THE JACOBSTHAL NUMBERS: TWO RESULTS AND TWO QUESTIONS

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Abstract. We obtain here two results associated with the Jacobsthal numbers. The first concerns a convolution while the second is in connection with systems of complete residues. Each of these results raises a further question.

1. Introduction

The nth Jacobsthal number J_n may be obtained in a recursive manner by way of the recurrence relation

$$J_n = J_{n-1} + 2J_{n-2}, \tag{1.1}$$

where $J_0 = 0$ and $J_1 = 1$. An explicit form for J_n is given by the Binet-type formula

$$J_n = \frac{2^n - (-1)^n}{3}. \tag{1.2}$$

The sequence of Jacobsthal numbers arises in a number of combinatorial situations. For example, J_n gives, for $n \geq 2$, the number of ways of tiling a $3 \times (n-1)$ rectangle with 1×1 and 2×2 square tiles. See the entry for sequence A001045 in [4], and the many references cited there, for further information, including several alternative combinatorial interpretations.

In this paper we obtain two results associated with the Jacobsthal numbers, each of which leads naturally on to a further question. The first result is the evaluation of a convolution involving the Lucas and Jacobsthal numbers, while the second is associated with complete residue systems.

2. A Convolution

We consider here the sequence $\{c(n)\}_{n\geq 0}$ given by the following convolution comprising the Lucas and the Jacobsthal numbers:

$$c(n) = \sum_{k=0}^{n} L_k J_{n-k}.$$

It is straightforward to show that the ordinary generating functions for $\{L_n\}_{n\geq 0}$ and $\{J_n\}_{n\geq 0}$ are given by

$$G(x) = \frac{2-x}{1-x-x^2} \qquad \text{and} \qquad H(x) = \frac{x}{1-x-2x^2},$$

respectively. By the method of partial fractions we then obtain

$$G(x) = \frac{1}{1-\alpha x} + \frac{1}{1-\beta x},$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \beta = -\frac{1}{\alpha} = \frac{1-\sqrt{5}}{2}.$$

Similarly,

$$H(x) = \frac{1}{3(1-2x)} - \frac{1}{3(1+x)}.$$

From the above it follows that

$$3G(x)H(x) = \frac{1}{(1-\alpha x)(1-2x)} - \frac{1}{(1-\alpha x)(1+x)} + \frac{1}{(1-\beta x)(1-2x)} - \frac{1}{(1-\beta x)(1+x)},$$

which, after further use of partial fractions and a considerable amount of simplification, gives

$$G(x)H(x) = \frac{2}{1-2x} - \frac{1}{1+x} - \frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}.$$

Then, since $c(n)$ is the coefficient of x^n in $G(x)H(x)$, we have

$$\begin{aligned} c(n) &= 2 \cdot 2^n - (-1)^n - \alpha \cdot \alpha^n - \beta \cdot \beta^n \\ &= 2^{n+1} - (-1)^n - \alpha^{n+1} - \beta^{n+1} \\ &= 2^{n+1} + (-1)^{n+1} - L_{n+1}. \end{aligned}$$

Note that this may also be written as

$$c(n) = 3J_{n+1} + 2(-1)^{n+1} - L_{n+1}$$

or

$$c(n) = j_{n+1} - L_{n+1},$$

where j_n is the nth Jacobsthal-Lucas number [5]. Since the convolution takes such a simple form, we ask whether it is possible to obtain a purely combinatorial proof of this result. Incidentally, $\{c(n)\}_{n\geq 0}$ does not appear in [4].

3. Complete Residue Systems

Next, suppose that $\{a_n\}_{n\geq 0}$ is a sequence of non-negative integers. This sequence is said to possess a complete system of residues modulo m if, for each $i \in \{0, 1, 2, \ldots m-1\}$, there exists some j such that $a_j \equiv i \pmod{m}$. Burr [2] classified the moduli for which the Fibonacci numbers contain a complete system of residues, while Avila and Chen [1] obtained the corresponding classification for the Lucas numbers. We show here that $m = 2$ and $m = 3^k$ for each $k \geq 0$ are the only moduli for which the sequence of Jacobsthal numbers has a complete system of residues. For the sake of clarity, a number of lemmas will be utilized in order to prove our main result. If the Jacobsthal sequence possesses a complete system of residues modulo m, then we say that m is *J-complete*.

Lemma 3.1. *The only J-complete even positive integer is 2.*

Proof. The sequence $\{J_n\}_{n\geq 0}$ contains 0 and 1, so m is indeed J-complete when $m = 2$. Now suppose that m is an even integer such that $m \geq 4$. Using (1.1) in conjunction with the initial conditions $J_0 = 0$ and $J_1 = 1$, it is straightforward to show by induction that J_n is odd for each $n \in \mathbb{N}$. It is therefore the case that none of the even numbers from 2 to $m-2$ inclusive appear in the residue system for the Jacobsthal numbers modulo m. The truth of the lemma follows from this. ∎

Lemma 3.2. *No prime $p > 3$ is J-complete.*

Proof. Let p be a prime such that $p > 3$. Using (1.2) we have

$$\begin{aligned} J_{n+p-1} - J_n &= \frac{2^{n+p-1} - (-1)^{n+p-1}}{3} - \frac{2^n - (-1)^n}{3} \\ &= \frac{2^n\left(2^{p-1} - 1\right) - (-1)^n\left((-1)^{p-1} - 1\right)}{3} \\ &= \frac{2^n\left(2^{p-1} - 1\right)}{3}, \end{aligned} \tag{3.1}$$

on noting that $(-1)^{p-1} = 1$ since p is an odd prime. Since $J_{n+p-1} - J_n$ is an integer, and $3 \nmid 2^n$, it follows from (3.1) that $3 \mid \left(2^{p-1} - 1\right)$. However, by Fermat's Little Theorem [3], we have

$$2^{p-1} \equiv 1 \pmod{p}.$$

Thus, since $\gcd(3, p) = 1$, we know that for some $k \in \mathbb{N}$ it is the case that $2^{p-1} - 1 = 3kp$, which in turn implies that

$$\frac{2^n\left(2^{p-1} - 1\right)}{3} \equiv 0 \pmod{p},$$

and hence that $J_{n+p-1} - J_n \equiv 0 \pmod{p}$. Therefore the sequence of residues given by $\{J_n \pmod{p}\}_{n\geq 0}$ has a period that is a factor of $p-1$. From this it follows that no more than $p - 1$ distinct residues can appear. $\square$

Lemma 3.3. *For each $k \geq 0$,*

$$4^{3^k} \equiv 3^{k+1} + 1 \pmod{3^{k+2}}.$$

Proof. We proceed by induction. First, when $k = 0$, we have

$$4^{3^k} = 4^1 = 4 \quad \text{and} \quad 3^{k+1} + 1 = 3^1 + 1 = 4,$$

so the statement of the lemma is certainly true in this case. Next, let us assume that it is true for some $k \geq 0$. We then have

$$4^{3^k} = i \cdot 3^{k+2} + 3^{k+1} + 1$$

for some $i \in \mathbb{N}$. Thus,

$$\begin{aligned} 4^{3^{k+1}} &= \left(4^{3^k}\right)^3 \\ &= \left(i \cdot 3^{k+2} + 3^{k+1} + 1\right)^3 \\ &= i^3 3^{3(k+2)} + 3i^2 3^{2(k+2)}\left(3^{k+1} + 1\right) + 3i \cdot 3^{k+2}\left(3^{k+1} + 1\right)^2 + \left(3^{k+1} + 1\right)^3 \\ &= i^3 3^{3(k+2)} + i^2 3^{2k+5}\left(3^{k+1} + 1\right) + i \cdot 3^{k+3}\left(3^{k+1} + 1\right)^2 + \left(3^{k+1} + 1\right)^3 \\ &= j \cdot 3^{k+3} + \left(3^{k+1} + 1\right)^3 \end{aligned}$$

for some $j \in \mathbb{N}$. It now follows that

$$\begin{aligned} 4^{3^{k+1}} &\equiv \left(3^{k+1} + 1\right)^3 \pmod{3^{k+3}} \\ &\equiv 3^{3(k+1)} + 3^{2k+3} + 3^{k+2} + 1 \pmod{3^{k+3}} \\ &\equiv 3^{k+2} + 1 \pmod{3^{k+3}}, \end{aligned}$$

where the final congruence holds since 3^{k+3} divides both $3^{3(k+1)}$ and 3^{2k+3}, as required. $\square$

Lemma 3.4. *Let n be a non-negative even integer. Then*

$$J_{n+2\cdot 3^k} \equiv J_n + 3^k \pmod{3^{k+1}}.$$

Proof. First, it is straightforward to show, by induction, that $2^n \equiv 1 \pmod 3$ when n is even. Then, using the result of Lemma 3.3 and the Binet-type formula (1.2), we obtain

$$\begin{aligned} 2^n \equiv 1 \pmod 3 &\implies 2^n \cdot 3^{k+1} \equiv 3^{k+1} \pmod{3^{k+2}} \\ &\implies 2^n\left(4^{3^k} - 1\right) \equiv 3^{k+1} \pmod{3^{k+2}} \\ &\implies 2^{n+2\cdot 3^k} - (-1)^{n+2\cdot 3^k} \equiv 2^n - (-1)^n + 3^{k+1} \pmod{3^{k+2}} \\ &\implies 3J_{n+2\cdot 3^k} \equiv 3J_n + 3^{k+1} \pmod{3^{k+2}} \\ &\implies J_{n+2\cdot 3^k} \equiv J_n + 3^k \pmod{3^{k+1}}. \end{aligned}$$

□

Theorem 3.5. *The J-complete numbers are given by $m = 2$ and $m = 3^k$ for each $k \geq 0$.*

Proof. We showed in Lemma 3.1 that 2 is the only J-complete even positive integer. Furthermore, since if m is not J-complete then neither is any positive multiple of m, it follows from Lemma 3.2 that, in order to prove the theorem, it just remains to prove that all integers of the form 3^k, $k \geq 0$, are J-complete. We will do this by showing that the set

$$\mathcal{S}_k = \left\{J_0, J_2, J_4, \ldots, J_{2\left(3^k-1\right)}\right\}$$

is a complete residue system modulo 3^k for each $k \geq 0$. Proceeding by induction on k, it is clear that this is true for $k = 0$, so let us suppose now that it is true for some $k \geq 0$.

By the inductive hypothesis, for any element $x \in \left\{0, 1, 2, \ldots, 3^k - 1\right\}$ there exists a unique element $J_{2j} \in \mathcal{S}_k$ such that $x \equiv J_{2j} \pmod{3^k}$. It follows from Lemma 3.4 that J_{2j}, $J_{2j+2\cdot 3^k}$ and $J_{2j+4\cdot 3^k}$ are congruent modulo 3^{k+1} to x, $x + 3^k$ and $x + 2\cdot 3^k$, respectively. It is the case, therefore, that J_{2j}, $J_{2j+2\cdot 3^k}$ and $J_{2j+4\cdot 3^k}$ provide us with three distinct residues modulo 3^{k+1}. Repeating this for each $x \in \left\{0, 1, 2, \ldots, 3^k - 1\right\}$ allows us to exhibit all of the 3^{k+1} required residues, noting first that exactly 3^{k+1} distinct residues modulo 3^{k+1} will indeed be generated via this process, and second that the set thus obtained is $\mathcal{S}_{k+1}$. □

Let $\mathcal{F}$, $\mathcal{L}$ and $\mathcal{J}$ denote the sets of Fibonacci-, Lucas- and Jacobsthal-complete moduli, respectively. From [2] and [1], the elements of $\mathcal{F}$ and $\mathcal{L}$ take the forms

$$5^k, 2\cdot 5^k, 4\cdot 5^k, 3^j 5^k, 6\cdot 5^k, 7\cdot 5^k, 14\cdot 5^k$$

for $j \geq 1$ and $k \geq 0$, and

$$2, 4, 6, 7, 14, 3^k$$

for $k \geq 0$, respectively. It is interesting to note the following chain of strict inclusions:

$$\mathcal{J} \subsetneq \mathcal{L} \subsetneq \mathcal{F}.$$

A mathematical explanation for the right-hand inclusion was given in [1], and we ask here for a corresponding one for the left-hand inclusion.

References

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