad n \geqslant 0.$$

Proof. The proof follows the structure of the last proof.

`Description of the set` $\boldsymbol{C_{2n+1}}$: Let C_{2n+1} denote the set of tilings of $[2n+1]$ in which squares are painted black or white and squares covering even numbers may be circled. The right side of Identity 2 then gives the signed sum over all members of C_{2n+1} according to the number of dominos k; note that there are 8^{n-k} choices with regard to the first $n-k$ pairs of squares (squares must alternate between covering odd and then even numbers) and two choices with regard to the final square (which must cover an odd number).

`Description of the set` $\boldsymbol{C'_{2n+1}}$: Let $C'_{2n+1} \subseteq C_{2n+1}$ consist of those tilings λ which satisfy the following three conditions:

(i) λ contains no dominos;

(ii) No consecutive numbers $2i-1$, $2i$ are covered by ww for any i, $1 \leqslant i \leqslant n$;

(iii) No consecutive numbers $2i$, $2i+1$ are covered by bb for any i, $1 \leqslant i \leqslant n$.

`Involution of` $\boldsymbol{C_{2n+1} - C'_{2n+1}}$: Given $\lambda \in C_{2n+1} - C'_{2n+1}$, identify the smallest $j \geqslant 1$ for which one of the following holds:

(i) $2j-1$ and $2j$ are covered by either a domino or by ww;

(ii) $2j$ and $2j+1$ are covered by either a domino or by bb.

Switching to the other option in either case produces a sign-changing involution of $C_{2n+1} - C'_{2n+1}$.

`Cardinality of the set` C'_{2n+1}:] Note that $|C'_{2n+1}| = a_{n+1}$, as seen upon taking members of P_{2n+1} and replacing any domino whose initial segment covers an odd number with $b w$ and replacing any domino whose initial segment covers an even number with $b w$ (leaving any squares unchanged). □

Remark: By allowing the tilings to have varying lengths in the proofs above for the first two identities, one can also combinatorially explain

$$p_{2n} = 1 + 2\sum_{k=1}^{n} a_k = 1 + 4\sum_{k=1}^{n} 6^{n-k} \sum_{j=0}^{\lfloor \frac{k-1}{2} \rfloor} (-1)^j \binom{n-k+j}{j}$$

and

$$p_{2n} = 1 + 4\sum_{k=1}^{n} 8^{n-k} \sum_{j=0}^{k-1} (-1)^j \binom{2n-2k+j+1}{j}.$$

Identity 3.

$$a_n^2 = \sum_{k=1}^{2n} 2^{4n-2k} \sum_{j=1}^{k} (-1)^{j-1} \binom{4n-k-j}{4n-2k}, \qquad n \geqslant 1.$$

Proof. This proof also follows the structure of Identity 1's proof.

`Description of the set` $\boldsymbol{C_{4n}}$: Given $n \geqslant 1$, let $C_{k,j}$ comprise those Pell $(4n-2j)$-tilings containing exactly $k-j$ dominos, where $1 \leqslant j \leqslant k \leqslant 2n$. Define the sign of $\lambda \in C_{k,j}$ by $(-1)^{j-1}$. Note that $|C_{k,j}| = 2^{4n-2k}\binom{4n-k-j}{4n-2k}$ since members of $C_{k,j}$ contain $k-j$ dominos and thus $4n-2j-2(k-j) = 4n-2k$ squares for a total of $4n-k-j$ pieces in all. The right side of Identity 3 then gives the signed sum over all members of

$$C_{4n} := \bigcup_{1 \leqslant j \leqslant k \leqslant 2n} C_{k,j} \ .$$

`Description of the set` $\boldsymbol{C'_{4n}}$: Let $C'_{4n} \subseteq C_{4n}$ consist of those tilings having positive sign (i.e., belonging to $C_{k,j}$ for some j odd) and ending in a square.

`Involution of` $\boldsymbol{C_{4n}}$–$\boldsymbol{C'_{4n}}$: Define a sign-changing involution of $C_{4n} - C'_{4n}$ by adding a domino to the end of a tiling having negative sign (i.e., belonging to $C_{k,j}$ for some $j \geqslant 2$ even) and taking a domino away from the end of a tiling having positive sign (which must end in at least one domino, by assumption).

`Cardinality of the set` $\boldsymbol{C'_{4n}}$: We need to show $|C'_{4n}| = a_n^2$, $n \geqslant 1$. Observe first that members of C'_{4n} are synonymous with Pell $4n$-tilings ending in an odd number of dominos preceded by a square (upon adding j dominos to the end, where

j is odd). So write $\lambda \in C'_{4n}$ as $\lambda = \alpha c d^{2i+1}$, where $0 \leqslant i \leqslant n-1$, c is a square (either black or white), and $\alpha \in P_{4n-4i-3}$. From λ, we'll construct a member $(\lambda_1, \lambda_2) \in P_{2n-1} \times P_{2n-1}$ as follows. If α can be decomposed further as $\alpha_1\alpha_2$, where α_1 and α_2 are Pell tilings of lengths $2n-2i-1$ and $2n-2i-2$, respectively, then let $\lambda_1 = \alpha_1 d^i$ and $\lambda_2 = \alpha_2 c d^i$. On the other hand, if $\alpha = \alpha_1 d \alpha_2$, where α_1 and α_2 have lengths $2n-2i-2$ and $2n-2i-3$, respectively, then let $\lambda_1 = \alpha_1 c d^i$ and $\lambda_2 = \alpha_2 d^{i+1}$ (note that $i \leqslant n-2$ in this case). The mapping $\lambda \mapsto (\lambda_1, \lambda_2)$ may be reversed upon considering whether or not λ_1 ends in at least as many dominos as λ_2. $\square$

Examples of the mapping $\boldsymbol{\lambda \mapsto (\lambda_1, \lambda_2)}$:

$$
\begin{aligned}
n = 5, i = 2: &\quad (bd^2)(wdb)wd^5 \in P_{20} \mapsto \quad (bd^4, wdbwd^2) \in P_9 \times P_9;\\
n = 4, i = 1: &\quad (db^2)d(bd)wd^3 \in P_{16} \mapsto \quad (db^2wd, bd^3) \in P_7 \times P_7.
\end{aligned}
$$

Identity 4.

$$
a_n^2 = 4\sum_{k=1}^{2n-1} 8^{2n-k-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{4n-k-j}{4n-2k}, \qquad n \geqslant 1.
$$

Proof. This proof also follows the structure of Identity 1's proof.

`Description of the set` $\boldsymbol{C_{4n}}$: Given $n \geqslant 1$, let $C_{k,j}$ consist of the tilings of length $4n-2j$ containing exactly $k-j$ dominos in which squares are painted black or white and squares covering even numbers may be circled, except for the last such square, which isn't circled. Note that $|C_{k,j}| = 4 \cdot 8^{2n-k-1}\binom{4n-k-j}{4n-2k}$ since there are 8 choices (with regard to coloring and circling) for the first $2n-k-1$ pairs of squares and only 4 choices for the final pair of squares. The right side of Identity 4 then gives the signed sum over all members of

$$
C_{4n} := \bigcup_{1 \leqslant j \leqslant k \leqslant 2n-1} C_{k,j}
$$

according to the number of dominos $k-j$.

`Description of the set` $\boldsymbol{C'_{4n}}$: Let $C'_{4n} \subseteq C_{4n}$ consist of those tilings for which j is odd (i.e., $\lambda \in C_{k,j}$ for some j odd), there are no dominos (i.e., $k = j$), no two consecutive numbers $2i-1$ and $2i$ are covered by $w w$, and no two consecutive numbers $2i$ and $2i+1$ are covered by $b b$.

`Involution of` $\boldsymbol{C_{4n} - C'_{4n}}$: To define a sign-changing involution of $C_{4n} - C'_{4n}$, first apply the involution used in the proof of Identity 3 if it is the case that $\lambda \in C_{k,j}$, where either

(i) j is even, or

(ii) j is odd with the last piece in λ a domino.

On the other hand, if j is odd and the last piece in λ is a square, then apply the involution used to prove Identity 2. Note that the involution in the latter case would not change the final piece in λ, a square which isn't circled, by assumption; hence, it is well-defined.

`Cardinality of the set` $\boldsymbol{C'_{4n}}$: By prior reasoning, members of C'_{4n} are synonymous with members of P_{4n} ending in an odd number of dominos (which number a_n^2 by the previous proof) upon adding j dominos to the end and replacing all occurrences of $b\textcircled{w}$ on $2i-1$, $2i$ as well as all occurrences of $\textcircled{b}w$ on $2i$, $2i+1$ with dominos. (Note that the final square not being circled in a member of C'_{4n} ensures that the resulting member of P_{4n} ends in a square preceded by an odd number of dominos.) $\square$

Identity 5.

$$a_n^2 = 4\sum_{k=1}^{n} 6^{2n-2k} \sum_{j=1}^{k} (-1)^{k-j} \binom{2n-k-j}{2n-2k}, \qquad n \geqslant 1.$$

Proof. Again, we follow the structure of Identity 1's proof.

`Description of the set` $\boldsymbol{C_{2n}}$: Let $C_{k,j}$ consist of the $(2n-2j)$-tilings containing exactly $k-j$ dominos in which each square is marked with a member of $[6]$. One fourth the right-hand side then gives the signed sum over all members of

$$C_{2n} := \bigcup_{1\leqslant j\leqslant k\leqslant n} C_{k,j}\ .$$

`Description of the set` $\boldsymbol{C'_{2n}}$: Let $C'_{2n} \subseteq C_{2n}$ consist of those tilings in which there are no dominos (i.e., $k=j$), no two numbers $2i-1$, $2i$ are covered by squares marked with 1, 2, respectively, for any i, and no two numbers $2i$, $2i+1$ are covered by squares marked with 3, 4, respectively, for any i.

`Involution of` $\boldsymbol{C_{2n} - C'_{2n}}$: Apply the involution used in the proof of Identity 1.

`Cardinality of the set` $\boldsymbol{C'_{2n}}$: We now show $4|C'_{2n}| = a_n^2$. Starting with a black or white square and a member of C'_{2n} of length $2n-2j$, where $1 \leqslant j \leqslant n$, obtain $\alpha \in P_{4n-4j+1}$ as in the final part of the proof for Identity 1. To α, add a square of either color, followed by $2j-1$ dominos, to obtain $\beta \in P_{4n}$ ending in an odd number of dominos preceded by a square. Thus, members of C'_{2n} are in 1-to-4 correspondence with members of P_{4n} ending in an odd number of dominos, of which there are a_n^2, by the last part of the proof for Identity 3. $\square$

Identity 6.

$$(a_n + a_{n-1})^2 = 4\sum_{k=1}^{2n-1} 8^{2n-k-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{4n-k-j-2}{4n-2k-2}, \qquad n \geqslant 1.$$

Proof. Using the structure of Identity 1's proof, we have the following.

`Description of the set` $\boldsymbol{C_{4n-2}}$`:` Given $n \geqslant 1$, let $C_{k,j}$ consist of the tilings of length $4n-2j-2$ containing exactly $k-j$ dominos in which squares may be painted black or white and squares covering even numbers may be circled. One fourth of the right-hand side then gives the signed sum over all members of

$$C_{4n-2} := \bigcup_{1\leqslant j\leqslant k\leqslant 2n-1} C_{k,j} \quad .$$

`Description of the set` $\boldsymbol{C'_{4n-2}}$`:` Let $C'_{4n-2} \subseteq C_{4n-2}$ consist of the empty tiling as well as the non-empty tilings for which $k = j$ is odd (and so there are no dominos), no $\textcircled{b}b$ covers $2i$, $2i+1$ for any i, and no $w\textcircled{w}$ covers $2i-1$, $2i$ for any i except for possibly $i = 2n-j-1$ (and so we allow $w\textcircled{w}$ to cover the numbers $4n-2j-3, 4n-2j-2$).

`Involution of` $\boldsymbol{C_{4n-2} - C'_{4n-2}}$`:` Apply the involution used in the proof of Identity 4 to $C_{4n-2} - C'_{4n-2}$.

`Cardinality of the set` $\boldsymbol{C'_{4n-2}}$`:` To complete the proof, we need to show that $4|C'_{4n-2}| = (a_n + a_{n-1})^2$. Reasoning as in the last part of the proof for Identity 2, observe first that a non-empty tiling $\lambda \in C'_{4n-2}$ may be expressed as $\lambda = \alpha c$, where α is a Pell tiling of length $4n-2j-3$, there are four options for the square c (black or white, circled or uncircled), and j is odd, $1 \leqslant j \leqslant 2n-3$. This implies

$$|C'_{4n-2}| = 4|\bigcup_{i=1}^{n-1} P_{4i-1}| + 1.$$

Taking four copies of a member of P_{4i-1} and leaving one unchanged, adding a black square to one, adding a white square to another, and, to the last, either removing a final domino, removing a final white square, or changing a final black square to a domino shows that

$$4|P_{4i-1}| = |\bigcup_{m=0}^{3} P_{4i-m}|, \qquad 1 \leqslant i \leqslant n-1,$$

and thus

$$|C'_{4n-2}| = 4|\bigcup_{i=1}^{n-1} P_{4i-1}| + 1 = |\bigcup_{i=0}^{4n-4} P_i|.$$

(The empty member of C'_{4n-2} is mapped to the empty Pell tiling.) In [5], a bijection is given between $\bigcup_{i=0}^{4n-4} P_i$ and the set consisting of all ordered pairs of Pell $(2n-1)$-tilings ending in either a domino or in a black square, which number $(p_{2n-2} + p_{2n-3})^2 = \frac{1}{4}(p_{2n-1} + p_{2n-3})^2$. Combining this bijection with the preceding implies $4|C'_{4n-2}| = (p_{2n-1} + p_{2n-3})^2$, as desired. □

3. Generalizations

We generalize the Pell number identities of the prior section to ones involving Fibonacci polynomials (see, e.g., [3, p.141] or [9]). If a and b are indeterminates, then define the sequence of polynomials $g_n(a,b)$ by the recurrence

$$g_n(a,b) = ag_{n-1}(a,b) + bg_{n-2}(a,b), \qquad n \geqslant 2,$$

with initial conditions $g_0(a,b) = 1$, $g_1(a,b) = a$. When $a = b = 1$ and $a = 2$, $b = 1$, the $g_n(a,b)$ reduce, respectively, to the Fibonacci and Pell number sequences. When a and b are positive integers, the $g_n(a,b)$ count tilings of length n in which a square may be painted with one of a colors and a domino with one of b colors.

If $n \geqslant 1$, then let $c_n(a,b) := g_{2n-1}(a,b)$, where $c_0(a,b) := 0$. The algebraic arguments for Identities 1–6 of the prior section may be extended (we omit the details) to yield the following generalizations:

Identity 7.

$$c_{n+1}(a,b) = a\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k b^{2k}(a^2+2b)^{n-2k}\binom{n-k}{k}, \qquad n \geqslant 0.$$

Identity 8.

$$c_{n+1}(a,b) = a\sum_{k=0}^{n}(-1)^k b^{k}(a^2+4b)^{n-k}\binom{2n+1-k}{k}, \qquad n \geqslant 0.$$

Identity 9.

$$c_n^2(a,b) = \sum_{k=1}^{2n} a^{4n-2k}b^{k-1}\sum_{j=1}^{k}(-1)^{j-1}\binom{4n-k-j}{4n-2k}, \qquad n \geqslant 1.$$

Identity 10.

$$c_n^2(a,b) = a^2\sum_{k=1}^{2n-1} b^{k-1}(a^2+4b)^{2n-k-1}\sum_{j=1}^{k}(-1)^{k-j}\binom{4n-k-j}{4n-2k}, \qquad n \geqslant 1.$$

Identity 11.

$$c_n^2(a,b) = a^2\sum_{k=1}^{n} b^{2k-2}(a^2+2b)^{2n-2k}\sum_{j=1}^{k}(-1)^{k-j}\binom{2n-k-j}{2n-2k}, \qquad n \geqslant 1.$$

Identity 12.

$$(c_n(a,b)+bc_{n-1}(a,b))^2$$

$$= a^2 \sum_{k=1}^{2n-1} b^{k-1}(a^2+4b)^{2n-k-1} \sum_{j=1}^{k} (-1)^{k-j} \binom{4n-k-j-2}{4n-2k-2}, \qquad n \geqslant 1.$$

Identities 7–12 reduce, respectively, to Identities 1–6 when $a = 2$ and $b = 1$ and to identities for the odd Fibonacci number f_{2n-1} when $a = b = 1$.

By allowing squares to come in a colors and dominos to come in b colors, the arguments given above for Identities 1, 3 and 5 may be modified to provide combinatorial interpretations for Identities 7, 9 and 11, respectively, when a and b are positive integers (which proves them in general). We leave the details for the interested reader. On the other hand, we were unable to find combinatorial proofs for Identities 8, 10 or 12 in either the general case where a and b are positive integers or in the specific case where $a = b = 1$.

Remark: Let $d_n(a,b) := g_{2n}(a,b)$, $n \geqslant 0$. Using Identities 7–12 above for $c_n(a,b)$ along with the relations

$$d_n(a,b) = \frac{c_{n+1}(a,b) - bc_n(a,b)}{a}, \qquad n \geqslant 1,$$

and

$$c_n(a,b) = \frac{d_n(a,b) - bd_{n-1}(a,b)}{a}, \qquad n \geqslant 1,$$

one gets similar, though more complicated, formulas for $d_n(a,b)$.

4. Recounting Square Triangular Numbers

Let $T_n = \frac{a_n}{2}$, $n \geqslant 0$, and Y_n be the sequence given by the recurrence

$$Y_n = 6Y_{n-1} - Y_{n-2} + 2, \qquad n \geqslant 2, \tag{6}$$

with initial values $Y_0 = 0$, $Y_1 = 1$. The Diophantine equation (see, e.g., [7])

$$T^2 = \binom{Y+1}{2} \tag{7}$$

has as its solution the set of ordered pairs $(T, Y) = (T_n, Y_n)$, $n \geqslant 0$. This can quickly be seen upon multiplying both sides of (7) by 8 and letting $Z = 2Y + 1$ and $U = 2T$ to get

$$Z^2 - 2U^2 = 1, \tag{8}$$

which is the $d = 2$ case of Pell's equation.

In this section, we provide a bijective proof that the squares of the T_n are all triangular numbers, which justifies the solution to (7) and hence (8). Perhaps the argument can be modified to show that there are no other perfect square triangular numbers, which would supply a full combinatorial explanation of (7) and (8). It would also be desirable to generalize our argument for (8) and provide a combinatorial solution to Pell's equation.

Upon multiplying (7) by 8, we need to show, equivalently,

$$2a_n^2 = (2Y_n + 1)^2 - 1, \qquad n \geqslant 1. \tag{9}$$

To do so, we specify combinatorial structures enumerated by the left and right sides of (9) and then describe a bijection between them.

`The left side:` This clearly counts the ordered pairs in two copies of $P_{2n-1} \times P_{2n-1}$.

`The right side:` First consider the set $\mathcal{Y}_n$ consisting of Pell $(2n-1)$-tilings in which a square covering cell 1 may also be green and containing at least one black or white square, the first of which we require to be white. Note that $|\mathcal{Y}_1| = 1 = Y_1$ and $|\mathcal{Y}_2| = 8 = Y_2$. We use (6) to show $|\mathcal{Y}_n| = Y_n$ when $n \geqslant 3$.

Note that members of $\mathcal{Y}_n$, $n \geqslant 3$, may be formed from members of $\mathcal{Y}_{n-1}$ by either adding two squares to the end in one of four ways (as bb, bw, wb, or ww), adding a domino to the end, or inserting a domino just prior to the last piece. Members of $\mathcal{Y}_n$ ending in at least two dominos are clearly synonymous with members of $\mathcal{Y}_{n-2}$, and are counted twice in the preceding, hence the second term on the right side of (6). This accounts for all members of $\mathcal{Y}_n$, except for those whose first black or white square (which must be white) covers cell $2n-2$, and there are exactly two such tilings (a green square, followed by $d^{n-2}w^2$ or $d^{n-2}wb$).

If $n \geqslant 1$, then let $\mathfrak{Z}_n$ denote the set of Pell $(2n-1)$-tilings in which a square covering cell 1 may also be green. Note that $|\mathfrak{Z}_n| = 2|\mathcal{Y}_n| + 1 = 2Y_n + 1$ since two members of $\mathfrak{Z}_n$ may be obtained from each member of $\mathcal{Y}_n$ by allowing the first black or white square to be black and since the single tiling comprised of a green square followed by $n-1$ dominos is also allowed, which we denote by λ^*. Thus, the right side of (9) counts all members of $\mathfrak{Z}_n \times \mathfrak{Z}_n$, where we exclude from consideration the ordered pair (λ^*, λ^*).

`The bijection:` Let $\lambda = (\lambda_1, \lambda_2) \in \mathfrak{Z}_n \times \mathfrak{Z}_n$. If neither λ_1 nor λ_2 starts with a green square, apply the identity mapping to obtain an ordered pair belonging to the first copy of $P_{2n-1} \times P_{2n-1}$ described above. So assume at least one of $\{\lambda_1, \lambda_2\}$ begins with a green square. Note that such members of $\mathfrak{Z}_n \times \mathfrak{Z}_n$ number $p_{2n-2}^2 + 2p_{2n-2}p_{2n-1} = p_{2n-2}(p_{2n-2} + 2p_{2n-1}) = p_{2n-2}p_{2n}$. Thus, the remaining members of $\mathfrak{Z}_n \times \mathfrak{Z}_n$ may be identified with ordered pairs in $P_{2n-2} \times P_{2n}$ and normal tail-swapping (see, e.g., [3, p.7]) provides the needed near bijection between $P_{2n-2} \times P_{2n}$ and the second copy of $P_{2n-1} \times P_{2n-1}$. Note that only the ordered pair in $P_{2n-2} \times P_{2n}$ in which both coordinates are tilings consisting solely of dominos fails to be mapped and that this ordered pair corresponds to the excluded member (λ^*, λ^*) in $\mathfrak{Z}_n \times \mathfrak{Z}_n$. Hence, $2|P_{2n-1} \times P_{2n-1}| = |\mathfrak{Z}_n \times \mathfrak{Z}_n| - 1$, which is (9), as desired.

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