FIBONACCI AND LUCAS REPRESENTATIONS

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ABSTRACT. An identity which relates the Fibonacci and Lucas representations of integers to the Riemann zeta function is derived.

1. Introduction and Results

The Fibonacci numbers ($F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$) can be used as the base of a numeral system. Any nonnegative integer k can be written in the form

$$k = \sum_{i=1}^{\infty} s_i F_i \equiv (\dots s_2 s_1)_F, \tag{1}$$

where $s_i \in \{0, 1\}$. In fact, if no further constraint is applied to the partition, such a Fibonacci representation is not unique if $k > 0$, e.g.,

$$1 = F_1 = F_2, \qquad 2 = F_3 = F_2 + F_1, \qquad 3 = F_4 = F_3 + F_2 = F_3 + F_1, \dots.$$

Let $R(k)$ be the number of Fibonacci representations of k, which defines a rather irregular sequence (see Table 1) [9]. This sequence was first studied by Hoggatt and Basin [7], and later by Klarner [8]. However, subsequent studies were focused on a variant of $R(k)$ in which F_1 is excluded from the base [5, 2, 6].

Similarly, the Lucas numbers ($L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$) can also be used to represent nonnegative integers, i.e.,

$$k = \sum_{i=0}^{\infty} s_i L_i \equiv (\dots s_1 s_0)_L \tag{2}$$

where $s_i \in \{0, 1\}$. The Lucas representation for an integer is also generally not unique. Let $Q(k)$ be the number of Lucas representations of k, which appears to be as irregular as $R(k)$ as a function of k (see Table 2).

In this note we prove that

$$\#\{k : Q(k) = n\} = 3\#\{k : R(k) = n\} \equiv 3w(n) \tag{3}$$

and

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^s} = \frac{\zeta(s-1)}{2\zeta(s) - \zeta(s-1)} \tag{4}$$

if $\mathrm{Re}(s)$ is sufficiently large, where $\zeta(s)$ is the Riemann zeta function.

2. Proof of the Main Result

The proof of identity (3) and (4) is based on a matrix product expression for $R(k)$ and $Q(k)$. Such an expression was first obtained by Berstel for the above-mentioned variant of $R(k)$ [2], which can be generalized to $R(k)$ and $Q(k)$ by a minor modification.

According to a well-known theorem of Zeckendorf, the Fibonacci representation can be made unique if we require $s_1 = 0$ and $s_n s_{n+1} = 0$, i.e., F_1 is excluded from the base and no consecutive Fibonacci numbers are allowed in the summation [10]. Under this restriction, the binary string $S = \dots s_2 s_1$ is called the Zeckendorf code of $k = (S)_F$. The analog of a Zeckendorf code for the Lucas representation is the Brown code, which satisfies $s_0 s_2 = 0$ and $s_n s_{n+1} = 0$ [3]. The matrix product expressions of $R(k)$ and $Q(k)$ turn out to be dependent on the Zeckendorf and Brown codes of k.

Proposition 1. *If $k > 0$ and S is its Zeckendorf code, where the leading infinite substring of 0's is ignored, S can be uniquely written as $10\hat{S}t$, where $\hat{S}$ is a string composed of words $r = 10$, $l = 00$ and $c = 010$ and $t \in \{\epsilon, 0\} \equiv A_F$. Here ϵ denotes the empty string.*

Proof. By induction. □

Proposition 2. *If $k > 2$ and S is its Brown code, where the leading infinite substring of 0's is ignored, S can be uniquely written as $10\hat{S}t$, where $\hat{S}$ is a string composed of words $r = 10$, $l = 00$ and $c = 010$ and $t \in \{0, 00, 10, 01, 010, 001\} \equiv A_L$.*

Proof. By induction. □

We call $\hat{S}$ the essential part of the Zeckendorf or Brown code of k. The following theorem informs us how to calculate $R(k)$ or $Q(k)$ from $\hat{S}$.

Theorem 3. *If the essential part of the Zeckendorf or Brown code of k is $\hat{S} = \sigma_m \dots \sigma_1$, then $R(k)$ or $Q(k)$ is given by*

$$e^T M(\sigma_k) \cdots M(\sigma_1) e \equiv e^T M(\hat{S}) e \equiv g(\hat{S}) \tag{5}$$

where

$$M(r) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad M(l) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad M(c) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are three 2×2 matrices and $e = [1, 1]^T$.

The proof of this theorem is a little tedious and we give it in a separate section (see below). Noticing that $R(0) = 1$, $Q(0) = Q(1) = Q(2) = 1$, and $\#A_L = 3\#A_F$, we have the following proposition.

Proposition 4. *For an arbitrary positive integer n, $\#\{k : Q(k) = n\} = 3\#\{k : R(k) = n\}$.*

A key observation of Theorem 3 is that $M(c) = ee^T$, which allows us to write $g(\hat{S})$ as $g(\hat{S}_1)g(\hat{S}_2)$ if $\hat{S} = \hat{S}_1 c \hat{S}_2$. In other words, we have the following proposition.

Proposition 5. *If $\hat{S} = \tilde{S}_m c \tilde{S}_{m-1} c \dots \tilde{S}_1$, where each $\tilde{S}_j$ is a string composed only of r and l, then*

$$g(\hat{S}) = \prod_{j=1}^{m} g(\tilde{S}_j). \tag{6}$$

The action of $M(r)$ and $M(l)$ on $(m, n)^T$ gives $(m, m + n)^T$ and $(m + n, n)^T$, respectively. If (m, n) is understood as the rational number m/n, this is just the generating rule of the Calkin-Wilf tree of fractions [4]. In addition, $e = [1, 1]^T$ can be identified as the seed of the Calkin-Wilf tree. Therefore, when $\tilde{S}$ runs over all sequences composed only of r and l, $M(\tilde{S})e$ will produce all co-prime pairs $(m, n)^T$ once and only once, with exceptions $(0, 1)^T$ and $(1, 0)^T$. Consequently, we have the following proposition.

Proposition 6.

$$A_s \equiv \sum_{\tilde{S}} g(\tilde{S})^{-s} = \sum_{\substack{m+n>1 \\ (m,n)=1}} \frac{1}{(m+n)^s} = \sum_{k=2}^{\infty} \frac{\varphi(k)}{k^s} = \frac{\zeta(s-1)}{\zeta(s)} - 1 \tag{7}$$

where $\varphi(k)$ is the Euler totient function [1].

Combining all above facts, we finally have the following theorem.

Theorem 7.

$$\sum_{n=1}^{\infty} \frac{w(n)}{n^s} = \sum_{k=0}^{\infty} R(k)^{-s} = \frac{\zeta(s-1)}{2\zeta(s)-\zeta(s-1)}. \tag{8}$$

Proof.

$$\begin{aligned}\sum_{k=0}^{\infty} R(k)^{-s} &= 1 + \sum_{k=1}^{\infty} R(k)^{-s} = 1 + \sum_{\hat{S}} \sum_{t \in A_F} R((10\hat{S}t)_F)^{-s} \\ &= 1 + 2\sum_{\hat{S}} g(\hat{S})^{-s} = 1 + 2(A_s + A_s^2 + \cdots) \\ &= \frac{1+A_s}{1-A_s} = \frac{\zeta(s-1)}{2\zeta(s)-\zeta(s-1)}. \end{aligned} \tag{9}$$

□

3. Discussion

A formula for $w(n)$ is possible as follows. A multiplicative composition of n is a sequence $x_1, x_2, \ldots, x_k$ of integers (for some $k \geq 1$) satisfying

$$n = x_1 x_2 \cdots x_k, \qquad x_j \geq 2 \text{ for all } 1 \leq j \leq k.$$

Let X_n denote the set of all multiplicative compositions of n. For example, if n is a prime power p^m, then $\#X_n = 2^{m-1}$; if n is a product of distinct primes pq, then $\#X_n = 3$. Let $X_{n,k}$ denote the subset of X_n of sequences containing exactly k terms. By use of Proposition 5, it can be shown that

$$w(n) = 2\left[\varphi(n) + \sum_{X_{n,2}} \varphi(x_1)\varphi(x_2) + \sum_{X_{n,3}} \varphi(x_1)\varphi(x_2)\varphi(x_3) + \cdots\right].$$

For example, $w(p^m) = 2(p-1)(2p-1)^{m-1}$ and $w(pq) = 6(p-1)(q-1)$. The arithmetic function $w(n)$ fails to be multiplicative; standard techniques for computing the Dirichlet series corresponding to $w(n)$ do not apply.

Since the number of n-bit binary strings is 2^n while F_n, L_n grow like τ^n when $n \to \infty$, where $\tau = (\sqrt{5}+1)/2$ is the golden ratio, by averaging, $R(k)$ and $Q(k)$ increase as k^α when $k \to \infty$, where

$$\alpha = \frac{\log 2}{\log \tau} = 1.44042\ldots.$$

Thus, $\sum_k R(k)^{-s}$ and $\sum_k Q(k)^{-s}$ diverge for real $s < 1+\alpha$, which is slightly smaller than $2.47875\ldots$, the exact lower bound for convergence of the two series determined by the zero of the denominator of formula (8).

4. Proof of Theorem 3

The key point to proving Theorem 3 is to categorize the Fibonacci (or Lucas) representations into two classes according to whether the leading non-zero bit of the Zeckendorf (or Brown) code is used. The counts of the two classes form a 2-component vector, and Edson and Zamboni found that this vector can be generated from a simple iteration relation [6].

Let us begin with the Fibonacci representation. The proof is based on two facts. One is the inequality

$$\sum_{k=1}^{n} F_k = F_{n+2} - 1 < F_{n+2} \leq 2F_{n+1} \tag{10}$$

for $n \geq 1$. Suppose S_1 and S_2 are two arbitrary binary sequences that begin with 1. The first part of this inequality implies that, if $(S_1)_F = (S_2)_F$, then $|S_1|$ and $|S_2|$ are either equal or differ by one, where $|S|$ denotes the length of S as a binary string. In addition, the second part of this inequality implies that, if $(S_1)_F = (S_2)_F + F_{|S_2|}$, then $|S_1| \geq |S_2|$.

The other fact crucial to our proof is that, of all Fibonacci representations of a positive integer, the Zeckendorf code as a binary string is the one with the largest lexicographical order. Therefore, if $S = 10S'$ is the Zeckendorf code of a positive integer k and $S_1 = 1S''$ is an arbitrary Fibonacci representation of k, i.e., $(S_1)_F = k = (S)_F$, then $|S_1| = |S|$ or $|S| - 1$.

The above described properties of the Fibonacci representation lead naturally to the following definitions.

Definition 8. *Letting $S = 10\ldots$ be the Zeckendorf code of a positive integer, define*

$$R_0(S) = \#\{S' \in [0,1]^{|S|-1} : (S')_F = (S)_F\}, \tag{11}$$

$$R_1(S) = \#\{S' \in [0,1]^{|S|-1} : (1S')_F = (S)_F\}. \tag{12}$$

Using this notation, we have the following proposition.

Proposition 9. *If $S = 10S'$ is a Zeckendorf code, then*

$$R((S)_F) = R_0(S) + R_1(S) = e^T \begin{bmatrix} R_0(S) \\ R_1(S) \end{bmatrix}. \tag{13}$$

Moreover, from the definition and properties of the Zeckendorf code we have the following proposition.

Proposition 10. *If $10S'$ is a Zeckendorf code, then*

$$R_1(10S') = R((S')_F). \tag{14}$$

We then consider how R_0 and R_1 change when a Zeckendorf code $10S$ is expanded to $10rS$, $10lS$ or $10cS$. The iteration rules for R_1 can be readily derived as follows:

$$R_1(1010S) = R((10S)_F) = R_0(10S) + R_1(10S), \tag{15}$$

$$R_1(1000S) = R((00S)_F) = R((S)_F) = R_1(10S), \tag{16}$$

$$R_1(10010S) = R((010S)_F) = R((10S)_F) = R_0(10S) + R_1(10S). \tag{17}$$

For the iteration rules of R_0, note that

$$\begin{aligned} R_0(10S) &= \#\{S' \in [0,1]^{|S|} : (1S')_F = (10S)_F\} \\ &= \#\{S' \in [0,1]^{|S|} : (S')_F = F_{|S|} + (S)_F\}, \end{aligned}$$

hence we have the following,

$$\begin{aligned} R_0(1010S) &= \#\{S' \in [0,1]^{|S|+2} : (S')_F = F_{|S|+2} + (10S)_F\} \\ &= \#\{S' \in [0,1]^{|S|+1} : (1S')_F = F_{|S|+2} + (10S)_F\} \\ &= \#\{S' \in [0,1]^{|S|+1} : (S')_F = (10S)_F\} \\ &= R_0(10S), \end{aligned} \tag{18}$$

$$\begin{aligned} R_0(1000S) &= \#\{S' \in [0,1]^{|S|+2} : (S')_F = F_{|S|+2} + (00S)_F = (10S)_F\} \\ &= R((10S)_F) = R_0(10S) + R_1(10S), \end{aligned} \tag{19}$$

$$\begin{aligned} R_0(10010S) &= \#\{S' \in [0,1]^{|S|+3} : (S')_F = F_{|S|+3} + (010S)_F = (1000S)_F\} \\ &= R_0(1000S) = R_0(10S) + R_1(10S). \end{aligned} \tag{20}$$

Combining formulas 15–17 and 18–20, we have the following proposition.

Proposition 11. *If $10S$ is the Zeckendorf code of a positive integer, then*

$$\begin{bmatrix} R_0(10\sigma S) \\ R_1(10\sigma S) \end{bmatrix} = M(\sigma) \begin{bmatrix} R_0(10S) \\ R_1(10S) \end{bmatrix} \tag{21}$$

for $\sigma = r, l, c$.

Finally, we can readily check that $R_0(10) = 1$ (as $F_2 = F_1$), $R_1(10) = 1$ (as $F_2 = F_2$), $R_0(100) = 1$ (as $F_3 = F_2 + F_1$) and $R_1(100) = 1$ (as $F_3 = F_3$), i.e.,

$$\begin{bmatrix} R_0(10) \\ R_1(10) \end{bmatrix} = \begin{bmatrix} R_0(100) \\ R_1(100) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e. \tag{22}$$

Combining this formula with Propositions 9 and 11, Theorem 3 for the Fibonacci representation is proved.

We now consider the Lucas representation. Because similar inequalities hold for Lucas numbers, i.e.,

$$\sum_{j=0}^{k} L_j = L_{k+2} - 1 < 2L_{k+1} \tag{23}$$

for $k \geq 1$ and the Brown code is also constructed from a greedy algorithm for the largest lexicographical order, the same iteration rules hold for similarly defined Q_0 and Q_1. Thus we need only to verify that

$$\begin{bmatrix} Q_0(10t) \\ Q_1(10t) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{24}$$

for each $t \in A_L$, which is obviously true (see Table 3). Therefore we complete the proof of Theorem 3.

Table 1.

n	Zeckendorf code	segmentation	essential part	$R(n)$
1	10	10	ϵ	2
2	100	10 0	ϵ	2
3	1000	10 00	l	3
4	1010	10 10	r	3
5	10000	10 00 0	l	3
6	10010	10 010	c	4
7	10100	10 10 0	r	3
8	100000	10 00 00	ll	4
9	100010	10 00 10	lr	5
10	100100	10 010 0	c	4
11	101000	10 10 00	rl	5
12	101010	10 10 10	rr	4
13	1000000	10 00 00 0	ll	4
14	1000010	10 00 010	lc	6
15	1000100	10 00 10 10	lr	5
16	1001000	10 010 00	cl	6
17	1001010	10 010 10	cr	6
18	1010000	10 10 00 0	rl	5
19	1010010	10 10 010	rc	6
20	1010100	10 10 10 0	rr	4
21	10000000	10 00 00 00	lll	5
22	10000010	10 00 00 10	llr	7
23	10000100	10 00 010 0	lc	6
24	10001000	10 00 10 00	lrl	8
25	10001010	10 00 10 10	lrr	7
26	10010000	10 010 00 0	cl	6
27	10010010	10 010 010	cc	8
28	10010100	10 010 10 0	cr	6
29	10100000	10 10 00 00	rll	7
30	10100010	10 00 00 10	rlr	8
31	10100100	10 10 010 0	rc	6
32	10101000	10 10 10 00	rrl	7
33	10101010	10 10 10 10	rrr	5

Table 2.

n	Brown code	segmentation	essential part	$Q(n)$
1	10	—	—	1
2	1	—	—	1
3	100	10 0	ϵ	2
4	1000	10 00	ϵ	2
5	1010	10 10	ϵ	2
6	1001	10 01	ϵ	2
7	10000	10 00 0	l	3
8	10010	10 010	ϵ	2
9	10001	10 001	ϵ	2
10	10100	10 10 0	r	3
11	100000	10 00 00	l	3
12	100010	10 00 10	l	3
13	100001	10 00 01	l	3
14	100100	10 010 0	c	4
15	101000	10 10 00	r	3
16	101010	10 10 10	r	3
17	101001	10 10 01	r	3
18	1000000	10 00 00 0	ll	4
19	1000010	10 00 010	l	3
20	1000001	10 00 001	l	3
21	1000100	10 00 10 0	lr	5
22	1001000	10 010 00	c	4
23	1001010	10 010 10	c	4
24	1001001	10 010 01	c	4
25	1010000	10 10 00 0	rl	5
26	1010010	10 10 010	r	3
27	1010010	10 10 001	r	3
28	1010100	10 10 10 0	rr	4

Table 3.

t	$Q_0(10t)$	partition	$Q_1(10t)$	partition
0	1	$3 = L_1 + L_0$	1	$3 = L_2$
00	1	$4 = L_2 + L_1$	1	$4 = L_3$
01	1	$6 = L_2 + L_1 + L_0$	1	$6 = L_3 + L_0$
10	1	$5 = L_2 + L_0$	1	$5 = L_3 + L_1$
001	1	$9 = L_3 + L_2 + L_0$	1	$9 = L_4 + L_0$
010	1	$8 = L_3 + L_2 + L_1$	1	$8 = L_4 + L_1$

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