FIBONACCI AND LUCAS NUMBERS WHICH HAVE EXACTLY THREE PRIME FACTORS AND SOME UNIQUE PROPERTIES OF F_{18} AND L_{18}

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Abstract. Let F_n and L_n be the nth Fibonacci and Lucas numbers, respectively. Let $\omega(n)$ be the number of prime factors of n, $d(n)$ the number of positive divisors of n, $A(n)$ the least positive reduced residue system modulo n, and $\ell(n)$ the length of the longest arithmetic progressions contained in $A(n)$. On the occasion of attending the 18th Fibonacci Conference, we give some results concerning $\omega(F_n)$, $\omega(L_n)$, $d(F_n)$, and $d(L_n)$ which reveal a unique property of F_{18} and L_{18}. We also find the solutions to the equation $\ell(n) = 18$ and show a connection between them and F_{18}. Some examples and numerical data are also given.

1. Introduction

The Fibonacci numbers are defined by the recurrence $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ with the initial values $F_1 = F_2 = 1$. The Lucas numbers L_n are defined by the same recursive pattern as the Fibonacci sequence but with the values $L_1 = 1$ and $L_2 = 3$. These numbers are famous for possessing wonderful properties, see for example in [12] and [34] for additional references and history. The Fibonacci Association was formed in order to provide enthusiasts an opportunity to exchange ideas about Fibonacci numbers and other integer sequences. On the occasion of attending the 18th Fibonacci Conference, the author would like to share an idea on some unique properties of F_{18} and L_{18}. We remark that F_n and L_n are also defined for $n \leq 0$, but we will focus our attention only on the case $n > 0$.

For each $n \in \mathbb{N}$, let $\omega(n)$ be the number of distinct prime factors of n and $d(n)$ the number of positive divisors of n. Bugeaud, Luca, Mignotte, and Siksek [4] give a description of F_n for which $\omega(F_n) \leq 2$. In this article, we extend the investigation on $\omega(F_n)$, $\omega(L_n)$, $d(F_n)$, and $d(L_n)$, which leads us to see some unique properties of F_{18} and L_{18} (with some restrictions).

In addition, let $A(n) = \{a \in \mathbb{N} \mid 1 \leq a \leq n \text{ and } (a, n) = 1\}$ be the least positive reduced residue system modulo n and let $\ell(n)$ be the length of longest arithmetic progressions contained in $A(n)$. Recamán (see Guy's book [7, Chapter B40]) asks if $\ell(n) \to \infty$ as $n \to \infty$. Stumpf [32] gives an affirmative answer to this question. Pongsriiam [19] completely solves this problem by giving exact formulas for $\ell(n)$ for all $n \in \mathbb{N}$. In this article, we find all positive integers n satisfying $\ell(n) = 18$ and show a connection between them and F_{18}.

We organize this article as follows. In Section 2, we give some preliminaries and lemmas. In Section 3, we present the results on $\omega(F_n)$, $d(F_n)$, $\omega(L_n)$, and $d(L_n)$. Finally, we find the solutions to the equation $\ell(n) = 18$ in Section 4. For some recent results concerning the divisibility properties or Diophantine equations involving F_n and L_n, we refer the reader to [17, 18, 20, 21, 22, 23, 24, 25] and references therein. Throughout this article, the letters p and q with or without subscript denote a prime.

2. Preliminaries and Lemmas

In this section, we recall some useful lemmas for the reader's convenience. In particular, Lemmas 2.1, 2.2, 2.3, and Theorem 2.4 will be used throughout this article, sometimes without reference.

Lemma 2.1. *Let m and n be positive integers and $d = (m, n)$. Then*

(i) $(F_m, F_n) = F_d$.

(ii) $(L_m, L_n) = \begin{cases} L_d, & \text{if } \frac{m}{d} \text{ and } \frac{n}{d} \text{ are odd,} \\ 2, & \text{if } \frac{m}{d} \text{ or } \frac{n}{d} \text{ is even and } 3 \mid d, \\ 1, & \text{otherwise.} \end{cases}$

(iii) $(F_m, L_m) = \begin{cases} 2, & \text{if } 3 \mid m, \\ 1, & \text{otherwise.} \end{cases}$

(iv) $F_{2m} = F_m L_m$.

(v) *For $m \neq 2$, $F_m \mid F_n$ if and only if $m \mid n$.*

(vi) *For $m \neq 1$, $L_m \mid L_n$ if and only if $m \mid n$ and $\frac{n}{m}$ is odd.*

(vii) (Binet's formula) *For every $n \geq 1$, we have $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $L_n = \alpha^n + \beta^n$ where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$.*

(viii) *For every $n \geq 2$, $L_n = F_{n+1} + F_{n-1} = F_n + 2F_{n-1} = 2F_n + F_{n-3}$.*

Proof. These are well-known results. For proofs see, for example, [12, 34]. □

We write $z(n)$ to denote the order (or the rank) of appearance of n in the Fibonacci sequence which is defined as the smallest positive k such that $n \mid F_k$. For some recent results on $z(n)$, see [8, 9, 10, 26, 29] and references therein. Basic properties of $z(n)$ are the following.

Lemma 2.2. *Let $p \neq 5$ be a prime and let m and n be positive integers. Then the following statements hold.*

(i) *$n \mid F_m$ if and only if $z(n) \mid m$.*

(ii) *$z(p) \mid p + 1$ if and only if $p \equiv 2$ or $-2 \pmod 5$ and $z(p) \mid p - 1$, otherwise.*

(iii) $\gcd(z(p), p) = 1$.

Proof. These are also well known. See [18, Lemma 1] for more details. □

Recall that the p-adic valuation (or p-adic order) of n, denoted by $v_p(n)$, is the exponent of p in the prime factorization of n. The formulas for $v_p(F_n)$ and $v_p(L_n)$ are as follows.

Lemma 2.3. (Lengyel [14]) *Let n be a positive integer. Then*

$$v_2(F_n) = \begin{cases} 0, & if\ n \equiv 1, 2 \pmod 3, \\ 1, & if\ n \equiv 3 \pmod 6, \\ v_2(n) + 2, & if\ n \equiv 0 \pmod 6; \end{cases}$$

$$v_2(L_n) = \begin{cases} 0, & if\ n \equiv 1, 2 \pmod 3, \\ 2, & if\ n \equiv 3 \pmod 6, \\ 1, & if\ n \equiv 0 \pmod 6; \end{cases}$$

$v_5(F_n) = v_5(n)$, $v_5(L_n) = 0$, *and if* p *is a prime,* $p \neq 2$, *and* $p \neq 5$, *then*

$$v_p(F_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } n \equiv 0 \pmod{z(p)}, \\ 0, & \text{if } n \not\equiv 0 \pmod{z(p)}; \end{cases}$$

$$v_p(L_n) = \begin{cases} v_p(n) + v_p(F_{z(p)}), & \text{if } z(p) \text{ is even and } n \equiv \frac{z(p)}{2} \pmod{z(p)}, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose that (u_n) is the Fibonacci or Lucas sequence. A prime p is said to be a primitive divisor of u_n if $p \mid u_n$ but p does not divide u_m for any $m < n$. Then a special case of the primitive divisor theorem of Carmichael can be stated as follows.

Theorem 2.4. (Carmichael [5]) *The Fibonacci number* F_n *has a primitive divisor for every* $n \neq 1, 2, 6, 12$, *and the Lucas number* L_n *has a primitive divisor for every* $n \neq 1, 6$.

There is a long history about primitive divisors and the most remarkable results in this topic are given by Bilu, Hanrot, and Voutier [1], by Stewart [31], and by Kunrui [13]. Nevertheless, Theorem 2.4 is good enough in our situation.

Bugeaud, Luca, Mignotte, and Siksek [4, proof of Theorem 5] show that $\omega(F_n) \geq d(n) - 4$ for all $n \in \mathbb{N}$. We extend this to the following lemma.

Lemma 2.5. *Let* m *be a positive integer and let* $x(m)$ *be the number of elements in the set* $\{1, 2, 6, 12\} \cap \{d : d \mid m\}$. *Then the following statements hold:*

(i) $\omega(F_m) \geq d(m) - x(m)$.
(ii) *If* $\omega(m) \geq 3$, *then* $\omega(F_m) \geq 5$.
(iii) $\omega(F_{2^m}) = m - 1$ *for* $m < 6$ *and* $\omega(F_{2^m}) \geq m$ *for* $m \geq 6$.

Proof. We use Lemma 2.1(v) and Theorem 2.4 throughout the proof without further reference. If $k \mid m$ and $k \notin \{1, 2, 6, 12\}$, then $F_k \mid F_m$ and F_k has a primitive divisor. By writing all such k in an increasing order as $k_1 < k_2 < \cdots < k_\ell$, we see that $\omega(F_m) \geq \ell = d(m) - x(m)$. This proves (i). Next assume that $\omega(m) \geq 3$. If $2 \nmid m$ or $3 \nmid m$, then at least one of 2 or 6 is not a divisor of m, so $x(m) \leq 3$ and $\omega(F_m) \geq d(m) - x(m) \geq 2^{\omega(m)} - 3 \geq 5$. Suppose that $2 \mid m$ and $3 \mid m$. Since $\omega(m) \geq 3$, there exists a prime $p \neq 2, 3$ such that $p \mid m$. Then $6p \mid m$, so $F_{6p} \mid F_m$ and

$$\omega(F_m) \geq \omega(F_{6p}) \geq d(6p) - 3 = 5.$$

Next we prove (iii). If $m < 6$, then $\omega(F_{2^m})$ can be obtained by using the table distributed by the Fibonacci Association [2]. In addition, $\omega(F_{2^6}) = 6$ and for $m > 6$, we have the chain of divisibility $F_{2^6} \mid F_{2^7} \mid \cdots \mid F_{2^{m-1}} \mid F_{2^m}$ and each of them has a primitive prime divisor. Therefore if $m > 6$, then

$$\omega(F_{2^m}) \geq \omega(F_{2^{m-1}}) + 1 \geq \omega(F_{2^{m-2}}) + 2 \geq \cdots \geq \omega(F_{2^6}) + m - 6 = m.$$

□

Using an intricate combination of Baker's method, modular method, and computer verification, Bugeaud, Mignotte, and Siksek [3] were able to determine all perfect powers in the Fibonacci and Lucas sequences. We state their result in the following theorem.

Theorem 2.6. (Bugeaud, Mignotte, and Siksek [3]) *The only solutions to the equation*

$$F_n = y^m \quad \text{in integers } m \geq 2,\ n \geq 0,\ y \geq 0$$

are given by $n = 0, 1, 2, 6$, *and* 12 *which correspond respectively to* $F_n = 0, 1, 1, 8$, *and* 144. *Moreover, the only solutions to the equation* $L_n = y^m$ *with* $m \geq 2$, $n \geq 0$, $y \geq 0$ *are given by* $n = 1$ *and* 3 *which correspond respectively to* $L_n = 1$ *and* 4.

Next we recall exact formulas for the length of longest arithmetic progressions contained in the least positive reduced residue systems.

Lemma 2.7. (Pongsriiam [19, Theorem 3.1]) *Suppose that $n > 1$ is squarefree and p is the largest prime factor of n. Then*

$$\ell(n) = \begin{cases} p-1, & \text{if } n \text{ is a prime;} \\ \frac{p+1}{2}, & \text{if } n = 2p,\ p \geq 3, \text{ and } p \equiv 3 \pmod 4; \\ p - \left\lfloor \frac{p^2}{n} \right\rfloor - 1, & \text{otherwise.} \end{cases}$$

Lemma 2.8. (Pongsriiam [19, Theorem 3.2 and Theorem 3.3]) *Suppose that $n > 1$ is not squarefree, p is the largest prime factor of n, and $\gamma(n) = \prod_{p \mid n} p$. Then*

$$\ell(n) = \max\left\{\frac{n}{\gamma(n)}, p-1\right\}.$$

3. Main Results I: On $\omega(F_n)$, $d(F_n)$, $\omega(L_n)$, and $d(L_n)$

Bugeaud, Luca, Mignotte, and Siksek [4, Lemma 3.1 and Theorem 4] give a description of the integers n satisfying $\omega(F_n) \leq 2$. We extend it to the case $\omega(F_n) = 3$ in the next theorem.

Theorem 3.1. *The only solutions to the equation $\omega(F_n) = 3$ are given by*

$$n = 16, 18, \text{ or } 2p \text{ for some prime } p \geq 19, \tag{3.1}$$

$$n = p, p^2, p^3 \text{ for some prime } p \geq 5, \tag{3.2}$$

$$n = pq \text{ for some distinct primes } p, q \geq 3. \tag{3.3}$$

Proof. Assume that $\omega(F_n) = 3$. By Lemma 2.5(ii), we obtain $\omega(n) \leq 2$. Therefore $n = 2^a$, p^a, $2^a p^b$, or $p^a q^b$ for some distinct odd primes p, q and positive integers a, b.

Case 1. $n = 2^a$. By Lemma 2.5(iii), we obtain $3 = \omega(F_n) \geq a - 1$, so $a \leq 4$. By Lemma 2.5(iii) again, $a - 1 = \omega(F_n) = 3$. So $a = 4$ and we obtain $n = 2^4$ as a solution to $\omega(F_n) = 3$.

Case 2. $n = p^a$. By Lemma 2.5(i), we have $3 = \omega(F_n) \geq d(p^a) - x(p^a) = a + 1 - 1 = a$. So $a \leq 3$ and thus $n = p$, p^2, or p^3. We also check that $\omega(F_3)$, $\omega(F_9)$, $\omega(F_{27}) \neq 3$. So $p \geq 5$. This case corresponds to (3.2).

Case 3. $n = 2^a p^b$. If $a \geq 2$ and $p \geq 5$, then $4p \mid n$, so $F_{4p} \mid F_n$ and we obtain by Lemma 2.5(i) that $\omega(F_n) \geq \omega(F_{4p}) \geq d(4p) - 2 = 4$, which is not the case. So $a = 1$ or $p = 3$. From this point on, we apply Lemma 2.5 without further reference.

Case 3.1. $a = 1$. If $p \geq 5$, then $3 = \omega(F_n) = \omega(F_{2p^b}) \geq d(2p^b) - 2 = 2b$, which implies $b = 1$ and so $n = 2p$. Suppose $p = 3$. If $b = 1$, then $n = 6$ and $\omega(F_n) = 1 \neq 3$. If $b \geq 3$, then $54 \mid n$ and $\omega(F_n) \geq \omega(F_{54}) = 6$, which is not the case. So $b = 2$ and we see that $\omega(F_n) = \omega(F_{18}) = 3$. We also check that $\omega(F_{2p}) \neq 3$ if $p < 19$. Therefore this case leads to the solutions $n = 2p$ with $p \geq 19$ or $n = 18$, which correspond to (3.1).

Case 3.2. $a > 1$ and $p = 3$. If $a \geq 3$, then $24 \mid n$ and $\omega(F_n) \geq \omega(F_{24}) = 4$, which is not the case. So $a = 2$. Then if $b \geq 2$, then $36 \mid n$ and $\omega(F_n) \geq \omega(F_{36}) = 5$, a contradiction. If $b = 1$, then $n = 12$ and $\omega(F_n) = \omega(F_{12}) = 2 \neq 3$. So there is no solution in this case.

Case 4. $n = p^a q^b$. Then $3 = \omega(F_n) \geq d(p^a q^b) - 1 \geq a + b + 1 \geq 3$. So $a + b = 2$, which implies $a = b = 1$ and $n = pq$. This corresponds to (3.3) and the proof is complete. □

Example 3.2. By considering the table distributed by the Fibonacci Association [2], we obtain some examples of all the possibilities given in Theorem 3.1 as follows.

(i) $\omega(F_n) = 3$ and $n = 2p$: $n = 2 \times 19$, $F_n = 37 \times 113 \times 9349$; $n = 2 \times 23$, $F_n = 139 \times 461 \times 28657$; $n = 2 \times 29$, $F_n = 59 \times 19489 \times 514229$.
(ii) $\omega(F_n) = 3$ and $n = p, p^2, p^3$: $n = 37$, $F_n = 73 \times 149 \times 2221$; $n = 7^2$, $F_n = 13 \times 97 \times 6168709$; $n = 5^3$, $F_n = 5^3 \times 3001 \times 158414167964045700001$.
(iii) $\omega(F_n) = 3$ and $n = pq$: $n = 3 \times 5$, 3×7, 3×11, 5×7 and $F_n = 2 \times 5 \times 61$, $2 \times 13 \times 421$, $2 \times 89 \times 19801$, $5 \times 13 \times 141961$, respectively.

We also remark that not all the integers in (3.1), (3.2), (3.3) give a solution to $\omega(F_n) = 3$. For instance, from the table [2] and computer verification, we obtain the following.

(iv) $n = 2p$ and $\omega(F_n) \neq 3$: $n = 2 \times 37$, $F_n = 73 \times 149 \times 2221 \times 54018521$.
(v) $n = p, p^2, p^3$ and $\omega(F_n) \neq 3$: $n = 7$, $F_n = 13$; $n = 5^2$, $F_n = 5^2 \times 3001$; $n = 7^3$, $F_n = 13 \times 97 \times 5449038756620509 \times 6168709 \times 46649 \times m$ where m is a positive integer with 42 digits.
(vi) $n = pq$ and $\omega(F_n) \neq 3$: $n = 3 \times 19$, $F_n = 2 \times 37 \times 113 \times 797 \times 54833$.

A more precise description of the Fibonacci numbers satisfying $\omega(F_n) = 3$ is given in the next theorem.

Theorem 3.3. *Assume that $\omega(F_n) = 3$ and $n = p_1p_2$ where $p_1 < p_2$ are odd primes. Then $F_{p_1} = q_1$, $F_{p_2} = q_2$, and $F_n = q_1^{a_1} q_2 q_3^{a_3}$ where q_1, q_2, q_3 are distinct primes, q_3 is a primitive divisor of F_n, $a_3 \geq 1$ and $a_1 \in \{1, 2\}$. Furthermore $a_1 = 2$ if and only if $q_1 = p_2$.*

Proof. Since F_{p_1} and F_{p_2} divide F_n and each of them has a primitive divisor, we obtain $3 = \omega(F_n) \geq \omega(F_{p_1}) + \omega(F_{p_2}) + 1 \geq 3$, which implies $\omega(F_{p_1}) = 1 = \omega(F_{p_2})$. By Theorem 2.6, $F_{p_1} = q_1$ and $F_{p_2} = q_2$ where q_1, q_2 are primes. Since $p_1 < p_2$, we obtain $q_1 < q_2$. In addition, since $q_1, q_2 \mid F_n$ and $\omega(F_n) = 3$, we see that $F_n = q_1^{a_1} q_2^{a_2} q_3^{a_3}$ where q_3 is a primitive divisor of F_n, and a_1, a_2, a_3 are positive integers. It remains to show that $a_2 = 1$ and $a_1 \in \{1, 2\}$.

Case 1. $p_1 \geq 7$. So $q_1, q_2 > 5$. By Lemma 2.2, we know that $z(q_1) = p_1$, $z(q_2) = p_2$, and $(q_1, p_1) = 1 = (q_2, p_2)$. In addition, $q_2 = F_{p_2} > p_2 > p_1$. Therefore we obtain by Lemma 2.3 that $a_2 = v_{q_2}(F_n) = v_{q_2}(p_1p_2) + v_{q_2}(F_{z(q_2)}) = 1$ and $a_1 = v_{q_1}(F_n) = v_{q_1}(p_1p_2) + v_{q_1}(F_{z(q_1)}) = v_{q_1}(p_2) + 1 = 1$ or 2, and $a_1 = 2$ if and only if $q_1 = p_2$.

Case 2. $p_1 = 5$. Then $q_1 = 5$ and similar to Case 1, we obtain $q_2 > p_2 > p_1$ and $F_n = 5^{a_1} q_2^{a_2} q_3^{a_3}$. Then $a_1 = v_5(F_n) = v_5(n) = 1$ and $a_2 = v_{q_2}(F_n) = v_{q_2}(p_1p_2) + 1 = 1$.

Case 3. $p_1 = 3$. Then $q_1 = 2$. If $p_2 = 5$, then $F_n = F_{15} = 2 \times 5 \times 61$, $q_2 = 5$, $q_3 = 61$, and $a_1 = a_2 = a_3 = 1$. If $p_2 \geq 7$, then $q_2 > p_2 > p_1$, $a_2 = v_{q_2}(F_n) = v_{q_2}(p_1p_2) + 1 = 1$, $a_1 = v_{q_1}(F_n) = v_2(F_n) = 1$. This completes the proof. $\square$

Suppose $\omega(F_n) = 3$ and $n = 2p$ where $p \geq 19$. Then by the method similar to the proof of Theorem 3.3, we obtain that F_p or L_p is a prime and $F_n = p_1 p_2^{a_2} p_3^{a_3}$, where p_1, p_2, p_3 are distinct primes and $(a_2, a_3) = 1$. The other cases of Theorem 3.1 can also be further analyzed in a similar way. To give results analogous to Theorem 3.1 for the Lucas numbers, we first prove the following lemma.

Lemma 3.4. *Let $m \in \mathbb{N}$ and $y(m)$ the number of elements in the set $\{1, 6\} \cap \{d : d \mid m\}$. Then the following statements hold.*

(i) *$\omega(L_m) \geq d(m/2^{v_2(m)}) - y(m)$.*
(ii) *If $\omega(m) \geq 3$, then $\omega(L_m) \geq 4$.*

Proof. If $k \mid m$, m/k is odd, and $k \neq 1, 6$, then $L_k \mid L_m$ and L_k has a primitive divisor. The conditions $k \mid m$ and m/k is odd means that $k = 2^{v_2(m)} k_1$ where k_1 is odd and k_1 divides $m/2^{v_2(m)}$. This implies (i). Next assume that $\omega(m) \geq 3$. If m is odd, then we obtain by (i)

that
$$\omega(L_m) \geq d(m) - 1 \geq 2^{\omega(m)} - 1 \geq 7.$$
So we suppose $m = 2^a p_2^b p_3^c k$ where $p_2 < p_3$ are odd primes, a, b, c are positive integers, and k is odd. If $p_2 > 3$ or $a > 1$ or $b > 1$, then L_{2^a}, $L_{2^a p_2^b}$, $L_{2^a p_3^c}$, and L_m are distinct divisors of L_m, each of which has a primitive divisor, and thus $\omega(L_m) \geq 4$. Therefore it remains to consider the case $p_2 = 3$ and $a = b = 1$. Then $m = 6p_3^c k$, and L_6, $L_{2p_3^c}$, and L_m are divisors of L_m. In addition, $L_{2p_3^c}$ and L_m have a primitive divisor. Therefore $\omega(L_m) \geq \omega(L_6) + 2 = 4$. This completes the proof. □

Theorem 3.5. *If $\omega(L_n) = 3$, then n satisfies one of the following conditions: $n = 2^a$ for some $a > 7$, $n = p$, p^2, p^3 for some odd prime p, $n = 2^a p$, $2^a p^2$ for some odd prime p and positive integer a, $n = pq$ for some distinct odd primes p, q. In addition, if $\omega(L_n) = 3$ and $n = 9 \cdot 2^a$ for some $a \geq 1$, then $n = 18$.*

Proof. Since $\omega(L_n) = 3$, we obtain by Lemma 3.4 that $\omega(n) \leq 2$. Therefore $n = 2^a, p^a, 2^a p^b$, or $p^a q^b$ for some distinct odd primes p, q and positive integers a, b. Since the proof is similar to that of Theorem 3.1, we give fewer details. Since $\omega(L_{2^a}) < 3$ for $a \leq 7$, we see that if $n = 2^a$, then $a > 7$. If $n = p^a$, then $\omega(L_n) = \omega(L_{p^a}) \geq d(p^a) - 1 = a$, so $n = p$, p^2, p^3. Next assume that $n = 2^a p^b$. Since $L_{2^a p^k} \mid L_{2^a p^b}$ for all $k = 0, 1, \ldots, b$, we see that $\omega(L_n) \geq b + 1$, so $b \leq 2$ and $n = 2^a p$ or $2^a p^2$. If $n = p^a q^b$, then $\omega(L_n) \geq d(p^a q^b) - 1 \geq a + b + 1$, and thus $a = b = 1$ and $n = pq$. This proves the first part. For the second part, suppose for a contradiction that $\omega(L_n) = 3$, $n = 9 \cdot 2^a$, but $a \geq 2$. Then 2, L_{2^a}, $L_{2^a \cdot 3}$, L_n divide L_n and each of them has a primitive divisor. Therefore $\omega(L_n) \geq 4$, a contradiction. So $a = 1$ and thus $n = 18$. □

Example 3.6. By using the table [2] and computer verification, we obtain some examples of all the possibilities given in Theorem 3.5 as follows.

(i) $\omega(L_n) = 3$ and $n = 2^a$: $n = 2^8$ and $L_n = 34303 \times 73327699969 \times p$ where p is a prime with 39 digits.
(ii) $\omega(L_n) = 3$ and $n = p$, p^2, p^3: $n = 59$, $L_n = 709 \times 8969 \times 336419$; $n = 5^2$, $L_n = 11 \times 101 \times 151$; $n = 3^3$, $L_n = 2^2 \times 19 \times 5779$.
(iii) $\omega(L_n) = 3$ and $n = 2^a p$, $2^a p^2$: $n = 2 \times 11, 2^2 \times 3, 2^3 \times 5, 2 \times 3^2$ and $L_n = 3 \times 43 \times 307$, $2 \times 7 \times 23$, $47 \times 1601 \times 3041$, $2 \times 3^3 \times 107$, respectively.
(iv) $\omega(L_n) = 3$ and $n = pq$: $n = 3 \times 5$, $L_n = 2^2 \times 11 \times 31$.

As in Theorem 3.1 and Example 3.2, not all integers of the form given in Theorem 3.5 give a solution to $\omega(L_n) = 3$; such the integers can be obtained from the table [2].

Next we present a unique property of F_{18}. If p and $p+2$ are primes, then we call p and $p+2$ twin primes. Since 17 and 19 divide F_{18}, we say that F_{18} has twin prime factors. In addition, F_n has no twin prime factors for $n < 18$. So F_{18} is the smallest Fibonacci number which has twin prime factors. In fact, F_{18} is the only even Fibonacci number which has exactly three prime factors, two of which are twin primes. To show this, we first prove the following lemma.

Lemma 3.7. *The following statements hold.*

(i) *For every $n \geq 5$, $2F_n(F_n + 2)^2 < F_{3n}$.*
(ii) *For every $n \geq 7$, $F_{3n} < 2F_n(F_n - 2)^3$.*

Proof. We let $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$, and we apply Lemma 2.1 throughout the proof. We first observe that
$$F_{3n} = \frac{\alpha^{3n} - \beta^{3n}}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\alpha - \beta}\left(\alpha^{2n} + (\alpha\beta)^n + \beta^{2n}\right) = F_n(L_{2n} + (-1)^n).$$

So (i) and (ii) are, respectively, equivalent to

$$2(F_n+2)^2 < L_{2n} + (-1)^n \quad \text{and} \quad L_{2n} + (-1)^n < 2(F_n-2)^3.$$

For $n \geq 5$, we have

$$\begin{aligned} L_{2n} + (-1)^n &\geq L_{2n} - 1 = 2F_{2n} + F_{2n-3} - 1 > 2F_{2n} \\ &= 2F_nL_n = 2F_n(F_n + 2F_{n-1}) = 2F_n^2 + F_n(4F_{n-1}) \\ &\geq 2F_n^2 + 12F_n > 2F_n^2 + 8F_n + 8 = 2(F_n+2)^2. \end{aligned}$$

For $n \geq 7$, we have

$$\begin{aligned} L_{2n} + (-1)^n &\leq L_{2n} + 1 \leq (\alpha^n - \beta^n)^2 + 3 = 5F_n^2 + 3 < 6F_n^2, \\ 2(F_n-2)^3 &= 2F_n^2\left(F_n - 6 + \frac{12}{F_n} - \frac{8}{F_n^2}\right) > 2F_n^2(F_7 - 6) > 6F_n^2, \end{aligned}$$

and so $L_{2n} + (-1)^n < 2(F_n-2)^3$, as required. □

Theorem 3.8. *F_{18} is the only even Fibonacci number which has exactly three prime factors where two of the prime factors are twins.*

Proof. Since $F_{18} = 2^3 \times 17 \times 19$, we see that F_{18} is even, $\omega(F_{18}) = 3$, and 17 and 19 are twin prime factors of F_{18}. Suppose that F_n is even, $\omega(F_n) = 3$, and there are twin primes p, $p+2$ dividing F_n. Since F_n is even, $3 \mid n$. Then by Theorem 3.1, $n = 18$ or $n = 3q$ for some prime $q \geq 5$. Consider the case $n = 3q$. If $q = 5$, then $F_n = F_{15}$ which does not have twin prime factors. So $q \geq 7$. By Theorem 3.3 and the assumption $p(p+2) \mid F_{3q}$, we see that

(i) $F_q = p$ and $F_{3q} = 2p(p+2)^a$ for some $a \geq 1$,
or (ii) $F_q = p+2$ and $F_{3q} = 2(p+2)p^b$ for some $b \geq 1$.

If $a \leq 2$ or $b \leq 2$, then $F_{3q} \leq 2F_q(F_q+2)^2$, which contradicts Lemma 3.7(i). If $a > 2$ and $b > 2$, then $F_{3q} \geq 2F_q(F_q-2)^3$, which contradicts Lemma 3.7(ii). So $n = 3q$ is not possible. Hence $n = 18$ and the proof is complete. □

Next we show a joint unique property of F_{18} and L_{18}. From the table [2], we see that $n = 18$ is the only positive integer $n \leq 150$ satisfying $\omega(F_n) = \omega(L_n) = 3$ and $d(F_n) = d(L_n) = 16$. The range $n \leq 150$ can be extended further by using a computer. In fact, this problem is connected to the existence or nonexistence of the prime p such that $v_p(F_{z(p)}) > 1$. Wall [35] observed that $v_p(F_{z(p)}) = 1$ for all $p < 10^4$. Mcintosh and Roettger [16], and Dorais and Klyve [6] extended the range $p < 10^4$ to $p < 2 \times 10^{14}$ and to $p < 9.7 \times 10^{14}$, respectively. For the most update information on the range of such primes p, see the PrimeGrid Project [30]. Z. H. Sun and Z. W. Sun [33] also showed that if p is odd and $v_p(F_{z(p)}) = 1$, then the first case of Fermat's last theorem holds for the exponent p. For a survey on the conjecture that $v_p(F_{z(p)}) = 1$ for all p and other related problems, we refer the reader to Klaška [11].

If the above conjecture is true, then F_{18} and L_{18} are the only Fibonacci and Lucas numbers which satisfy $\omega(F_n) = \omega(L_n) = 3$ and $d(F_n) = d(L_n) = 16$. To show this, we need the following lemma.

Lemma 3.9. *Let b be a positive integer. Assume that $v_p(F_{z(p)}) = 1$ for all primes p. Then the following statements hold.*

(i) *If $p \neq 5$, then F_{p^b} is squarefree.*
(ii) *If $p = 5$, then $F_{p^b} = 5^b m$ where m is squarefree and $5 \nmid m$.*
(iii) *If $p \neq 3$, then L_{p^b} is squarefree.*

(iv) *If $p = 3$, then $L_{p^b} = 4m$ where m is odd and squarefree.*
(v) *If $p \neq 3$, then F_{2p} is squarefree. If $p \neq 3, 5$, then F_{2p^b} is squarefree.*

Proof. For (i), let $p \neq 5$ and let q be a prime, $a \geq 1$, and $q^a \mid F_{p^b}$. We know that $2 \mid F_n$ if and only if $3 \mid n$. So if $q = 2$, then $p = 3$ and $a \leq v_q(F_{p^b}) = v_2(F_{3^b}) = 1$. Similarly, if $q = 5$, then $p = 5$, which is not the case we are considering. So assume that $q \neq 2, 5$. Then by Lemmas 2.2 and 2.3, we have $(q, z(q)) = 1$, $p^b \equiv 0 \pmod{z(q)}$, and $a \leq v_q(F_{p^b}) = v_q(p^b) + v_q(F_{z(q)})$. Since $1 < z(q) \mid p^b$ and $(q, z(q)) = 1$, we see that $(q, p) = 1$. Hence $v_q(p^b) = 0$ and $a = 1$. In any case, we have $a = 1$. This shows that F_{p^b} is squarefree.

For (ii), let $p = 5$. Then by Lemma 2.3, we have $v_5(F_{p^b}) = v_5(p^b) = b$. So it remains to show that $F_{p^b}/5^b$ is squarefree. Let q be a prime, $a \geq 1$, and $q^a \mid (F_{p^b}/5^b)$. Then $q \neq 2, 5$, $q^a \mid F_{p^b}$, and we can use the same argument as in Case 1 to obtain $a \leq v_q(F_{p^b}) = v_q(p^b) + v_q(F_{z(q)}) = 1$. So $a = 1$. This proves (ii). For (iii), let $p \neq 3$ and let q be a prime, $a \geq 1$, and $q^a \mid L_{p^b}$. Then by Lemma 2.3, we obtain $q \neq 2, 5$, $z(q)$ is even, $p^b \equiv \frac{z(q)}{2} \pmod{z(q)}$ and

$$a \leq v_q(L_{p^b}) = v_q(p^b) + v_q(F_{z(q)}).$$

By an argument similar to that in Case 1, we obtain $1 = (q, z(q))$ and $1 < \frac{z(q)}{2} \mid p^b$ and thus $(q, p) = 1$. Therefore $v_q(p^b) = 0$ and $a = 1$, as required. The proof of (iv) is similar to those of (ii) and (iii), so we leave the details to the reader. For (v), if $p \neq 3$, then we have $F_{2p} = F_p L_p$ where F_p and L_p are squarefree by (i) and (iii) and are coprime by Lemma 2.1. Similarly, if $p \neq 3, 5$, then $F_{2p^b} = F_{p^b} L_{p^b}$ where F_{p^b} and L_{p^b} are squarefree and coprime. This proves (v). □

Theorem 3.10. *Suppose $v_p(F_{z(p)}) = 1$ for all p. Then $\omega(F_n) = 3$ implies $d(F_n) = 8, 12, 16$. Moreover,*

$$\omega(F_n) = 3 \text{ and } d(F_n) = 16 \text{ if and only if } n = 18 \text{ or } 125.$$

Proof. Suppose $\omega(F_n) = 3$. Then by Theorem 3.1, n satisfies (3.1), (3.2), or (3.3). We first consider the case when n satisfies (3.3). By Theorem 3.3, we obtain that $F_n = q_1^{a_1} q_2 q_3^{a_3}$ where q_3 is a primitive divisor of F_n and $a_1 \in \{1, 2\}$. By the assumption that $v_p(F_{z(p)}) = 1$ for all p, we have $a_3 = 1$. Therefore $d(F_n) = 8$ or 12. For (3.2), if $n = p$, p^2, p^3 with $p > 5$, then we obtain by Lemma 3.9 that F_n is squarefree, and so $d(F_n) = 8$. If $n = 5, 5^2, 5^3$, then $\omega(F_n) = 1, 2, 3$, respectively. So we only need to consider the case $n = 5^3$ and we obtain $d(F_{125}) = 16$. For (3.1), we have $F_{16} = 3 \times 7 \times 47$ and $d(F_{16}) = 8$; $F_{18} = 2^3 \times 17 \times 19$ and $d(F_{18}) = 16$; F_{2p} is squarefree (by Lemma 3.9) and so $d(F_{2p}) = 8$. In any case, $d(F_n) \in \{8, 12, 16\}$. The other statement also follows from the above proof. □

To obtain an analogue of Theorem 3.10 for L_n, we first prove the following results.

Lemma 3.11. *Let p be an odd prime. Then p is a primitive divisor of L_n if and only if p is a primitive divisor of F_{2n}.*

Proof. This is probably well-known but we cannot find a reference for it, so we give a proof for completeness. Suppose p is a primitive divisor of L_n. Since $F_{2n} = F_n L_n$, $p \mid F_{2n}$. Suppose for a contradiction that $p \mid F_m$ for some $m < 2n$. By Lemmas 2.2 and 2.3, $z(p) \mid m$ and $z(p)$ is even, so m is even. Let $m = 2^a m_1$, where $a \geq 1$, m_1 is odd, and $2^{a-1} m_1 = \frac{m}{2} < n$. Since $F_m = F_{m_1} L_{m_1} L_{2m_1} L_{4m_1} \cdots L_{2^{a-1} m_1}$ and p is a primitive divisor of L_n, we obtain $p \mid F_{m_1}$. So $z(p) \mid m_1$ which contradicts the fact that $z(p)$ is even and m_1 is odd. For the converse, suppose p is a primitive divisor of F_{2n}. Since $F_{2n} = F_n L_n$ and $p \nmid F_n$, we have $p \mid L_n$. If $p \mid L_m$ for some $m < n$, then $p \mid F_{2m}$, which is a contradiction. So p is a primitive divisor of L_n. □

Theorem 3.12. *Suppose that $v_p(F_{z(p)}) = 1$ for all p. Then the following statements hold.*

(i) *If $\omega(L_n) = 3$ and $n = 2^a p^2$ for some $a \geq 1$ and $p \geq 5$, then $L_n = p_1^{a_1} p_2 p_3$ where p_1, p_2, p_3 are distinct odd primes, p_3 is a primitive divisor of L_n, and $a_1 \in \{1, 3\}$. In this case, $a_1 = 3$ if and only if $L_{2^a} = p$.*

(ii) *If $\omega(L_n) = 3$ and $n = 2^a p$ for some $a \geq 1$ and $p \geq 5$, then $L_n = p_1^{a_1} p_2^{a_2} p_3$ where p_1, p_2, p_3 are distinct odd primes, and $\{a_1, a_2\} = \{1\}$ or $\{1, 2\}$.*

Proof. The proof of this theorem is similar to that of Theorem 3.3, so we give fewer details. For (i), we have $L_{2^a} \mid L_{2^a p} \mid L_n$ and each of them has a primitive divisor. So $L_{2^a} = p_1$, $L_{2^a p} = p_1^{b_1} p_2$, and $L_n = p_1^{a_1} p_2^{a_2} p_3$, where p_2 is a primitive divisor of $L_{2^a p}$ and p_3 is a primitive divisor of L_n. Now we only need to show that $a_1 \in \{1, 3\}$ and $a_2 = 1$. Since $L_{2^a} = p_1$, we see that $p_1 \neq 2, 5$. So we have

$$a_1 = v_{p_1}(L_n) = v_{p_1}(2^a p^2) + v_{p_1}(F_{z(p_1)}) = v_{p_1}(p^2) + 1 = 1 \text{ or } 3, \text{ and } a_1 = 3 \Leftrightarrow p_1 = p \Leftrightarrow L_{2^a} = p.$$

In addition, $p_2 \neq 2, 5$ and $p_2 + 1 \geq z(p_2) = 2^{a+1} p \geq 2p - 1 > p + 1$, where the equality $z(p_2) = 2^{a+1} p$ is obtained by Lemma 3.11. Therefore $a_2 = v_{p_2}(L_n) = v_{p_2}(2^a p^2) + 1 = 1$. This proves (i). The proof of (ii) is the same as (i), so we leave the details to the reader. □

Theorem 3.13. *Assume that $\omega(L_n) = 3$ and $n = p_1 p_2$ for some odd primes $p_1 < p_2$. Then the following statements hold.*

(i) *If $p_1 \geq 5$, then $L_{p_1} = q_1$, $L_{p_2} = q_2$, and $L_n = q_1^{a_1} q_2 q_3^{a_3}$ where q_1, q_2, q_3 are distinct odd primes, q_3 is a primitive divisor of L_n, $a_3 \geq 1$, and $a_1 \in \{1, 2\}$. In addition, $a_1 = 2$ if and only if $q_1 = p_2$.*

(ii) *If $p_1 = 3$, then $L_n = 4 q_2 q_3^{a_3}$ where q_2 and q_3 are distinct odd primes, q_3 is a primitive divisor of L_n, and $a_3 \geq 1$.*

Proof. The proof of this theorem is the same as that of Theorem 3.3. So we leave the details to the reader. □

Theorem 3.14. *Assume that $\omega(L_n) = 3$ and $n = 2^a \cdot 3$ for some $a \geq 1$. Then L_{2^a} is a prime and $L_n = 2 L_{2^a} p^b$ where $b \geq 1$ and p is a primitive prime divisor of L_n.*

Proof. Since 2, L_{2^a}, and L_n divide L_n and each of them has a primitive divisor, we can obtain the desired result in the same way as other similar theorems. □

Theorem 3.15. *Suppose that $v_p(F_{z(p)}) = 1$ for all p. If $\omega(L_n) = 3$, then $d(L_n) = 8, 12, 16$. Moreover, $\omega(L_n) = 3$ and $d(L_n) = 16$ occurs only when $n = 18$ or $n = 2^a p^2$ for some $p \geq 5$ such that $p = L_{2^a}$.*

Proof. Assume that $\omega(L_n) = 3$. By Theorem 3.5, $n = 2^a$, p, p^2, p^3, $2^a p$, $2^a p^2$, pq for some distinct odd primes p, q and $a \geq 1$. If $n = 2^a$, p, p^2, p^3 with $p \geq 5$, then we obtain by Lemma 3.9 that L_n is squarefree and so $d(L_n) = 8$. If $n = 3$, 3^2, 3^3, then $\omega(L_n) = 1, 2, 3$, respectively, so we only need to consider the case $n = 3^3$. We have $L_{27} = 2^2 \times 19 \times 5779$ and so $d(L_{27}) = 12$. So it remains to consider the cases $n = 2^a p$, $2^a p^2$, pq.

Case 1. $n = pq$. Then by Theorem 3.13, $L_n = q_1^{a_1} q_2 q_3^{a_3}$ where $a_1 \in \{1, 2\}$ and by the assumption that $v_p(F_{z(p)}) = 1$ for all p, we also obtain $a_3 = 1$. Therefore $d(L_n) = 8, 12$.

Case 2. $n = 2^a p$. This case is similar to Case 1. We apply Theorem 3.12 to obtain $d(L_n) = 8$ or 12.

Case 3. $n = 2^a p^2$. If $p = 3$, then we obtain by Theorem 3.5 that $n = 18$, and so $L_n = 2 \times 3^3 \times 107$ and $d(L_n) = 16$. So suppose $p \geq 5$. Then by Theorem 3.12, $L_n = p_1^{a_1} p_2 p_3$, $a_1 \in \{1, 3\}$, and $a_1 = 3$ if and only if $L_{2^a} = p$. Then $d(L_n) = 8$ or 16, and $d(L_n) = 16$ if and only if $L_{2^a} = p$. This completes the proof. □

Corollary 3.16. *Suppose that $v_p(F_{z(p)}) = 1$ for all p. Then $\omega(F_n) = \omega(L_n) = 3$ and $d(F_n) = d(L_n) = 16$ if and only if $n = 18$.*

Proof. This follows immediately from Theorems 3.10 and 3.15. □

The integers $n \leq 300$ such that $\omega(F_n) = \omega(L_n)$ and $d(F_n) = d(L_n)$ are $n = 1$, 4, 5, 7, 10, 11, 13, 14, 17, 18, 26, 46, 47, 58, 73, 77, 85, 89, 103, 107, 121, 139, 167, 179, 181, 187, 205, 221, 233, 241, 247, 253, 257, 262, 269, 273, 281, 293, 295. More details are given in Table 1. The author will also upload more data on $\omega(F_n)$, $\omega(L_n)$, $d(F_n)$, and $d(L_n)$ on his ResearchGate account [27, 28] which will be freely downloadable to everyone.

4. Main Results II: The solutions to $\ell(n) = 18$.

In this section, we find the solutions to the equation $\ell(n) = 18$ and show a connection between them and F_{18}. For convenience, we sometimes write $P(n)$ to denote the largest prime factor of n if $n \geq 2$, and define $P(1) = 1$. In addition, we let $\gamma(n) = \prod_{p|n} p$.

Theorem 4.1. *Let n be a positive integer. Then $\ell(n) = 18$ if and only if n satisfies one of the following conditions:*

$$n = 19, 74, 115, \tag{4.1}$$

$$n = 19A \text{ where } A \text{ is a positive divisor of } \prod_{p \leq 17} p,\ \omega(A) \geq 2, \text{ and } A \neq 6, 10, 14, 15, \tag{4.2}$$

$$n = 19m\gamma(m)B_m \text{ where } m = 2, 3, \ldots, 17 \text{ and } B_m \text{ is a positive divisor of } \frac{\prod_{p \leq 17} p}{\gamma(m)}, \tag{4.3}$$

$$n = 108C \text{ where } C \text{ is a positive divisor of } \prod_{5 \leq p \leq 19} p. \tag{4.4}$$

Proof. By using Lemmas 2.7 and 2.8, it is not difficult to verify the converse of this theorem. We show the details for (4.2) and (4.3) as follows. Suppose n satisfies (4.2). Then n is squarefree, n is not a prime, $n \neq 2p$ for any prime p, and $n > 19^2$. So we obtain by Lemma 2.7 that

$$\ell(n) = 19 - \left\lfloor \frac{19^2}{n} \right\rfloor - 1 = 18,$$

as required. Suppose n satisfies (4.3). Then n is not squarefree, $P(n) - 1 = 18$, and

$$\frac{n}{\gamma(n)} = \frac{19m\gamma(m)B_m}{19\gamma(m)B_m} = m < 18.$$

Therefore we obtain by Lemma 2.8 that $\ell(n) = 18$. The verification is similar for those n satisfying (4.1) or (4.4).

Now suppose that $\ell(n) = 18$. Obviously, $n > 1$ and we will show that n satisfies one of the conditions (4.1) to (4.4). We divide our calculations into several cases and apply Lemmas 2.7 and 2.8 repeatedly without further reference.

Case 1. n is prime. Then $n - 1 = \ell(n) = 18$, so $n = 19$, which satisfies (4.1).

Case 2. $n = 2p$ where p is an odd prime. If $p \equiv 3 \pmod 4$, then we obtain $\frac{1}{2}(p + 1) = \ell(n) = 18$, which implies $p = 35$ contradicting the fact that p is a prime. So $p \equiv 1 \pmod 4$. Then

$$p - \left(\frac{p-1}{2}\right) - 1 = p - \left\lfloor \frac{p^2}{n} \right\rfloor - 1 = \ell(n) = 18,$$

which implies $p = 37$. This leads to $n = 74$ which is in (4.1).

Case 3. $n = p_1 p$ where $3 \le p_1 < p$. Then

$$p - \left\lfloor \frac{p}{p_1} \right\rfloor - 1 = p - \left\lfloor \frac{p^2}{n} \right\rfloor - 1 = \ell(n) = 18.$$

Therefore

$$p - \left\lfloor \frac{p}{p_1} \right\rfloor = 19. \tag{4.5}$$

Since $3 \le p_1 < p$, we obtain $1 \le \left\lfloor \frac{p}{p_1} \right\rfloor \le \left\lfloor \frac{p}{3} \right\rfloor$. Therefore $p - \left\lfloor \frac{p}{p_1} \right\rfloor < p$ and $p - \left\lfloor \frac{p}{p_1} \right\rfloor \ge p - \left\lfloor \frac{p}{3} \right\rfloor > \frac{2p}{3}$. From this and (4.5), we see that $19 < p < 28.5$. The only prime p satisfying this inequality is $p = 23$. Substituting $p = 23$ in (4.5), we obtain $\left\lfloor \frac{23}{p_1} \right\rfloor = 4$. So $4 \le \frac{23}{p_1} < 5$. The only prime p_1 satisfying this inequality is $p_1 = 5$. This leads to $n = 5 \cdot 23 = 115$.

Case 4. $n = p_1 p_2 \cdots p_k$ where $k \ge 3$ and $2 \le p_1 < p_2 < \cdots < p_k$. For convenience, let $p = p_k$ and $A = \prod_{i=1}^{k-1} p_i$. Then $18 = \ell(n) = p - \left\lfloor \frac{p^2}{n} \right\rfloor - 1 = p - \left\lfloor \frac{p}{A} \right\rfloor - 1$. So $p - \left\lfloor \frac{p}{A} \right\rfloor = 19$. Since $k \ge 3$, we obtain $A \ge (2)(3) = 6$ and so

$$19 = p - \left\lfloor \frac{p}{A} \right\rfloor \ge p - \left\lfloor \frac{p}{6} \right\rfloor \ge p - \frac{p}{6} = \frac{5p}{6}.$$

In addition, $p \ge p - \left\lfloor \frac{p}{A} \right\rfloor$. Therefore $19 \le p \le \frac{(6)(19)}{5}$. The only prime in this range is $p = 19$. Since $19 = p - \left\lfloor \frac{p}{A} \right\rfloor = 19 - \left\lfloor \frac{19}{A} \right\rfloor$, we see that $A > 19$. This leads to the solutions $n = 19A$ where A is a divisor of $\prod_{p<19} p$, $\omega(A) \ge 2$, and $A > 19$, which correspond to (4.2).

Cases 1 to 4 give the solutions to $\ell(n) = 18$ in squarefree numbers n. Next we consider the case when n is not squarefree. Recall again that we write $v_p(m)$ to denote the exponent of p in the prime factorization of m.

Case 5. n is not squarefree and $P(n) - 1 > \frac{n}{\gamma(n)}$. Then $18 = \ell(n) = P(n) - 1$. So $P(n) = 19$. Therefore n is of the form $n = 2^{a_2} 3^{a_3} \cdots 19^{a_{19}} = \prod_{p \le 19} p^{a_p}$, where $a_{19} \ge 1$, $a_p \ge 0$ for $2 \le p < 19$, and $a_p > 1$ for some p. For convenience, let $b_p = \max\{a_p - 1, 0\}$. Then

$$18 = P(n) - 1 > \frac{n}{\gamma(n)} = \prod_{p \le 19} p^{b_p}.$$

This implies that $a_{19} = 1$, $a_p > 1$ for some $p < 19$, and $\prod_{p \le 17} p^{b_p} < 18$. So we only need to check for the solutions when $\prod_{p \le 17} p^{b_p} = 2, 3, \ldots, 17$. We see that $\prod_{p \le 17} p^{b_p} = 2$ implies $a_2 = 2$ and $a_p \in \{0, 1\}$ for $3 \le p \le 17$, which leads to the solutions $n = 2^2 \cdot B \cdot 19$ where B is a divisor of $\prod_{3 \le p \le 17} p$. In general, suppose $m \in \{2, 3, \ldots, 17\}$ and $\prod_{p \le 17} p^{b_p} = m$. If $p \mid m$, then $a_p - 1 = b_p = v_p(m)$, so $a_p = v_p(m) + 1$. If $p \nmid m$, then $b_p = 0$ and therefore $a_p = 0$ or 1. This implies that

$$n = 19 \prod_{p \le 17} p^{a_p} = (19) \left(\prod_{\substack{p \le 17 \\ p \mid m}} p^{v_p(m)+1} \right) \left(\prod_{\substack{p \le 17 \\ p \nmid m}} p^{a_p} \right) = 19 \cdot m\gamma(m) \cdot B_m,$$

where B_m is a divisor of $\frac{1}{\gamma(m)} \prod_{p \le 17} p$. This leads to the solutions given in (4.3).

Case 6. n is not squarefree and $P(n) - 1 < \frac{n}{\gamma(n)}$. Then $\frac{n}{\gamma(n)} = \ell(n) = 18$ and $P(n) - 1 < 18$. So $P(n) < 19$. Similar to Case 5, we see that $n = 2^2 \cdot 3^3 \cdot \prod_{5 \le p \le 17} p^{a_p}$, where $a_p = 0$ or 1 for each $p = 5, 7, \ldots, 17$.

Case 7. n is not squarefree and $P(n) - 1 = \frac{n}{\gamma(n)}$. Then $\frac{n}{\gamma(n)} = P(n) - 1 = \ell(n) = 18$. Similar to Case 6, we obtain $n = 2^2 \cdot 3^3 \cdot \prod_{5 \le p \le 19} p^{a_p}$, where $a_p = 0$ or 1 for $5 \le p < 19$ and $a_{19} = 1$.

Combining Cases 6 and 7, we obtain the integers n of the form

$$n = 2^2 \cdot 3^3 \cdot \prod_{5 \le p \le 19} p^{a_p} = 108C \text{ where } C \text{ is a divisor of } \prod_{5 \le p \le 19} p.$$

This completes the proof. □

A surprising fact is that F_{18} is one of the solutions to the equation $\ell(n) = 18$. Moreover, F_{18} is the only Fibonacci number satisfying this equation. It is straightforward to check all the solutions given in Theorem 4.1 and see that $F_{18} = 2^3 \cdot 17 \cdot 19$ satisfies (4.3) with $m = 4$ and $B_m = 17$, and that the other solutions are not Fibonacci numbers. Since there are quite a lot of solutions, this may take time. Therefore we give a shorter proof in the following corollary.

Corollary 4.2. $\ell(F_m) = 18$ *if and only if* $m = 18$.

Proof. By Lemma 2.8, the converse can be easily checked. So we suppose $\ell(F_m) = 18$. Observe that if $\ell(n) = 18$ and n does not satisfy (4.1), then $P(n) \le 19$. Since $19, 74$, and 115 are not Fibonacci numbers and $\ell(F_m) = 18$, we obtain $P(F_m) \le 19$. By using the table [2], it is easy to see that each prime $p \le 19$ is a divisor of a Fibonacci number F_m for some $m \le 18$. In addition, by Theorem 2.4, if $m > 18$, then F_m has a primitive prime divisor larger than 19. Since $P(F_m) \le 19$, we see that $m \le 18$. By considering the table [2] again, it is easy to see that $F_1, F_2, F_3, \ldots, F_{17}$ are not $19, 74, 115$, and are not divisible by 19 or 108. So we obtain by Theorem 4.1 that they are not solutions to $\ell(n) = 18$. Hence $m = 18$ only. □

Conclusion: F_{18} is the only Fibonacci number which is a solution to the equation $\ell(m) = 18$. In addition, F_{18} is the only even Fibonacci number which has exactly three prime factors, two of which are twins. Furthermore, if $v_p(F_{z(p)}) = 1$ for all p, then F_{18} and L_{18} are the only Fibonacci and Lucas numbers satisfying $\omega(F_n) = \omega(L_n) = 3$ and $d(F_n) = d(L_n) = 16$.

Remark. We obtained referee's comments and suggestions which we would like to add in this article. Luca and Stănică [15] give some heuristics about the number of prime factors of members of Lucas sequences. Using those heuristics, it would seem that perhaps of the cases presented in Theorem 3.1, only the case $n = p$ might have a chance to lead to infinitely many examples of Fibonacci numbers F_n with $\omega(F_n) = 3$. Indeed, take $n = 2p$. Then $\omega(F_{2p}) = 3$ means that each of F_p and L_p has at most two prime factors. The heuristic is that this would happen for a random p with probability $\ll (\log p)^{O(1)}/p$ for each F_p and L_p, so assuming these events are independent their joint probability would be $\ll (\log p)^{O(1)}/p^2$. Since

$$\sum_{p \ge 2} \frac{(\log p)^{O(1)}}{p^2} < \infty,$$

it would seems that there are only finitely many $n = 2p$ with $\omega(F_n) = 3$. A similar analysis can be carried out for $n = p^2$, $n = p^3$, and $n = pq$. This heuristic can also be made about the conclusion of Theorem 3.5.

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n	factorization of n	$\omega(F_n)$	$\omega(L_n)$	$d(F_n)$	$d(L_n)$
1	1	0	0	1	1
4	2^2	1	1	2	2
5	5	1	1	2	2
7	7	1	1	2	2
10	$2 \cdot 5$	2	2	4	4
11	11	1	1	2	2
13	13	1	1	2	2
14	$2 \cdot 7$	2	2	4	4
17	17	1	1	2	2
18	$2 \cdot 3^2$	3	3	16	16
26	$2 \cdot 13$	2	2	4	4
46	$2 \cdot 23$	3	3	8	8
47	47	1	1	2	2
58	$2 \cdot 29$	3	3	8	8
73	73	2	2	4	4
77	$7 \cdot 11$	4	4	16	16
85	$5 \cdot 17$	4	4	16	16
89	89	2	2	4	4
103	103	3	3	8	8
107	107	2	2	4	4
121	11^2	2	2	4	4
139	139	3	3	8	8
167	167	2	2	4	4
179	179	3	3	8	8
181	181	3	3	8	8
187	$11 \cdot 17$	4	4	16	16
205	$5 \cdot 41$	6	6	64	64
221	$13 \cdot 17$	4	4	16	16
233	233	3	3	8	8
241	241	3	3	8	8
247	$13 \cdot 19$	5	5	32	32
253	$11 \cdot 23$	5	5	32	32
257	257	3	3	8	8
262	$2 \cdot 131$	4	4	16	16
269	269	4	4	16	16
273	$3 \cdot 7 \cdot 13$	9	9	768	768
281	281	3	3	8	8
293	293	2	2	4	4
295	$5 \cdot 59$	9	9	512	512

TABLE 1. The integers $n \in [1, 300]$ such that $\omega(F_n) = \omega(L_n)$ and $d(F_n) = d(L_n)$