UNIVERSAL ARITHMETICK;

OR, A

TREATISE

OF

ARITHMETICAL COMPOSITION

AND

RESOLUTION.

Written in LATIN by Sir ISAAC NEWTON.

Translated by

The late Mr. RALPHSON; and Revised and Corrected by Mr. CUNN.

To which is added, a

TREATISE upon the MEASURES of RATIOS,

By JAMES MAGUIRE, A. M.

The whole illustrated and explained,

In A SERIES of NOTES,

By the Rev. THEAKER WILDER, D. D.

SENIOR FELLOW of Trinity College, Dublin.

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TO HIS GRACE

JOHN,

DUKE OF BEDFORD,

MARQUIS OF TAVISTOCK, &c.

CHANCELLOR of the UNIVERSITY of DUBLIN,

THIS WORK IS DEDICATED,

BY HIS GRACE's MOST HUMBLE

AND DUTIFUL SERVANT

THEAKER WILDER.

PREFACE.

THE Occaſion of my being engaged in a Work of this Kind was as follows.

Upon the Death of Mr. Maguire, I was appointed, being then a Junior Fellow, to ſucceed him as Teacher of the Mathematicks to the Under-graduates of the Univerſity. And the many and obvious Advantages attending an early Initiation into this Study, having ſome Time ſince determined the Right Honourable the Provoſt, and the Senior Fellows, to order Euclid's Elements to be taught by the Tutors in their private Lectures, and to be made a Part of the quarterly Examinations; the Munificence alſo of the Governors of the Schools founded by Eraſmus Smith, Eſq; having added to the Salary of the Donegal Lecturer of the Graduates, and made a Profeſſor of Mathematicks; I conſented to the putting down my Office, and to the converting its Emoluments to the eſtabliſhing of proper Aſſiſtants to the Profeſſor, for the Inſtruction of the Under-graduates.

In the Courſe of my Attendance upon the Duty of that Employment, I experienced the Neceſſity which there was of illuſtrating this Work of the Author; and drew up moſt of the following Notes.

In doing this I conſulted the moſt celebrated Writers, and transferred from them what ſeemed moſt to conduce to this Deſign. I found many who had illuſtrated ſome particular Parts of this Work, as 'sGraveſende, Reyneau, Bernoulli, Maclaurin, Colſon, Campbell, &c. &c. but none whom I know of, except Caſtilioneus, who had written a regular and continued Comment upon it.

I had alſo the Uſe of three Manuſcripts of the late Mr. Maguire.

The firſt is an unfiniſhed Treatiſe upon Arithmetick; containing Remarks and Criticiſms, collected from different Authors, particularly from Wells (whom he ſometimes juſtly cenſures) from Jones's Synopſis Palmariorum Matheſeos, from Kerſey, Wallis, Dodſon upon Wingate, &c. &c. Many Things alſo are his own, particularly the Proof of the Rules of finding compound Diviſors (Art. L. of this Work) from the Nature of the algebraical Operations. And I prefer this Method of Proof in that Place to thoſe of Bernoulli and Maclaurin, becauſe it does not conſider the Dividend as an Equation: Yet that of Maclaurin (Art. CXL, *a.*) has a peculiar Elegance and Propriety, when the Dividend is an Equation.

The ſecond is an unfiniſhed Treatiſe of Equations, drawn up, ſo far as it goes, in a moſt elegant and clear, though conciſe Method. It contains among other Things, ſome Strictures upon our Author (Art. CXXXVI. *f.*); ſome Objections alſo which other Writers had made (Art. CLIII. CLIV); and the Proofs of Art. CXXXIII. IV. V. and of Numb. 257, 258. It appears that he kept his Eye conſtantly upon our Author, and perhaps deſigned theſe two Treatiſes as a Comment upon him.

The third is a complete Treatiſe upon the Meaſures of Ratios; in which the Reader will find the whole Doctrine of Logarithms moſt accurately laid open, together with ſome very curious Strictures upon Wallis, Briggs, and Halley. This may very properly be added to a Treatiſe of Univerſal Arithmetick, and was probably ſo deſigned to be, if Mr. Maguire had lived to have finiſhed the two former Parts.

Theſe Treatiſes are in Latin: They are now in the Preſs, under the Care of the Profeſſor of

Mathe-

Mathematicks. The Use of them was given to me by John and Bridget, the Brother and Sister of the Author James Maguire: And to the Use of his Representatives, the Profits (if any) of this Work are by Deed conveyed; the Losses, if any, are to be sustained solely by me.

As to Castilioneus, I dislike chiefly three Things in his Book.

First, beside the great Errors of the Press, and which (Tome I. Page 54. Tome II. Page 18, 29, 32, 33, 34, 106, 180, 204, &c. &c.) are insuperable to young Students, he is unnecessarily prolix.

Secondly, he does not pay a proper Regard to the Method of Notation used by his Author: For altho' a Person may put what Symbols he pleases, provided he is constant in their Use, to denote particular Coefficients, Quantities, Operations, &c. &c. yet it will occasion much unnecessary Trouble to the Student, if the Commentator uses a Method of Notation different from that of his Author.

Lastly, the Price and Bulk of his Book is too great in Respect of its Utility. This is occasioned, not only by the Additions from other Authors, although the Substance of them is mostly thrown into his foregoing Notes, but also by his increasing the Number of Schemes to two Thirds more than it originally was. Our Author gave geometrical Questions as Exercises for the Student, supposing him already well versed in Geometry, and in those other Sciences upon which their Solutions depend: It seems, therefore, a superfluous Undertaking in the Commentator, to draw Solutions and Constructions from Principles different from those which the Author used; and to explain not so much what the Author has done, as what he might have done.

I have

I have endeavoured to avoid these Inconveniencies: and whenever I have been obliged, by adhering to the Order of the Author, to cite any Thing in Proof of another, although the Thing cited is itself afterwards to be proved; Care is taken, that it shall not depend upon that, in whose Support it had been cited.

I have every where supposed the Student to be well versed in Euclid's Elements, and to be Master of common Arithmetick, so far at least as it is generally taught in Schools: If he is not, I would recommend to his Study, antecedent to this, Wingate's Arithmetick, as it has been altered and improved by Kersey, Shelly, and Dodson.

Having determined to publish these Notes in English, that they might be of more universal Use to such as want Assistances of this Kind, I connected them with the Translation which goes under the Name of Mr. Ralphson: And finding that there has been generally annexed to this Translation, the Method of resolving Equations by Dr. Halley, I substituted in its Place the Methods of Approximation by Mr. Maclaurin; because these contain the Method of deducing Halley's, and all other Theorems for that Purpose. To this there is added a Translation of the beforementioned Treatise of the Measures of Ratios, so that the whole Collection seems to approach to the Idea of an universal Arithmetick.

Si quid novisti rectius istis,
Candidus imperti; si non, his utere mecum.

Trinity-College, Dublin,
September 1, 1768.

UNIVERSAL ARITHMETIC;

OR, A

TREATISE

OF

ARITHMETICAL COMPOSITION

AND

RESOLUTION.

Of Notation.

COMPUTATION *is either performed by* Numbers, *as in Vulgar Arithmetic, or by* Species, *as usual among Algebraists. They are both built on the same Foundations, and aim at the same End,* viz. Arithmetic *Definitely and Particularly,* Algebra *Indefinitely and Universally*; (*a*) so that almost all Expressions that are found out by this Computation, and particularly Conclusions, may be called *Theorems.* But Algebra is particularly excellent in this, that whereas in Arithmetic Questions are only resolved by proceeding from given Quantities to the Quantities sought, Algebra proceeds in a retrograde Order, from the Quantities sought, as if they were given, to the Quantities given, as if they were sought, to the End that we

(*a*) In numeral Operations, the Figures are changed and displaced for others, leaving no Vestige of the Operations; but in Algebra, the Symbols or Species remain unchanged, and exhibit all Operations to the Eye.

we may some Way or other come to a Conclusion or Equation, from which one may bring out the Quantity sought. And after this Way the most difficult Problems are resolved, the Resolutions whereof would be sought in vain from only common Arithmetic. Yet Arithmetic in all its Operations is so subservient to Algebra, as that *they* seem *both* but to *make one perfect Science of Computing*; and therefore I will explain them both together.

Whoever goes upon this Science, must first understand the Signification of the Terms and Notes, and learn the fundamental Operations, *viz.* Addition, Subtraction, Multiplication, and Division; Extraction of Roots, Reduction of Fractions, and of Radical Quantities, and the Methods of Ordering the Terms of Equations, and *exterminating the unknown Quantities*, where there are more than one. Then let the Learner proceed to exercise himself in these Operations, by bringing Problems to Equations; and, lastly, let him consider the Nature and Resolution of Equations.

Of the Signification of some Words and Notes.

Article I. *By* Number *we understand*, not so much a Multitude of Unities, as *the abstracted Ratio of any Quantity, to another Quantity of the same Kind, which we take for Unity.* And this *is threefold*; *integer*, *fracted*, and *surd*: *An* Integer, *is what is measured by Unity*; *a* Fraction, *that which a submultiple Part of Unity measures*; *and a* Surd, *to which Unity is incommensurable.* (*b*)

II. Every

I. (*b*) See Eucl. V. Def. 1. 2. VII. Def. 3. 4. 5. X. Def. 2. *Quantities having a common Measure are called commensurable*, or as Number to Number; *Quantities which have not a common Measure are incommensurable*, and are not as Number to Number; that is, their Magnitude is not to be expressed in common numeral Notation. For, if any one Quantity be called rational, all those commensurate with it are rational; and all those which are incommensurable with it must be irrational. Now *Unity being either a Multiple or Submultiple of all Numbers, every Quantity, which is neither a Multiple or Submultiple of Unity, such as the Roots of Integers, which are not Integers,* (N° 161.) *must be incommensurate to Unity*, and irrational.

II. Every one underſtands the Notes of *whole Numbers*, (0, 1, 2, 3, 4, 5, 6, 7, 8, 9) and the Values of thoſe Notes when more than one are ſet together. But as Numbers placed on the left Hand, next before Unity, denote Tens of Units, in the ſecond Place Hundreds, in the third Place Thouſands, &c. ſo Numbers ſet in the firſt Place after Unity, denote tenth Parts of an Unit, in the ſecond Place hundredth Parts, in the third Place thouſandth Parts, &c. and theſe are called *Decimal Fractions*, becauſe they always decreaſe in a Decimal Ratio; and to *diſtinguiſh the Integers from the Decimals, we place a Comma, or a Point, or a ſeparating Line.* Thus the Number 732 ∠ 569 denotes ſeven hundred thirty-two Units, together with five tenth Parts, ſix centeſimal, or hundredth Parts, and nine milleſimal, or thouſandth Parts of Unity; which are alſo written thus, 732,569; or thus, 732.569; and ſo the Number 57104,2083 fifty ſeven thouſand one hundred and four Units, together with two tenth Parts, eight thouſandth Parts, and three ten thouſandth Parts of Unity; and the Number 0,064 denotes ſix centeſimal and four milleſimal Parts. The Notes of Surds and fracted Numbers are ſet down in the following Pages. (*c*)

III. *When*

II. (*c*) Notation teaches how to expreſs in Characters any Number propoſed in Words; and, converſely, how to enunciate any Number propoſed in Characters.

1. *Every Number conſiſts of a Numerator and a Denominator; and the Value of any Number is the Product of its Numerator multiplied by its Denominator. The Numerator denotes the Multitude or Quantity, and the Denominator the diſtinguiſhing Name, of what is numbered.*

2. *The Numerators are diſtinguiſhed by their Shape in Integers, and are called Figures. The Denominators are known from the Claſs or Place of the Numerator or Figure.* For,

3. *In Integers, the Denominator of each Figure encreaſes in a decuple Ratio from the right Hand to the left*; the greateſt Denominator being always firſt enunciated in reading from the left Hand to the right. *The firſt* Claſs or *Place is that of Units; the ſecond, that of Tens; and the third, that of Hundreds.*

4. To

III. *When the Quantity of any Thing is unknown, or looked upon as indeterminate, so that we cannot express it in Numbers, we denote it by some* Species, *or by some Letter.* And

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4. To avoid the Trouble of inventing new Names for every Class or Place, *the Classes were distributed into Ternaries*, and a proper Name given to the two first Ternaries; and in enunciating a Number, the three first Names of the Classes are jointly repeated with that of the Ternary. *The first Ternary is that of Units; the second, that of Thousands.*

This Distribution, and these Names, were sufficient for civil Use; as most Numbers which are of Use in common Business are generally confined within six Places. But, when the Sciences began to extend themselves, and Mathematical Calculations were applied in every Inquiry in which they could have Place, the Confusion which necessarily must arise, in Numbers containing a long Series of Places, from the frequent Repetition of the Names of the two first Ternaries, became evident.

5. Then a new *Distribution of the Places* was introduced *into Senaries, called Periods*; containing six Places each, or two Ternaries, which were thence called *Semiperiods. The Name of the first Period is that of Units; of the second, that of Millions; of the third, that of Billions; and so on; each Period being denominated from the Number of preceding Periods*; so the sixth Period is that of Quintillions. In enunciating any Number, the three Names of the Classes, and the two Names of the Semiperiods, are jointly repeated before that of the Period.

6. *In Integers*, because the Denominators increase in a constant decuple Ratio from right to left; therefore *the Denominator of any Figure* must be Ten, so often multiplied into itself, as the Figure is exclusively distant from the Place of Units; that is, *that Power of Ten, whose Index is the Number of Places to the left Hand of the Place of Units exclusive:* It can *therefore* be always known from the Place of the Figure, and *there is no need of writing it down.* Hence the Value of any Figure is its Product into such a Power of Ten as its Distance from the Place of Units indicates (N°. 1.) 7. A

And if we consider known Quantities as indeterminate, we denote them, for Distinction sake, with the initial Letters

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7. A Submultiple of Unity being the Measure of a Fraction, (Art. I.) the Fraction must be denominated from that Submultiple; therefore *the Denominator of a Fraction must be Unity divided into a certain Number of equal Parts. When there is a constant Ratio of this Division, the Denominators of the Fractions will be in a constant Ratio*; and thus *Decimals* are Fractions whose Numerators increase in a decuple Ratio: *Sexagesimals* are Fractions whose Denominators increase in a sixty-fold Ratio: *But when there is no constant Ratio of the Division, the Fractions are called* vulgar, *their Denominators having no constant Ratio.*

8. *Hence*, there being no constant Ratio of *the Denominators in vulgar Fractions*, the Denominators *must be constantly wrote down along with the Numerators.* Also *in Sexagesimals*, the Ratio of the Denominators, 'tho' constant, being not discoverable from the Distance of the Numerator from the Place of Units, *the Denominators*, if not wrote down, must yet be *denoted by* some *Symbols of a sexagesimal Division annexed to each Numerator.* But *in Decimals, the Ratio of the Denominators being discoverable from their Distance from the Place of Units, the Denominators need not be wrote.* For,

9. *In Decimals, the Denominators encrease in a decuple Ratio from left to right*; and the Denominator of any Decimal is Unity divided by a Power of Ten, whose Index is the Number of Places from that of Units. The Value also of a Decimal is the Figure divided by that Power of Ten; (N° 1. 7.) and consequently *the Values of Decimals are in a subdecuple Ratio from left to right.* For,

10. To multiply any Figure into Unity divided by any Number, being the same Thing as to divide the Figure by that Number, (N° 145.) and the Value of a Fraction being the Product of its Numerator into its Denominator, (N° 1.) that is, into Unity divided by some Number, (N° 7.) it follows universally, that *the Value of any Fraction is the Quote of the Numerator divided by the Denominator.*

Letters of the Alphabet, as *a*, *b*, *c*, *d*, and the unknown ones by the final ones, *x*, *y*, *z*, &c. Some substitute Consonants

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11. *The Value of Decimals, as well as of Integers, increases uniformly in a decuple Ratio from right to left.* For the Value of each being the Quote of the Numerator divided by the Power of Ten, whose Index is the Number of Places from that of Units towards the right, (N° 9.) this Value must decrease (Eucl. VII. 19. c. 1.) as the Denominators increase; that is, towards the right; (N° 9.) therefore it increases towards the left; that is, in the same Manner and Ratio as the Value of Integers. (N° 3.)

12. *Decimals, and Numbers mixed of Integers and Decimals, may be enunciated after the Manner of Integers, by giving them all one common Denomination,* viz. *that of the last Figure on the right.* For thus every Numerator and Denominator is multiplied into the same Power of Ten, viz. that whose Index is the Number of Places on the right of the Place of Units: Therefore the Value of each Figure continues unchanged. (Eucl. VII. 17.)

13. *The Distance of any Figure from the Place of Units exclusive is called its Index, or Exponent.* It determines the Number of Cyphers in its Denominator; that is, the Number of continued Multiplications of Unity into Ten, or the Number of continued Divisions of Unity by Ten, in order to produce its Denominator. Now as the Place of Units must belong to Integers, *the Index in Integers will be the whole Number of Places of Integers less one*; that is, *the Number of Places to the left of the Place of Units*; that is, *the Number of Places to the right of the Figure itself.* But *in Decimals the Index will always be equal to the whole Number of Places of Decimals.*

14. Because the Denominators of Integers are produced by Multiplication, (6.) but those of Decimals by Division, (9.) and that Multiplication is contrary to Division; it follows, that if an Integer and Decimal, having the same Numerator, and being equidistant from the Place of Units, be multiplied into each other, the Place and Denominator of the Product will be that of Units; and consequently the Index

Consonants or great Letters for known Quantities, and Vowels

Index or Distance of the Product from the Place of Units can be nothing. Whence *putting* o *(Cypher) the Exponent of the Place of Units, as the Indices of Integers are usually supposed to be affirmative, the Indices of Decimals must be negative.* Thus

4 3 2 1 0 — 1 — 2 — 3 — 4
1 2 3 4 5 ∠ 6 7 8 9

(See also N° 76.)

15. The Value of every Figure in any Rank of Numbers, how large soever, is readily found by the following Rule. *Set a Point under the Figure in the Place of Units, and under every sixth Figure on each Hand of the Place of Units exclusive; so shall the 1st, 2d, 3d,* &c. *Point stand under Units, Millions, Billions, Trillions,* &c. *respectively*; that is, *distinguish the Periods*; (5. 9.) *then a Comma, placed after every third Figure from a Point inclusive, will distinguish the Semiperiods:* (4. 9.) As is evident from the following Example.

Suppose the Number given is
1 2 3 4 5 6 7 8 9 1 2 3 4 5 ∠ 6 7 8 9 0 1 2 3 4 5 6 7 8;
when distributed according to the above Rule, it will be
1 2 3 4 5, 6 7 8 9 1 2, 3 4 5 ∠ 6 7, 8 9 0 1 2 3, 4 5 6 7 8;
• • • • •

and is to be enunciated, 12 Billions, 345,678 Millions, 912,345 Units; 67,890 hundred thousandth Parts, 123, 456 millioneth Parts, and 78 billioneth Parts.

16. *In Integers and Decimals there can be but nine distinct Numerators, or significant Figures.* For each Figure of a Number being multiplied or divided by some Power of Ten, the greatest Figure must be less than Ten, lest the Number of Places in the Product or Quote should exceed those in the Power of Ten, by which it is multiplied or divided; which if it came to pass, the decuple Ratio could not be preserved in the Places, nor any Denominator discovered from the Distance of the Figure from the Place of Units.

Vowels or little Letters for the unknown ones. (*d*)

IV. *Quantities are either* Affirmative, *or greater than than nothing*; *or* Negative, *or less than nothing.* Thus in human Affairs, Poſſeſſions or Stock may be called affirmative Goods, and Debts negative ones. And ſo in local Motion, Progreſſion may be called affirmative Motion, and Regreſſion negative Motion; becauſe the firſt augments, and the other diminiſhes the Length of the Way made. And after the ſame Manner in Geometry, if a Line drawn any certain Way be reckoned for Affirmative, then

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17. *In vulgar Fractions the Number of different Numerators will be unlimited*, becauſe Unity can be divided by an infinite Variety of Numbers: The nine ſignificant Figures however, with the Cypher, by being eſtimated in a decuple Ratio, are capable of expreſſing every one of that unlimited Number.

18. Becauſe a Number may conſiſt of Figures, whoſe Denominators are Terms of the decuple Ratio taken at different Intervals, and not in Succeſſion, therefore *that the Denominators may always be expounded by the Diſtances of the Figures from the Place of Units, there will be Occaſion for an inſignificant Symbol or Cypher*, that the Places void of Numerators being filled up by it, the Progreſſion of the Denominators may be indicated by the Diſtances from the Place of Units. *Hence Cyphers to the left of Integers, and to the right of Decimals*, cannot alter their Value, and *are uſeleſs: But*, being placed *to the right of Integers, increaſe their Value, and to the left of Decimals, diminiſh their Value, each in a decuple Ratio.*

III. (*d*) The Method of ſtating Law Queſtions under general Names, was by Civilians ſtiled Species; *Vieta* transferred this Title to his Invention of denoting known Quantities by Conſonants, and unknown Quantities by Vowels. The *known Quantities are now uſually repreſented by the former, and the unknown by the latter Letters of the Alphabet: Alſo like Quantities are repreſented by the ſame Letters, and under the ſame Power; and unlike Quantities by different Letters; or, if by the ſame Letters, they muſt be under different Powers.*

IV. (*) In

then a Line drawn the contrary Way may be taken for Negative. As if A B [*See* Fig. 1.] be drawn to the right, and B C to the left; and A B be reckoned Affirmative, then B C will be Negative; becaufe in the drawing it diminifhes A B, and reduces it either to a fhorter, as A C, or to none, if C chances to fall upon the Point A, or to lefs than none, if B C be longer than A B from which it is taken. *A negative Quantity is denoted by the Sign* —; *the Sign* + *is prefixed to an* affirmative *one*; *and* $\mp$ *denotes an uncertain Sign, and* $\pm$ *a contrary uncertain one.* (*)

V. *In an Aggregate of Quantities, the Note* + *fignifies, that the Quantity it is prefixed to, is* to be added; *and the Note* —, *that it is* to be fubtracted. And we ufually exprefs thefe Notes by the Words *Plus* (on *more*) and *Minus* (or *lefs.*) Thus 2+3, or 2 more 3, denotes the Sum of the Numbers 2 and 3, that is, 5. And 5—3, or 5 lefs 3, denotes the Difference which arifes by fubducting 3 from 5, that is, 2. And —5+3 fignifies the Difference which arifes from fubducting 5 from 3, that is —2; and 6—1+3 makes 8. Alfo $a+b$ denotes the Sum of the Quantities a and b, and $a-b$ the Difference which arifes by fubducting b from a; and $a-b+c$ fignifies the Sum of that

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IV. (*) In Geometry, Lines are reprefented by a Line; Triangles by a Triangle; and other Figures by a Figure of the fame Kind; But, in Algebra, Quantities are reprefented by the fame Letters of the Alphabet; and various Signs have been imagined for reprefenting their Relations, Affections, and Dependencies. *In Geometry the Reprefentations are more natural*; *in Algebra more arbitrary.* The former are like the firft Attempts towards the Expreffion of Objects, which was by drawing their Refemblances; the latter correfpond more to the prefent Ufe of Languages and Writing. Thus *the Evidence of Geometry is fometimes more fimple and obvious*; *but the Ufe of Algebra more extenfive, and often more ready*; efpecially fince the mathematical Sciences have acquired fo vaft an Extent, and have been applied to fo many Inquiries.

In

that Difference and of the Quantity c. Suppose if a be 5, b 2, and c 8, then $a+b$ will be 7, and $a-b$ 3, and $a-b+c$ will be 11. Also $2a+3a$ is $5a$, and $3b-2a-b+3a$ is $2b+a$; for $3b-b$ makes $2b$, and $-2a+3a$ makes a, whose Aggregate, or Sum, is $2b+a$, and so in others. These Notes + and — are called *Signs*; and when neither is prefixed, the Sign + is always to be understood. (*e*)

VI. Multiplication, *properly so called, is that which is made by Integers, as seeking a new Quantity, so many Times greater than the Multiplicand, as the Multiplier is greater than Unity.* But, for want of a better Word, *that is also called* Multiplication, *which is made use of in Fractions and Surds,*

In those Sciences, *it is not barely Magnitude that is the Object of Contemplation: But there are many Affections and Properties of Quantities, and Operations to be performed upon them, that are necessary to be considered.* In estimating the Ratio or Proportion of Quantities, Magnitude only is considered: But the Nature and Properties of Figures depend on the Position of the Lines that bound them, as well as on their Magnitude. In treating of Motion, the Direction of Motion, as well as its Velocity, and the Direction of Powers which generate or destroy Motion, as well as their Forces, must be regarded. In Optics, the Position, Brightness, and Distinctness of Images, are of no less Importance than their Bigness; and the like is to be said of other Sciences. *It is necessary therefore that other Symbols be admitted into* Algebra, *beside the Letters and Numbers which represent the Magnitude of Quantities.*

V. (*e*) Quantities connected by the Signs + or — make one *compound Quantity*, whose *Terms* are the single or simple Quantities so connected. A compound Quantity of two Terms is called a *Binome*, whether it be the Sum or the Difference of two Quantities: When it is necessary to distinguish them, the Sum shall be called a *Binominal*, and the Difference a *Residual*. A compound Quantity of more Terms than two is called a *Multinome*; such are *Trinomes*, *Quadrinomes*, &c. If the Number of Terms be indefinite, the compound Quantity is an *Infinitinome*.

VI. (*f*) 19. Hence

Surds, to find a new Quantity in the same Ratio *(whatever it be) to the Multiplicand, as the Multiplier has to Unity.* (*f*) Nor is *Multiplication made* only by abstract Numbers, but also *by concrete Quantities*, as by Lines, Surfaces, Local Motion, Weights, *&c. as far as these being related to some known Quantity of their Kind, as to Unity, may express the Ratios of Numbers, and supply their Place.* As if the Quantity A be to be multiplied by a Line of 12 Foot, supposing a Line of 2 Foot to be Unity, there will be produced by that Multiplication 6 A, or six Times A, in the same Manner as if A were to be multiplied by the abstract Number 6; for 6 A is in the same Ratio to A, as a Line of 12 Foot has to a Line of 2 Foot. And so if you were to multiply any two Lines, A C [*See* Fig. 2.] and A D by one another, take A B for Unity, and draw B C, and parallel to it D E, and A E will be the Product of this Multiplication; because it is to A D as A C to the Unity A B. Moreover, *Custom has obtained, that the Genesis or Description of a Surface, by a Line moving at right Angles upon another Line, should be called the Multiplication of those two Lines.* For though a Line, however multiplied, cannot become a Surface, and consequently this Generation of a Surface by Lines is very different from Multiplication, yet they agree in this, that the Number of Unities in either Line, multiplied by the Number of Unities in the other, produces an abstracted Number of Unities in the Surface comprehended under those Lines, if the superficial Unity be defined as it is used to be, *viz.* a Square whose Sides are linear Unities. As if the right Line [Fig. 3.] A B

VI. (*f*) 19. Hence it follows, that *as Unity is to the Multiplier, so is the Multiplicand to the Product.* (Eucl. VII. Def. 15.) The Product is the Aggregate of the Multiplicand alone, whence *the Multiplier is an absolute Number*, or Multiple of Unity: Hence the Operation in *Surds* and in *Fractions* is not properly Multiplication. And *in Fractions*, because *the Multiplier is a Submultiple of Unity, the Product* is so far from being a Multiple of the Multiplicand, that it *must be* a Submultiple of, that is, *less than, the Multiplicand.*

VII. (*g*) The

A B consist of four Unities, and A C of three, then the Rectangle A D will consist of four times three, or 12 square Unities, as from the Scheme will appear. *And there is the like Analogy of a Solid and a Product made by the continual Multiplication of three Quantities.* And *hence* it is, that the Words to *multiply into*, the *Content*, a *Rectangle*, a *Square*, a *Cube*, a *Dimension*, a *Side*, and the like, which are *Geometrical Terms*, are *applied to Arithmetical Operations*. For by a *Square*, or *Rectangle*, or a *Quantity of two Dimensions*, we do not always understand a Surface, but most commonly a Quantity of some other Kind, which is produced by the Multiplication of two other Quantities, and very often a Line which is produced by the Multiplication of two other Lines. And so we call a *Cube*, or a *Parallelopiped*, or a *Quantity of three Dimensions*, that which is produced by two Multiplications. We say likewise the *Side* for a *Root*, and use *Draw into* instead of *Multiply*; and so in others.

VII. *A Number prefixed immediately before any Species, denotes that Species to be so often to be taken.* Thus 2 *a* denotes two *a*'s, 3 *b* three *b*'s, 15 *x* fifteen *x*'s. (*g*)

VIII. *Two or more Species immediately connected together, denote a* Product *or Quantity made by the Multiplication of all the Species together.* Thus *a b* denotes a Quantity made by multiplying *a* by *b*; and *a b x* denotes a Quantity made by multiplying *a* by *b*, and the Product again by *x*. As suppose, if *a* were 2, and *b* 3, and *x* 5, then *a b* would be 6, and *a b x* would be 30. (*h*)

IX. *Among*

VII. (*g*) The Number prefixed is called the *Coefficient*, or Cofactor: And *if a Quantity is without one, it is implied that Unity is its Coefficient*; which causing no Alteration, is therefore generally omitted.

VIII. (*h*) When the Factors are single Letters, or Figures and single Letters, every Sign of Multiplication is generally omitted, and the Product is any Combination of the Letters.

IX. (*i*) Besides

IX. *Among Quantities multiplying one another, the Sign* ×, *or the Word* by *or* into, *is made use of to denote the Product sometimes.* Thus 3×5, or 3 by or into 5, denotes 15; but *the chief Use of these Notes is when compound Quantities are multiplied together.* As if $y-2b$ were to multiply $y+b$, the Way is to draw a Line over each Quantity, and then write them thus, $\overline{y-2b}$ into $\overline{y+b}$, or $\overline{y-2b}\times\overline{y+b}$. (*i*)

X. Division *is properly that which is made use of for integer or whole Numbers, in finding a new Quantity so much less than the Dividend, as Unity is than the Divisor.* But *by Analogy*, the Word may also be used *when a new Quantity is sought, that shall be in any such* Ratio *to the Dividend, as Unity has to the Divisor; whether that Divisor be a Fraction or surd* Number, or other Quantity of any other Kind. Thus to divide the Line [*See* Fig. 4.] A E by the Line A C, A B being Unity, you are to draw E D parallel to C B, and A D will be the Quotient. Moreover, *it is called Division*, by reason of a certain Similitude, *when a Rectangle is* applied *to a given Line as a Base, in order thereby to know the Heighth.* (*k*)

XI. *One*

IX. (*i*) Besides the oblique Cross ×, and the Words *by*, and *into*; a *full Point* is also interposed between the Factors, to denote their Product. Sometimes also *compound Factors*, instead of having Lines continued over their Terms, are *included within Hooks*, and some Mark of Multiplication interposed. But in all Cases

20. *The Product is the same whatever the Order of the Factors may be.* (Eucl. VII. 16.)

X. (*k*) 21. Hence, *as the Divisor is to the Dividend, so is Unity to the Quote.* The Dividend is an Aggregate of the Divisor alone; *the Quote* therefore is *an absolute Number*, or Multiple of Unity. In the Operation of *Surds* and *Fractions*, it is not, strictly speaking, Division; and *in* the Case of *Fractions*, as the Divisor is a Submultiple of Unity, so *the Dividend* is a Submultiple of, that is, *is less than, the Quote.*

XI. (*l*) See

XI. *One Quantity below another, with a Line interposed, denotes a* Quotient, *or a Quantity arising by the Division of the upper Quantity by the lower.* Thus $\frac{6}{2}$ denotes a Quantity arising by dividing 6 by 2, that is 3; and $\frac{5}{8}$ a Quantity arising by the Division of 5 by 8, that is one eighth Part of the Number 5. And $\frac{a}{b}$ denotes a Quantity which arises by dividing a by b; as suppose a was 15, and b 3, then $\frac{a}{b}$ would denote 5. Likewise thus $\frac{ab-bb}{a+x}$ denotes a Quantity arising by dividing $ab-bb$ by $a+x$. And so in others. *These Sorts of Quantities are called* Fractions; *and the upper Part is called by the Name of the* Numerator, *and the lower is called the* Denominator. (*l*)

XII. *Sometimes the Divisor is set before the divided Quantity, and separated from it by a Mark resembling an Arch of a Circle.* Thus to denote the Quantity which arises by the Division of $\frac{axx}{a+b}$ by $a-b$, it may be wrote thus, $\overline{a-b})\frac{axx}{a+b}$. (*m*)

XIII. Although we commonly denote Multiplication by the immediate Conjunction of the Quantities, yet *an Integer before a Fraction, denotes the Sum of both.* Thus $3\frac{1}{2}$ denotes three and a half. (*n*)

XIV. *If a Quantity be multiplied by itself, the Number of* Facts *or Products is, for Shortness sake, set at the Top of the Letter.* Thus, for aaa, we write a^3; for $aaaa$, a^4; for $aaaaa$, a^5; and for $aaabb$ we write a^3bb, or a^3b^2: As

XI. (*l*) See Article XXXVII. N° 143, &c.

XII. (*m*) The Dividend is also sometimes set before the Divisor, with the Mark ÷ interposed.

XIII. (*n*) And is called *a mixed Number.*

XIV. (*o*) The

As suppose, if a were 5, and b be 2, then a^3 will be $5\times5\times5$ or 125, and a^4 will be $5\times5\times5\times5$ or 625, and a^3b^2 will be $5\times5\times5\times2\times2$ or 500. Where note, that *if a Number be written immediately between two Species, it always belongs to the former*; thus the Number 3 in the Quantity a^3bb, does not denote that bb is to be taken thrice, but that a is to be thrice multiplied by itself. (*o*) Note, moreover, that *these Quantities are said to be of so many* Dimensions, *or of so high a* Power *or* Dignity, *as they consist of Factors or Quantities multiplying one another*; *and the Number set on forwards at the Top of the Letter, is called the Index of those Powers or Dimensions*; thus aa is of two Dimensions, or of the 2d Power; and a^3 of three, as the Number 3 at the Top denotes. aa is also called a *Square*, a^3 a *Cube*, a^4 a *Biquadrate*, or *squared Square*, a^5 a *Quadrato-Cube*, a^6 a *Cubo-Cube*, a^7 a *Quadrato-Quadrato-Cube* or *Squared-Squared-Cube*, and so on: And the Quantity a, by whose Multiplication by itself these Powers are generated, is called their *Root*, *viz.* it is the Square Root of the Square aa, the Cube Root of the Cube aaa, &c. (*p*)

XV. But

XIV. (*o*) The Multiplication of a Quantity by itself, is called *Involution*. *The Sign of Involution is* ☉. When a compound Quantity is signified to be involved, *the Index of the Power is set at the End of a Line continued over all its Terms*. In all Cases, because Unity (VII. *g*) is supposed to be continually a Cofactor, *the Index is equal to the Number of Multiplications by which the Power is produced*, and *equal to the Number of Factors, excluding Unity*.

(*p*) *A single Quantity* is said to be of one Dimension. *A simple Product of Factors of one Dimension* is of so many Dimensions as there are literal Factors. The Factors are called the Roots of the Product; and if the Factors are equal, the Product is usually called a Power. *A simple Product of Factors of different Dimensions* is of so many Dimensions as there are Units in the Sum of the Exponents of the Factors. *A compound Product, which contains Quantities all known*, is of so many Dimensions as its highest Term. And *a compound Product which contains an unknown Quantity* is of so many Dimensions, as there are Units in the highest Index of the unknown in any of its Terms.

XVI. (*q*) The

XV. But when *a Root multiplied by itself, produces a Square, and that Square, multiplied again by the Root, produces a Cube*, &c. *it will be* (by the Definition of Multiplication) *as Unity to the Root, so that Root to the Square, and that Square to the Cube*, &c. And consequently *the square Root of any Quantity will be a mean Proportional between Unity and that Quantity, and* the Cube Root *the first of two mean Proportionals, and* the Biquadratick Root *the first of three, and so on.* Wherefore *Roots* are *known* by these *two Properties*, or Affections; *first*, that *by multiplying themselves they produce the superior Powers*; *2dly*, that *they are mean Proportionals between those Powers and Unity.* Thus 8 is the Square Root of the Number 64; and 4, the Cube Root of it, is hence evident, because 8×8 and $4 \times 4 \times 4$ make 64; or because as 1 is to 8, so is 8 to 64; and 1 is to 4, as 4 to 16, and as 16 to 64. And hence, if the Square Root of any Line as A B [*See* Fig. 5.] is to be extracted, produce it to C, and let B C be Unity; then upon A C describe a Semicircle, and at B erect a Perpendicular, meeting the Circle in D; then will B D be the Root, because it is a mean Proportional between A B and Unity B C.

XVI. *To denote the Root of any Quantity, we use to prefix this Note* $\surd$ *for a Square Root*; *and this* $\surd 3:$ *if it be a Cube Root*; *and this* $\surd 4:$ *for a Biquadratic Root*, &c. Thus $\surd 64$ denotes 8; and $\surd 3: 64$ denotes 4; and $\surd aa$ denotes a; and $\surd ax$ denotes the Square Root of ax; and $\surd 3: 4axx$ the Cube Root of $4axx$. As if a be 3, and x 12; then $\surd ax$ will be $\surd 36$, or 6; and $\surd 3: 4axx$ will be $\surd 3: 1728$, or 12. And *when these Roots cannot be extracted, the Quantities are called* Surds; as $\surd ax:$ or *Surd Numbers*, as $\surd 12$. (*q*)

There

XVI. (*q*) The Extraction of a Root is called *Evolution. The Sign of Evolution is* ɯ. *The Sign of Irrationalty is* $\surd$ with the Index of the Root set over it. *A simple Surd* consists of one irrational Term. *A compound or universal Surd* contains more irrational Terms than one.

(*r*) The

There are ſome, that, to denote the Square or firſt Power, make uſe of q, and of c for the Cube, $q\,q$ for the Biquadrate, and $q\,c$ for the Quadrato-Cube, *&c.* After this Manner for the Square, Cube, and Biquadrate of A, they write A q, A c, A $q\,q$, &c. And for the Cube Root of $a\,b\,b - x^3$, they write $\surd\, c : a\,b\,b - x^3$. Others make uſe of other Sorts of Notes, but they are now almoſt out of Faſhion. (*r*)

XVII. *The Mark* $=$ *ſignifies, that the Quantities on each Side of it are equal.* Thus $x=b$ denotes x to be equal to b.

The Note $::$ *ſignifies, that the Quantities on both Sides of it are* proportional. Thus $a.b :: c.d$ ſignifies, that a is to b as c to d; and $a.b.e :: c.d.f$ ſignifies, that a, b, and e, are to one another reſpectively, as c, d, and f, are among themſelves; or that a to c, b to d, and e to f, are in the ſame *Ratio.*

Laſtly, *the Interpretation of any Marks or Signs that may be compounded out of theſe, will eaſily be known by Analogy.* Thus $\frac{3}{4}a^3bb$ denotes three quarters of a^3bb, and $3\frac{a}{c}$ ſignifies thrice $\frac{a}{c}$, and $7\surd ax$ ſeven times $\surd ax$. Alſo $\frac{a}{b}x$ denotes the Product of x by $\frac{a}{b}$; and $\frac{5ee}{4a+9e}z^3$ denotes the Product made by multiplying z^3 by $\frac{5ee}{4a+9e}$, that is the Quotient ariſing by the Diviſion of $5ee$ by $4a+9e$; and

(*r*) The Notation by the Sign of Irrationality called the *Vinculum*, with the Index ſet above it, is chiefly followed: But *the Form of Surds beſt ſuited to the ſeveral Operations, is that, where the Index of the Surd is a Fraction, whoſe Numerator denotes the Power to which the Quantity is ſuppoſed to be raiſed, and whoſe Denominator denotes the Root to be extracted from the ſaid Power of that Quantity.* Thus $x^{\frac{5}{3}} = \sqrt[3]{x^5}$; and $\overline{x^2+ab}^{\frac{1}{3}} = \sqrt[3]{x^2+ab}$. See N° 78.

and $\frac{2a^3}{9c}\sqrt{ax}$, that which is made by multiplying $\sqrt{ax}$ by $\frac{2a^3}{9c}$; and $\frac{7\sqrt{ax}}{c}$ the Quotient arising by the Division of $7\sqrt{ax}$ by c; and $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ the Quotient arising by the Division of $8a\sqrt{cx}$ by the Sum of the Quantities $2a+\sqrt{cx}$. And thus $\frac{3axx-x^3}{a+x}$ denotes the Quotient arising by the Division of the Difference $3aax-x^3$ by the Sum $a+x$, and $\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Root of that Quotient, and $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ denotes the Product of the Multiplication of that Root by the Sum $2a+3c$. Thus also $\sqrt{\frac{1}{4}aa+bb}$ denotes the Root of the Sum of the Quantities $\frac{1}{4}aa$ and bb, and $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root of the Sum of the Quantities $\frac{1}{2}a$ and $\sqrt{\frac{1}{4}aa+bb}$, and $\frac{2a^3}{aa-zz}\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ denotes the Root multiplied by $\frac{2a^3}{aa-zz}$. And so in other Cases.

But Note, that *in complex Quantities of this Nature, there is no Necessity of giving a particular Attention to*, or bearing in your Mind, *the Signification of each Letter*; it will suffice in general to understand, *e. g.* that $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ signifies the Root of the Aggregate or Sum of $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}$; whatever that Aggregate may chance to be, when Numbers or Lines are substituted in the room of Letters. And thus it is as sufficient to understand, that $\frac{\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}}{a-\sqrt{ab}}$ signifies the

Quotient

Quotient arising by the Division of the Quantity $\sqrt{\frac{1}{2}a+\sqrt{\frac{1}{4}aa+bb}}$ by the Quantity $a-\sqrt{ab}$, as much as if those Quantities were simple and known, though at present one may be ignorant what they are, and not give any particular Attention to the Constitution or Signification of each of their Parts; which I thought I ought here to admonish, lest young Beginners should be deterred in the very Beginning, by the Complexness of the Terms.

Of Addition.

XVIII. THE *Addition of Numbers, where they are not very compounded, is manifest of itself.* Thus it is at first Sight evident, that 7 and 9 or 7+9 make 16, and that 11+15 make 26. But *in more compounded Numbers, the Business is performed by writing the Numbers in a Row downwards, or one under another, and singly collecting the Sums of the Columns.* (*a*) As if the Numbers 1357 and 172 are to be added, write either of them (suppose 172) under the other 1357, so that the Units of the one, *viz.* 2, may exactly stand under the Units of the other, *viz.* 7, and the other Numbers of the one exactly under the correspondent ones of the other, *viz.* the Place of Tens under Tens, *viz.* 7 under 5, and that of Hundreds, *viz.* 1, under the Place of Hundreds of the other, *viz.* 3. Then beginning at the right hand, say 2 and 7 make 9, which write underneath. Also 7 and 5 make 12, the last of which two Numbers, *viz.* 2, write underneath, and reserve in your Mind the other, *viz.* 1, to be added to the two next Numbers, *viz.* 1 and 3.

$$\begin{array}{r}1357\\172\\\hline 1529\end{array}$$

XVIII. (*a*) *Begin with the Column of Units, and if their Sum is under Ten, set it underneath: But if equal to Ten, or Tens, set a Cypher underneath: And if greater than Ten, or Tens, set the Excess underneath: And for every Ten carry an Unit to the next Column.* Proceed thus through all.

1 and 3. Then say 1 and 1 make 2, which being added to 3 they make 5, which write underneath, and there will remain only 1, the first Figure of the upper Row of Numbers, which also must be writ underneath; and then you have the whole Sum, *viz.* 1529.

Thus, to add the Numbers 87899+13403+885+1920 into one Sum, write them one under another, so that all the Units may make one Column, the Tens another, the Hundreds a third, and the Places of Thousands a fourth, and so on. Then say 5 and 3 make 8, and 8+9 make 17; then write 7 underneath, and the 1 add to the next Rank, saying 1 and 8 make 9, 9+2 make 11, and 11+9 make 20; and having writ the 0 underneath, say again, as before, 2 and 8 make 10, and 10+9 make 19, and 19+4 make 23, and 23+8 make 31; then reserving 3 in your Memory, write down 1 as before, and say again 3+1 make 4, 4+3 make 7, and 7+7 make 14, wherefore write underneath 4; and lastly say 1+1 make 2, and 2+8 make 10, which in the last Place write down, and you will have the Sum of them all, 104107. (*b*)

$$\begin{array}{r} 87899 \\ 13403 \\ 1920 \\ 885 \\ \hline 104107 \end{array}$$

XIX. *After the same Manner we also add Decimals*, as in the following Example may be seen:

$$\begin{array}{l} 630,953 \\ \ \ 51,0807 \\ 305,25 \\ \hline 987,2837 \end{array}$$ (*c*)

XX. *Addition*

(*b*) *Quantities of different Denominations*, whether they be Numbers or Species, whether Integers or Fractions, whether Powers or Roots, whether rational or irrational, *cannot properly be added*, that is, *united into one Sum*; *because the Aggregate cannot be of any certain Denomination*: They can therefore be only connected with their Signs.

XIX. (*c*) *Numbers whose Denominators are in any other Ratio may be added after the Manner of those, by substituting in the Place of Ten the Number by which the Denominators increase.*

XXI. (*) That

XX. *Addition is performed in Algebraic Terms or Species, by connecting the Quantities to be added with their proper Signs, and moreover by uniting into one Sum those that can be so united.* Thus a and b make $a+b$; a and $-b$ make $a-b$; $-a$ and $-b$ make $-a-b$; $7a$ and $9a$ make $7a+9a$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $-a\sqrt{ac}+b\sqrt{ac}$, or $b\sqrt{ac}-a\sqrt{ac}$; for it is all one, in what Order soever they are written.

XXI. *Affirmative Quantities, which agree in Species, are united together, by adding the prefixed Numbers that are multiplied into those Species.* Thus $7a+9a$ make $16a$. And $11bc+15bc$ make $26bc$. Also $3\frac{a}{c}+5\frac{a}{c}$ make $8\frac{a}{c}$, and $2\sqrt{ac}+7\sqrt{ac}$ make $9\sqrt{ac}$, and $6\sqrt{ab-xx}+7\sqrt{ab-xx}$ make $13\sqrt{ab-xx}$. And, in like Manner, $6\sqrt{3}+7\sqrt{3}$ make $13\sqrt{3}$. Moreover $a\sqrt{ac}+b\sqrt{ac}$ make $\overline{a+b}\sqrt{ac}$, by adding together a and b as Numbers multiplying $\sqrt{ac}$. And so $\overline{2a+3c}\sqrt{\frac{3axx-x^3}{a+x}}+3a\sqrt{\frac{3axx-x^3}{a+x}}$ make $\overline{5a+3c}\sqrt{\frac{3axx-x^3}{a+x}}$ because $2a+3c$ and $3a$ make $5a+3c$. (*)

XXII. *Affirmative Fractions, that have the same Denominator, are united by adding their Numerators.* Thus $\frac{1}{5}+\frac{2}{5}$ make $\frac{3}{5}$, and $\frac{2ax}{b}+\frac{3ax}{b}$ make $\frac{5ax}{b}$; and thus $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}+\frac{17a\sqrt{cx}}{2a+\sqrt{cx}}$ make $\frac{25a\sqrt{cx}}{2a+\sqrt{cx}}$, and $\frac{aa}{c}+\frac{bx}{c}$ make $\frac{aa+bx}{c}$.

XXIII. *Negative*

XXI. (*) That is, *Surds are united by uniting their rational Coefficients,* when being reduced to their lowest Terms, they agree in their radical Part, or have the same Denomination; for in this Case they are commensurable to each other, having the Ratio of their rational Coefficients. (Eucl. VII. 18. Cor.)

 XXIV. (*d*) The

XXIII. *Negative Quantities are added after the same Way as Affirmative.* Thus -2 and -3 make -5; $-\frac{4ax}{b}$ and $-\frac{11ax}{b}$ make $-\frac{15ax}{b}$; $-a\sqrt{ax}$ and $-b\sqrt{ax}$ make $-\overline{a-b}\sqrt{ax}$.

XXIV. *But when a Negative Quantity is to be added to an Affirmative one, the Affirmative must be diminished by the Negative one.* Thus 3 and -2 make 1; $\frac{11ax}{b}$ and $-\frac{4ax}{b}$ make $\frac{7ax}{b}$; $-a\sqrt{ac}$ and $b\sqrt{ac}$ make $\overline{b-a}\sqrt{ac}$. And note, *That when the Negative Quantity is greater than the Affirmative, the Aggregate or Sum will be Negative.* (*d*) Thus 2 and -3 make -1; $-\frac{11ax}{b}$ and $\frac{4ax}{b}$ make $-\frac{7ax}{b}$; and $2\sqrt{ac}$ and $-7\sqrt{ac}$ make $-5\sqrt{ac}$.

In

XXIV. (*d*) *The Sum of similar Quantities, affected by contrary Signs, must be the Excess of the greater:* For, if they are simple, it is their Difference which we seek; and, if they are compound, it is the Sum of the Differences of their Terms.

Again, To add a Negative is to take away an Equal Positive; therefore to add a Negative to a Positive is to make one, and destroy the other, so far as they are equal; and consequently the Sum must be nothing when they are equal, and affected by the Sign of the greater when they are unequal.

22. Hence it follows universally, that *the Sum of two Quantities added to their Difference is equal to double the greater:* For the less is destroyed in the Addition by the Signs being contrary; and the greater is doubled by their being the same. Thus let a be the greater, and b the less, then their Sum is $a+b$, and their Difference is $a-b$; but $\overline{a+b}+\overline{a-b}=2a$. See N° 36.

(*e*) When

In the Addition of a greater Number of Quantities, or more compound ones, it will be convenient to obſerve the Method or Form of Operation we have laid down above in the Addition of Numbers. As if $17ax-14a+3$, and $4a+2-8ax$, and $7a-9ax$, were to be added together, diſpoſe them ſo in Columns, that the Terms that contain the ſame Species may ſtand in a Row one under another, *viz.* the Numbers 3 and 2 in one Column, the Species $-14a$, and $4a$, and $7a$, in another Column, and the Species $17ax$, and $-8ax$, and $-9ax$, in a third. Then I add the Terms of each Column by themſelves, ſaying 2 and 3 make 5, which I write underneath; then $7a$ and $4a$ make $11a$, and moreover $-14a$ make $-3a$, which I alſo write underneath; laſtly, $-9ax$ and $-8ax$ make $-17ax$, to which $17ax$ added make 0. And ſo the Sum comes out $-3a+5$. (*e*)

$$\begin{array}{r} 17ax-14a+3 \\ -8ax+\ \ 4a+2 \\ -9ax+\ \ 7a \\ \hline *-\ \ 3a+5 \end{array}$$

After the ſame Manner the Buſineſs is done in the following Examples:

$$\begin{array}{r} 12x+\ 7a \\ 7x+\ 9a \\ \hline 19x+16a \end{array} \qquad \begin{array}{r} 11bc-7\sqrt{ac} \\ 15bc+2\sqrt{ac} \\ \hline 26bc-5\sqrt{ac} \end{array} \qquad \begin{array}{r} -\dfrac{4ax}{b}+6\sqrt{3}+\dfrac{1}{5} \\ +\dfrac{11ax}{b}-7\sqrt{3}+\dfrac{2}{5} \\ \hline \dfrac{7ax}{b}-\ \sqrt{3}+\dfrac{3}{5} \end{array}$$

$$\begin{array}{r} -6xx+\tfrac{3}{7}x \\ 5x^3+\tfrac{5}{7}x \\ \hline 5x^3-6xx+\tfrac{8}{7}x \end{array} \qquad \begin{array}{r} aay+2a^3-\dfrac{a^4}{2y} \\ -2ayy-4aay-\ a^3\phantom{-\dfrac{a^4}{2y}} \\ y^3+2ayy-\tfrac{1}{2}aay\phantom{-\ a^3-\dfrac{a^4}{2y}} \\ \hline y^3\quad *-3\tfrac{1}{2}aay+\ a^3-\dfrac{a^4}{2y} \end{array}$$

$5x^4$

(*e*) When there are many Quantities, both affirmative and negative, of the ſame Species, to be added, it will be moſt convenient to find ſeparately the Sums of thoſe which are like effected, and then to find the Exceſs of thoſe Sums.

 (*f*) It

$$\begin{array}{l} 5x^4+2ax^3 \\ -3x^4-2ax^3+8\tfrac{1}{2}a^3\sqrt{aa+xx} \\ -2x^4+5bx^3-20a^3\sqrt{aa-xx} \\ \quad\;\; -4bx^3-7\tfrac{1}{2}a^3\sqrt{aa+xx} \\ \hline * \; bx^3+a^3\sqrt{aa+xx}-20a^3\sqrt{aa-xx}. \end{array} \; (f)$$

(*f*) It will be of Use to explain the Generation and Properties of *Figurate Numbers*, so far as we may have Occasion hereafter to make Use of them.

23. *A Series of Numbers which arises from adding a Rank*

of	Units or Figurates of the 1st Order - -	are Figurates of the	2d	Order.
	Figurates of the 2d Order - -		3d	
	Figurates of the 3d Order - -		4th	
	Figurates of the 4th Order - -		5th	
	Figurates of the 5th Order, *&c.*		6th	

Whence the Figurates

of the	1st Order	are	1	1	1	1	1	1,	&c.
	2d Order		1	2	3	4	5	6,	&c.
	3d Order		1	3	6	10	15	21,	&c.
	4th Order		1	4	10	20	35	56,	&c.
	5th Order		1	5	15	35	70	126,	&c.
	6th Order		1	6	21	56	126	252,	&c.

And consequently *a figurate Number of any Order is the Sum of all the Figurates of the next preceding Order so far*; *or* it is *the Sum of the preceding Figurate of the same Order, and of the corresponding Term or Figurate of the preceding Order.* Note, that *Figurates of the* 2d *Order are called Laterals.*

24. *Let* m *and* n *be any two Integers, then the* m*th Term of the* n*th Order is the* n*th Term of the* m*th Order:* For they are the Sums of the same Numbers. (23.) Thus the sixth Term of the fourth Order is 56; which is the fourth Term of the sixth Order. Thus also the 2d, 3d, 4th, *&c.* Terms of the second Order are the second Terms of the 2d, 3d, 4th, *&c.* Orders respectively. Now *the Terms of the second Order being the Laterals*, and *the second Term of each Order being called the Exponent of that Order*; consequently *the Laterals are the Exponents of the Orders.*

25. In

25. In any Order, *the Sum of any given Number of Terms is equal to the Product of the subsequent Term into the given Number, divided by the Exponent of that Order:* For the Exponent being the Sum of Unity and of the Exponent of the preceding Order, (23.) and each subsequent Term being the Sum of its antecedent and of the corresponding Term of the preceding Order, (23.) if any Term be taken so often as there have been Additions, that is, if it is multiplied into the given Number of antecedent Terms, the Product and the Exponent will be Equimultiples of the Sum of the antecedent Terms, and of Unity; therefore (Eucl. V. Def. 5.) the Sum of the antecedent Terms is to this Product, as Unity to the Exponent; and consequently the Sum of the antecedent Terms is equal to the Product of the subsequent Term into the Number of the antecedent, divided by the Exponent. (Eucl. VII. 19.) Thus the Sum of the first five Terms of the third Order, *viz.* $1+3+6+10+15$ will be the sixth Term multiplied by five, and divided by three, *viz.* $=\frac{21\times5}{3}=35$; and universally putting S for the Sum, e for the Exponent, and z for the Term subsequent to the last of the given Number, which call n, we have $S=\frac{z\times n}{e}$ or $S=z\times\frac{n}{e}$

26. In any Order, *the Sum of any given Number of Terms is equal to the Product of the last of the given Number of Terms multiplied into the Quote, which arises by dividing the Sum of the Exponent, and of the given Number of Terms less one, by the Exponent:* For putting the last of the given Number of Terms, l, the Sum of the given Number of Terms less this last will be $S-l$, and $S-l=l\times\frac{n-1}{e}$; (N°. 25.) wherefore, ($Se-le=l\times\overline{n-1}$; whence $Se=l\times\overline{n-1}+e$; whence) we have $S=l\times\frac{n-1+e}{e}$. (Art. LXVII. &c.) So $1+3+6+10+15\,[=\frac{21\times5}{3}$ (N°. 25.)$]=\frac{15\times\overline{5-1+3}}{3}$ $=\frac{15\times7}{3}=35$.

27. *If*

27. *If* n *be put for any Number of Terms, then the* n*th Figurate in every Order, proceeding uniformly from the first, will be Unity and the following Products*; viz. *those, which will arise from the continual Multiplication of Fractions, whose Numerators are the given Number, and the given Number increased uniformly by the ascending Laterals; and whose Denominators are the same ascending Laterals beginning from Unity:* For the Sum in every Series is $l \times \frac{n-1+e}{e}$; (26.) but this Sum is the nth Term of the next subsequent Order; (24.) wherefore, by substituting successively in the general Expression $l \times \frac{n-1+e}{e}$ the given Number for n, and the Laterals for e, (24.) we shall have the nth Sums and Terms successively throughout the Orders. Now in the first Order, by this Substitution, we have $1 \times \frac{n-1+1}{1} = n$ the Sum of n Terms in the first, and the nth Term of the second Order, (or l, in the general Expression $l \times \frac{n-1+e}{e}$). In the second Order, therefore by substituting $\frac{n}{1}$ for l, and 2 for e, we shall have $\frac{n}{1} \times \frac{n-1+2}{2} = \frac{n}{1} \times \frac{n+1}{2}$ the Sum of n Terms in the second, and the nth Term of the third Order; and substituting $\frac{n}{1} \times \frac{n+1}{2}$ for l, and 3 for e; the general Expression $l \times \frac{n-1+e}{e}$ will become $\frac{n}{1} \times \frac{n+1}{2} \times \frac{n-1+3}{3}$ $= \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}$ the Sum of n Terms in the 3d, and the nth Term of the 4th Order; and so on. Thus the 6th Term

Term is 1; $\frac{6}{1}=6$; $1\times\frac{6}{1}\times\frac{7}{2}=21$; $1\times\frac{6}{1}\times\frac{7}{2}\times\frac{8}{3}=56$; $1\times\frac{6}{1}\times\frac{7}{2}\times\frac{8}{3}\times\frac{9}{4}=126$; &c. in the 1st, 2d, 3d, 4th, 5th, &c. Orders.

28. *If* n *be put the Exponent of any Order, the Terms of that Order will be uniformly, Unity,* $\frac{n}{1}$, $\frac{n}{1}\times\frac{n+1}{2}$, $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}$, $\frac{n}{1}\times\frac{n+1}{2}\times\frac{n\times 2}{3}\times\frac{n+3}{4}$, &c. For the *n*th Terms throughout the Orders are these Products; (N° 27.) but the *n*th Term throughout each Order is the 1st, 2d, 3d, &c. Terms of the *n*th Order; (24.) therefore these Products are the Terms of the *n*th Order.

29. *If there be taken three figurate Numbers, being successive Terms of any* nth *Order, or the* nth *Figurate in three successive Orders,* (24.) *the Square of the middle Term exceeds the Product of the adjacent Terms:* For the Terms of the 2d Order being generated from the continued Addition of Units, those of the 3d Order from the continued Addition of those of the 2d Order or the Laterals, those of the 4th Order from the continued Addition of the Terms of the 3d Order, and so on, the Extremes of three will differ more and more from an Equality with each other in each succeeding Order; and consequently will differ more and more from an Equality with half their Sum in each succeeding Order: Now the middle Term will be equal to their half Sum in the second Order, they being generated from the continued Addition of Equals, or Units: In the 3d Order the middle Term will be less than half their Sum, but cannot deviate so much from an Equality with it, as each Extreme differs from it, because they are generated from the continued Addition of Laterals. And so on, in the superior Orders; the middle Term will deviate more and more from an Equality with half the Sum of the Extremes; but always deviate less from an Equality with it, than either Extreme deviates from an Equality with it: But the Square of the half Sum has the greatest Ratio to the Square of the whole Sum;

Sum; and the more unequal the Parts are, the less is the Ratio which their Product has to the Square of the Sum; and the nearer any Part is to an Equality with the half Sum, the greater is the Ratio which its Square has to the Square of the Sum: (Eucl. II. 4.) Therefore the Square of the middle Term (which deviates less from the half Sum) will have a greater Ratio to the Square of the Sum, than the Product of the Extremes (which deviate more from an Equality with the half Sum) has to it; and therefore the Square of the middle Term is greater than the Product of the Extremes. (Eucl. V. 10.) *Hence the Square of any Lateral exceeds the Product of any adjacent ones by Unity.*

30. It follows also, that *the Ratio of the middle Term to either Extreme, is greater than the Ratio of the other Extreme to the middle Term*; (Eucl. VII. 10.) and consequently, that *if each subsequent Figurate of the nth Order be divided by the antecedent*, or if in the nth Figurate throughout the Orders, (24.) *the subsequent be divided by the antecedent, the Quotes or Fractions will continually decrease.*

31. *If throughout the Orders, beginning at the first, there be taken Figurates, so as* n, *the Number of Terms in the first, shall continually decrease by Unity, those Figurates will be generated by the continued Multiplication of Fractions, whose Numerators are the Laterals continually decreasing from* n, *and whose Denominators are the same Laterals continually increasing from Unity*; that is, Unity, and the Products

$$\frac{n}{1},\ \frac{n}{1}\times\frac{n-1}{2},\ \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3},\ \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4},\ \&c.$$

For, as when n was a constant Quantity, Unity only, and that once only, was to be subducted from n, and the Laterals were to be successively added (27.) in the general Expression $S=l\times\frac{n-1+e}{e}$; so here (by Supposition) a second Unit is to be successively subducted, and the Laterals to be successively added; whence it comes to pass, that the odd Laterals are successively subducted, *viz.* 1, 3, 5, 7, &c. but the Laterals in the natural Progression only are added, *viz.* 1, 2, 3, 4, &c. whence the

Numerators

Numerators are n, $n-1$, $n-2$, &c. (for $n=n-1+1$, $n-1=n-3+2$, $n-2=n-5+3$, $n-3=n-7+4$, &c.) and the Denominators continue as before in Nº 27. Thus in the general Expression $S=l\times\frac{n-1+e}{e}$, we have $S=1\times\frac{n-1+1}{1}=n$, the nth Figurate of the 2d Order: Then for the $n-1$th Term of the 3d Order, we are to substitute for l, $\frac{n}{1}$: and for $\frac{n-1+e}{e}$, not $\frac{n-1+2}{2}$, but $\frac{n-3+2}{2}=\frac{n-1}{2}$: whence we have $\frac{n}{1}\times\frac{n-1}{2}$, for the $n-1$th Term of the 3d Order: and substituting for l; $\frac{n}{1}\times\frac{n-1}{2}$: and for $\frac{n-1+e}{e}$, not $\frac{n-1+3}{3}$, but $\frac{n-5+3}{3}=\frac{n-2}{3}$; we have $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, for the $n-2$th Term of the fourth Order; and so on. Thus, if we would know what the Figurates are in the several Orders, beginning at the 6th Term, so as 6 should continually decrease by Unity, they will be $1:\frac{6}{1}=6:\frac{6}{1}\times\frac{5}{2}=15:\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}=20:\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}=15:\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}=6:\frac{6}{1}\times\frac{5}{2}\times\frac{4}{3}\times\frac{3}{4}\times\frac{2}{5}\times\frac{1}{6}=1$: that is, 1. 6. 15. 20. 15. 6. 1.

32. *In the Figurates* 1, $\frac{n}{1}$, $\frac{n}{1}\times\frac{n-1}{2}$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, *&c. the Figurates will increase, while the excess of* n *above the odd Laterals to be subducted is positive :* For the Numerators being the Aggregates of this Excess and the

Laterals

Laterals, and the Denominators being the same Laterals: while this Excess is positive, the Numerators will exceed the Denominators: And consequently the Products will increase. *But when this Excess becomes negative*, it will diminish the Exponents of the Orders: Their Aggregates therefore, the Numerators, will become less than the Denominators: And therefore the Products will *decrease*. Now because those Aggregates are the Numerators decreasing from *n*, that is, the Laterals decreasing from *n*, (31.) and because the Denominators are the same Laterals increasing to *n*, the Values of the Numerators and Denominators will be interchanged when that Excess becomes negative; and therefore the decreasing Products will be the same as before, that is, *the Figurates equidistant from Unity in each Extreme will be equal.* And because *when* n *is an odd Number*, the middle Fraction must have its Numerator and Denominator the same, it will make no Difference in the Products: And therefore *there will then be two middle Products equal, and greatest, and adjacent. But when* n *is an even Number, there will be* two Fractions in the middle, whose Numerators and Denominators will be reciprocally the same: There will be a Difference therefore in every Product: And consequently but *one greatest Product; and that in the middle.* Now because those Figurates are generated by Addition, in the same Manner as those in N° 29. *the Square of the middle one of any three in Succession shall be greater than the Product of the Extremes.* And lastly, the Ratio of the middle one to one Extreme is greater than the Ratio of the other Extreme to the middle one: And therefore, *if each subsequent Figurate be divided by its antecedent, the Quotes or Fractions will continually decrease.* Thus $\frac{6}{1} \sqsubset \frac{5}{2} \sqsubset \frac{4}{3} \sqsubset \frac{3}{4} \sqsubset \frac{2}{5} \sqsubset \frac{1}{6}$.

33. *If the Series of Figurates* $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. *be continued to* n *Terms, and if the generating Fractions* $\frac{n}{1}, \frac{n-1}{2}, \frac{n-2}{3}, \frac{n-3}{4}$,

&c.

&c. be divided, each subsequent one by its antecedent, and the Quotes be placed over the Figurates generated from them respectively; then the Square of any Figurate multiplied into its corresponding Quote, placed over it, will be equal to the Product of the two Figurates which are adjacent to it on each Side: For the Figurates being generated from the continual Multiplication, but the Quotes from the Division, as above, of the same Fractions, the Ratio of the Numerator to the Denominator, in each Quote, will be the Reciprocal of the Ratio of the Square of the Figurate underneath to the Product of the adjacent Figurates; and therefore those Ratios will together compound the Ratio of Equality. (Eucl. V. Def. 20.) Let the generating Fractions be $\frac{e}{a}, \frac{d}{b}, \frac{c}{c}, \frac{b}{d}, \frac{a}{e}$, then will the Figurates be $1, \frac{e}{a}, \frac{ed}{ab}, \frac{ed}{ab}, \frac{e}{a}, 1$, and the Quotes will be $\frac{ad}{eb}, \frac{b}{d}, \frac{b}{d}, \frac{ad}{eb}$; and the Quotes placed over the Figurates will stand thus,

$$\frac{ad}{eb}, \frac{b}{d}, \frac{b}{d}, \frac{ad}{eb},$$

$$1, \frac{e}{a}, \frac{ed}{ab}, \frac{ed}{ab}, \frac{e}{a}, 1:$$

Now the Ratio of the Square of $\frac{e}{a}$ to $\frac{ed}{ab} \times 1$ is $\frac{eeab}{aaed}(149.) = \frac{eb}{ad}$, which is the Reciprocal of $\frac{ad}{eb}$; also the Ratio of the Square of $\frac{ed}{ab}$ to $\frac{ed}{ab} \times \frac{e}{a}$ is $\frac{e^2d^2a^2b}{e^2a^2b^2d}(149.) = \frac{d}{b}$, the Reciprocal of the Quote $\frac{b}{d}$.

34. *Hence, if different Numbers are substituted for* n, *whereby different Series of Figurates will result from the different Series of generating Fractions, and each subsequent generating Fraction be divided by the antecedent, the Quotes in every Series will be so many Theorems, for shewing the Ratio of the Square of the corresponding Figurate, to the Product of the adjacent Extremes:* For these Ratios are the Reciprocals

Reciprocals of the Ratios of the Numerators to the Denominators in the corresponding Quotes. Thus, put $n=2$, the Fractions are $\frac{2}{1}$, $\frac{1}{2}$, the Figurates 1, 2, 1, and the Quote $\frac{1}{4}$; therefore $\frac{1}{4}$ of the Square of the middle Term is equal to the Product of the adjacent Figurates; or the Square is equal to quadruple the Product. If $n=3$, the Fractions being $\frac{3}{1}$, $\frac{2}{2}$, $\frac{1}{3}$, and the Figurates 1, 3, 3, 1, and the Quotes $\frac{1}{3}$, $\frac{1}{3}$; therefore $\frac{1}{3}$ the Square of the 2d or 3d Figurate is equal to the Product of the adjacent ones; or the Square is triple the Product. Put $n=4$, then the Fractions are $\frac{4}{1}$, $\frac{3}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, the Figurates 1, 4, 6, 4, 1, and the Quotes $\frac{3}{8}$, $\frac{4}{9}$, $\frac{3}{8}$; whence $\frac{3}{8}$ the Square of the 2d and 4th Figurate is equal to the Product of the adjacent ones; and $\frac{4}{9}$ the Square of the 3d is equal to the Product of the 2d and 4th, &c. &c.

35. *The Series of Figurates or Products* $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$, &c. *continued to* n *Terms, exhibits the Number of Combinations of which* n *Number of Things is capable:* For in two Things there is but 1 Binary; Add a third, this is to be combined with the former 2; the Binaries therefore in 3 Things are $1+2$: Add a fourth, this is to be combined with the former 3; the Binaries therefore of 4 Things are $1+2+3$: Add a fifth, this is to be combined with the former 4; the Binaries therefore of 5 Things are $1+2+3+4$; and so on; that is, putting n for the Number of Things, the Number of Binaries will be the Sum of $n-1$ Laterals, or the $n-1$th Term of the 3d Order of Figurates; that is, (24. 31.) $\frac{n}{1}\times\frac{n-1}{2}$.

In three Things there is but 1 Ternary: Add a fourth, this is to be combined with the 3 Binaries of the former 3; the Ternaries therefore of 4 Things are $1+3$: Add a fifth, this is to be combined with the 6 Binaries of the former 4; the Ternaries therefore of 5 Things are $1+3+6$: Add a sixth, this is to be combined with the 10 Binaries of the former 5; the Ternaries therefore of 6 Things are

1+3

1 + 3 + 6 + 10, and so on; that is, putting n for the Number of Things, the Number of Ternaries will be the Sum of $n - 2$ Figurates of the 3d, or the $n - 2$th Figurate of the fourth Order; that is, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$. After the same Manner, the Number of Quaternaries are $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$; of Quinaries $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$; and so on continually, still increasing the Number of Combinations by all those of the next inferior Order; because as the Number of Things increases, the last can be combined with all the next inferior Combinations of the former Things: and still lessening the Number of Things by Unity, as the Exponent of the Combination increases.

Of SUBTRACTION.

XXV. THE *Invention of the Difference of Numbers that are not too much compounded, is of itself evident*; as if you take 9 from 17, there will remain 8. *But in more compounded Numbers, Subtraction is performed by subscribing or setting underneath the Subtrahend, and subtracting each of the lower Figures from each of the upper ones.* Thus to subtract 63543 from 782579, having subscribed 63543, say 3 from 9 and there remains 6, which write underneath; and 4 from 7, and there remains 3, which write likewise underneath; then 5 from 5 and there remains nothing, which in like manner set underneath; then 3 comes to be taken from 2; but because 3 is greater than 2, you must borrow 1 from the next Figure 8, which together with 2, make 12, from which 3 may be taken and there will remain 9, which write likewise underneath; and then when besides 6 there is also 1 to be taken from 8, add the 1 to the 6, and the Sum 7 being taken from 8, there will be left 1, which in like Manner write underneath (*a*). Lastly, when in the lower Rank of Numbers there remains nothing to be taken from 7, write underneath the 7, and so you have the Difference 719036.

782579
63543
719036

But especial Care is to be taken, that the Figures of the Subtrahend be placed or subscribed in their proper or homogeneous Places; viz. the Units of the one under the Units of the other, and the Tens under Tens, and likewise the Decimals under the Decimals, &c. as we have shewn in Addition (*b*). Thus to take the Decimal 0,63 from the Integer 547, they are not to be disposed thus $\begin{smallmatrix}547\\0{,}63\end{smallmatrix}$, but thus $\begin{smallmatrix}547\\ 0{,}63\end{smallmatrix}$; so that the 0, which supplies the Place of Units in the Decimal, must be placed under the Units of the other Number. Then 0 being understood to stand in the empty Places of the upper Number, say, 3 from 0, which since it cannot be, 1 ought to be borrowed from the foregoing Place, which will make 10, from which 3 is to be taken, and there remains 7, which write underneath. Then that 1 which was borrowed added to 6 make 7, and this is to be taken from 0 above it; but since that cannot be, you must again borrow 1 from the foregoing Place to make 10, then 7 from 10 leaves 3, which in like Manner is to be writ under-

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XXV. (*a*) For it is manifest, that by increasing the Minuend in the right-hand Place by Ten, and the Subtrahend in the left-hand Place by an Unit, that an equal Addition is made to each (3); and that therefore the Residue is not changed. The same Thing is also done, if instead of adding the borrowed Unit to the Subtrahend in the left-hand Place, the minuend Figure in the left-hand Place is diminished by an Unit before Subduction, as in reality it is, by the borrowing from it the Ten added to its right-hand Figure.

(*b*) *Quantities of Different Denominations cannot be subducted from each other, because the Residue could not be of any one certain Denomination.* Their Difference therefore can only be shewn by the Sign of the Subducend.

underneath; then that 1 being added to 0, makes 1, which 1 being taken from 7 leaves 6, which again write underneath. Lastly, write the two Figures 54 (since nothing remains to be taken from them) underneath, and you will have the Remainder 546,37 (*c*).

547
0,63
546,37

For Exercise sake, we here set down some more Examples, both in Integers and Decimals.

1673	1673	458074	35,72	46,5003	308,7
1541	1580	9205	14,32	3,078	25,74
132	93	448869	21,4	43,4223	282,96

XXVI. *If a greater Number is to be taken from a less, you must first subtract the less from the greater, and then prefix a negative Sign to the Remainder.* As if from 1541 you are to subtract 1673, on the contrary, I subtract 1541 from 1673, and to the Remainder 132 I prefix the Sign — (*d*).

XXVII. *In Algebraic Terms, Subtraction is performed by connecting the Quantities, after having changed all the Signs of the Subtrahend; and by uniting those together which can be united,* as we have done in Addition (*e*). Thus

+ 7

(*c*) *Numbers whose Denominators increase in any other Ratio may be subducted after the Manner of those in the Decimals, by substituting the Number by which the Denominators increase in the Place of Ten.*

XXVI. (*d*) For this Residue thus affected with a negative Sign — 132, being added to the Subducend + 1673, will restore the Minuend + 1541.

XXVII. (*e*) *The Sign of the Subducend is always to be changed into its contrary.* For if the Subducend be positive and simple, it ought to have the Sign of Subduction, that is, its Sign is changed into Negative; and if the Subducend be compounded of an Affirmative and Negative, their Difference only ought to be subducted; therefore, having subducted the affirmative Term, too much, by the Quantity of the negative Term, has been subducted, and the Residue is too small by that Quantity;

$+7a$ from $+9a$ leaves $9a-7a$ or $2a$; $-7a$ from $+9a$ leaves $+9a+7a$, or $16a$; $+7a$ from $-9a$ leaves $-9a-7a$, or $-16a$; and $-7a$ from $-9a$ leaves $-9a+7a$, or $-2a$; so $3\frac{a}{c}$ from $5\frac{a}{c}$ leaves $2\frac{a}{c}$; $7\sqrt{ac}$ from $2\sqrt{ac}$ leaves $-5\sqrt{ac}$ (*f*); $\frac{2}{9}$ from $\frac{5}{9}$ leaves $\frac{3}{9}$; $-\frac{4}{7}$ from $\frac{3}{7}$ leaves $\frac{7}{7}$; $-\frac{2ax}{b}$ from $\frac{3ax}{b}$ leaves $\frac{5ax}{b}$; $\frac{8a\sqrt{cx}}{2a+\sqrt{cx}}$ from $\frac{-17a\sqrt{cx}}{2a+\sqrt{cx}}$ leaves $\frac{-25a\sqrt{cx}}{2a+\sqrt{cx}}$; $\frac{aa}{c}$ from $\frac{bx}{c}$ leaves $\frac{bx-aa}{c}$; $a-b$ from $2a+b$ leaves $2a+b-a+b$, or $a+2b$; $3az-zz+ac$ from $3az$ leaves $3az-3az+zz-ac$ or $zz-ac$; $\frac{2aa-ab}{c}$ from $\frac{aa+bb}{c}$ leaves $\frac{aa+ab-2aa+ab}{c}$, or $\frac{-aa+2ab}{c}$; and $\overline{a-x}\sqrt{ax}$ from $\overline{a+x}\sqrt{ax}$ leaves $\overline{a+x-a+x}\sqrt{ax}$, or $2x\sqrt{ax}$, and so in others. But where Quantities consist

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tity; therefore, to restore the Residue to its just Magnitude, the negative Term ought to be added to it, that is, the negative Term must become Affirmative.

36. Hence, *the Difference of two Quantities, subducted from their Sum, is equal to double the less*; for the greater is destroyed, and the less doubled in adding them after the Change of their Signs. Thus if a be the greater, and b the less, then their Difference is $a-b$, and their Sum $a+b$; but $\overline{a+b}-\overline{a-b}=2b$. *See* N° 22.

(*f*) *Surds are subducted by subducting their rational Coefficients*, when being reduced to their lowest Terms they agree in their radical Part; for they are then as Number to Number, and their Ratio is that of the Coefficients, by which Coefficients the Subduction can be made.

sist of more Terms, the Operation may be managed as in Numbers, as in the following Examples (*g*):

12 X

37. *(g) Four Quantities are in arithmetical Proportion, when the Difference between the two former is equal to the Difference between the two latter. Quantities are said to be in arithmetical Progression, when they increase or decrease continually by equal Differences.*

38. In arithmetical Progression, *when the Number of Terms is even, the Sum of the Extremes is equal to the Sum of every two mean Terms equidistant from them; and when the Number of Terms is odd, the Sum of the Extremes is double the middle Term.* For since the Terms are equidifferent (37), the Second will exceed, or be deficient from, the first, as much as the last exceeds, or is deficient from, the Penultimate; therefore the Sum of the Extremes equals the Sum of the Second and Penultimate, the Excess of the one making up the Defect of the other. The same Reasoning holds good in every Pair of Terms equidistant from the Extremes; and therefore in those also which are adjacent to the middle Term, when their Number is odd. Now one of these exceeds the middle Term as much as the other is deficient of it (37); if therefore, the Defect of one be compensated by the Excess of the other, the three Terms will be equal: and consequently, the Sum of the adjacent Terms, and therefore of every other equidistant Pair, will be double the middle Term.

39. *The last Term is equal to the first, increased if the Progression ascends, but diminished if the Progression descends, by the Product of the common Difference multiplied into, either the Number of Terms less one, or the Number of Means more one.* For the last Term exceeds, or is exceeded, by the first, by the common Difference so often taken, as there are Terms after the first; that is, so often as there are Terms less one, or Means more one.

40. Hence, *the common Difference is equal to the Difference of the Extremes divided by, either the Number of Terms less one, or the Number of Means more one.* Consequently, the

$$\begin{array}{rrr} 12x + 7a & 15bc + 2\sqrt{ac} & 5x^3 + \frac{5}{7}x \\ 7x + 9a & -11bc + 7\sqrt{ac} & 6x^2 - \frac{3}{7}x \\ \hline 5x - 2a & 26bc - 5\sqrt{ac} & 5x^3 - 6xx + \frac{8}{7}x. \end{array}$$

$$\begin{array}{l} \frac{11ax}{b} - 7\sqrt{3} + \frac{2}{5} \\ \frac{4ax}{b} - 6\sqrt{3} - \frac{1}{5} \\ \hline \frac{7ax}{b} - \sqrt{3} + \frac{3}{5} \end{array}$$

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the Extremes being given, any Number of Means may be found: For by dividing their Difference by the Number of Means sought more one, the common Difference is found, and this continually added to the less Extreme exhibits the Means.

41. *The Sum of any Series is equal to the Quote, which is had by dividing the Product of the Sum of the Extremes into the Number of Terms, by Two.* For the Sum of the Series contains the equal Sums of the Pairs of equidistant Terms only half so many Times as there are Terms (38).

42. *In the Progression of Laterals, or Figurates of the second Order, the Sum of any Series is equal to half the Product arising from multiplying the last Term into the next greater.* For both first Term and common Difference being Unity, the last Term is the Number of Terms, and the next greater is the Sum of the Extremes. *See* N° 25, 26.

43. *In the Progression of odd Laterals, whose common Difference is Two, the Sum of the Series is always a square Number; to wit, the Square of the Number of Terms.* For Unity being the first Term, and Two the common Difference, the Sum of the Extremes is double the Number of Terms; whence their Product would be the Product of the Number of Terms into double itself; consequently, half this Product, which is the Sum of the Series (41), is the Product of the Number of Terms into itself, and therefore a Square.

2

44. If

44. *If a given Quantity is added to, or subducted from, every Term of an arithmetical Progression, the Progression will continue unaltered, with the same common Difference.* For an equal Addition, or Subduction, although it alters the Magnitude of the Terms, will not alter their Relation to Difference.

45. *If the same Quantity be multiplied into every Term of an arithmetical Series, the Products will be in arithmetical Progression; and the common Difference of the Terms will be the Product of the given Quantity into the former common Difference.* For as each Term consisted of the preceding Term and of the common Difference, any given Multiple of any Term must be the same Multiple of the preceding Term and of the common Difference; therefore the common Difference of the Products will be that Multiple of the former common Difference. After the same Manner, if the Terms of an arithmetical Series be divided by a given Quantity, the Quotes will be in arithmetical Progression, whose common Difference will be the Quote of the former Difference divided by that Quantity.

46. *If in the Binome* $nx \pm a$ *the Terms* 3, 2, 1, 0, — 1 — 2 — 3, &c. viz. *of the descending lateral Progression, be substituted for the unknown Quantity* x: *the resulting Numbers will be in arithmetical Progression, whose common Difference will be* n, *the Coefficient of* x *in the former Member of the Binome.* For the Products arising, from the successive Substitution of the Laterals for x, that is, from the successive Multiplication of n into the Laterals, will be be in a Progression, whose common Difference is $n \times 1$ (45); that is, n; and the second Member $\pm a$ of the Binome cannot alter the Progression (44).

Of MULTIPLICATION.

XXVIII. NUMBERS *which arise or are produced by the Multiplication of any two Numbers, not greater than* 9, *are to be learnt and retained in the Memory:* As that 5 into 7 makes 35, and that 8 by 9 make 72, and *then the Multiplication of greater Numbers is to be performed after the Rule of these Examples.*

If 795 is to be multiplied by 4, write 4 underneath, as you see here. Then say, 4 into 5 makes 20, whose last Figure, *viz.* 0, set under the 4, and reserve the former 2 for the next Operation. Say moreover, 4 into 9 makes 36, to which add the former 2, and there is made 38, whose latter Figure 8 write underneath as before, and reserve the former 3. Lastly, say, 4 into 7 makes 28, to which add the former 3, and there is made 31, which being also set underneath, you will have the Number 3180, which comes out by multiplying the whole 795 by 4.

795
4
——
3180

Moreover, if 9043 be to be multiplied by 2305, write either of them, *viz.* 2305 under the other 9043 as before, and multiply the upper 9043 first by 5, after the Mannner shewn, and there will come out 45215; then by 0, and there will come out 0000: Thirdly, by 3, and there will come out 27129: Lastly, by 2, and there will come out 18086. Then *dispose these Numbers so coming out in a descending Series, or under one another, so that the last Figure of every lower Row shall stand one Place nearer to the Left-hand than the last of the next superior Row. Then add all these together, and there will arise* 20844115, *the Number that is made by multiplying the whole* 9043 *by the whole* 2305.

9043
2305
——
45215
0000
27129
18086
——
20844115

XXIX. *In the same manner Decimals are multiplied by Integers or other Decimals, or both,* as you may see in the following Examples:

72,4	50,18	3,0925
29	2,75	0,0132
——	——	——
6516	25060	78050
1448	35126	117075
——	10036	39025
2099,6	——	——
	137,9950	0,05151300

But Note, in the Number coming out, or the Product, so many Figures must be cut off to the Right-hand for Decimals, as there are decimal Figures both in the Multiplier and the Multi-

Multiplicand. And if by Chance there are not so many Figures in the Product, the deficient Places must be filled up to the Left-hand with o's, as here in the third Example (*a*).

XXIX. (*a*) 47. *In the Multiplication of Integers, of Decimals, and of mixed Numbers, the Sum of the Indices of the Factors is the local Index of the right-hand Figure of the Product.* For the Index of each Factor being the Number of Multiplications, or of Divisions of Unity by Ten, in its Denominator (13, 14); to multiply, or to divide the Denominator of the Multiplicand by Ten, is to add Unity to, or to take Unity from, its Index; and to multiply, or to divide the Denominator of the Multiplicand by any other Number as a Denominator, is to add the Index of the Multiplier to, or to subduct it from, the Index of the Multiplicand: But in the Multiplication of Integers by Integers, and of Decimals by Decimals, the Denominator of the Multiplicand is multiplied by the Denominator of the Multiplier; and in the Multiplication of Integers by Decimals, the Denominator of the former is divided by the Denominator of latter: Therefore in the former Case, the Sum of the Exponents is the Exponent of the Product; and in the latter Case, their Difference. Now the Index of a decimal Multiplier being Negative (14) it must be changed in Subduction into Affirmative, and added (XXVII); therefore, in all Cases, the Sum of the Exponents of the Factors, when made both of the same Affection, is the Exponent of the Product.

48. *Decimals are considered in Multiplication as Integers,* the only Difference being that in the former the Numerators are divided, and in the latter multiplied, by that Power of Ten, whose Index is the Number of Places from Unity (6. 9. 13); wherefore *the Numerators are to be multiplied as Integers: and as to the Multiplication of the Denominators, in all Products, of Decimals by Decimals, or by mixed Numbers, so many Places are to be allotted on the Right-hand for Decimals, as there are decimal Places in both Factors taken together.* For the Index of the right-hand Figure in Decimals, is always their whole Number of Places (13); and the Exponent of the right-hand Figure

XXX. *Simple Algebraic Terms are multiplied by multiplying*

Figure of an Integer Factor, is always Cypher (14): Therefore the Index of the right-hand Figure of the Product, to wit, the Sum of the Indices of the right-hand Figures in both Factors (47); that is to say, the Number of decimal Places in the Product, is the Sum of the decimal Places in both Factors.

49. Hence, *if the Number of Places in the Product is less than the Number of Decimals in both Factors, which will be the Case when the Cyphers on the Left-hand of a pure decimal Multiplier are neglected in the Operation, the Defect must be supplied by Cyphers added on the Left-hand.*

50. *The Right-hand Figure of every particular Product ought to be set directly under the multiplying Figure, and then the Place of every succeeding Figure will follow in Order.* For the Index of the right-hand Figure of the Multiplicand, is either Cypher (14) or is considered as such during the Operation (48); consequently the Index of the right-hand Figure of each particular Product will be the Index of the multiplying Figure (47, 48). The particular Products, being thus ranged, will have their Figures of the same Class in the same perpendicular Series ready for Addition (XVIII).

51. *If there are Cyphers between the significant Figures of the Multiplier, they may be entirely neglected,* for their sole Use being to determine the Places of the significant Figures to the Left (18); this End will be answered by placing the right-hand Figure of each particular Product by the significant Figures directly under the Multiplier (50).

52. *Cyphers on the Right-Hand of either Factor are neglected in the Operation. In Decimals they are to be totally expunged, as being useless* (18), *but they are to be restored to the Product of Integers*; because they serve to determine the Place of the Right-hand significant Figure of the Product, whose Exponent being always the Sum of the Indices of the right-hand Figures of both Factors (47); will be in this Case the Number of Cyphers.

53. Hence, *an Integer is multiplied by any Power of Ten, by annexing the Cyphers of the Power to the Integer* (52).

But

plying the Numbers into the Numbers, and the Species into the Species,

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But *a Decimal or mixed Number is multiplied by any Power of Ten, by moving the Seperatrix so many Places to the Right-hand, as there are Cyphers in the Power, or as there are Units in the Exponent of the Power.* For the Multiplication is performed at length by annexing the Cyphers of the Power of Ten on the Right-hand of the Multiplicand, and then cutting off so many Places for Decimals, as there were Places before in the Multiplicand (48), and lastly, by expunging the added Cyphers, being useless in Decimals (18); but this is equivalent to giving so many Places from the Decimals to the Integers, or, to moving the Seperatrix so many Places to the Right, as there are Units in the Index of the Power of Ten; and the Trouble of expunging is avoided.

Now, though the Number of Places in the Decimals is diminished, yet their Value is so many times more Decuple of their former, by being raised so many Places higher to the Left (11); and the negative Exponent of the right-hand Figure of the Decimal Factor being diminished by the Addition of the affirmative Index of the Power (XXIV), is the Index of the right-hand Figure of the Product, which being negatively less, is affirmatively greater (47).

From this, *the common Rule for reducing Decimals into Sexagesimals*, is deduced, to wit, Multiply the Decimals by sixty; because this Multiplication not only multiplies by Six, but also moves the Seperatrix one Place to the Right.

54. *If the Product of the left-hand Figures of two Integers, either alone or augmented by an Increment from the Product of the adjacent Figures, consists of two Places; or if the left-hand Figure of this Product is less than the left-hand Figure of either Factor, then the whole Product will consist of as many Places as there are Places in both Factors, otherwise of one Place less.* For the Exponent of the left-hand Figure in each Factor being the Number of Places in each less One (13), the Exponent of the right-hand Figure of their Product will be the Number of Places in both Factors less Two (48); to which must be added the Place of Units, whose Exponent is Cypher in each Factor; whence

Species, and by making the Product Affirmative, if the Factors

whence the Exponent of the Product of the left-hand Figures, when this Product is a single Figure, is the Number of Places in both Factors less One; and when it consists of two Figures, the Exponent of the left-hand Figure is the Sum of the Places in both Factors: Now when this Product is a single Figure, it must be greater than either Factor; and when it consists of two Figures, the Figure to the Left must be less than either Factor; wherefore the Number of Places will fall One short, when it is greater than either Factor.

55. *Because it may be sufficient, especially in Decimals, to find only an assigned Part of a Product.* 1. *Set the Place of Units of the less Number (which make the Multiplier) under that Place of the greater whose Index is equal to the Index of the designed right-hand Figure of the assigned Part of the Product; that is, to the Number of Figures to be cut off in Integers, or to be retained in Decimals.* 2. *Set the Rest of the Figures of the Multiplier in a contrary Order.* 3. *Begin every Multiplication, at that Figure of the greater, which stands over the multiplying Figure; having Regard to the Increment, which would have arisen from the foregoing Figures of the Multiplicand. Lastly, Set the right-hand Figures of every particular Product under one another; and then the Sum of these particular Products, will be the required Part of the Product.* For the Index of the right-hand Figure of every Product, is the Sum of the Indices of the Factors; and by inverting the Order of the Figures of the Multiplier, the Sums of the Indices of the corresponding Place of the Factors will be equal among themselves; and therefore equal to the Index of the right-hand Figure of the required Part of the Product: But Products whose Indices are equal belong to the same Place; they, therefore, must be set under each other: And their Sum must be the required Part of the Product.

56. *In every Multiplication, whether by Figures or by Species, every Part of the Multiplicand must be multiplied by every Part of the Multiplier.* For if Equals be multiplied into Equals, the Products are equal (Eucl. I. Def. 6.); but every whole is equal to all its Parts taken together (Eucl. I. Def.

Factors are both Affirmative, or both Negative; and Negative if otherwise (b).

Thus

Def. 9); therefore the Product arising from multiplying the Whole by the Whole is equal to the Product arising from the Multiplication of all the Parts of one by all the Parts of the other Factor.

XXX. (*b*) 57. *When the Sign of the Multiplier is Affirmative* (+), *the Signs of the Product are the same with those of the Multiplicand.* For the Multiplicand is then so often added to itself as there are Units in the Multiplier (VI); if therefore all the Signs of the Multiplicand are Affirmative, the Sum of its Terms is so often added to itself, by preserving the Signs of the Multiplicand (XX); and if any of its Terms are Negative, their Differences are so often added to themselves, by preserving the Signs of the Multiplicand (XXIII). Now *the Signs of the Multiplicand being to be preserved, when the Multiplier is Affirmative; it follows, that* + *into* + *makes* +: *and that* — *into* + *makes* —.

58. *When the Sign of the Multiplier is Negative* (—), *the Signs of the Product are contrary to those of the Multiplicand.* For a negative Term in the Multiplier shews that it is the Difference between two Quantities by which the Multiplicand is to be added to itself (XXIV); that therefore, having made the Product by an affirmative Term in the Multiplier, this Product is too great, and is to be diminished by the Product of the same Multiplicand by this negative Term of the Multiplier; that is, that the latter Product is to be subducted from the former; and that therefore this latter Product must have its Sign changed (XXVII); but the Sign of this latter Product when produced by an affirmative Multiplier was the same with that of the Multiplicand (57); therefore, when changed, it must be contrary. Now *the Signs of the Multiplicand being to be changed in the Product by a negative Multiplier, it follows, that* + *into* — *makes* —; *and that* — *into* — *makes* +.

59. Because, that + into + makes + (57): and that — into — makes also + (58): and because — into

Thus $2a$ into $3b$, or $-2a$ into $-3b$ make $6ab$, or $6ba$: For it is no Matter in what Order they are placed *. Thus also $2a$ by $-3b$, or $-2a$ by $3b$ make $-6ab$. And thus, $2ac$ into $8bcc$ make $16abccc$, or $16abc^3$; and $7axx$ into $-12aaxx$ make $-84a^3x^4$; and $-16cy$ into $31ay^3$ make $-496acy^4$; and $-4z$ into $-3\sqrt{az}$ make $12z\sqrt{az}$. And so 3 into -4 make -12, and -3 into -4 make 12.

XXXI. *Fractions are multiplied, by multiplying their Numerators by their Numerators, and their Denominators by their Denominators.* Thus $\frac{2}{5}$ into $\frac{3}{7}$ make $\frac{6}{35}$; and $\frac{a}{b}$ into $\frac{c}{d}$ make $\frac{ac}{bd}$; and $2\frac{a}{b}$ into $3\frac{c}{d}$ make $6 \times \frac{a}{b} \times \frac{c}{d}$, or $6\frac{ac}{bd}$; and $\frac{3acy}{2bb}$ into $\frac{-7cyy}{4b^3}$ make $\frac{-21accy^3}{8b^5}$; and $\frac{-4z}{c}$ into $\frac{-3\sqrt{az}}{c}$ make $\frac{12z\sqrt{az}}{cc}$; and $\frac{a}{b}x$ into $\frac{c}{d}xx$ make $\frac{ac}{bd}x^3$. Also 3 into $\frac{2}{5}$ make $\frac{6}{5}$ as may appear, if 3 be reduced to the Form of a Fraction, *viz.* $\frac{3}{1}$ by making use of Unity for the Denominator. And thus $\frac{15aaz}{cc}$ into $2a$ make $\frac{30a^3z}{cc}$. Whence note by the Way, that $\frac{ab}{c}$ and $\frac{a}{c}b$ are the same; as also $\frac{abx}{c}$, $\frac{ab}{c}x$, and $\frac{a}{c}bx$, also $\frac{\overline{a+b}\sqrt{cx}}{c}$ and $\frac{a+b}{c}\sqrt{cx}$; and so in others (*d*.)

XXXII. *Ra-*

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into + makes — (57): and, that + into — makes also — (58): *It follows universally, that similar Signs in the Factors make an affirmative Sign; and that dissimilar Signs in the Factors make a negative Sign in the Product.*

* *See* N° 20.

XXXI. (*d*) 60. *Fractions, whose Numerators are equal, are to each other inversely as their Denominators.* For any equal

XXXII. *Radical Quantities of the ſame Denomination* (that is, if they are both Square Roots, or both Cube Roots, or both Biquadratic Roots, &c.) *are multiplied by multiplying*

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equal Number of Parts muſt be to each other as the Magnitudes of thoſe Parts; but the Magnitudes are inverſely as the Denominators. (Eucl. VII. 19.)

61. *Fractions, whoſe Denominators are equal, are to each other as their Numerators.* For Parts whoſe Magnitude is the ſame muſt be to each other as their Number.

62. Therefore, *univerſally, Fractions are to each other as their Numerators directly, and as their Denominators inverſely.*

63. As Unity is to the Multiplier, ſo is the Multiplicand to the Product (19); but as Unity is to a Fraction, ſo is its Denominator to its Numerator (XXI); therefore, as the Denominator of the multiplying Fraction is to its Numerator, ſo is the Multiplicand, whether Integer or Fraction, to the Product; and conſequently *in all Caſes of Multiplication by a Fraction, the Multiplicand is to be multiplied by its Numerator, and divided by its Denominator* (Eucl. VI. 12.) as before in the Multiplication of Decimals (48).

64. *A Fraction is multiplied by an Integer by multiplying the Integer into the Numerator, and by ſubſcribing the Denominator under the Product* (XI); *or by dividing the Denominator by the Integer, and by ſubſcribing the Quote under the Numerator* (XI). For it is the ſame Thing to increaſe a given Number of Parts in a given Ratio, the Magnitude of the Parts being unchanged, as to increaſe the Magnitude of the Parts in that Ratio, their Number being unchanged; and their Magnitude is always increaſed in any Ratio, by diminiſhing the Diviſor or Denominator in that Ratio (60): But by multiplying the Integer into the Numerator, their Number is increaſed in the Ratio of the Integer to Unity, their Magnitude being the ſame (VI); and by dividing the Denominator by it, the Magnitude of the Parts is increaſed in the ſame Ratio, their Number being unchanged.

65. *The Product of two Fractions is found* [there being always a Multiplication of both Numerators and Denominators

tiplying the Terms together under the same radical Sign. Thus $\sqrt{3}$ into $\sqrt{5}$ make $\sqrt{15}$; and $\sqrt{ab}$ into $\sqrt{cd}$ make $\sqrt{abcd}$; and $\sqrt[3]{5ayy}$ into $\sqrt[3]{7ayz}$ make $\sqrt[3]{35aay^3z}$; and $\sqrt{\frac{a^3}{c}}$ into $\sqrt{\frac{abb}{c}}$ make $\sqrt{\frac{a^4bb}{cc}}$ that is * $\frac{aab}{c}$. And $2a\sqrt{az}$ into $3b\sqrt{az}$ make $6ab\sqrt{aazz}$, that is $6aabz$;

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minators (47), but that by the Denominators being equivalent to a Division (64. 63)] First, *By multiplying the Numerators, for a new Numerator; and the Denominators, for a new Denominator.* For thus the Multiplicand is multiplied by the Numerator of the Multiplier, and divided by its Denominator, because the Product of the Denominators is subscribed (XI).

66. Secondly, *By dividing the Numerator and Denominator of the Multiplicand respectively, by the Numerator and Denominator of the Reciprocal of the Multiplier.* For thus the Multiplicand is multiplied by the Numerator of the Multiplier, because its Denominator is divided by it (64); and it is divided by the Denominator of the Multiplier. *See* N° 145.

67. Thirdly, *By dividing the Product of the Denominators by the Numerator of the Multiplier, and subscribing this Quote to the Numerator of the Multiplicand.* For thus the Multiplicand is multiplied into the Numerator of the Multiplier; because its Denominator (in the Product of both Denominators) is divided by it; and it is divided by the Denominator of the Multiplier, because the Product of both Denominators (reduced by the foregoing Division) is subscribed to its Numerator.

68. Lastly, *By dividing the Product of the Numerators by the Denominator of the Multiplier, and subscribing the Denominator of the Multiplicand to the Quote.* For thus the Multiplicand is multiplied into the Numerator of the Multiplier, and divided by its Denominator, because the Product of the Numerators is divided by it.

Note, that *the first Method is the best*, being not liable to compound Fractions.

* *See the Chapter of* Notation.

$6aabz$; and $\frac{3xx}{\sqrt{ac}}$ into $\frac{-2x}{\sqrt{ac}}$ make $\frac{-6x^3}{\sqrt{aacc}}$, that is $\frac{-6x^3}{ac}$; and $\frac{-4x\sqrt{ab}}{7a}$ into $\frac{-3dd\sqrt{5cx}}{10ee}$ make $\frac{12ddx\sqrt{5abcx}}{70aee}$ (*e*).

XXXIII. *Quan-*

XXXII. (*e*) *Powers and Radicals* are either of the same Quantity, or of different Quantities: and they are either of the same, or of different Exponents, or Denominations. In all Cases *they must be brought to the same Exponent before they can be multiplied or divided*; otherwise the Product or Quote could have no certain Exponent.

69. *The Product of Powers and of Radicals of the same Exponent, but of different Quantities, has the same Exponent with the Factors:* so that the Product of such Powers is the same Power of the Product of their Roots: and the Product of such Roots is the same Root of their Product. Thus $a^m \times b^m = a^m b^m = ab|^m = \overline{ab}^m$.

As to the Multiplication and Involution of Powers and Radicals of the same Quantity; it will be necessary to attend more distinctly to their Generation, and also to some Properties of geometrical Progression.

70. *Four Quantities are in geometrical* Proportion *when the Ratio of the First to the Second is equal to the Ratio of the Third to the Fourth*; and *Quantities are in geometrical* Progression, *when they increase, or decrease, by equal Ratios:* and consequently, the Terms increase by a common Multiplier, if the Progression ascends; and decrease by a common Divisor, if the Progression descends.

71. In a geometrical Progression, *when the Number of Terms is even, the Product of the Extremes is equal to the Product of every two mean Terms equidistant from them; and when the Number of Terms is odd, it is equal to the Square of the middle Term.* For the Ratio of the first to the second Term being equal to the Ratio of the Penultimate to the last, these four Terms will be proportional (70); and therefore the Product of the Extremes is equal to the

XXXIII. *Quantities that consist of several Parts, are multiplied*

Product of the second and penultimate (Eucl. VII. 19). The same Reasoning holds good in every Pair of Terms equidistant from the Extremes; and therefore in those also which are adjacent to the middle Term, when their Number is odd: But the Square of the Middle is equal to the Product of the adjacent Terms (Eucl. VII. 20); and therefore to that of every Pair of Terms equidistant from it.

72. *The last Term is equal to the Product (if the Progression ascends) and to the Quote (if the Progression descends) which arises, by multiplying in the former Case, and by dividing in the latter Case, the first Term by that Power of the common Ratio, whose Exponent is the Number of Terms less One, or the Number of Means more One.* For it exceeds the first Term, or is exceeded by it, by the common Ratio so often multiplied into itself, as there are Terms after the First; that is, as there are Terms less One, or Means more One.

73. Hence, *the common Ratio is equal to that Root of the Quote of the Extremes, whose Exponent is either the Number of Terms less One, or the Number of Means more One:* Whence *the Extremes being given, any Number of mean Proportionals will be found.* For by extracting from the Quote of the Extremes, the Root, whose Index is the Number of Means sought more One, the common Ratio is found; and thence any Number of Means.

74. *If two descending Progressions, consisting of three Terms each, have the middle Term the same; according as the first Term of the former is greater or less than the first Term of the latter, so the last Term of the latter shall be greater or less respectively, than the last Term of the former.* For the Products of the Extremes in each being equal to the same Square, shall be equal to each other: and therefore the Extremes shall be reciprocally proportional (Eucl. VII. 19).

75. *The same Progressions being supposed, the Difference of the first Terms shall be greater than the Difference of the last.* For in the reciprocal Proportion in N° 74, as Antecedent

tiplied by multiplying all the Parts of the one into all the Parts of

tecedent is to Consequent, so is the Difference of the Antecedents to the Difference of the Consequents (Eucl. V. 19); but the Antecedent being greater than the middle Term, is greater than the Consequent; wherefore the Difference of the Antecedents is greater than the Difference of the Consequents.

76. *Unity, the Root, the Square, the Cube,* &c. *of the same Quantity are in geometrical Progression* (XV): But each Term of this Progression consists of the Number of Factors in the last antecedent Term, and of One more; therefore *the Exponents of these Terms, which indicate the Number of Factors in each* (XIV), *differ by Unity, are equidifferent, and in arithmetical Progression* (37). Now the Square, Cube, &c. are produced by the continued Multiplication of Unity into the Root; wherefore putting Cypher, *the Exponent* of Unity; then Unity must be the Exponent of the Root, and 2, 3, &c. the Exponents of the Square, Cube, &c. respectively; that is, *the ascending Laterals will be the Indices of the Powers ascending*. Now the Powers below Unity are the Reciprocals of those above Unity, and consequently are Unity divided by those Powers; their Denominators will therefore have the same Exponents as their Reciprocals; that is, the same Laterals: But *the Value of any Quantity in the Denominator of a Fraction being the Reciprocal of it in the Numerator, it may be transposed above the Line if the Sign of its Exponent be changed*; consequently *the Powers of any Quantity below Unity may be expressed as Integers, with the negative Laterals for their Exponents.* Thus the Series $\frac{1}{x^5}, \frac{1}{x^4}, \frac{1}{x^3}, \frac{1}{x^2}, \frac{1}{x}, 1, x, x^2, x^3, x^4, x^5$, may be expressed by $x^{-5}, x^{-4}, x^{-3}, x^{-2}, x^{-1}, 1, x, x^2, x^3, x^4, x^5$.

77. *If between the Terms of any geometrical Series any Number of mean Proportionals be found* (73), *and also the same Number of arithmetical Means be found between the ponents of those Terms* (40); *the arithmetical Means shall Exponents respectively of the geometrical Means.* Thus, two mean Proportionals be found between the

of the other, as is shewn in the Multiplication of Numbers.

Terms of the above Series, and interpolated, it will be x^{-5}, $x^{\frac{-14}{3}}$, $x^{\frac{-13}{3}}$, x^{-4}, $x^{\frac{-11}{3}}$, $x^{\frac{-10}{3}}$, x^{-3}, $x^{\frac{-8}{3}}$, $x^{\frac{-7}{3}}$, x^{-2}, $x^{\frac{-5}{3}}$, $x^{\frac{-4}{3}}$, x^{-1}, $x^{\frac{-2}{3}}$, $x^{\frac{-1}{3}}$, 1, $x^{\frac{1}{3}}$, $x^{\frac{2}{3}}$, x, $x^{\frac{4}{3}}$, $x^{\frac{5}{3}}$, x^2, $x^{\frac{7}{3}}$, $x^{\frac{8}{3}}$, x^3, $x^{\frac{10}{3}}$, $x^{\frac{11}{3}}$, x^4, $x^{\frac{13}{3}}$, $x^{\frac{14}{3}}$, x^5, &c.

78. *Surds therefore are mean Proportionals,* (XV) *or Powers with fractional Exponents, whether Affirmative or Negative, whose Numerators denote the Power to which the Quantity is raised, and whose Denominators denote the Root to be extracted from that Power.* Wherefore they may be expressed either fractionwise, or by the Note of Irrationality with integer Exponents. Thus the last Series may be expressed by the Vinculum, or Note of Irrationality, x^{-5}, $\sqrt[3]{x^{-14}}$, $\sqrt[3]{x^{-13}}$, x^{-4}, $\sqrt[3]{x^{-11}}$, $\sqrt[3]{x^{-10}}$, x^{-3}, $\sqrt[3]{x^{-8}}$, $\sqrt[3]{x^{-7}}$, x^{-2}, $\sqrt[3]{x^{-5}}$, $\sqrt[3]{x^{-4}}$, x^{-1}, $\sqrt[3]{x^{-2}}$, $\sqrt[3]{x^{-1}}$, 1, $\sqrt[3]{x}$, $\sqrt[3]{x^2}$, x, $\sqrt[3]{x^4}$, $\sqrt[3]{x^5}$, x^2, $\sqrt[3]{x^7}$, $\sqrt[3]{x^8}$, x^3, $\sqrt[3]{x^{10}}$, $\sqrt[3]{x^{11}}$, x^4, $\sqrt[3]{x^{13}}$, $\sqrt[3]{x^{14}}$, x^5, &c.

79. *The Product of Powers or Radicals of the same Quantity is found by adding their Exponents*; but their fractional Exponents must be reduced to the same Denominator, before they can be added. For as the Ratios of those Terms to Unity make, if added together, the Ratio of the Product to Unity, so the Exponents of those Ratios added together, make the Exponent of the compound Ratio.

80. Whence, *if the Sum of the Exponents is nothing, the Product of the Radicals is Unity; and if when they are fractional their Sum is an Integer, the Product of the Radicals is rational.*

81. *Powers and Radicals of the same Quantity are divided by subducting their Exponents.* For the Ratio of the Quote to Unity is the Excess of the Ratio of the Dividend to Unity above the Ratio of the Divisor to Unity, therefore

bers. Thus, $c-x$ into a make $ac-ax$, and aa +

aa +

therefore the Exponent of the Ratio of the Quote to Unity will be the Excess of the Exponent of the Ratio of the Dividend to Unity, above the Exponent of the Ratio of the Divisor to Unity.

82. Hence *if the Difference of the Exponents is* 0, *or an Integer*; *the Quote will be Unity*, *or Rational*, *respectively*.

83. *Powers and Radicals of the same Quantity are involved by multiplying their Exponents by the Exponent of the Power required.* For as the Ratio of the Power or Radical to be involved to Unity, multiplied into itself so often as there are Units in the Exponent of the Power required, makes the Ratio of the Power required to Unity; so the Exponent of the Ratio of the Power or Radical to be involved to Unity, so often added to itself as there are Units in the Exponent of the Power required, (that is, multiplied by it) will give the Exponent of the Ratio of the Power required to Unity.

84. Hence, *if the Exponent of the Power required is equal to, or a Multiple of, the Denominator or Name of the Surd, the Product of the Exponents being an Integer, the Power of the Surd will become rational.*

85. *Powers and Radicals of the same Quantity are evolved by dividing their Exponents by the Name, or Exponent of the Root required.* For the Ratio of the Root to Unity is that submultiple of the Ratio of the Power to be evolved to Unity, which the Index of the Root denotes: Consequently, the Index of the Power to be evolved, divided by the Index of the Root required, gives the Index of the Ratio of the Root to Unity.

86. Hence, *if the Name of the Root required is a Divisor of the Exponent of the Quantity to be evolved, the Quote being Integer the Root will be rational*, that is, a lower perfect Power in the same Series, but *if it be not a Divisor, the Quote being fractional, the Root will be a Surd*, or mean Proportional between some perfect Powers in the Series.

87. *The Product of an odd Number of negative Factors is negative: and the Product of an even Number of negative Factors is Affirmative.* For the negative Sign of the

 Multi-

$+ 2ac - bc$ into $a - b$ make $a^3 + 2aac - aab - 3bac$

Multiplicand is changed into Affirmative in the first Multiplication (58), and the affirmative Sign of the first Product is changed into Negative in the second Multiplication, and so on; that is, the Product is positive and negative, as the Number of negative Factors is even and odd.

88. *The Powers of a negative Quantity, whose Exponents are odd, are Negative; and those whose Exponents are even, are Affirmative.* For the Exponent being odd, the Number of Factors is odd (XIV), and the Power negative (87); and the Exponent being even, the Number of Factors is even, and the Power Affirmative. Consequently, *the Powers of a negative Quantity beginning with the Square, are alternately Affirmative and Negative.* Hence

89. *A negative Square, or any negative Power whose Exponent is even, is an impossible Quantity; and therefore its Root is imaginary: but a negative Power whose Index is odd is possible, and its Root real* (88).

90. *The Powers of a quadratic Radical, whose Exponents are even, are rational* (84), *and Affirmative, whether the Radical be Affirmative, or Negative* (88); *but the Powers, whose Exponents are odd, are Irrational; and Affirmative if the Radical is Affirmative. but Negative if it is Negative* (88).

91. *If an imaginary Radical* (89) *be supposed to be involved, the Powers whose Exponents are even are all rational* (84); *and Negative and Affirmative alternately: and the Powers whose Exponents are odd, are all Irrational; but Affirmative and Negative alternately.* For the Root being imaginary, the Square must be Negative; otherwise, a real Product would arise from imaginary Factors, which is absurd; and if the Root has any Coefficient, whether Affirmative or Negative, its Square will be Affirmative (88); and consequently the Product of those Squares will be Negative (58): Now the Square being Negative, the Biquadrate will be Affirmative (88); and the Square being Negative and the Biquadrate Affirmative, the Cubocube will be Negative (88). And so on, alternately, Negative and Affirmative. Again, the Square and the Root being both Negative, the Cube will be Affirmative (88);

$3bac + bbc$. (*f*) For $aa + 2ac - bc$ into $-b$ make $-aab - 2acb + bbc$, and into a make $a^3 + 2aac - abc$, (*g*) the Sum whereof is $a^3 + 2aac - aab - 3abc + bbc$. A Specimen of this Sort of Multiplication, together with other like Examples, you have underneath (*h*):

$$\begin{array}{r} aa + 2ac - bc \\ a - b \\ \hline -aab - 2abc + bbc \\ a^3 + 2aac - abc \quad\quad\;\; \\ \hline a^3 + 2aac - aab - 3abc + bbc \; (i) \end{array}$$

$$a + b$$

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(88); and the Root being Negative and the Cube Affirmative, the Quadratocube will be Negative: And so on, Affirmative and Negative alternately. Thus $-1\sqrt{-a^2} \times -1\sqrt{-a^2} = 1 \times -a^2 = -a^2$; and $-1\sqrt{-a^2} \times -a^2 = a^2\sqrt{-a^2}$; and $-1\sqrt{-a^2} \times a^2\sqrt{-a^2} = -a^2 \times -a^2 = a^4$; and $-1\sqrt{-a^2} \times a^4 = -a^4\sqrt{-a^2}$; and $-1\sqrt{-a^2} \times -a^4\sqrt{-a^2} = a^4 \times -a^2 = -a^6$, &c.

92. *If an affirmative Square is supposed to have an imaginary Root, or a negative Square any Root, they must each have two Roots, whose Coefficients have contrary Signs.* For if the Roots of the former were both Negative, the Square would have been Negative (91): And if those of the latter were both Affirmative, the Square would have been Affirmative (88): But supposing the Signs of the Roots to be contrary, in each Case the Product of the Coefficients with contrary Signs will be Negative; therefore the whole Product in the former Case will be Affirmative, and in the latter, Negative. Thus $-1\sqrt{-a^2} \times 1\sqrt{-a^2} = -1 \times -a^2 = a^2$: And $-1\sqrt{a^2} \times 1\sqrt{a^2} = -1 \times a^2 = -a^2$.

XXXIII. (*f*) See Number 56. (*g*) See Numb. 58.

93. (*h*) Because every Term of one Factor is multiplied into every Term of the other, *the Number of Terms in the Product, before the similar Terms are united, will be the Product of the Number of Terms in each*; and *if the Terms in each Factor are of the same Dimensions, that is, homogeneous, the Terms also of the Product will be homogeneous.*

$$\begin{array}{r} a+b \\ a+b \\ \hline ab+bb \\ aa+ab \\ \hline aa+2ab+bb \end{array}$$

(i) The Learner ought to consider attentively the Multiplication of Binomes; and the Generation of Powers by the Involution of any Binomial, or Residual Root.

94. *If* n *Number of Binomes be multiplied into each other, and the Terms of the Product be ranged, and aggregated together, according to the Number of the first Members combined in each; then the Number of first Members combined in each Term will be, from first to last Term, in arithmetical Progression, whose common Difference is Unity, descending from* n; *viz.* n, n—1, n—2, &c. n—n=o; *and the Number of second Members will be in the same Progression, ascending from* o *to* n; *and the Number of the same Combinations in the Aggregates will be* 1, n, $\frac{n}{1}\times\frac{n-1}{2}$, $\frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, &c. *continued to* n *Terms; and the whole Number of Terms will be* n+1. For let the two Binomes A+*a* and B+*b* be multiplied the one into the other; it is manifest that A+*a* into B, makes AB+*a*B; and into *b*, makes *b*A+*ab*: these being ranged, and aggregated according to the Number of first Members, will be AB+*a*B
+*b*A+*ab*; in which the Number of first Members combined in the Terms is 2, 1, 0; that is, *n*, *n*—1, *n*—*n*, =*o*: but the Terms must be homogeneous, because the Binomes are homogeneous (93.); therefore the Number of second Members combined are 0, 1, 2. Also the second Term of the first Series is a similar Combination with the first of the second Series, they are therefore to be aggregated, and their Number is 2, or *n*; to wit, the Number of Times which two Things can be taken singly (35); and as there are no other similar Combinations, there must be three Terms,

viz.

$$\begin{array}{r} a+b \\ a-b \\ \hline -ab-bb \\ aa\ +ab\qquad \\ \hline aa\quad *-bb \end{array}$$

yy +

viz. $n+1$. Now let this Product be multiplied into $C+c$; it is plain that the Series of Terms from the Multiplication by C will be $ABC+aBC$
$+bAC+abC$; and that the Series of Terms from the Multiplication by c will be $cAB+caB$
$+cbA+abc$: so that the second Term of the first is a similar Combination with the first of the second Series, and the last of the first a similar Combination with the Penultimate of the second, whence they are to be respectively aggregated after this Manner,

$$\begin{array}{l} ABC+aBC \\ \quad +bAC+abC \\ \quad +cAB+caB \\ \qquad\quad +cbA+abc. \end{array}$$

Now the first Term of the first, and last of the second Series cannot be aggregated, being dissimilar to each other, and to every other Term; so that the Number of Terms is 4; to wit, $n+1$; and the Number of similar Combinations in each aggregated Term, is $1, \frac{3}{1}, \frac{3}{1}\times\frac{2}{2}, \frac{3}{1}\times\frac{2}{2}\times\frac{1}{3}$; viz. $1, n, \frac{n}{1}\times\frac{n-1}{n}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$ (35): and the Number of first Members combined are 3, 2, 1, 0; viz. $n, n-1, n-2, n-n$: and consequently the Number of second Members are 0, 1, 2, 3 (93). After the same Manner, if there are more Binomes, because in every subsequent Multiplication of a foregoing Product by a Binome, the second and following Terms of the first Series of Products, viz. by the first Member of the Multiplier, are always similar Combinations with the first and following (except the last) of the second Series respectively, they are to be aggregated; and the whole

4 Number

$$\begin{array}{r}
yy + 2ay - \frac{1}{2}aa \\
yy - 2ay + aa \\
\hline
aayy + 2a^3y - \frac{1}{2}a^4 \\
-2ay^3 - 4aayy + a^3y \quad \\
y^4 + 2ay^3 - \frac{1}{2}aayy \qquad\qquad \\
\hline
y^4 \quad * \quad -3\frac{1}{2}aayy + 3a^3y - \frac{1}{2}a^4
\end{array}$$

2 a x

Number of Terms will always be $n+1$; the Number of first Members will be n in the first Term, and decrease by Unity, and none in the last; and consequently, there will be no second Member in the first Term, and the second Members will be combined, one by one, two by two, 3 by 3, &c. in the 2d, 3d, 4th, &c. Terms respectively; and their Number in the last Term will be n; and the Number of similar Combinations in the Terms will be $1, n, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ &c. (35) continued to n Terms.

95. *Let the first Members of the Binomes, whose Number is* n, *become equal to each other; viz. let* A = B = C &c. = x; *and let the Terms of the Product be ranged according to the Dimensions of* x; *it is manifest*, that, *the Terms of the Product will become the Powers of* x, *combined with the Combinations, that is, multiplied into the Products, of the second Members, one by one, two by two,* 3 *by* 3, &c. *in the* 2*d*, 3*d*, 4*th*, *&c. Terms respectively*; that, *the Indices of* x *will be* n, n − 1, n − 2, *&c.* n − n; *and* that, *the Dimensions of the Product will be denominated from* x^n. *The Combinations of the second Members are called the literal Coefficients of the Powers of* x; *the Number of similar Combinations in the Terms are called the numeral Coefficients, or Unciæ; to wit,* $1, \frac{n}{1}, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$ *&c; the Uncia of the second Term being always* n, *the Dimensions of the Product*. Thus putting A = B = C = x; the foregoing Product will become $x^3 \begin{array}{l} +a \\ +b \\ +c \end{array} x^2 \begin{array}{l} +ab \\ +ac \\ +bc \end{array} x + abc$.

96. *Hence, if* x *the first Member, which is common to the Binome Factors, is found in the last Term of a Product, whose*

$$\frac{2ax}{c} - \sqrt{\frac{a^3}{c}}$$

$$3a + \sqrt{\frac{abb}{c}}$$

$$\frac{2ax}{c}\sqrt{\frac{abb}{c}} - \frac{aab}{c}$$

$$\frac{6aax}{c} - 3a\sqrt{\frac{a^3}{c}}$$

$$\frac{6aax}{c} - 3a\sqrt{\frac{a^3}{c}} + \frac{2ax}{c} - \sqrt{\frac{aab}{c}} - \frac{aab}{c}.$$

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whose Terms are ranged according to the Dimensions of x; *then, the second Member of one Binome is nothing: and so often as* x *is found in the last Term, so many Binomes there are, whose second Members are* = 0, *and the Product* of the Dimensions of x in the highest Term *wants so many of its last Terms, as there are Units in the lowest Index of* x. For if the Product was compleat, the Index of x would have decreased by Unity to nothing, that is, there would have been so many more Terms as there are Units in the lowest Index of x.

97. *Let the first Member of the Binome Factors be* x, *as before, and the second Members be unequal Numbers; the Terms of the Product (ranged always by the Dimensions of* x) *will be its Powers whose Indices decrease by Unity, multiplied, into the second Members*, one by one in the second; into the Products of two in the third; into the Products of three in the fourth; and so on; the last Term being the Product of all the second Members, and into which x does not enter; *But the second Members being Numbers, the similar Combinations of them will be united into one Sum, so that the Unciæ, and the Coefficients of the Terms which were heretofore called literal, are now the Sum of the second Members in the second Term, the Sum of their Products by two in the third; the Sum of their Products by three in the fourth; and so on; and consequently grow greater and greater in the subsequent Terms: and the last Term will be the Product of them all, purely numeral.* Thus if $a = 2$, $b = 3$, $c = 4$, the above Product will be $x^3 + 9x^2 + 26x + 24$; in which $9x^2 = 2x^2 + 3x^2 + 4x^2$, the Sum of the three Combi-

Combinations of 2, 3, and 4, singly with x^2; and $26x = (6x=)\overline{2\times3\times x} + (8x=)\overline{2\times4\times x} + (12x=)\overline{3\times4\times x}$, the Sum of the three Products of 2, 3, and 4, viz. two Factors in each Product combined with x; and the last Term is $24 = 2\times3\times4$, the Product of them all together. Now if any of the second Members have contrary Signs, the similar Combinations are to be summed by Art. xxiv. so that the Sums may not increase in the subsequent Terms.

98. *Let now both* first *and* second *Members of the Binomes, whose Number is* n, *be equal, and being expressed by Species, be multiplied; that is, let the Binome* $x \pm a$ *be involved to the Power, whose Index is* n: *the Terms of the Power will consist of the Powers of* x, *whose Indices decrease from* n *to* 0 *combined with the Powers of* a, *whose Indices increase from* 0 *to* n; *and also with the Unciæ* $1, \frac{n}{1}, \frac{n}{1}\times\frac{n-1}{2}, \frac{n}{1}\times\frac{n-1}{2}\times\frac{n-2}{3}$, *&c. continued to* n *Terms, respectively.* For the second Members having become equal, as well as the first, their Combinations will become also the Powers of one Member; that is, the Indices of a will increase as those of x decrease, so that their Sum in each Term will be equal to n (93), so that what were heretofore called the literal Coefficients of the Terms are now become the Powers of a, and the Unciæ, or numeral Coefficients (which could not be expressed in Numbers when the second Members were all different (95) and their similar Combinations were particularly wrote down in the Aggregates; nor, when the second (97) Members being numeral, their similar Combinations were united,) may now be expressed in Numbers, to abbreviate the Expression; and it is manifest, that these *Unciæ will first increase and then decrease, and be the same when they decrease, as when they increased* (32). Thus if $a = b = c = a$, the above Product will be the Cube of $x \pm a$; and will be $xxx\begin{array}{l}\pm a\\ \pm a\\ \pm a\end{array}xx\begin{array}{l}+aa\\ +aa\\ +aa\end{array}x \pm aaa$, or writing the Unciæ $xxx \pm 3a.xx + 3aax \pm aaa$: or $x^3 \pm 3ax^2 + 3a^2x \pm a^3$ (Art. XIV. 59. 88).

99. *Let*

99. *Let the above Binome* x ± a *have its second Member changed into some Number, and be raised to the Power* n. *It is manifest that the Terms will consist of the Powers of* x *decreasing from* n *to* o, *combined with the Powers of the Number increasing from* o *to* n; *and which are multiplied respectively into the Unciæ*, $1, \frac{n}{1}, \frac{n}{1} \times \frac{n-1}{2}$, &c. for (97) the Sums of the Products are now the Sums of Powers, and those Powers being equal to each other in every Term, their Sum will be equal to the Product of one into their Number: *and the Sums will grow greater and greater, as the Powers grow greater; though their Number the Uncia grows less.*

Thus, if $a = 4$, the Cube of $x \pm 4$ will be $x^3 \begin{array}{l} \pm 4 \\ \pm 4 \\ \pm 4 \end{array} x^2 \begin{array}{l} +16 \\ +16 \\ +16 \end{array} x \pm 4 \times 4 \times 4$ (59, 88.); now it is plain that the Sum $\pm 4 \pm 4 \pm 4 =$ to the Uncia 3 multiplied into the 2d Member ± 4, and that the Sum $16 + 16 + 16 =$ to the Uncia 3 into the Square $+16$ of ± 4; wherefore the cube is $x^3 \pm 12x^2 + 48x \pm 64$ (59. 88.)

100. Wherefore universally *putting* x ± a *indefinitely for any Binome to be raised to any Power whose Index is any Affirmative Integer* n, *the following Invention of our Author, and which is called the* Binomial Theorem, *is an universal Module for raising it to that Power without the Labour of Multiplication*; $x^n \pm \frac{n}{1} \times x^{n-1} a + \frac{n}{1} \times \frac{n-1}{2} \times x^{n-2} a^2 \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times x^{n-3} a^3 + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times x^{n-4} a^4$ *&c.* ...... $\pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \ldots \frac{n-n}{n} \times x^{n-n} a^n$.

101. Because in the Series $\frac{n}{1}, \frac{n}{1} \times \frac{n-1}{2}, \frac{n}{1} \times \frac{n-1}{2} \times$

$\times \frac{n-2}{3}$ &c. of Fractions for finding the Unciæ, the Numerators are the same decreasing Laterals with the Indices of *x*, yet so that the Index of *x* in any Term is by One less than the corresponding Numerator (for $\frac{n}{1}$ is the Fraction, which multiplied into Unity, gives the Coefficient of the second Term, in which Term the Index of *x* is only $n - 1$, and so on) and because the Denominators are exactly the same increasing Laterals with the Indices of *a*; therefore *the same Series may be enunciated thus, in any Term, if the Uncia of the preceding Term be multiplied by the Index of* x *more Unity, and the Product be divided by the Index of* a, *the Quote will be the Uncia of that Term.* And because in any Term the Index of *a* is equal to the Number of preceding Terms, and the Index of *x* is equal to the Number of subsequent Terms; *the same Series is also thus enunciated, in any Term, if the preceding Uncia be multiplied into the Number of Terms following more One, and the Product be divided by the Number of preceding Terms, the Quote will be the Uncia of that Term.* And because the Index of *a* may be called the Index of the Coefficient, as it marks its Place, whether it be 1st, 2d, &c. and because the Index of *x* is called the Index of the Term; *the same Series is also enunciated thus, in any Term, if the preceding Uncia is multiplied into the Index of the Term more One, and divided by the Index of the Coefficient, the Quote is the Uncia of that Term.* Now because *these Unciæ must decrease and be the same as before,* when the Numerators and Denominators of the generating Fractions become equal, or change their Value (32); therefore the same Thing will happen, *when the Indices of* x *and* a *become equal or change their Value in the same Term; whence it will be sufficient to find the Unciæ for half the Number of Terms: Now the Number of Terms is given, being always* $n + 1$ (94); and the Series must terminate, *n* being by Supposition finite, and affirmative. *If the Terms of the Power be numbered* from the other End of the Series, viz. *from* a *to* x; it is evident that *the same Things will be demonstrable* by substituting *a* for *x*, and *x* for *a*.

102. *Let the Power to which the Binome* $x \pm a$ *is to be raised be an affirmative mean Proportional, or one which is above*

above Unity; that is, a Power whose Index is the affirmative Fraction $\frac{m}{n}$ (78); *then the Module for the Unciæ of Number* (100) *will be changed into* $1, \frac{\frac{m}{n}}{1}, \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n} - 1}{2}, \frac{\frac{m}{n}}{1} \times \frac{\frac{m}{n} - 1}{2} \times \frac{\frac{m}{n} - 2}{3}$, &c. *that is,* by reducing the Fractions to a more simple Form, *into* $1, \frac{m}{n}, \frac{m}{n} \times \frac{m-n}{2n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n}, \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n}$, &c. [*For* $\frac{\frac{m}{n} - 1}{2} = \frac{m-n}{2n}$, and $\frac{\frac{m}{n} - 2}{3} = \frac{m-2n}{3n}$, &c. (Art. LIX.)] *and therefore the whole Module will become*

$$x^{\frac{m}{n}} \pm \frac{m}{n} x^{\frac{m-n}{2n}} a - \frac{m}{n} \times \frac{m-n}{2n} x^{\frac{m-2n}{3n}} a^2 \pm \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3-n} x^{\frac{m-3n}{4n}} a^3 - \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} \times \frac{m-3n}{4n} x^{\frac{m-4n}{5n}} a^4, \text{ \&c.}$$

103. *Let the Power, to which the Binome* $x \pm a$ *is to be raised, be a mean Proportional below Unity,* that is, *a Power whose Index is a negative Fraction* $-\frac{m}{n}$ (78); *the Expression of the Unciæ will be the same, but the Module will become*

$$x^{-\frac{m}{n}} \mp \frac{m}{n} x^{\frac{-m-n}{2n}} a + \frac{m}{n} \times \frac{m-n}{2n} x^{\frac{-m-2n}{3n}} a^3 \mp \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3n} x^{\frac{-m-3n}{4n}} a^3 + \frac{m}{n} \times \frac{m-n}{2n} \times \frac{m-2n}{3a} \times \frac{m-3n}{4n} x^{\frac{-m-4n}{5n}} a^4, \text{ \&c.}$$

104. *Let the Power, to which the Binome* $x \pm a$ *is to be raised, be a Divisor, or the Denominator of a Fraction; that is, Let the Index be the negative Integer* $-n$ (76); then the Expression of the Unciæ will be $1, \frac{n}{1}, \frac{n}{1} \times \frac{n+1}{2}, \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3}, \frac{n}{1} \times \frac{n+1}{2} \times \frac{n+2}{3} \times \frac{n+3}{4}$, (28) &c. for

$-\frac{n}{1}\times-\frac{n-1}{2}=\frac{n}{1}\times\frac{n+1}{2}$; &c. and the Module becomes

$x^{-n} \mp \frac{n}{1} x^{-n-1} a + \frac{n}{1}\times\frac{n+2}{2} x^{-n-2} a^2 \mp \frac{}{1}\times\frac{n+1}{2}\times\frac{n+2}{2} x^{-n-3}$

$a^3 + \frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3}\times\frac{n+3}{4} x^{-n-4} a^4$, &c.

105. Becauſe that to extract any Root from any given Quantity is the ſame Thing, as to raiſe that Quantity to a Power, whoſe Index is a Fraction, the Denominator of which is the Name of the Root (76); and becauſe that to divide any Quantity by an Infinitinome, is the ſame as to multiply it by the Reciprocal of the Diviſor (145); *the Theorems of Numbers* 102, 103, 104, *are Modules for the Operations of raiſing Infinitinomes to Powers, and of extracting infinite Roots, in which the Expreſſion will always be an infinite Series.* For the Denominator (102, 103.) of the fractional Index being greater than the Numerator, the Exponents of the Terms muſt become Negative, and increaſe continually; and neither the Denominator nor any multiple of it can ever meaſure the Difference between the Numerator and Denominator, nor the Difference between the Numerator and a leſs Multiple of the Denominator; whence the Unciæ muſt increaſe continually: And when the Index is a negative Integer $-n$ (104), the continual Subduction of the Laterals from it, increaſes the Indices of the Terms negatively; whence the Operation for the Unciæ (104) becomes equivalent to the Generation of Figurates of the *n*th Order; and the Unciæ being thoſe Figurates (28) will increaſe ſine fine.

106. *When the Index of the Power is an affirmative Fraction, all the Indices of* x, *except the Firſt, are Negative; and when the Index of the Power is Negative, whether it be Integer or Fractional, all the Indices of* x *are Negative; but in all Caſes the Indices of* a *are Affirmative. If the Power be* $\overline{x+a}|^{-\frac{m}{n}}$ *The Terms in all the odd Places after the Firſt will be Negative, and the Terms in all even Places Affirmative:* For all the generating Fractions except the Firſt being Negative (105), the Unciæ beginning at the ſecond Term (101) will be alternately Affirmative and Negative (59); that is, Affirmative in the even, and Negative

Negative in the odd Places. *If the Power be* $\overline{x-a}|^{\frac{m}{n}}$ *all the Terms after the First will be Negative:* For the Terms exclusive of the Unciæ are Affirmative and Negative alternately beginning at the First (59); but the Unciæ are Affirmative and Negative alternately beginning at the Second (105, 59); wherefore all the Terms after the First (59) are Negative. *If the Power be* $\overline{x+a}|^{\frac{-m}{n}}$, *or* $\overline{x+a}|^{-n}$ *the Terms in the odd Places will be Affirmative, and in the even Places Negative*; for all the Indices of x being Negative, the generating Fractions are all Negative; wherefore the Unciæ beginning at the first Term are Affirmative and Negative alternately. *If the Power be* $\overline{x-a}|^{\frac{-m}{n}}$ *or* $\overline{x-a}|^{-n}$ *all the Terms will be Affirmative.* For as well the Terms exclusive of the Unciæ, as the Unciæ, beginning from the first Term, will be Affirmative and Negative alternately; wherefore all their Products, the Terms, are Affirmative (59).

107. Because $x^{n-1} = \frac{x^n}{x}$ (XVI), and $x^{n-2} = \frac{x^n}{x^2}$, and $x^{n-3} = \frac{x^n}{x^3}$, &c. the form of $\overline{x \pm a}|^n$ in Number 100 may be changed into $x^n \pm \frac{n}{1}\frac{x^n a}{x} + \frac{n}{1} \times \frac{n-1}{2} \cdot \frac{x^n a^2}{x^2} \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \frac{x^n a^3}{x^3}$, &c. if then we put $P = x$, and $Q = \frac{a}{x}$, so that $Q^2 = \frac{a^2}{x^2}$, $Q^3 = \frac{a^3}{x^3}$, &c, and $x \pm a = P \pm PQ$: the form of $\overline{x \pm a}|^n$ will be $\overline{P \pm QP}|^n = P^n \pm \frac{n}{1} P^n Q + \frac{n}{1} \times \frac{n-1}{2} P^n Q^2 \pm \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} P^n Q^3$, &c. Lastly, if we put $P^n = A$; and the 2d, 3d, &c. Terms of this last Ex-

pression

preſſion = to B, C, D, reſpectively; then $\overline{x \pm a}^n = \overline{P \pm QP}^n = A \pm \frac{n}{1} QA + \frac{n-1}{2} QB \pm \frac{n-2}{3} QC \pm \frac{n-3}{4} QD \pm \frac{n-4}{5} QE$, &c. an Expreſſion comprehending all Caſes.

108. *Any Polynome may be involved to any Power by diſtinguiſhing it into two Parts, conſidering it as a Bin̄ome, and raiſing it to the Power required, by Number* 107; *and then, by the ſame in the Place of the Powers of theſe compound Parts, by ſubſtituting their Values.* Thus $\overline{a+b+c}^3 = \overline{a+b+c}^3 = \overline{a+b}^3 + 3c \times \overline{a+b}^2 + 3c \times \overline{a+b} + c^3 = a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$.

109. *In the Power* $\overline{x \pm a}^n$ *the Square of any Term is greater than the Product of the two Terms which are adjacent on each Side.* For the Square of the Species Part of it is equal to the Product of the adjacent Species (XIV. 71); but the Square of the middle Uncia is greater than the Product of the adjacent ones (32); ſo that the Square of the whole middle Term is greater than the Product of the adjacent Terms: *Alſo if each ſubſequent Term be divided by the next antecedent Term, the Quotes will grow leſs continually* (32): *Alſo in the Square of* $x \pm a$ *the Square of the middle Term is quadruple the Product of the Extremes* (33): *And in the Cube of it, the Square of either middle Term is triple the Product of the adjacent Terms: And in the Biquadrate of it, the Squares of the ſecond and fourth Terms, are* $\frac{8}{3}$ *of the Product of the Terms adjacent to them*; *and the Square of the Third is* $\frac{9}{4}$ *of the Product of the Second and Fourth*, &c.

110. *In the Power* $\overline{x - a}^n$, *if* n *be odd, the laſt Term will be Negative; if even, Affirmative.* For n is the Number of negative Factors, and according as that Number is odd, or even (87, 88), ſo the Power a^n, or laſt Term, is Negative or Affirmative.

111. The Terms of every *Product by binomial Factors* being all Affirmative; and thoſe of every *Product of reſidual Factors* being alternately Affirmative and Negative (59);

(59); and the Number of Terms in each being $n+1$, One more than the Number of Factors (94): It follows that *the Number of Successions in the former Product, and of Alternations in the latter, shall be equal to* n, *the Number of Factors.*

112. *Let any Product of Binomes, having their second Members numeral, be multiplied by a Binome whose second Member is also numeral; then* 1. *if the second Member of the Multiplier is less than the Quote, which arises by dividing any Term of the Multiplicand by its next antecedent Term, the corresponding Term of the Product and all its Terms subsequent will have their Signs the same with the Signs of the respective Terms of the Multiplicand;* 2. *If the second Member of the Multiplier is equal to that Quote, the corresponding Term of the Product will vanish, and the Signs of the Terms adjacent on each Side will be contrary;* 3. *If the second Member of the Multiplier is greater than that Quote, the corresponding Term of the Product and all its Terms subsequent will have their Signs contrary to those of the respective Terms of the Multiplicand.*

Let the Multiplier be a Binomial: The Signs of the Terms in both Series of Products by the Members of the Multiplier being successively the same with those of the Multiplicand (57); and the second Term and all the subsequent Terms of the former Series being to be added respectively to the first and subsequent Terms (except the last) of the second Series (94), to make the Sums or Terms of the Product; in those Sums the Sign of the greater Term in either Series must prevail, when they are contrary (XXIV); and they must be contrary, when there is an Alternation of Signs in the Terms of the Multiplicand (94): An Alternation therefore in this Case is always supposed. Putting then the second Term of the binomial Multiplier $\left\{\begin{matrix}\text{less}\\\text{greater}\end{matrix}\right\}$ than that Term of the Multiplicand to which the Alteration is made, divided by its immediate Antecedent from which the Alteration begins, it shall also be $\left\{\begin{matrix}\text{less}\\\text{greater}\end{matrix}\right\}$ than every succeeding Term of the Multiplicand divided by its immediate Antecedent (109); therefore its Products into that Term from which the Alternation begins, and into every succeeding Term of the Multiplicand, will be $\left\{\begin{array}{c}\text{less}\\ \text{greater}\end{array}\right\}$ than the similar Terms of the first Series, to which they are respectively to be added (94); therefore the Signs of the Sums will be the $\left\{\begin{array}{c}\text{same}\\ \text{contrary}\end{array}\right\}$ successively with those of the Terms of the first Series (XXIV), that is, with those of the Terms of the Multiplicand (57). Putting the second Member of the binomial Multiplier equal to that Term of the Multiplicand to which the Alternation is made, divided by its immediate Antecedent from which the Alternation begins, its Product into that antecedent Term will be equal to the similar Term of the first Series, to which it is added (94): Therefore their Sum, that is the corresponding Term of the Product, vanishes (XXIV): Now the second Member, being equal to any Term divided by the preceding, must be less than every one of the preceding Terms divided by its immediate Antecedent (109); wherefore as before the Signs of the preceding Terms of the Product are the same respectively with those of the Multiplicand; also it must be greater than every one of the succeeding Terms of the Multiplicand divided by its immediate Antecedent (109); therefore as before the Signs of the Sums or Terms of the Product will be the same with those of the second Series (XXIV); that is, contrary to those of the Multiplicand (59). Now because when a Term vanishes, the Signs of the Terms antecedent to it are the same with those of the Multiplicand, and the Signs of the Terms subsequent to it are the contrary; and that there is an Alternation in the Signs of the Terms of the Multiplicand (94); it follows, that the Signs of the adjacent Terms must be contrary. Now let the Multiplier be a Residual: The Signs of the Terms in the first Series are the same (57), and those in the Second the contrary respectively, to the Signs of the Terms of the Multiplicand (58); therefore when there is a Succession of Signs in the Multiplicand, the corresponding Terms in each Series, which being similar are to be added, have contrary Signs, and the Sign of the Sum will be that of the greater; supposing therefore a Succession of Signs in the Multiplicand, and putting the second Member of the residual Multiplier equal to any Term of the Multiplicand to which the Succession of Signs is made divided by its Antecedent from which the Succession of Signs begins, its Product into that Antecedent will be equal to the Term of the first Series to which it is added; therefore the Sum is $= o$; and being equal to one Quote, it will be less than all the antecedent Quotes, and greater than all the subsequent Quotes (109); therefore the Signs of the former Series will prevail in all the Terms antecedent, and in all the subsequent, the Signs of the latter; therefore when there is a Succession of Signs in the Multiplicand, and consequently in the former Series, the Signs of the respective Terms of the Product will be contrary to those of the Multiplicand; and consequently the Signs of the Terms adjacent to that which vanishes must be contrary, the Sign of the subsequent Term being contrary to that of the subsequent Term of the Multiplicand; and the Signs in the Multiplicand being in Succession (94, 59).

113. *If any given Product of Binomes, whose second Members are Numeral, is multiplied into a Binomial, whose second Member is also Numeral, one Succession of Signs, and one only, will be added to the Terms of the Product, the Number of Alternations remaining the same as in the Multiplicand; and if the given Product is multiplied into a Residual, one Alternation of Signs, and one only, will be added in the Product; and the Number of Successions will remain the same as in the Multiplicand.* Suppose the Multiplier a Binomial, the Signs of each Series are the same respectively with those of the Mutliplicand (57); and the last Term of the Product has the same Sign with the last of the Multiplicand, and by this Term alone the Number of Terms is increased (94); now if the Signs of the first Series, or Multiplicand, prevail throughout, there is one Succession, and one only, added (the Number of Alternations remaining as in the Multiplicand) to wit, from the Penultimate to the last Term. But if the Signs of the second Series prevail in any Sum, they must prevail in all the subsequent Terms, except the last (112); and an Alternation in the Multiplicand has become a Succession in the Pro-

duct (112), whereby the Successions are increased by One only, and the Alternations diminished by One only; but the Loss of this Alternation is restored by an Alternation from the Penultimate to the Ultimate, which always retains the Sign of the Multiplicand: And if any Term of the Product vanishes, because the Signs of the adjoining Terms must be contrary (112), by putting either Sign for it, there will be one Succession and one Alternation, and consequently one Succession only, added to the Number in the Multiplicand, the Number of Alternations being the same. Suppose the Multiplier to be a Residual: The Sign of the last Term of the second Series, that is, of the Product remains unalterably the contrary to that of the last Term of the Multiplicand (94, 58), and this Term only increases the Number of Terms in the Product; Now the Signs of the first Series being the same, and those of the second respectively contrary to those of the Multiplicand, if the Signs of the first Series prevail, there is an Alternation, and one only from the Penultimate to the last of the Product added, the Successions remaining as in the Multiplicand: If the Signs of the second Series prevail in any Sum, a Succession in the Multiplicand is become an Alternation in the Product, and the Signs of the second Series prevail in all the subsequent Terms of the Product; the Alternations therefore are increased, and the Successions diminished by one only, but the Loss of this Succession is compensated by a Succession from the Penultimate to the last. And if any Term of the Product vanishes, because the adjacent Terms must have contrary Signs, if the vanished Term be supposed to be affected with either Sign, it will add both an Alternation and a Succession; consequently one Alternation, and one only, is added to the Number in the Multiplicand, the Number of Successions being the same.

114. Hence, universally *in every Product of Binomes, whose second Members are numeral, there are as many Alternations of Signs as there are Residual Factors, and as many Successions as Binomial Factors; and conversely. Supposing that, when any Term vanishes, that Term is affected with either Sign.* Thus, to illustrate No. 112 and 113 together, if the Binome $x \pm 3$ be multiplied into $x \pm 2$, there will emerge

emerge the Products $x^2 \pm 5x + 6$ (59); and if these be multiplied by $x \mp a$, making a less than $\frac{6}{5}$, the Signs in the Products will be the same as in the Multiplicand, except that of the last Term, which, when the Multiplier is $x - a$, shall be the contrary. Let $x \mp a = x \mp 1$, and there will emerge $x^3 \pm 4x^2 + x \mp 6$; but if $x \mp a = x \mp \frac{6}{5}$, the Penultimate shall vanish, and the Signs of the adjacent Terms be contrary, as $x^3 \mp \frac{19}{5} x^2 \mp \frac{36}{5}$: Now making a greater than $\frac{6}{5}$, and less than 5, as $x \mp 4$, the Sign of the second Term of the Product will be the same with the Sign of the second Term in the Multiplicand, and the Signs of the subsequent Terms will be contrary, viz. $x^3 \pm x^2 - 14x \mp 24$. Let $x^2 \pm 5x + 6$ now be multiplied into $x \mp 5$, and the second Term will vanish, and the Signs of the subsequent Term will be contrary to those in the Multiplicand, viz. $x^3 - 19x \mp 30$; and if a be made greater than 5, the Sign of the second Term will also be contrary to that in the Multiplicand; let it be $x \mp 6$, and we shall have $x^3 \mp x^2 - 24x \mp 36$; in all which Cases, the Number of Alternations and Successions are respectively equal to the Number of residual and binomial Factors.

115. *In the Square of any Binome, the Sum of the Squares of the Members is greater than the Sum of the Products of the Members.* For each Product is a geometrical Mean between the Squares (Eucl. VIII. 17): Whence the Sum of the Squares is greater than the Sum of the Products (Eucl. V. 25.)

116. *If the Binome* $\sqrt{\mathrm{a}} \pm \sqrt{\mathrm{b}}$ *expounded in Numbers, (*$\sqrt{\mathrm{a}}$ *being greater than* $\sqrt{\mathrm{b}}$*) be involved to a Power whose Index is* n, *the Terms of the Power can be united alternately; and if the Sums of the alternate Terms be connected with the Sign of the second Member of the Root, the Power will be a Binome, whose greater Member contains* $\sqrt{\mathrm{a}}$, *and the less* $\sqrt{\mathrm{b}}$, *when*

when n *is odd*; *but when* n *is even*, *the greater Member will be rational* (90), *and the less will contain* $\surd$ a b. Let n be odd, the first Term consists solely of the n^{th} Power of $\surd a$, that is, of $\surd a \times a^{\frac{n-1}{2}}$; and the other Terms in the odd Places consist of the Powers of $\surd a$, whose Indices are either $n-2$ or n less a Multiple of two, that is, of the rational Powers of $\surd a$ into $\surd a$, and of the Powers of $\surd b$ whose Indices are even, and which therefore are rational; therefore the Terms in the odd Places consist of the rational Powers of $\surd a$ and of $\surd b$ multiplied into $\surd a$ only; that is, the only irrational Part of the Terms in the odd Places is $\surd a$, and they therefore will be entirely rational where n is even; they are therefore similar in both Cases, and may be united: Now the second Term consists of the Power of $\surd a$, whose Index is $n-1$, and the other Terms in the even Places contain the Powers of $\surd a$, whose Indices are n less the odd Laterals, multiplied either into $\surd b$, or, into the Products of the rational Powers of $\surd b$ into $\surd b$; so that the Powers of $\surd a$ in the even Places are rational, and the only irrational Parts of the Terms in the even Places is $\surd b$; but if n is even, the irrational Part of the Terms in the even Places will be $\surd a b$; because the Powers of $\surd a$ will also be irrational as well as the Powers of $\surd b$; consequently, the Terms in the even Places also are similar, and agree in their irrational Part $\surd b$, when n is odd; or in $\surd a b$, when n is even, and can be united; and the Sums, connected as above, will form the Binome described above. Hence, n *being odd*, *if* $\surd$ a *be put rational*, *the first Member*; *but if* $\surd$ b, *the second Member of the binome Power*, *will be rational*; *and* n *being even*, *if* $\surd$ a *be put rational*, *the irrational Part of the irrational Member of the binome Power will be only* $\surd$ b, *instead of* $\surd$a b; *and so if* $\surd$ b *is put rational*, *it will be only* $\surd$ a *instead of* $\surd$ a b.

117. Hence it follows, that *any given Binome may be taken for the* n[th] *Power of a binome Root*, *conceiving the Members of the given Binome to be the Sums of the Terms of the* n[th] *Power united alternately*, *and connected with the Sign of the second Member of the Root*; also that *the Difference of the Members of the given Binome is the* n[th] *Power*

of

of the Difference of the Members of the Root: And that consequently multiplying the Sum into the Difference, the Difference of the Squares of the Members of the given Binome is the n^{th} *Power of the Difference of the Squares of the Members of the Root.* For universally let *the Sum or Difference of the Quantities* x *and* a *be raised to the* n^{th} *Power, and let the Sum of the Terms in the odd Places be called* A, *and the Sum of the Terms in the even Places be called* B, *then the Difference of the Squares of* A *and* B *shall be equal to the* n^{th} *Power of the Difference of the Squares of* x *and* a; *for* $A + B = \overline{x+a}|^n$ and $A - B = \overline{x-a}|^n$ therefore $\overline{A+B} \times \overline{A-B}$, that is, $A^2 - B^2$ (Eucl. II. 5.) $= \overline{x+a}|^n \times \overline{x-a}|^n$, that is, $\overline{x+a \times x-a}|^n$, that is (Eucl. II. 5.) $\overline{x^2 - a^2}|^n$.

118. *Any binome Surd in Numbers, both Members having the same Index* $\frac{1}{n}$, *as* $x^{\frac{1}{n}} \pm a^{\frac{1}{n}}$ *being given, let* m *be the least Integer which* $\frac{1}{n}$ *will measure, then shall* $x^{m-\frac{1}{n}} \mp x^{m-\frac{2}{n}} a^{m} + x^{m-\frac{3}{n}} a^{2m} \mp x^{m-\frac{4}{n}} a^{3m}$, *&c. continued to* $\frac{m}{n}$ *terms, be a compound Surd, which multiplied by the proposed Surd, gives a rational Product* $x^m - a^m$; *and when* $\frac{m}{n}$ *is odd, the Product* $x^m + a^m$. For by Numbers 59 and 112, all the middle Terms of the Product will vanish, and the extreme Terms (80, 90) will be rational; and if the Surd is the Binomial $x^{\frac{1}{n}} + a^{\frac{1}{n}}$, and $\frac{m}{n}$ odd, the last Term of the compound Surd will be Affirmative; wherefore the Product will be $x^m + a^m$ (59). Thus if $x^{\frac{3}{4}} + a^{\frac{3}{4}}$ be given, then $\frac{1}{n} = \frac{3}{4}$; therefore

fore $n = 3$, and $\frac{m}{n} = \frac{12}{3} = 4$; therefore the compound Surd is $x^{3-\frac{3}{4}} - x^{3-\frac{6}{4}}a^{\frac{3}{4}} + x^{3-\frac{9}{4}}a^{\frac{6}{4}} - x^{3-3}a^{\frac{9}{4}} = x^{\frac{9}{4}} - x^{\frac{6}{4}}a^{\frac{3}{4}} + x^{\frac{3}{4}}a^{\frac{6}{4}} - a^{\frac{9}{4}}$; whence the Product is $(x^{\frac{9}{4}} \times x^{\frac{3}{4}} = x^{\frac{12}{4}} =)\ x^3\ (-a^{\frac{9}{4}} \times a^{\frac{3}{4}} = -a^{\frac{12}{4}} =) -a^3$. And by the same Reasoning

119. *Any binome Surd whose Numbers have different Indices being given, as* $x^{\frac{1}{n}} \pm a^{\frac{1}{l}}$, *let* m *be the least Integer, which* $\frac{1}{n}$ *and* $\frac{n}{l}$ *will measure, then shall* $x^{m-\frac{1}{n}} \mp x^{m-\frac{2}{n}}a^{\frac{1}{l}} + x^{m-\frac{3}{n}}a^{\frac{2}{l}} \mp x^{m-\frac{4}{n}}a^{\frac{3}{l}}$, *&c. continued to* $\frac{m}{n}$ *Terms give a compound Surd, which multiplied by the proposed shall give a rational Product* $x^m \mp a^{\frac{ml}{n}}$. Thus if $x^{\frac{1}{2}} - a^{\frac{1}{3}}$ be given, then $\frac{1}{n} = \frac{1}{2}$, and $\frac{n}{l} = \frac{3}{2}$; therefore $m = 3$; and $x^{3-\frac{1}{2}} + x^{3-1}a^{\frac{1}{3}} + x^{3-\frac{3}{2}}a^{\frac{2}{3}} + x^{3-2}a + x^{3-\frac{5}{2}}a^{\frac{4}{3}} + x^{3-\frac{6}{2}}a^{\frac{5}{3}} = x^{\frac{5}{2}} + x^2a^{\frac{1}{3}} + x^{\frac{3}{2}}a^{\frac{2}{3}} + xa + x^{\frac{1}{2}}a^{\frac{4}{3}} + a^{\frac{5}{3}}$: Whence the Product is $(x^{\frac{5}{2}} \times x^{\frac{1}{2}} = x^{\frac{6}{2}} =)\ x^3\ (-a^{\frac{1}{3}} \times a^{\frac{5}{3}} = -a^{\frac{6}{3}} =) -a^2$.

120. *A Product of even Dimensions, all whose Terms are rational, may be had from the Multiplication of Binomes, whose second Members are Radicals, whether the Radicals be all real or all imaginary, or some real and some imaginary, if they be taken equal in Pairs, and with contrary Signs; but to have a Product of odd Dimensions, all whose Terms shall be rational, one rational Factor at least is requisite.* For when the

the Dimensions are even, the first Term and all the Terms whose Indices are even will be rational, and the Terms whose Indices are odd, which only can be affected with the radical Sign, will vanish upon Account of the Equality of those Radicals which have contrary Signs (XXIV); and this Product being multiplied into a Binome, having a radical Member, must have all its Terms after the first Irrational. Thus $\overline{x-\sqrt{a^2}}\times\overline{x+\sqrt{a^2}}$, is x^2-a^2; and $\overline{x-\sqrt{b^2}}\times\overline{x+\sqrt{b^2}}$, is x^2-b^2; and $\overline{x-\sqrt{a^2}}\times\overline{x+\sqrt{a^2}}\times\overline{x-\sqrt{b^2}}\times\overline{x+\sqrt{b^2}}$, is $x^4 \begin{smallmatrix}-a^2\\-b^2\end{smallmatrix} x^2 + a^2b^2$; and $\overline{x-\sqrt{-a^2}}\times\overline{x+\sqrt{-a^2}}$, is x^2+a^2 (92); and $\overline{x-\sqrt{-b^2}}\times\overline{x+\sqrt{-b^2}}$, is x^2+b^2 (92); and $\overline{x-\sqrt{-b^2}}\times\overline{x+\sqrt{-b^2}}\times\overline{x-\sqrt{-a^2}}\times\overline{x+\sqrt{-a^2}}$, is $x^4 \begin{smallmatrix}+a^2\\+b^2\end{smallmatrix} x^2 + x^4$; and $\overline{x-\sqrt{a^2}}\times\overline{x+\sqrt{a^2}}\times\overline{x-\sqrt{-b^2}}\times\overline{x+\sqrt{-b^2}}$, is $x^4 \begin{smallmatrix}-a^2\\+b^2\end{smallmatrix} x + a^2b^2$; and any of those Products into $x \pm c$, will give a rational Product; but into $x \pm \sqrt{c}$, or into $x \pm \sqrt{-c}$, an irrational one.

121. *The same Product will arise from the Multiplication of Binomes, whose first Members have Coefficients different from Unity, and from the Multiplication of them after they are divided by their Coefficients, if the fractional Coefficients of the Product are made integral by multiplying all the Terms by the Product of the Denominators.* Thus $\overline{bx+a}\times\overline{dx-c} = bdx^2 \begin{smallmatrix}+ad\\-cb\end{smallmatrix} x - ca$; and $\overline{x+\frac{a}{b}}\times\overline{x-\frac{c}{d}}$

$$= x^2 \begin{smallmatrix}+\frac{a}{b}\\-\frac{c}{d}\end{smallmatrix} x - \frac{ac}{bd} = bdx^2 \begin{smallmatrix}+ad\\-cb\end{smallmatrix} x - ca.$$

122. *A Product of Binomes, which has all its Terms, may be squared thus: Let the Indices of the Terms be the Laterals descending from double the Index of the highest Term of the Product*

Product (83); *then to find the Coefficients, multiply into each other the Coefficients of those Terms of the Product, the Sum of whose Indices is equal to the Index of the Term in the Square; also, into each other, the Coefficients of all the Pairs of the intermediate equidistant Terms; then the Products doubled, and added to the Square of an odd intermediate Coefficient, are the Coefficients of the Terms in the Square.* Thus to square $x^2 + 5x + 6$; the Terms will be x^4, x^3, &c. the Coefficient of x^4 will be 1, the Square of that of x^2; the Coefficient of x^3 will be 1×5 doubled, viz. 10; the Coefficient of x^2 will be 1×6 doubled, more the Square of 5, viz. $12 + 25 = 37$; the Coefficient of x will be 5×6 doubled, viz. 60; and the last Term will be the Square of 6; the Square then is $x^4 + 10x^3 + 37x^2 + 60x + 36$. For the Products of the similar Parts unite in Pairs (Eucl. II. 4.), and the Products of the Coefficients of the Terms, the Sum of whose Indices is the Index of the Term in the Square, are similar; therefore, when doubled, will make its Coefficient; and the Square of the Coefficient of any odd Term being double the Product of equal Coefficients (Eucl. II. 4.), must not be doubled, but united singly with the Products which are similar with it.

123. *The Coefficients of the Terms of the above-mentioned Square, in the odd Places, contain a Square joined with doubled Products; those in the even Places contain doubled Products, and no Square:* For in the odd Places the Indices are even, therefore the Number of Coefficients of the Product, compounding those Coefficients, are odd; and therefore there is then a Square in the Composition of each (122); and in the even Places the Indices are odd, whence the Number of compounding Coefficients is even, and no Square enters the Composition.

124. *The Number of Terms in the Square of a Product as above, when none but doubled Products are united, is equal to the Product of Half the Number of Terms in the Product multiplied into that Number more Unity.* For the whole Number is the Square of that Number (93); from which subducting its Root, viz. the Terms containing Squares, the Difference is the Number of Products, therefore Half this Difference (because they unite in Pairs) added to the Root,

Root, that is, Half the Number of Terms multiplied into the Whole more one, is the Number of distinct Terms in the Square.

125. *Hence, in a given Square, it is known if any other Terms than those containing doubled Products have coalesced.* For putting the whole Number of Terms in the Root n, the Number in the Square is $\frac{n}{2} \times \overline{n+1}$ (124); substituting therefore for n, the Laterals 2, 3, 4, &c. successively, if the resulting Product is not the Number in the given Square, some Terms containing Squares are united; therefore, subducting the given Nnmber of Terms from the Number resulting next greater by Substitution, half the Difference more One will be the Number of Terms which have united (36).

Of DIVISION.

XXXIV. D*IVISION is performed in Numbers, by seeking how many times the Divisor is contained in the Dividend, as often subtracting, and writing so many Units in the Quotient; and by repeating that Operation upon Occasion, as often as the Divisor can be subtracted.*

Thus, to divide 63 by 7, seek how many times 7 is contained in 63, and there will come out precisely 9 for the Quotient; and consequently $\frac{63}{7}$ is equal to 9. Moreover, to divide 371 by 7, prefix the Divisor 7, and beginning at the first Figures of the Dividend, coming as near them as possible, say how many times 7 is contained in 37, and you will find 5; then writing five in the Quotient, subtracting 5 × 7, or 35, from 37, and there will remain 2, to which set the last Figure of the Dividend, *viz.* 1; and then 21 will be the remaining Part of the Dividend for the next Operation; say therefore as before, how many times 7 is contained in 21? and the Answer will be 3; wherefore writing 3 in the Quotient, take 3 × 7, or 21, from 21, and there will remain 0. Whence

```
7) 371 (53
   35
   ——
    21
    21
    ——
     0
```

6. Whence it is manifest, that 53 is precisely the Number, that arises from the Division of 371 by 7.

And thus to divide 4798 by 23, first beginning with the initial Figures 47, say, how many times is 23 contained in 47? Answer 2; wherefore write 2 in the Quotient, and from 47 subtract 2 × 23, or 46, and there will remain 1, to which join the next Number of the Dividend, viz. 9, and you will have 19 to work upon next. Say therefore, how many times is 23 contained in 19? Answer 0; wherefore write 0 in the Quotient; and from 19 subtract 0 × 23, or 0, and there remains 19; to which join the last Number 8, and you will have 198 to work upon next. Wherefore in the last Place say, how many times is 23 contained in 198 (which may be guessed at from the first Figures of each, 2 and 19, by taking notice how many times 2 is contained in 19)? I answer 8; wherefore write 8 in the Quotient, and from 198 subtract 8 × 23, or 184, and there will remain 14 to be farther divided by 23; and so the Quotient will be $208\frac{14}{23}$. And if this Fraction is not liked, you may continue the Division in Decimal Fractions as far as you please, by adding always a Cypher to the remaining Number. Thus to the Remainder 14 add 0, and it becomes 140. Then say, how many times 23 in 140? Answer 6; write therefore 6 in the Quotient; and from 140 subtract 6 × 23, or 138, and there will remain 2; to which set a Cypher (or 0) as before. And thus the Work being continued as far as you please, there will at length come out this Quotient; viz. 208,6086, &c.

```
23) 4789 (208,6086, &c.
    46
    ---
     19
     00
    ---
     198
     184
     ---
      140
      138
      ---
        20
        00
       ---
        200
        184
        ---
         160
```

XXXV.

XXXV. *After the same Manner the decimal Fraction* 3,5218 *is divided by the decimal Fraction* 46,1, *and there comes out* 0,07639, &c. *Where note, that there must be so many Figures cut off in the Quotient, for Decimals, as there are more in the last Dividend than in the Divisor:* As in this Example 5, because there are 6 in the last Dividend, *viz.* 0,004370, and 1 in the Divisor 46,1 *(a).*

```
46,1) 3,5218 (0,07639
      322,7
      -----
        2948
        2766
        ----
         1820
         1383
         ----
          4370
```

We

XXXV. *(a)* 126. *In the Division of Integers, of Decimals, and of mixed Numbers, the Excess of the Index of the dividend Figure above the Index of the dividing Figure is the Index of the Figure in the Quote.* For the Index of the Dividend and Divisor Figures being the Number of Multiplications, or Divisions, of Unity by Ten in the Denominator of each (13), to divide, or to multiply, the Denominator of the Dividend by Ten, is to subduct Unity from, or to add Unity to, its Exponent; and to divide, or to multiply, the Denominator of the Dividend by any other Denominator, is to subduct the Index of the Divisor from, or to add it to, the Index of the Dividend; but in the Division of Integers by Integers, and of Decimals by Decimals, the Denominator of the Dividend is divided by the Denominator of the Divisor; and in the Division of Integers by Decimals, the Denominator of the Integer is multiplied by the Denominator of the Decimal; therefore, in the former Case, the Difference of the Indices, and in the latter, the Sum of the Indices, is the Index of the Quote; now the Index of a Decimal Divisor is negative (14), and must therefore, in Addition, be subducted (XXIV); therefore, in all Cases, the Difference of the Indices of the Dividend and Divisor, is the Index of the Quote.

a 127. *The*

We have here subjoined more Examples, for Clearness Sake, viz.

```
9043) 20844115 (2305        72,4) 2099,6 (29
      18086                       1448
      -----                       ----
       27581                      6516
       27129                      6516
       -----                      ----
        45215                        0
        45215
        -----
            0
```

50,18

---

127. *The Index and Place of the highest Figure in any Quote, is the Index of that Figure of the Dividend, under which the Place of Units in the Divisor falls, when first placed under the Dividend.* For the Index of each Figure in the Quote, being the Excess of the Index of the dividend Figure above that of the dividing Figure (126), and the Index of the lowest Figure of the Divisor being Cypher, the Index, and Place, and Class, of each Figure in the Quote, is the same with that of the lowest Figure in each particular Dividend; therefore the Index of the highest Figure in the Quote, is the same with that of the lowest Figure of the first Dividend; that is, the same with that of the Figure of the whole Dividend, under which the Place of Units in the Divisor falls, when it is subscribed.

128. Hence, *there must be as many Places in the Quote, as there are particular Dividends*; for the highest Figure in the Quote being of the same Place, or Class, with the lowest Figure of the first Dividend, so many Figures must follow in the Quote, as there are Dividends; *that is, as Figures follow in the Dividend, after that Figure under which the Place of Units in the Divisor falls.*

129. Hence, *if the Divisor is not contained in any particular Dividend, a Cypher must be wrote in the Quote*; to keep

```
50,18) 137,995 (2,75      0,0132) 0,051513 (3,9025
       10036                      396
       -----                      ----
        37635                      1191
        35126                      1188
        -----                      ----
         25090                      330
         25090                      264
         -----                      ---
             0                      660
                                    660
                                    ---
                                      0
```

XXXVI.

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keep up the Number of Places equal to the Number of Dividends, and Figures succeeding to that of the Dividend, under which the Place of Units in the Divisor first fell.

130. *Decimals, in Division, are considered as Integers*; the only Difference between them being, that in the former the Numerators are divided, but in the latter they are multiplied, by that Power of Ten, whose Index is the Number of Places from Unity (13); therefore *their Numerators are divided, as in Integers.* As to the Division of their Denominators, in all Divisions of Decimals, or of mixed Numbers, by Decimals, *the Exponent of the first Figure to the Left-hand, that is, of the highest Figure of the Quote, is the Excess of the Index of the lowest Figure of the first particular Dividend, above the Index of the lowest Figure of the Divisor* (126); *that is, the Sum of the Indices of the last Figures to the Right in the Divisor and Quote, is equal to the Index of the last Figure to the Right in the Dividend; that is, the Number of decimal Places in the Divisor and Quote together, is the Number of Places in the Dividend, however they may have been increased by the Addition of Cyphers in the Operation.*

XXXVI. *In Algebraick Terms, Division is performed by the Resolution of what is compounded by Multiplication.* Thus *ab* di-

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131. Hence, *if the Dividend contains decimal Places, but the Divisor none, the Number of Places of Decimals in the Quote will be the same as in the Dividend; and if the Divisor has decimal Places, but the Dividend none, an equal Number of decimal Cyphers may be added to the Dividend;* whence there are no Decimals in the Quote.

132. *If both Divisor and Dividend have Cyphers on the Right-hand, they may be entirely expunged, provided the Number expunged in each is equal* (Eucl. VII. 18.); *and if there are Cyphers on the Right of the Divisor, and none on that of the Dividend, the Cyphers and an equal Number of the Right-hand Figures may be cut off during the Operation; but they must be restored to the Residue, if there is any; and if there is none, these Figures are the Residue.* For the Index of the highest, or first, Figure to the Left in the Quote, is the Index of the lowest, or last, to the Right of the first Dividend lessened by the Index of the lowest, or last, significant Figure to the Right of the Divisor; that is, by the Number of Cyphers (18); therefore the Cyphers being cut off, an equal Number of Figures must be cut off from the Right of the Dividend; and as they cannot contain the Divisor, which consists of more Places, they must be either the Residue, or a Part of the Residue.

133. Hence, *an Integer is divided by any Power of Ten, by cutting off from the Right-hand so many Places for Decimals, as there are Units in the Index of the Power:* For the negative Index of the last Figure in this Quote, will be equal to the Index of the only significant Figure of the Divisor (130), whence their Sum is Cypher, that is, the Index of the Right-hand Figure of the Dividend. And *a mixed Number, or Decimal, is divided by any Power of Ten, by removing the Separatrix so many Places to the Left, as there are Units in the Index of the Power.* For Unity making no Difference between the Figures of the Dividend and of the Quote, it will suffice, in order to know

4 the

ab divided by a gives for the Quotient b, $6ab$ divided by $2a$ gives $3b$; and divided by $-2a$ gives $-3b$. $-6ab$

---

the Quote, to diminish the Index of the Right-hand Figure of the Dividend, by the Index of the Power; that is, by the Number of Cyphers; that is, to augment its negative Value, by moving the Line so many Places to the Left, as the negative Indices increase to the Right; and the affirmative Index of the Divisor being added to the negative Index of the Right-hand Figure of this Quote, will reduce it (XXIV); and the Sum will be the Exponent of the Right-hand Figure of the Dividend. Hence is deduced the *Rule for the Reduction of Sexagesimals into Decimals*; to wit, *divide the Sexagesimals by* 60; for the Division by 60 not only divides by 6, but also moves the Line one Place to the Left.

134. *If the Divisor is not an aliquot Part of the Dividend, the Division may be terminated by annexing the Fraction mentioned Art* XXXVII. *or it may be continued until either an accurate Quote is had, or that it runs out into an infinite Series. It is continued by joining Cyphers to the Right Hand of the Residue* (53); *and the Figures in the Quote resulting from continuing the Division, become a Decimal Fraction, of so many Places, as Cyphers have been annexed to the Residue.* For thus, both Dividend and Divisor are multiplied into the same Power of Ten (53), and consequently the Value of the Quote is not altered (Eucl. VII. 17).

135. *An accurate Quote will be found by this Continuation of the Division; that is, the vulgar Fraction annexed and reduced to its lowest Terms will be reducible to an accurate Decimal, and the Division will terminate, if the Divisor can be measured by the Numbers* 2 *or* 5. For 2 and 5 being the only Measures of Ten, if they measure the Divisor, the Divisor will also measure some Power of Ten, and some Multiple of some Power of Ten, and consequently the Product of the Residue into some Power of Ten (Eucl. VII. 32.); wherefore the Division will terminate, and an accurate Quote is had in Decimals.

 136. *The*

— $6ab$ divided by $2a$ gives — $3b$; and divided by — $2a$ gives $3b$. $16abc^3$ divided by $2ac$ gives $8bcc$. — $84a^3x^4$ divided

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136. *The Quote will run into an infinite Series; that is, the vulgar Fraction, in its lowest Terms, will not be reducible to an accurate Decimal, when the Divisor cannot be measured by* 2 *or* 5. For 2 and 5 being the only Measures of Ten, and being also Prime to the Divisor, the Divisor and its Powers are Prime to Ten and its Powers (Eucl. VIII. 6.); but it is also Prime to the Dividend (Eucl. VII. 24.); therefore the Divisor and its Powers are Prime to the Multiples of Ten and its Powers; and therefore to the Product of the Dividend into Ten, or into any Power of Ten (Eucl. VII. 26.); and therefore cannot divide it accurately; for if it could, a Number would measure a Number to which it is Prime, which is impossible (Eucl. VII. 21.).

137. *When the Quote runs into an infinite Series, if there happens any Residue equal to a former one, the same Figures must return and circulate in the Quote.* For Cyphers being always added to the Residue, that is, the Dividend being always the same Multiple of the Residue, when any Residue is equal to a former one, the Dividend must be equal to a former Dividend, and thence the Quote equal also to a former one; and consequently the subsequent Residue equal to the former subsequent Residue, the subsequent Dividend equal to the former subsequent Dividend; and so the subsequent Quote equal to the former subsequent Quote, and so on.

138. *The Number of periodical Figures will never be more than the Divisor less Unity; that is, the Denominator of the vulgar Fraction less Unity.* For the Residue being always less than the Divisor, it may be any Number less by Unity than the Divisor; but in so many Divisions at most, as there are Units in the Divisor, one of the former Residues must return again; and therefore the same Quote must also return, and continue the Circulation.

139. I

divided by $-12aaxx$ gives $7axx$ (*b*). Likewise $\frac{6}{35}$ divided

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139. *It being generally sufficient to have an assigned Number of Decimals in the Quote, the Division will be usefully abbreviated, especially when the Divisor is an indefinite Number, by keeping only so many Figures, or one more, to the Left-hand of both Dividend and Divisor, as the required Decimals and Places of Integers in the Quote together make*; (126) *but it will be necessary to keep one Place more in the Dividend than in the Divisor, if the Left-hand Figure of the Dividend be less than that of the Divisor* (54); *then divide the Dividend, and the Residues continually unaugmented by so many Figures of the Divisor to the Left, as are found to be contained in each Residue, always writing a Cypher before the Quotient Figure for every Place above one, that the Divisor is shortened for the Division of any particular Residue.* For these Cyphers, if the Divisor is shortened of more Places than one, must be inserted in order to keep the true Values, and Places, of the significant Figures of the Quote (18); also, by lessening the Divisor and not increasing the Residues, both Dividend and Divisor are diminished in the same Proportion; therefore the Quote will be true, except perhaps in the lowest Place or two to the Right-hand.

140. *In every Division, whether by Figures or Species, the Product of the Quote into the Divisor is equal to the Dividend*; that is, the Dividend is equal to the Sum of all the Products of the whole Divisor into all the Parts of the Quote; but the Parts of the Quote are an Integer and a Fraction, whose Numerator is the Residue and Denominator the Divisor; therefore the Product of the Divisor into the integral and Fractional Parts of the Quote, is equal to the Dividend.

XXXVI. (*b*) 141. Because that which was compounded by Multiplication is resolved by Division, the Product in Multiplication being the Dividend in Division; and because in Multiplication, similar Signs gave an affirmative Product, and dissimilar Signs a negative one; therefore in Division, *similar Signs in the Dividend and* divided by $\frac{2}{5}$ gives $\frac{3}{7}$. $\frac{ac}{bd}$ divided by $\frac{a}{b}$ gives $\frac{c}{d}$. $\frac{-21accy^3}{8b^5}$ divided by $\frac{3acy}{2bb}$ gives $\frac{-7cyy}{4b^3}$. $\frac{6}{5}$ divided by 3 gives $\frac{2}{5}$; and reciprocally $\frac{6}{5}$ divided by $\frac{2}{5}$ gives $\frac{3}{1}$, or 3. $\frac{30a^3z}{cc}$ divided by $2a$ gives $\frac{15aaz}{cc}$; and reciprocally divided by $\frac{15aaz}{cc}$ gives $2a$. Likewise $\sqrt{15}$ divided by $\sqrt{3}$ gives $\sqrt{5}$. $\sqrt{abcd}$ divided by $\sqrt{cd}$ gives $\sqrt{ab}$ (c). $\sqrt{a^3c}$ by $\sqrt{ac}$ gives $\sqrt{aa}$, or a. $\sqrt[3]{35aay^3z}$ divided by $\sqrt[3]{5ayy}$ gives $\sqrt[3]{7ayz}$. $\sqrt{\frac{a^3bb}{cc}}$ divided by $\frac{a^2}{c}$ gives $\sqrt{\frac{abb}{c}}$. $\frac{12ddx\sqrt{5abcx}}{70aee}$ divided by $\frac{-3dd\sqrt{5cx}}{10ee}$ gives $\frac{-4x\sqrt{ab}}{7a}$. And so $\overline{a+b}$

and Divisor, will make the Quote affirmative; but dissimilar ones, negative; that is, $+a) +ab (+b$, $+a) -ab (-b$, $-a) +ab (-b$, $-a) -ab (+b$.

(c) *Powers and Radicals must be brought to the same Name before they can be divided*, otherwise the Quote could have no certain Index.

142. *The Quote of Powers and of Radicals, of the same Denomination but of different Quantities, has the same Index with the Dividend and Divisor*; so that the Quote of such Powers is the same Power of the Quote of their Roots; and the Quote of such Roots is the same Root of their Quote. Thus $a^m) b^m (\overline{\frac{b}{a}}|^m$; for $a^m) b^m = \frac{b^m}{a^m} = \overline{\frac{b}{a}}|^m$. For the Division of Powers and Radicals of the same Quantity, see Numbers 81 and 82; and for the Division by a compound Surd, see Number 159.

(d) For

$\overline{a+b}\sqrt{ax}$ divided by $a+b$ gives $\sqrt{ax}$; and reciprocally divided by $\sqrt{ax}$ gives $a+b$. And $\frac{a}{a+b}\sqrt{ax}$ divided by $\frac{1}{a+b}$ gives $a\sqrt{ax}$, or divided by a gives $\frac{1}{a+b}\sqrt{ax}$, or $\frac{\sqrt{ax}}{a+b}$; and reciprocally divided by $\frac{\sqrt{ax}}{a+b}$ gives a. *But in Divisions of this Sort you are to take care, that the Quantities divided by one another be of the same Kind,* viz. that Numbers be divided by Numbers, and Species by Species, Radical Quantities by Radical Quantities, Numerators of Fractions by Numerators, and Denominators by Denominators; also in Numerators, Denominators, and Radical Quantities, the Quantities of each Kind must be divided by homogeneous ones, or Quantities of the same Kind (*d*).

XXXVII. *Now if the Quantity to be divided cannot be thus resolved by the Divisor proposed, it is sufficient, when both the Quantities are Integers, to write the Divisor underneath, with a Line between them* (*e*). Thus to divide ab by

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(*d*) For the Dividend, is an Aggregate of the Divisor alone (21).

XXXVII. (*e*) 143. *This Fraction truly expresses the Quote.* For the Numerator being some of the equal Parts into which an Unit is divided, and the Denominator, all those equal Parts, or the Unit itself, (7) the Fraction is to Unity, as the Numerator to the Denominator; but the Dividend is to the Divisor, as the Quote is to Unity (21); wherefore the Fraction and the Quote have the same Ratio to Unity, and are equal (Eucl. V. 9.)

Hence we have the Origin of the Notation of Vulgar Fractions, in which the Numerator, or Number above the Line, is placed there to shew, that it is a Dividend less than the Divisor, or Denominator placed below; so that by c, write $\frac{ab}{c}$; and to divide $\overline{a+b}\sqrt{cx}$ by a, write $\frac{\overline{a+b}\sqrt{cx}}{a}$, or $\frac{a+b}{a}\sqrt{cx}$. And so $\sqrt{ax-xx}$ divided by $\sqrt{cx}$ gives $\frac{\sqrt{ax-xx}}{\sqrt{cx}}$, or $\sqrt{\frac{ax-cx}{cx}}$. And $\overline{aa+ab}\sqrt{aa-2xx}$ divided by $\overline{a-b}\sqrt{aa-xx}$ gives $\frac{aa+ab}{a-b}\sqrt{\frac{ax-2xx}{aa-xx}}$. And $12\sqrt{5}$ divided by $4\sqrt{7}$ gives $3\sqrt{\frac{5}{7}}$.

XXXVIII.

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that a Fraction is called an improper one, if the Numerator is equal to, or greater than the Denominator; because it ought to express some only of the equal Parts, whose whole Number is subscribed; and if the Division was made, the Quote would not be fractional, but integral, or mixed.

144. *Now, because the Ratio of the Numerator to the Denominator will continue unvaried, if they are both multiplied or both divided by the same Number, it follows, that the same Fraction may be expressed an infinite Number of Ways*, by equal Multiplication; but the most commodious Form is that which consists of the lowest Terms, and it is found by an equal Division of them both; hence also it appears, that *the Terms of equal Fractions are proportional* (Eucl. VII. 18.); *and conversely, that if the Terms are proportional, the Fractions are equal.*

145. *As a Fraction is to Unit, so is Unit to the reciprocal Fraction.* For the Fraction is to Unit, as its Numerator to its Denominator; but Unit is to the reciprocal Fraction, as the Numerator to the Denominator; whence, as the Fraction is to Unit, so is Unit to the reciprocal Fraction. Hence, *a Multiplier may be found, when a Divisor is given; and also a Divisor, when a Multiplier is given:* By which Method a Variety of useful Rules are derived, for Ease and Expedition, in common and mercantile Accompts.

XXXVIII.

XXXVIII. *But when these Quantities are Fractions, multiply the Numerator of the Dividend into the Denominator of the Divisor, and the Denominator into the Numerator, and the first Product will be the Numerator, and the latter the Denominator of the Quotient* (*f*). Thus to divide $\frac{a}{b}$ by $\frac{c}{d}$ write

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XXXVIII. (*f*) 146. As the Divisor is to Unity, so is the Dividend to the Quote (21); but as a Fraction is to Unity, so is its Numerator to its Denominator; therefore, as the Numerator of the dividing Fraction is to its Denominator, so is the Dividend, whether Integer or Fraction, to the Quote; and therefore, *in all Cases of Division by a Fraction, the Dividend is to be multiplied by the Denominator, and divided by the Numerator of the dividing Fraction*, as in Number 126.

147. Hence, *a Fraction is divided by an Integer* (which is a Fraction whose Denominator is Unity) *either by dividing the Numerator, or by multiplying the Denominator of the Fraction by the Integer.* For it is the same thing to diminish any given Number of Parts in any given Ratio, the Magnitude of the Parts being unchanged; or, to diminish the Magnitude of the Parts in the same Ratio, their Number being unchanged; but their Magnitude is always diminished in any Ratio, by increasing the Divisor or Denominator in that Ratio (60); therefore, by multiplying the Integer into the Denominator, that Magnitude of the Parts is diminished in the Ratio of the Integer to Unity, their Number being unchanged: And by dividing the Numerator by it, the Number of the Parts is diminished in the same Ratio, their Magnitude being unchanged.

148. *The Quote of two Fractions is found* [there being always a Division by the Numerator of the Divisor, and a Multiplication by the Denominator of the Divisor (146, 147).] *First by dividing the Numerator of the Dividend by the Numerator of the Divisor, and the Denominator of the Dividend by the Denominator of the Divisor.* For the Dividend is multiplied by the Denominator of the Divisor,

write $\frac{ad}{bc}$, that is, multiply *a* by *d* and *b* by *c*. In like manner,

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in having its own Denominator divided by it; (146, 147) and the Dividend is directly divided by the Numerator.

149. *Secondly, by multiplying the Numerator of the Dividend by the Numerator, and the Denominator of the Dividend by the Denominator of the Reciprocal of the Divisor* (145). For thus the Dividend is multiplied directly by the Denominator of the Divisor, and it is divided by the Numerator, because its own Denominator is multiplied by it (147).

Now it is evident, that this Method of Division is equivalent to a Division of the Numerators (after the Fractions have been reduced to one common Denomination) and expunging the common Denomination, or rather an Abbreviation of this Operation; (LIX.) and that the *Rule* commonly given *for dividing Fractions which have the same Denominator, directing the Division to be made by the Numerators, and to expunge the common Denominator*, is a further Abbreviation of this, and saves the Trouble of reducing the Quote to lower Terms, by dividing its Terms by this common Denominator (144). In like Manner, the Rule for Multiplication, Number 65, is an Abbreviation of a Multiplication by the Numerators, (after the Fractions have been reduced to a common Denominator) and subscribing the common Denominator; and the Direction to subscribe the common Denominator, when the given Factors have one, to the Product of their Numerators, is a further Abbreviation.

150. *Thirdly, by multiplying the Quote of the Denominators, by the Numerator of the Divisor*: For thus the Dividend is multiplied by the Denominator of the Divisor, because its own Denominator is divided by it (147); and it is divided by the Numerator of the Divisor, because its own Denominator, in the Quote of the Denominators, is multiplied by it.

151. *Lastly, by multiplying the Quote of the Numerators, by the Denominator of the Divisor*; For thus the Dividend is

manner, $\frac{3}{7}$ by $\frac{5}{4}$ gives $\frac{12}{35}$. And $\frac{3a}{4c}\sqrt{ax}$ divided by $\frac{2c}{5a}$ gives $\frac{15aa}{8cc}\sqrt{ax}$, and divided by $\frac{2c\sqrt{aa-xx}}{5a\sqrt{ax}}$ gives $\frac{15a^3x}{8cc\sqrt{aa-xx}}$. After the same manner, $\frac{ad}{b}$ divided by c $\left(\text{or by } \frac{c}{1}\right)$ gives $\frac{ad}{bc}$. And c $\left(\text{or } \frac{c}{1}\right)$ divided by $\frac{ad}{b}$ gives $\frac{bc}{ad}$. And $\frac{3}{7}$ divided by 5 gives $\frac{3}{35}$. And 3 divided by $\frac{5}{4}$ gives $\frac{12}{5}$. And $\frac{a+b}{c}\sqrt{cx}$ divided by a gives $\frac{a+b}{ac}\sqrt{cx}$. And $\overline{a+b}\sqrt{cx}$ divided by $\frac{a}{c}$ gives $\frac{ac+bc}{a}\sqrt{cx}$. And $2\sqrt{\frac{axx}{c}}$ divided by $3\sqrt{cd}$ gives $\frac{2}{3}\sqrt{\frac{axx}{cca}}$; and divided by $3\sqrt{\frac{cd}{x}}$ gives $\frac{2}{3}\sqrt{\frac{ax^3}{ccd}}$. And $\frac{1}{5}\sqrt{\frac{7}{11}}$ divided by $\frac{1}{2}\sqrt{\frac{3}{7}}$ gives $\frac{2}{5}\sqrt{\frac{49}{33}}$, and so in others.

XXXIX. *A Quantity compounded of several Terms, is divided by dividing each of its Terms by the Divisor.* Thus $aa+3ax-xx$ divided by a gives $a+3x-\frac{xx}{a}$. *But when the Divisor consists also of several Terms, the Division is*

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is multiplied by the Denominator of the Divisor, because the Quote of the Numerators is multiplied by it; and it is directly divided by the Numerator of the Divisor. The second Method is most useful, being free from compound Fractions.

is performed as in Numbers. Thus to divide $a^3 + 2aac - aab - 3abc + bbc$ by $a - b$, say, how many Times is a contained in a^3, viz. the first Term of the Divisor in the first Term of the Dividend? Answer aa. Wherefore write aa in the Quotient; and having subtracted $a - b$ multiplied into aa, or $a^3 - aab$ from the Dividend, there will remain $2aac - 3abc + bbc$ yet to be divided. Then say again, how many Times a in $2aac$? Answer $2ac$. Wherefore write also $2ac$ in the Quotient, and having subtracted $a - b$ into $2ac$, or $2aac - 2abc$ from the aforesaid Remainder, there will yet remain $- abbc + bc$. Wherefore say again, how many Times a in $- abc$? Answer $- bc$, and then write $- bc$ in the Quotient; and having, in the last Place, subtracted $+ a - b$ into $- bc$, viz. $- abc + bbc$ from the last Remainder, there will remain nothing; which shews that the Division is at an end, and the Quotient coming out $aa + 2ac - bc$.

XL. *But that these Operations may be duly reduced to the Form which we use in the Division of Numbers, the Terms both of the Dividend and the Divisor must be disposed in Order, according to the Dimensions of that Letter which is judged most proper for the Operation*; so that those Terms may stand first, in which that Letter is of most Dimensions, and those in the second Place whose Dimensions are next highest; and so on to those wherein that Letter is not at all involved, or into which it is not at all multiplied, which ought to stand in the last Place. Thus in the Example we just now brought, if the Terms are disposed according to the Dimensions of the Letter a, the following Diagram will shew the Form of the Work, viz.

$$
\begin{array}{l}
a - b)\, a^3 \begin{array}{l} + 2aac \\ - aab \end{array} - 3abc + bbc \;(aa + 2ac - bc \\
\quad\quad\; \underline{a^3 - aab} \\
\quad\quad\; 0 + 2aac - 3abc \\
\quad\quad\quad\quad\; \underline{2aac - 2abc} \\
\quad\quad\quad\quad\quad\quad 0 - abc + bbc \\
\quad\quad\quad\quad\quad\quad\;\; \underline{- abc + bbc} \\
\quad\quad\quad\quad\quad\quad\quad\quad 0 \quad\; 0
\end{array}
$$

Where

Where may be seen, that the Term a^3, or a of three Dimensions, stands in the first Place of the Dividend, and the Terms $\begin{array}{l} 2aac \\ -aab \end{array}$, in which a is of two Dimensions, stand in the second Place, and so on. The Dividend might also have been writ thus;

$$a^3 \begin{array}{l} +2c \\ -b \end{array} aa - 3bca + bbc.$$

Where the Terms that stand in the second Place are united, by collecting together the Factors of the Letter, according to which the Order is made (*g*). And thus if the Terms were to be disposed according to the Dimensions of the Letter *b*, the Business must be performed as in the following Diagram, the Explication whereof we shall here subjoin.

$$\begin{array}{l}
-b+a)\ cbb \begin{array}{l} -3ac \\ -\ \ aa \end{array} b \begin{array}{l} +a^3 \\ +2aac \end{array} \left(-cb \begin{array}{l} +2ac \\ +\ \ aa \end{array}\right. \\
\qquad\quad \underline{cbb - acb} \\
\qquad\quad 0 \begin{array}{l} -2ac \\ -\ \ aa \end{array} b \begin{array}{l} +a^3 \\ +2aac \end{array} \\
\qquad\quad\ \ \begin{array}{l} -2ac \\ -\ \ aa \end{array} b \begin{array}{l} +2aac \\ +a^3 \end{array} \\
\qquad\qquad\quad 0 \qquad\ 0
\end{array}$$

Say,

XL. (*g*) By ranging both Dividend and Divisor according to the Dimensions of the same Letter, it will readily be found, how often the first Term of the Divisor is contained in the first Term of the Dividend; also, the Products of the Terms of the Divisor into each particular Term of the Quote, as they are found, will come readily under the similar Terms of the Dividend, in order to be subducted from them.

The Number of Terms in the Quote of a compound algebraic Quantity, divided by another, is the Number of Terms of the Dividend divided by the Number of Terms in the Divisor, provided that none of the Terms have been united, or destroyed by contrary Signs; for the Number of Terms in the Dividend, is the Product of the Numbers in the Divisor and Quote.

Say, How many Times is $-b$ contained in cbb? Answer $-cb$. Wherefore having writ $-cb$ in the Quotient, subtract $\overline{-b+a}\times -cb$, or $bbc-abc$, and there will remain in the second Place $\begin{matrix}-2ac\\-aa\end{matrix}b$. To this Remainder add, if you please, the Quantities that stand in the last Place, viz. $\begin{matrix}a^3\\2aac\end{matrix}$, and say again, how many Times is $-b$ contained in $\begin{matrix}-2ac\\-aa\end{matrix}b$? Answer $\begin{matrix}+2ac\\+aa\end{matrix}$. These therefore being writ in the Quotient, subtract $-b+a$ multiplied by $\begin{matrix}+2ac\\+aa\end{matrix}$ or $\begin{matrix}-2ac\\-aa\end{matrix}b$ $\begin{matrix}+2aac\\+a^3\end{matrix}$, and there will remain nothing. Whence it is manifest, that the Division is at an End, the Quotient coming out $-cb+2ac+aa$, as before.

And thus, if you were to divide $aay^4-aac^4+yyc^4+y^6-2y^4cc-a^6-2a^4cc-a^4yy$ by $yy-aa-cc$. I order or place the Quantities according to the Dimensions of the Letter y, thus:

$$yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc.\\-aac^4\end{matrix}$$

Then I divide as in the following Diagram. Here are added other Examples, in which you are to take Notice, that where the Dimensions of the Letter, which this Method of ordering ranges, does not always proceed in the same Arithmetical Progression, but sometimes interrupted, in the defective Places this Mark $*$ is put.

$$yy\begin{matrix}-aa\\-cc\end{matrix}\Big)\,y^6\begin{matrix}+aa\\-2cc\end{matrix}y^4\begin{matrix}-a^4\\+c^4\end{matrix}yy\begin{matrix}-a^6\\-2a^4cc.\\-aac^4\end{matrix}\Big(\,y^4\begin{matrix}+2aa\\-cc\end{matrix}y^2\begin{matrix}+a^4\\+aacc\end{matrix}$$

$$\overline{y^6\begin{matrix}-aa\\-cc\end{matrix}y^4.}$$

$-2a^4$

$$0 \begin{matrix}\pm 2aa\\ -cc\end{matrix}\, y^4 \begin{matrix}-2a^4\\ -aacc\\ +c^4\end{matrix}\, y^2$$

$$\begin{array}{ll} 0 & \begin{matrix}+a^4\\ +aacc\end{matrix}\, y^2 \\ & \begin{matrix}+a^4\\ +aacc\end{matrix}\, y^2 \begin{matrix}-a^6\\ -2a^4cc\\ -aac^4\end{matrix} \\ \hline & \quad 0 \qquad\qquad 0 \end{array}$$

$$\begin{array}{l} a+b)\ aa * -bb\ (a-b \\ \quad aa+ab \\ \hline \quad 0-ab \\ \quad -ab-bb \\ \hline \quad\ 0 \qquad 0\ (b). \end{array}$$

yy — 2ay

(b) 152. Becaufe Divifion refolves what Multiplication has compounded, therefore *if any Quantity is divifible by another, whether fimple, or binome, or trinome, &c. that Divifor was a Factor, or Root, in its Compofition.*

153. *If a Quantity, having all its Terms rational, is divifible by a Binome, one of whofe Members is irrational, it fhall be divifible alfo by the fame Binome having its irrational Member affected with the contrary Sign; and according as that Member is real or imaginary, fo muft the Member of the contrary Binome be* (120).

154. *If the higheft Term of a Dividend has no Coefficient but Unity, and no fractional Term, none of its binome Divifors can have Coefficients different from Unity, or their fecond Members fractional; but if the higheft Term has a Coefficient different from Unity, either one, or all fuch Divifors have Coefficients differing from Unity; or one, or all of them, have their fecond Members fractional* (121).

155. *If algebraic Divifion is not terminated by a Fraction, as in Art.* XXXVII. *it will run into an infinite Series; and by obferving the firft three or four Terms, the Law, which the Terms obferve, will be known; by which Means, without any more Divifion, the Quote or Series may be continued on; and*

thefe

$$yy - 2ay + aa)\ y^4 * - 3\tfrac{1}{2}aayy + 3a^3y - \tfrac{1}{2}a^4\ (yy + 2ay - \tfrac{1}{2}aa$$
$$\underline{y^4 - 2ay^3 + aayy}$$
$$0 + 2ay^3 - 4\tfrac{1}{2}aayy$$
$$\underline{+ 2ay^3 - 4\ aayy + 2a^3y}$$
$$0 - \tfrac{1}{2}aayy + a^3y$$
$$\underline{- \tfrac{1}{2}aayy + a^3y - \tfrac{1}{2}a^4}$$
$$0 \quad 0 \quad 0$$

---

these Series being, in general Symbols or Species, are universal Rules for expressing all Quantities by infinite Series, an infinite Number of Ways. Thus,

A. $x-1)\ x^4\ (x^3 + x^2 + x + 1 + x^{-1} + x^{-2} + x^{-3}$, &c. $= \frac{x^4}{x-1}$ the Series descends, or ascends, according as x is greater or less than Unity; the common Divisor or Multiplier being x.

$$\underline{x^4 - x^3}$$
$$\underline{x^3 - x}$$
$$\underline{x^2 - x}$$
$$x, \&c.$$

B. $x+1)\ x^4\ (x^3 - x^2 + x - 1 + x^{-1} - x^{-2}$, &c. the same the Terms being alternately affirmative and negative.

$$\underline{x^4 + x^3}$$
$$\underline{- x^3 - x^2}$$
$$\underline{+ x^2 + x}$$
$$- x, \&c.$$

C. $a-b)\ x\ (\frac{xb^0}{a} + \frac{xb}{a^2} + \frac{xb^2}{a^3} + \frac{xb^3}{a^4}$, &c. $= xa^{-1} + xba^{-2} + xb^2a^{-3} + xb^3a^{-4}$ (76), &c.

$$\underline{x - \frac{xb}{a}}$$
$$\frac{xb}{a} - \frac{xb^2}{a^2}$$
$$\underline{\frac{xb}{a} - \frac{xb^2}{a^2}}$$
$$\underline{\frac{xb^2}{a^2}}$$
$$\frac{xb^2}{a^2}$$

&c. the Terms all positive, the common Multiplier $\frac{b}{a}$.

D. $a + b)$

$$aa + ab\sqrt{2} + bb)\quad a^4 \quad * \quad * \quad * \quad + b^4 \quad (aa - ab\sqrt{2} + bb$$

$$\begin{array}{l} a^4 + a^3b\sqrt{2} + aabb \\ \hline -\,a^3b\sqrt{2} - aabb \\ -\,a^3b\sqrt{2} - 2aabb - ab^3\sqrt{2} \\ \hline +\,aabb + ab^3\sqrt{2} \\ +\,aabb + ab^3\sqrt{2} + b^4 \\ \hline 0 \quad 0 \quad 0 \end{array}$$

Some

D. $a+b)\; x\,(\frac{xb^0}{a} - \frac{xb}{a^2} + \frac{xb^2}{a^3} - \frac{xb^3}{a^4}$, &c. $= xa^{-1}$ $- xba^{-2} + xb^2a^{-3} - xb^3a^{-4}$ the same, the Terms alternately affirmative and negative.

156. *If in* D *we put the Numerator of any Fraction for* x, *and its Denominator less Unity for* a, *and Unity for* b, *we have all vulgar Fractions, except* $\frac{1}{2}$, *expressed by an infinite Series.* For $\frac{1}{2} = \frac{1}{1+1} = 1 + 1 + 1$, &c.

Thus, $\frac{1}{3} = \frac{1}{2+1} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$, &c.
$= \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \frac{1}{32} - \frac{1}{64}$, &c.

Thus, $\frac{2}{3} = \frac{2}{2+1} = \frac{2}{2} - \frac{2}{4} + \frac{2}{8} - \frac{2}{16} + \frac{2}{32} - \frac{2}{64}$, &c.
$= 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32}$.

Thus, $\frac{3}{4} = \frac{3}{3+1} = \frac{3}{3} - \frac{3}{9} + \frac{3}{27} - \frac{3}{81} + \frac{3}{243} - \frac{3}{729}$, &c.
$= 1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \frac{1}{81} - \frac{1}{243}$.

157. *If in* C *we substitute for* x *the Numerator of a vulgar Fraction, and for* a *the Denominator more Unity, and*

for

Some begin Division from the last Terms, but it comes to the same Thing, if, inverting the Order of the Terms, you

---

for b *Unity, we have another Law for expressing all vulgar Fractions.*

Thus, $\frac{1}{2} = \frac{1}{3-1} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81}$, &c. therefore $\frac{2}{2} = 1 = \frac{2}{3} + \frac{2}{9} + \frac{2}{27} + \frac{2}{81}$, &c.

Thus, $\frac{2}{3} = \frac{2}{4-1} = \frac{2}{4} + \frac{2}{16} + \frac{2}{64} + \frac{2}{256}$, &c. therefore $\frac{3}{3} = 1 = \frac{3}{4} + \frac{3}{16} + \frac{3}{64} + \frac{3}{256}$, &c.

Thus, $\frac{3}{4} = \frac{3}{5-1} = \frac{3}{5} + \frac{3}{25} + \frac{3}{125} + \frac{3}{625}$, &c. therefore $\frac{4}{4} = 1 = \frac{4}{5} + \frac{4}{25} + \frac{4}{125} + \frac{4}{625}$, &c.

158. Whence it appears, that if x be put universally for any Integer, then

$$\frac{x-1}{x} + \frac{x-1}{x^2} + \frac{x-1}{x^3} + \frac{x-1}{x^4}, \text{ \&c.} = 1$$

$$\frac{2x-2}{x} + \frac{2x-2}{x^2} + \frac{2x-2}{x^3} + \frac{2x-2}{x^4}, \text{ \&c.} = 2$$

$$\frac{3x-3}{x} + \frac{3x-3}{x^2} + \frac{3x-3}{x^3} + \frac{3x-3}{x^4}, \text{ \&c.} = 3, \text{ \&c.}$$

which gives *an universal Rule for expressing all Integers by infinite Series, an infinite Number of Ways.*

But all these Series are at once derived from the binomial Theorem: For all Fractions may be considered as compounded of Radicals, multiplied either into rational or surd Quantities; thus $\frac{x}{a+b} = x \times \overline{a+b}|^{-1}$: and $\frac{x}{\overline{a-b}|^{\frac{1}{2}}} = x \times \overline{a-b}|^{-\frac{1}{2}}$; wherefore having a general Ex-you begin from the firft. There are alfo other Methods of

---

Expreffion in Number 107, to which particular Cafes may be compared, we may thence derive the particular Series refulting from Divifion. Thus (fee C and D in 155) $\frac{x}{a \pm b} = x \times \overline{a \pm b}^{-1} = A \pm \frac{n}{1} QA + \frac{n-1}{2}$ QB, &c. $= x - \frac{m}{n} \mp ax - \frac{m-n}{2n}$, &c. (103) $= xa^{-1} \mp xba^{-2} + xb^2a^{-3}$, &c. (155).

159. *When a binome Surd is to be divided by another, the Quote will be expreffed in the moft fimple Form, by multiplying both Numerator and Denominator, by that Surd which multiplied into the Denominator gives a rational Product* (118). Thus, $\frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}} = \frac{\sqrt{20} + \sqrt{12}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} = \frac{\sqrt{100} + 2\sqrt{60} + 6}{5 - 3} = \frac{16 + 2\sqrt{60}}{2} = 8 + 2\sqrt{15}$.

And in general, when any Quantity is divided by a binome Surd as $x^n \pm a^l$, *where* n *and* l *reprefent any Fractions whatfoever, take* m *the leaft integer Number, which is meafured by* n *and* $\frac{n}{l}$, *multiply both Numerator and Denominator by* $x^{m-n} + x^{m-2n} a^l + x^{m-3n} a^{2l}$, *&c. and the Denominator of the Product will become rational* (119); *and equal to* $x^m - a^{\frac{ml}{n}}$; *then divide all the Members of the Numerator, by this rational Quantity, and the Quote arifing will be that of the propofed Quantity divided by the binome Surd, in its leaft Terms.*

Thus,

of dividing, but it is sufficient to know the most easy and commodious.

Of EXTRACTION *of* ROOTS.

XLI. *WHEN the Square Root of any Number is to be extracted, it is first to be noted with Points in every other Place, beginning from Unity; then you are to write down such a Figure for the Quotient, or Root, whose Square shall be equal to, or nearest, less than the Figure or Figures to the first Point. And then subtracting that Square, the other Figures of the Root will be found one by one, by dividing the Remainder by the double of the Root as far as extracted, and each Time taking from that Remainder the Square of the Figure that last came out, and the Decuple of the aforesaid Divisor augmented by that Figure.*

Thus to extract the Root out of 99856, first point it after this Manner, $\dot{9}9\dot{8}5\dot{6}$; then seek a Number whose Square shall equal the first Figure 9, viz. 3, and write it in the Quotient; and then having subtracted from 9, 3×3, or 9, there will remain 0; to which set down the Figures to the next Point, viz. 98 for the following Operation. Then, taking no Notice of the last Figure 8, say, How many times is the Double of 3, or 6, contained in the first Figure 9? Answer 1; wherefore having writ 1 in the Quotient, subtract the Product of 1×61, or 61, from 98, and there will remain 37, to which connect

```
99856 (316
9
-----
098
 61
-----
 3756
 3756
-----
    0
```

Thus, $\dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} = \dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} \times \dfrac{\sqrt[3]{16}+2+\sqrt[3]{4}}{\sqrt[3]{16}+2+\sqrt[3]{4}} =$

$\dfrac{\sqrt[3]{20}}{\sqrt[3]{4}-\sqrt[3]{2}} \times \dfrac{2\sqrt[3]{2}+2+\sqrt[3]{4}}{2\sqrt[3]{2}+2+\sqrt[3]{4}} = \dfrac{2\sqrt[3]{40}+2\sqrt[3]{20}+\sqrt[3]{80}}{2}$

$= 2\sqrt[3]{5}+\sqrt[3]{20}+\sqrt[3]{10}.$

connect the last Figures 56, and you will have the Number 3756, in which the Work is next to be carried on. Wherefore also neglecting the last Figure of this, viz. 6, say, How many times is the Double of 31, or 62, contained in 375, (which is to be guessed at from the initial Figures 6 and 37, by taking Notice how many times 6 is contained in 37?) Answer 6; and writing 6 in the Quotient, subtract 6 x 626, or 3756, and there will remain 0; whence it appears that the Business is done; the Root coming out 316.

And so if you were to extract the Root out of 22178791, first having pointed it, seek a Number, whose Square (if it cannot be exactly equalled) shall be the next less Square to 22, the Figures to the first Point, and you will find it to be 4. For 5 x 5, or 25, is greater than 22; and 4 X 4, or 16, less; wherefore 4 will be the first Figure of the Root. This, therefore, being writ in the Quotient, from 22 take the Square 4 X 4, or 16, and to the Remainder 6 adjoin moreover the next Figures 17, and you will have 617, from whose Division by the Double of 4 you are to obtain the second Figure of the Root, viz. neglecting the last Figure 7, say, how many times is 8 contained in 61? Answer 7; wherefore write 7 in the Quotient, and from 617 take the Product of 7 into 87, or 609, and there will remain 8; to which join the two next Figures 87, and you will have 887, by the Division whereof by the Double of 47, or 94, you are to obtain the third Figure; as say, How many times is 94 contained in 88? Answer 0; wherefore write 0 in the

22178791 (4709,43637, &c.
16
617
609
88791
84681
411000
376736
3426400
2825649
60075100
56513196
356190400
282566169
73624231

Quotient, and adjoin the two last Figures 91, and you will have 88791, by whose Division by the Double of 470, or 940, you are to obtain the last Figure, viz. say, How many times 940 in 8879? Answer 9; wherefore write 9 in the Quotient, and you will have the Root 4709.

But since the Product 9 × 9409 or 84681 subtracted from 88791, leaves 4110, that is a Sign *that* the *Number* 4709 *is not the Root* of the *Number* 22178791 *precisely*, but that it is a little less. And *in this Case, and in others like it, if you desire the Root should approach nearer, you must carry on the Operation in Decimals, by adding to the Remainder two Cyphers in each Operation.* Thus the Remainder 4110 having two Cyphers added to it, becomes 411000; by the Division whereof by the Double of 4709, or 9418, you will have the first Decimal Figure 4. Then having writ 4 in the Quotient, subtract 4 × 94184, or 376736 from 411000, and there will remain 34264. And so having added two more Cyphers, the Work may be carried on at Pleasure, the Root at length coming out 4709,43637, &c.

XLII. *But when the Root is carried on half-way, or above, the rest of the Figures may be obtained by Division alone.* As in this Example, if you had a mind to extract the Root to nine Figures, after the five former 4709,4 are extracted, the four latter may be had, by dividing the Remainder by the Double of 4709,4 (*a*).

And

XLII. (*a*) The Division by the Double of the Root found to Half the Number of Places, differs from the whole Operation of Extraction in this only, that the Square of each Figure added to the Root is not now subducted from the Resolvend; now, as in both Operations, the additional Figure is always determined by the first Figures to the Left of the Resolvend and of the doubled Root, the same Figures will be determined both ways for so many Places as there were Figures rightly determined before, but no further; for after just so many subsequent Divisions, those Figures of the Resolvend,

from

And after this Manner, if the Root of 32976 was to be extracted to five Places in Numbers: After the Figures are pointed, write 1 in the Quotient, as being the Figure whose Square 1 × 1, or 1, is the greatest that is contained in 3 the Figure to the first Point; and having taken the Square of 1 from 3, there will remain 2; then having set the two next Figures, viz. 29 to it, (viz. to 2) seek how many times the Double of 1, or 2, is contained in 22, and you will find indeed that it is contained more than ten times; but you are never to take your Divisor ten times, no, nor nine times in this Case; because the Product of 9 × 29, or 261, is greater than 229, from which it would be to be taken. Wherefore say only 8: And then having writ 8 in the Quotient, and subtracted 8 × 28, or 224, there will remain 5; and having set down to this the Figures 76, seek how many times the Double of 18, or 36, is contained in 57, and you will find 1, and so write 1 in the Quotient; and having subtracted 1 × 361, or 361 from 576, there will remain 215. Lastly, to obtain the remaining Figures, divide this Number 215 by the Double of 181, or 362, and you will have the Figures 59, which being writ in the Quotient, you will have the Root 181,59.

```
 . . .
 32976 (181,59
 -
 1
 ---
 2) 229
    224
 -------
 36) 576
     361
 -------
 362) 215 (59, &c.
```

After the same way Roots are also extracted out of Decimal Numbers. Thus the Root of 329,76 is 18,159; and the Root of 3,2976 is 1,8152; and the Root of 0,032976 is 0,18159, and so on. But the Root of 3297,6 is 57,4247; and the Root of 32,976 is 5,74247.

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from which the Squares of the added Figures were not subducted, will become the first toward the Left-hand of the Resolvend, and being too great will (when divided by the doubled Root) determine the succeeding Figures of the Root too great.

Power, &c. and then such a Figure is to be writ in the Quotient, whose greatest Power (i. e. whose Cube, if it be a cubic Power, or whose Quadrato-Cube, if it be the fifth Power, &c.) shall either be equal to the Figure or Figures before the first Point, or the next less; and then having subtracted that Power, the next Figure will be found by dividing the Remainder augmented by the next Figure of the Resolvend, by the next greatest Power of the Quotient, multiplied by the Index of the Power, to be extracted; that is, by the triple Square of the Quotient, if the Root be a cube one; or by the quintuple Biquadrate, i. e. five times the Biquadrate, if the Root be of the fifth Power, &c. And having again subtracted the greatest Power of the whole Quotient from the first Resolvend, the third Figure will be found by dividing that Remainder angmented by the next Figure of the Resolvend, by the next greatest Power of the whole Quotient multiplied by the Index of the Power to be extracted; and so on, in infinitum.

Thus to extract the Cube Root of 13312053, the Number is first to be pointed after this Manner, viz. 13312053. Then you are to write in the Quotient the Figure 2, whose Cube 8 is the next less Cube to the Figures 13 (which is not a perfect Cube Number), or to the first Point; and having subtracted that Cube, there will remain 5; which being augmented by the next Figure of the Resolvend 3, and divided by the triple

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commensurable. But though they are incommensurable with Unity, *they are commensurable in Power with it*; because their Powers are Integers, i. e. Multiples of an Unit; they may also be sometimes commensurate one to another, when they have a common Measure, by which being divided they have rational Coefficients combined with the Root of that common Measure; for their Ratio is then effable by Numbers, to wit, by those Coefficients.

163. *The Difference between an irrational Root, and the next greater and next less rational Roots, of the same Name with it, is less than Unity:* For the rational Roots of the same Name, which are the next greater and less, are Laterals (161), and differ but by Unity.

triple Square of the Quotient 2, by seeking how many times 3 x 4, or 12, is contained in 53, it gives 4 for the second Figure of the Quotient. But since the Cube of the Quotient 24, viz. 13824 would come out too great to be subtracted from the Figures 13312 that precede the second Point, there must only 3 be writ in the Quotient. Then the Quotient 23 being in a separate Paper or Place multiplied by 23 gives the Square 529, which again multiplied by 23 gives the Cube 12167, and this taken from 13312, will leave 1145; which augmented by the next Figure of the Resolvend 0, and divided by the triple Square of the Quotient 23, viz. by seeking how many times 3 x 529, or 1587, is contained 11450, it gives 7 for the third Figure of the Quotient. Then the Quotient 237, multiplied by 237, gives the Square 56169, which again multiplied by 237 gives the Cube 13312053, and this taken from the Resolvend leaves 0. Whence it is evident, that the Root sought is 237.

```
                      13312053 (237
Subtract the Cube     8
                      —
         12) rem.     53 (4 or 3
Subtract Cube         12167
       1587) rem.     11450 (7
Subtract Cube         13312053
            Remains   0
```

And so to extract the Quadrato-Cubical Root of 36430820, it must be pointed over every fifth Figure, and the Figure 3, whose Quadrato-Cube (or fifth Power) 243 is the next less to 364, viz. to the first Point must be writ in the Quotient. Then the Quadrato-Cube 243 being subtracted from 364, there remains 121, which augmented by the next Figure of the Resolvend, viz. 3, and divided by five times the Biquadrate of the Quotient, viz. by seeking how many times 5 x 81, or 405, is contained in 1213, it gives 2 for the second Figure. That

```
          36430820 (32,5
          243
    405)  1213 (2
          33554432
5242880)  2876388,0 (5
```

Quotient 32 being thrice multiplied by itself, makes the Biquadrate 1048576; and this again multiplied by 32, makes the Quadrato-Cube 33554432, which being subtracted from the Resolvend leaves 2876388. Therefore 32 is the Integer Part of the Root, but not the exact Root; wherefore, if you have a mind to prosecute the Work in Decimals, the Remainder, augmented by a Cypher, must be divided by five times the aforesaid Biquadrate of the Quotient, by seeking how many times 5×1048576, or 5242880, is contained in 2876388,0, and there will come out the third Figure, or the first Decimal 5. And so by subtracting the Quadrato-Cube of the Quotient 32,5 from the Resolvend, and dividing the Remainder by five times its Biquadrate, the fourth Figure may be obtained. And so on *in infinitum*.

XLIV. *When the Biquadratic Root is to be extracted, you may extract twice the Square Root, because* $\sqrt{}^4$ *is as much as* $\sqrt{}^2 \times 2$. *And when the Cubo-Cubic Root is to be extracted, you may first extract the Cube Root, and then the Square Root of that Cube Root, because* $\sqrt{}^6$ *is the same as* $\sqrt{}^2 \times 3$; whence some have called these Roots not Cubo-Cubic ones, but Quadrato-Cubes, and the same is to be observed in other Roots, whose Indexes are not prime Numbers.

XLV. *The Extraction of Roots out of simple Algebraic Quantities, is evident, even from the Notation itself*; as that $\sqrt{aa}$ is a, and that $\sqrt{aacc}$ is ac, and that $\sqrt{9aacc}$ is $3ac$, and that $\sqrt{49a^4xx}$ is $7aax$. And also, that $\sqrt{\frac{a^4}{cc}}$, or $\frac{\sqrt{a^4}}{\sqrt{cc}}$, is $\frac{aa}{c}$; and that $\sqrt{\frac{a^4bb}{cc}}$, is $\frac{aab}{c}$; and that $\sqrt{\frac{9aazz}{25bb}}$, is $\frac{3az}{5b}$; and that $\sqrt{\frac{4}{9}}$ is $\frac{2}{3}$; and that $\sqrt{}^3 \frac{8b^6}{27a^3}$, is $\frac{2bb}{3a}$; and that $\sqrt{}^4 aabb$, is $\sqrt{ab}$. Moreover, that $b\sqrt{aacc}$, or b into $\sqrt{aacc}$, is b into ac, or abc. And that $3c\sqrt{\frac{9aazz}{25bb}}$, is $3c \times \frac{3az}{5b}$; or $\frac{9acz}{5b}$.

$\frac{9acz}{5b}$. And that $\frac{a+3x}{c}\sqrt{\frac{4bbx^4}{81aa}}$, is $\frac{a+3x}{c}\times\frac{2bxx}{9a}$; or $\frac{2abxx+6bx^3}{9ac}$. (*d*)

XLVI. I say, these are all evident, because it will appear, at first Sight, that the proposed Quantities are produced by multiplying the Roots into themselves (as aa from $a\times a$, $aacc$ from ac into ac, $9aacc$ from $3ac$ into $3ac$, &c.) But *when Quantities consist of several Terms, the Business is performed as in Numbers.* Thus, to extract the Square Root out of $aa+2ab+bb$, in the first Place, write the Root of the first Term aa, viz. a in the Quotient, and having subtracted its Square $a\times a$, there will remain $2ab+bb$ to find the Remainder of the Root by. Say, therefore, How many times is the Double of the Quotient, or $2a$, contained in the first Term of the Remainder $2ab$? I answer b; therefore write b in the Quotient, and having subtracted the Product of b into $\overline{2a+b}$, or $2ab+bb$, there will remain nothing. Which shews that the Work is finished, the Root coming out $a+b$ (*e*).

$$\begin{array}{l} aa+2ab+bb\,(a+b \\ aa \\ \overline{} \\ 0 \quad +2ab+bb \\ \quad\;\; \overline{+2ab+bb} \\ \quad\;\;\;\; 0 \quad\;\; 0 \end{array}$$

And

XLV. (*d*) The Roots of single Quantities are extracted by dividing their Indices by the Number denominating their Root (85). If the Index of the Root be a Divisor of the Index of the Power, the Root will be rational; otherwise, irrational (86). If the Index of the Root be even, the Root may be positive or negative, if the Power is positive (88). If the Power is negative, no Root with an even Index can be assigned (89). If the Index of the Root be odd, the Root will have the same Sign with the Power or given Quantity (88).

XLVI. (*e*) *The general Theorem, Number* 107, *for the Involution of Binomials, will serve also for their Evolution;* because

And thus, to extract the Root out of $a^4 + 6a^3b + 5a^2bb - 12ab^3 + 4b^4$, first, set in the Quotient the Root of the first Term a^4, viz. aa, and having subtracted its Square $aa \times aa$, or a^4, there will remain $6a^3b + 5aabb - 12ab^3 + 4b^4$ to find the Remainder of the Root. Say, therefore, How many times is $2aa$ contained in $6a^3b$? Answer $3ab$; wherefore write $3ab$ in the Quotient, and having subtracted the Product of $3ab$ into $2aa + 3ab$, or $6a^3b + 9aabb$, there will yet remain $-4aabb - 12ab^3 + 4b^4$ to carry on the Work. Therefore say again, How many times is the Double of the Quotient, viz. $2aa + 6ab$ contained in $-4aabb - 12ab^3$, or, which is the same Thing, say, How many times is the Double of the first Term of the Quotient, or $2aa$, contained in the first Term of the Remainder $-4aabb$? Answer $-2bb$. Then having writ $-2bb$ in the Quotient, and subtracted the Product $-2bb$ into $2aa + 6ab - 2bb$, or $-4aabb - 12ab^3 + 4b^4$, there will remain nothing. Whence it follows, that the Root is $aa + 3ab - 2bb$.

$$a^4 + 6a^3b + 5aabb - 12ab^3 + 4b^4 \; (aa + 3ab - 2bb$$
$$a^4$$
$$\overline{0.}$$
$$+6a^3b + 9aabb$$
$$\overline{0 \quad -4aabb}$$
$$-4aabb - 12ab^3 + 4b^4$$
$$\overline{0 \qquad 0 \qquad 0}$$

And thus the Root of the Quantity $xx - ax + \frac{1}{4}aa$ is $x - \frac{1}{2}a$; and the Root of the Quantity $y^4 + 4y^3 - 8y + 4$ is $yy + 2y - 2$; and the Root of the Quantity

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because to extract any Root of a given Quantity, is the same Thing as to raise that Quantity to a Power, whose Exponent is a Fraction which has its Denominator equal to the Number which expresses the Name of the Root to be extracted (78).

tity $16a^4 - 24aaxx + 9x^4 + 12bbxx - 16aabb + 4b^4$ is $3xx - 4aa + 2bb$, as may appear by the Diagrams underneath:

$$xx - ax + \tfrac{1}{4}aa \;(x - \tfrac{1}{2}a$$
$$\underline{xx}$$
$$0$$
$$\underline{-ax + \tfrac{1}{4}aa}$$
$$0 \quad 0$$

$$9x^4 \begin{matrix} -24aa \\ +12bb \end{matrix} x^2 \begin{matrix} +16a^4 \\ -16aab^2 \\ +4b^4 \end{matrix} \left(3x^2 \begin{matrix} -4aa \\ +2bb \end{matrix}\right.$$
$$\underline{9x^4}$$
$$0$$
$$\underline{\begin{matrix} -24aa \\ +12bb \end{matrix} x^2 \begin{matrix} +16a^4 \\ -16a^2b^2 \\ +4b^4 \end{matrix}}$$
$$0 \quad 0$$

$$y^4 + 4y^3 * - 8y + 4 \;(yy + 2y - 2$$
$$\underline{y^4}$$
$$0$$
$$\underline{4y^3 + 4yy}$$
$$0 \quad -4yy$$
$$\underline{-4yy - 8y + 4}$$
$$0 \quad 0 \quad 0$$

XLVII. If you would extract the Cube Root of $a^3 + 3aab + 3abb + b^3$, the Operation is performed thus:

$$a^3 + 3aab + 3abb + b^3 \;(a + b$$
$$\underline{a^3}$$
$$3aa)\; 0 + 3aab \;(b$$
$$\underline{a^3 + 3aab + 3abb + b^3}$$
$$0 \quad 0 \quad 0 \quad 0$$

Extract first the Cube Root of the first Term a^3, viz. a, and set it down in the Quotient: Then, subtracting its Cube a^3, say, How many times is its triple Square, or $3aa$, contained in the next Term of the Remainder $3aab$? and there comes out b; wherefore write

write b in the Quotient, and subtracting the Cube of the Quotient, there will remain o. Therefore $a + b$ is the Root.

After the same manner, if the Cube Root is to be extracted out of $z^6 + 6z^5 - 40z^3 + 96z - 64$, it will come out $zz + 2z - 4$. And so in higher Roots.

Of the REDUCTION *of* FRACTIONS, *and* RADICAL QUANTITIES.

THE Reduction of Fractions and Radical Quantities is of Use in the preceding Operations, and that either to the least Terms, or to the same Denomination.

Of the REDUCTION *of* FRACTIONS *to the least Terms.*

XLVIII. *FRACTIONS are reduced to the least Terms, by dividing the Numerators and Denominators by the greatest Divisor.* Thus the Fraction $\frac{aac}{bc}$ is reduced to a more Simple one $\frac{aa}{b}$ by dividing both aac and bc by c; and $\frac{203}{667}$ is reduced to a more simple one $\frac{7}{23}$ by dividing both 203 and 667 by 29; and $\frac{203aac}{667bc}$ is reduced to $\frac{7aa}{23b}$ by dividing by $29c$. And so $\frac{6a^3 - 9acc}{6aa + 3ac}$ becomes $\frac{2aa - 3cc}{2a + c}$ by dividing by $3a$. And $\frac{a^3 - aab + abb - b^3}{aa - ab}$ becomes $\frac{aa + bb}{a}$ by dividing by $a - b$. (*a*).

And

XLVIII. (*a*) Eucl. VII. 35.

And after this Method, the Terms after Multiplication or Division, may be for the most part abridged. As if you were to multiply $\frac{2ab^3}{3ccd}$ by $\frac{9acc}{bdd}$, or divide it by $\frac{bdd}{9acc}$ (*b*), there will come out $\frac{18aab^3cc}{3bccd^3}$, and by Reduction $\frac{6aabb}{d^3}$. But in these Cases, *it is better to abbreviate the Terms before the Operation, by dividing those Terms first by the greatest common Divisor, which you would be obliged to do afterwards* (*c*). Thus, in the Example before us, if I divide $2ab^3$ and bdd by the common Divisor b, and $3ccd$ and $9acc$ by the common Divisor $3cc$, there will come out the Fraction $\frac{2abb}{d}$ to be multiplied by $\frac{3a}{dd}$, or to be divided by $\frac{dd}{3a}$, there coming out $\frac{6aabb}{d^3}$ as above. And so $\frac{aa}{c}$ into $\frac{c}{b}$ becomes $\frac{aa}{1}$ into $\frac{1}{b}$, or $\frac{aa}{b}$. And $\frac{aa}{c}$ divided by $\frac{b}{c}$ becomes aa divided by b, or $\frac{aa}{b}$. And $\frac{a^3-axx}{xx}$ into $\frac{cx}{aa+ax}$ becomes $\frac{a-x}{x}$, into $\frac{c}{1}$, or $\frac{ac}{x}-c$. And 28 divided by $\frac{7}{3}$ becomes 4 divided by $\frac{1}{3}$, or 12.

Of

(*b*) Number 145 and 149.

(*c*) Hence the Rule, *Multiplicatio comparat heterologos Terminos, et multiplicat homologos: Divisio comparat homologos Terminos, et multiplicat heterologos.* That is, the heterologous Terms before Multiplication, and the homologous Terms before Division, are to be reduced to their lowest Expressions.

I

Of the INVENTION *of* DIVISORS.

XLIX. TO this Head may be referred the Invention of Divisors, by which any Quantity may be divided. *If it be a simple Quantity, divide it by its least Divisor, and the Quotient by its least Divisor, till there remain an indivisible Quotient, and you will have all the prime Divisors of that Quantity. Then multiply together each Pair of these Divisors, each Ternary or three of them, each Quaternary, &c. and you will also have all the compounded Divisors.* As, if all the Divisors of the Number 60 are required, divide it by 2, and the Quotient 30 by 2, and the Quotient 15 by 3, and there will remain the indivisible Quotient 5. Therefore the prime Divisors are 1, 2, 2, 3, 5; those composed of the Pairs 4, 6, 10, 15; of the Ternaries 12, 20, 30; and of all of them 60. Again, If all the Divisors of the Quantity $21abb$ are desired, divide it by 3, and the Quotient $7abb$ by 7, and the Quotient abb by a, and the Quotient bb by b, and there will remain the prime Quotient b. Therefore the prime Divisors are 1, 3, 7, a, b, b; and those composed of the Pairs 21, $3a$, $3b$, $7a$, $7b$, ab, bb; those composed of the Ternaries $21a$, $21b$, $3ab$, $3bb$, $7ab$, $7bb$, abb; and those of the Quaternaries $21ab$, $21bb$, $3abb$, $7abb$; that of the Quinaries $21abb$. After the same Way all the Divisors of $2abb - 6aac$ are 1, 2, a, $bb - 3ac$, $2a$, $2bb - 6ac$, $abb - 3aac$, $2abb - 6aac$ (*a*).

L. *If*

XLIX. (*a*) For the Product of all those prime Divisors being equal to the Dividend, and prime Quantities being prime to each other (Eucl. VII. 31), every other prime Quantity will be Prime to each of those Divisors, and consequently Prime to their Product (Eucl. VII. 30), that is, to the Dividend, and therefore cannot measure it; that is, the Dividend will admit no other prime Divisors, and therefore no other compound Divisors, but such as are compounded of its own prime Divisors.

L. *If after a Quantity is divided by all its simple Divisors, it remains still compounded, and you suspect it has some compounded Divisor, dispose it according to the Dimensions of any of the Letters in it, and in the Room of that Letter substitute successively three or more Terms of this Arithmetical Progression, viz.* 3, 2, 1, 0, —1, —2, *and set the resulting Terms together with all their Divisors, by the corresponding Terms of the Progression, setting down also the Signs of the Divisors, both affirmative and negative. Then set also down the Arithmetical Progressions which run through the Divisors of all the Numbers proceeding from the greater Terms to the less, in the Order that the Terms of the Progression* 3, 2, 1, 0, —1, —2 *proceed, and whose Terms differ either by Unity, or by some Number which divides the highest Term of the Quantity proposed. If any Progression of this Kind occurs, that Term of it which stands in the same Line with the Term* 0 *of the first Progression, divided by the Difference of the Terms, and joined with its Sign to the aforesaid Letter, will compose the Quantity by which you are to attempt the Division* (*b*).

L. (*b*) By Supposition the Divisor (when found) measures the Dividend; if, therefore, the same Number is substituted in both, the Number resulting from the Substitution in the Divisor will measure the Number resulting from the Substitution in the Dividend: But the Divisor being supposed to be of one Dimension, and a Binome, that is, of the Form $nx \pm a$, the Numbers resulting from the successive Substitution of the Laterals in its first Member in the Place of x, will form an arithmetical Progression, whose common Difference will be n, the Coefficient of the first Member (46); and they are also Divisors of the Numbers resulting from the Substitution of the same Laterals in the Dividend: But when Cypher is substituted for x in the Divisor, (its first Member being destroyed by the Multiplication with 0) the Number resulting must be a, its second Member; therefore that arithmetical Progression, whose common Difference is the Coefficient of the first Member of the Divisor, and whose Term, which is placed over-against Cypher, As if the Quantity be $x^3 - xx - 10x + 6$, by substituting, one by one, the Terms of this Progression 1. 0. — 1, for x, there will arise the Numbers — 4, 6, + 14, which, together with all their Divisors, I place right against the Terms of the Progression 1. 0. — 1. after this Manner.

1	4	1. 2. 4.	+ 4.
0	6	1. 2. 3. 6	+ 3.
— 1	14	1. 2. 7. 14	+ 2.

Then, because the highest Term x^3 is divisible by no Number but Unity, I seek among the Divisors a Progression whose Terms differ by Unity, and (proceeding from the highest to the lowest) decrease as the Terms of the lateral Progression 1. 0. — 1. And I find only one Progression of this Sort, viz. 4. 3. 2. whose Term therefore

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Cypher, is the second Member of the Divisor, must run through the Divisors of the Numbers resulting from the Substitution of the same Laterals for x in the Dividend. Now the first Member of the Divisor is to measure the first Term of the Dividend, and the common Difference of the Progression is to be the Coefficient of the first Member of the Divisor; therefore that Progression, running through the Divisors of the Numbers resulting from the Substitution of the Laterals in the Dividend, is to be chosen, whose common Difference measures the Coefficient of the first Term of the Dividend. But as it often happens that the Coefficient of the first Term of the Dividend will admit various Divisors, and consequently the Progressions, whose common Differences will all measure it, will be various; so among the Divisors to be formed according to this Rule, Trial must be made to distinguish the true one. It is evident, however, that if the Dividend admits a Divisor of this Form, it will be found among the Divisors framed according to this Rule; consequently if no Divisor of this Form can be found by the Rule, or none which will divide the Dividend, the Dividend admits not a Divisor of this Form.

fore $+3$ I chuse, which ſtands in the ſame Line with the Term 0 of the firſt Progreſſion 1. 0. — 1. and I attempt the Diviſion by $x+3$, and find it ſucceeds, there coming out $xx-4x+2$.

Again, if the Quantity be $6y^4-y^3-21yy+3y+20$, for y I ſubſtitute ſucceſſively 2. 1. 0. — 1. — 2. and the reſulting Numbers 30. 7. 20. 3. 34. with all their Diviſors, I place by them as follows.

2	30	1. 2. 3. 5. 6. 10. 15. 30	+ 10.
1	7	1. 7.	+ 7.
0	20	1. 2. 4. 5. 10. 20	+ 4.
— 1	31	1. 3.	+ 1.
— 2	34	1. 2. 17. 34	— 2.

And among the Diviſors I perceive there is this decreaſing arithmetical Progreſſion + 10. + 7. + 4. + 1. — 2. The Difference of the Terms of this Progreſſion, viz. 3, divides the higheſt Term of the Quantity $6y^4$. Wherefore I adjoin to the Letter y the Term + 4, which ſtands in the Row oppoſite to the Term 0, divided by the Difference of the Terms, viz. 3, and I attempt the Diviſion by $y+\frac{4}{3}$, or, which is the ſame Thing, by $3y+4$, and the Buſineſs ſucceeds, there coming out $2y^3-3yy-3y+5$.

And ſo, if the Quantity be $24a^5-50a^4+49a^3-140a^2+64a+30$; the Operation will be as follows.

2	42	1. 2. 3. 6. 7. 14. 21. 42	+ 3.	+ 3.	+ 7.
1	23	1. 23	+ 1.	— 1.	+ 1.
0	30	1. 2. 3. 5. 6. 10. 15. 30	— 1.	— 5.	— 5.
— 1	297	1. 3. 9. 11. 27. 33. 99. 297	— 3.	— 9.	— 11.

Here are three Progreſſions, whoſe Terms — 1. — 5. — 5, divided by the Differences of the Terms 2, 4, 6, give three Diviſors to be tried $a-\frac{1}{2}$, $a-\frac{5}{4}$, and $a-\frac{5}{6}$. And the Diviſion by the laſt Diviſor $a-\frac{5}{6}$, or $6a-5$, ſucceeds, there coming out $4a^4-5a^3+4aa-20a-6$.

LI, If no Divisor occur by this Method, or none that divides the Quantity proposed, we are to conclude, that that Quantity does not admit a Divisor of one Dimension. But perhaps it may, if it be a Quantity of more than three Dimensions (*c*), admit a Divisor of two Dimensions. And if so, that Divisor will be found by this Method. *Substitute in that Quantity for the Letter or Species as before, four or more Terms of this Progression* 3, 2, 1, 0. — 1. — 2. — 3. *Let all the Divisors of the Numbers that result be singly added to and subtracted from the Squares of the correspondent Terms of that Progression, multiplied into some Numeral Divisor of the highest Term of the Quantity proposed, and right against the Progression let be placed the Sums and Differences. Then note all the collateral Progressions which run through those Sums and Differences. Then suppose* $\mp$ C *to be a Term of such like Progressions that stands against the Term* 0 *of the first Progression, and* $\mp$ B *the Difference which arises by subducting* $\mp$ C *from the next superior Term which stands against the Term* 1 *of the first Progression, and* A *to be the aforesaid numeral Divisor of the highest Term, and* l *to be the Letter which is the proposed Quantity, then* All $\pm$ Bl $\pm$ C *will be the Divisor to be tried* (*d*).

Thus

LI. (*c*) 164. *If a compound Quantity of three Dimensions admits not a Divisor of one Dimension, it will not admit one of two; and universally, a Quantity which admits not Divisors, whose Indices are less than Half its own Index, will not admit Divisors whose Indices are greater than half its own Index*, and which would be the Indices of the Quotes, if it was divisible by such Divisors; for being indivisible by the Divisors, it will be so by the Quotes.

(*d*) If the same Number be substituted in both Divisor and Dividend, the Number resulting from the Substitution in the Divisor will divide the Number resulting from the Substitution in the Dividend: But the Divisor being of two Dimensions, that is, a Trinome $Al^2 \pm Bl \pm C$, its first Term is the Square of the Quantity represented by l, multiplied into some Submultiple A of the

Thus suppose the proposed Quantity to be $x^4 - x^3 - 5xx + 12x - 6$, for x I write successively 3, 2, 1, 0. — 1, — 2, and the Numbers that come out 39. 6. 1. — 6. — 21. — 26. I dispose or place together with their

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the Coefficient of the highest Term of the Dividend; and if the whole Divisor was subducted from its first Term, the Residue would be its second and third Terms with contrary Signs $\mp Bl \mp C$ (XXVII); and if the Laterals 2, 1, 0, — 1, — 2, &c. were successively substituted for l, in the first Member $\mp Bl$ of this Residue, the resulting Members would descend in an arithmetical Progression, whose common Difference is the Coefficient of the first Member of the Residue; and consequently, if, in order to find this common Difference, each Term of this Progression was subducted from the next Superior, the Residue would be the Coefficient of the second Member of the Divisor with a contrary Sign (46); and when 0 were substituted, the resulting Number would be the Number right against Cypher in the lateral Progression, to wit, the last Member of the Divisor, with a contrary Sign. Therefore if the same Laterals 2, 1, 0, — 1, — 2, be successively substituted for l in the Dividend; and if all the Divisors of the resulting Numbers be severally subducted from the Squares of the substituted Laterals, multiplied into the Coefficient of the first Term of the Divisor, to wit, the Submultiple of the Coefficient of the highest Term of the Dividend; there will be an arithmetical Progression in those Residues, whose common Difference being found by subducting any Term of it from the next Superior (consequently by subducting the Term right against Cypher in the lateral Progression from its next Superior) will be the Coefficient of the second Term of the Divisor with a contrary Sign; and whose Term, which is right against Cypher in the Progression of Laterals, is the last Term of the Divisor with a contrary Sign. Now because from the Substitution of a Lateral for l, a negative Number might emerge; and that to subduct a Negative is equivalent to adding it affirmatively;

their Divisors in the same Line with them, and I add and subtract the Divisors to and from the Squares of the Terms of the first Progression, multiplied by the Numeral Divisor of the Term x^4, which is Unity, viz, to and from the Terms 9. 4. 1. 0. 1. 4, and I dispose likewise the Sums and Differences on the Side. Then I write, as follows, the Progressions which occur among the same. Then I make use of the Terms of

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matively; therefore the affirmative Divisors of the resulting Numbers are not only to be subducted from the Squares of the substituted Laterals multiplied into the Submultiple of the Coefficient of the highest Term of the Dividend, but also to be added to them, that by this Method the negative Divisors may also be subducted from them. Now because it may be, that the common Difference of the Progression, which runs through the Residues, may not be the Coefficient of the second Term of the Divisor (as in Art. L), with its Sign changed; and moreover, because the Coefficient of the highest Term of the Dividend may admit various Divisors; and because the assumed Submultiple of it may not be the Coefficient of the first Term of the Divisor; therefore Trial is to distinguish among the Divisors, formed by the Rule, the true ones. However, the more Laterals there are substituted for the Letter in the Dividend, the fewer are the Divisors which are to be tried; for tho' various Progressions may occur, yet they will all be at last broken off, those only excepted, from which the true Divisors can be formed, by the preceding Rules. Because any Progression being indefinitely continued, and a Divisor being thereby formed, 'tis plain that any Number being substituted in the Divisor, the Number resulting would measure the Number resulting from the Substitution of the same Number in the Dividend; and that therefore the Divisor is a true one. Hence it follows, that if no Divisor can be found by this Method, the Quantity proposed will be indivisible by a Divisor of two Dimensions.

of these Progressions 2 and — 3, which stands opposite to the Term 0 in that Progression which is in the first Column, successively for $\mp$ C, and I make

3	39	1. 3. 13. 39	9	−30.−4. 6. 8. 10. 12. 22. 48	−4. 6
2	6	1. 2. 3. 6	4	−2. 1. 2. 3. 5. 6. 7. 10	−2. 3
1	1	1.	1	0. 2.	0. 0
0	6	1. 2. 3. 6	0	−6.−3.−2.−1. 1. 2. 3. 6	2.−3
−1	21	1. 3. 7. 21	1	−20.−6.−2. 0. 2. 4. 8. 22	4.−6
−2	26	1. 2. 13. 26	4	−22. 9. 2. 3. 5. 6. 17. 30	6.−9

use of the Differences that arise by subtracting these Terms from the superior Terms 0 and 0, viz. — 2 and + 3 respectively for $\mp$ B. Also Unity for A; and x for l. And so in the room of $All \pm Bl \pm C$, I have these two Divisors to try, viz. $xx + 2x - 2$, and $xx - 3x + 3$, by both of which the Business succeeds.

Again, if the Quantity $3y^5 - 6y^4 + y^3 - 8yy - 14y + 14$ be proposed, the Operation will be as follows. First I attempt the Business by adding and subtracting the Divisors to and from the Squares of the Terms of the Progression 2. 1. 0, — 1, making use of 1 for A, but the Business does not succeed. Where-

3	170		27		−7. 17
2	38	1. 2. 19. 38	12	−16.−7. 10. 11. 13. 14. 31. 50	−7.−11
1	10	1. 2. 5. 10	3	−7. 2. 1. 2. 4. 5. 8. 13	−7. 5
0	14	1. 2. 7. 14	0	−14.−7.−2.−1. 1. 2. 7. 14	−7.− 1
−1	10	1. 2. 5. 10	3	−7.−2. 1. 2. 4. 5. 8. 13	−7.− 7
−2	190		12	0	−7.−13

fore in the room of A, I make use of 3, the other numeral Divisor of the highest Term $3y^5$; and these Squares being multiplied by 3, I add and subtract the Divisors to and from the Products, viz. 12. 3. 0. 3, and I find these two Progressions in the resulting Terms, — 7. — 7. — 7. — 7, and 11. 5. — 1. — 7. For Expedition Sake, I neglected the Divisors of the outermost Terms 170 and 190. Wherefore, the Progressions being continued upwards and downwards, I take the next Terms,

Terms, viz. -7 and 17 at the Top, and -7, and -13 at Bottom, and I try if these being subducted from the Numbers 27 and 12, which stand against them in the fourth Column, their Differences divide those Numbers 170 and 190, which stand against them in the second Column. And the Difference between 27 and -7, that is, 34, divides 170; and the Difference of 12 and -7, that is, 19, divides 190. Also the Difference between 27 and 17, that is, 10, divides 170; but the Difference between 12 and -13, that is, 25, does not divide 190. Wherefore I reject the latter Progression. According to the former, $\mp C$ is -7, and $\mp B$ is nothing; the Terms of the Progression having no Difference. Wherefore the Divisor to be tried $All \pm Bl \pm C$ will be $3yy + 7$. And the Division succeeds, there coming out $y^3 - 2yy - 2y + 2$.

If after this way, there can be found no Divisor which succeeds, we are to conclude, that the proposed Quantity will not admit of a Divisor of two Dimensions. The same Method may be extended to the Invention of Divisors of more Dimensions, by seeking in the aforesaid Sums and Differences not arithmetical Progressions, but some others, the first, second, and third, &c. Differences of whose Terms are in arithmetical Progression: But the Learner ought not to be detained about them.

LII. *Where there are two Letters in the proposed Quantity, and all its Terms ascend to equally high Dimensions, put Unity for one of those Letters; then, by the preceding Rules, seek a Divisor, and compleat the deficient Dimensions of this Divisor, by restoring that Letter for Unity.*

As if the Quantity be $6y^4 - cy^3 - 21ccyy + 3c^3y + 20c^4$, where all the Terms are of four Dimensions; for c I put 1, and the Quantity becomes $6y^4 - y^3 - 21yy + 3y + 20$, whose Divisor, as above, is $3y + 4$; and having compleated the deficient Dimension of the last Term by a correspondent Dimension of c, you have $3y + 4c$ for the Divisor sought. So, if the Quantity be $x^4 - bx^3 - 5bbxx + 12b^3x - 6b^4$; putting 1 for b, and having found $xx + 2x - 2$ the Divisor of the resulting Quantity $x^4 - x^3 - 5xx + 12x - 6$, I compleat its deficient Dimensions by respective Dimensions fions of b, and so I have $xx+2bx+2bb$, the Divisor sought.

LIII. *When there are three or more Letters in the Quantity proposed, and all its Terms ascend to the same Dimensions, the Divisor may be found by the precedent Rules; but more expeditiously after this Way: Seek all the Divisors of all the Terms in which some one of the Letters is not, and also of all the Terms in which some other of the Letters is not; as also of all the Terms in which a third, fourth, and fifth Letter is not, if there are so many Letters; and so run over all the Letters: And in the same Line with those Letters place the Divisors respectively. Then see, if in any Series of Divisors going through all the Letters, all the Parts involving only one Letter can be as often found as there are Letters (excepting only one) in the Quantity proposed*; and likewise if the Parts involving two Letters *may be found as often as there are Letters (excepting two) in the Quantity proposed. If so; all those Parts taken together under their proper Signs will be the Divisor sought.*

As if there were proposed the Quantity $12x^3-14bxx+9cxx-12bbx-6bcx+8ccx+8b^3-12bbc-4bcc+6c^3$; the Divisors of one Dimension of the Terms $8b^3-12bbc-4bcc+6c^3$, in which x is not, will be found by the preceding Rules to be $2b-3c$, and $4b-6c$; and of the Terms $12x^3+9cxx+8ccx+6c^3$, in which b is not, there will be only one Divisor $4x+3c$; and of the Terms $12x^3-14bxx-12bbx+8b^3$, in which there is not c, there will be the Divisors $2x-b$ and $4x-2b$. I dispose these Divisors in the same Lines with the Letters x, b, c, as you here see;

x	$2b-3c$. $4b-6c$.
b	$4x+3c$.
c	$2x-b$. $4x-2b$.

Since there are three Letters, and each of the Parts of the Divisors only involve one of the Letters, those Parts ought to be found twice in the Series of Divisors. But the Parts $4b$, $6c$, $2x$, b of the Divisors $4b-6c$ and $2x-b$, only occur once, and are not found any where out of those Divisors whereof they are Parts. Wherefore

fore I neglect those Divisors. There remain only three Divisors $2b - 3c$, $4x + 3c$, and $4x - 2b$. These are in the Series going through all the Letters x, b, c, and each of the Parts $2b$, $3c$, $4x$, are found in them twice, as ought to be, and that with the same Signs, provided the Signs of the Divisor $2b - 3c$ be changed, and in its Place you write $-2b + 3c$. For you may change the Signs of any Divisor. I take therefore all the Parts of these, viz. $2b$, $3c$, $4x$, once apiece under their proper Signs, and the Aggregate $-2b + 3c + 4x$ will be the Divisor which was to be found. For if by this you divide the proposed Quantity, there will come out $3xx - 2bx + 2cc - 4bb$.

Again, if the Quantity be $12x^5 - 10ax^4 - 9bx^4 - 26a^2x^3 + 12abx^3 + 6bbx^3 + 24a^3xx - 8aabxx - 8abbxx - 24b^3xx - 4a^3bx + 6aabbx - 12ab^3x + 18b^4x + 12a^4b + 32aab^3 - 12b^5$; I place the Divisors of the Terms in which x is not, by x; and those Terms in which a is not, by a; and those in which b is not, by b, as you here see. Then I perceive that all

x	$b.\ 2b.\ 4b.\ aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb.$ $bb - 3aa.\ 2bb - 6aa.\ 4bb - 12aa.$
a	$4xx - 3bx + 2bb.\ 12xx - 9bx + 6bb.$
b	$x.\ 2x.\ 3x - 4a.\ 6x - 8a.\ 3xx - 4ax.\ 6xx - 8ax.$ $2xx + ax - 3aa.\ 4xx + 2ax - 6aa.$

those that are but of one Dimension are to be rejected, because the Simple ones, $b.\ 2b.\ 4b.\ x.\ 2x$, and the Parts of the compounded ones, $3x - 4a.\ 6x - 8a$, are found but once in all the Divisors; but there are three Letters in the proposed Quantity, and those Parts involve but one, and so ought to be found twice. In like manner, the Divisors of two Dimensions $aa + 3bb.\ 2aa + 6bb.\ 4aa + 12bb.\ bb - 3aa.$ and $4bb - 12aa$ I reject, because their Parts $aa.\ 2aa.\ 4aa.\ bb.$ and $4bb.$ involving only one Letter a or b, are not found more than once. But the Parts $2bb$ and $6aa$ of the Divisor $2bb - 6aa$, which is the only remaining one in the Line with x, and which likewise involve only one Letter, are found again, or twice, viz. the Part $2bb$ in the Divisor $4xx - 3bx + 2bb$, and the Part $6aa$ in the Divisor

visor $4xx + 2ax - 6aa$. Moreover, these three Divisors are in a Series standing in the same Lines with the three Letters x, a, b; and all their Parts $2bb$, $6aa$, $4xx$, which involve only one Letter, are found twice in them, and that under their proper Signs; but the Parts $3bx$, $2ax$, which involve two Letters, occur but once in them. Wherefore, all the divers Parts of these three Divisors, $2bb$, $6aa$, $4xx$, $3bx$, $2ax$, connected under their proper Signs, will make the Divisors sought, viz. $2bb - 6aa + 4xx - 3bx + 2ax$. I therefore divide the Quantity proposed by this Divisor, and there arises $3x^3 - 4axx - 2aab - 6b^3$.

LIV. *If all the Terms of any Quantity are not equally high, the deficient Dimensions must be filled up by the Dimensions of any assumed Letter; then having found a Divisor by the precedent Rules, the assumed Letter is to be blotted out.* As if the Quantity be $12x^3 - bxx + 9xx - 12bbx - 6bx + 8x + 8b^3 - 12b^2 - 4b + 6$; assume any Letter, as c, and fill up the Dimensions of the Quantity proposed by its Dimensions, after this Manner, $12x^3 - 14bxx + 9cxx - 12bbx - 6bcx + 8ccx + 8b^3 - 12bbc - 4bcc + 6c^3$. Then having found out its Divisor $4x - 2b + 3c$, blot out c; and you will have the Divisor required, viz. $4x - 2b + 3$.

Sometimes Divisors may be found more easily than by these Rules. As *if some Letter in the proposed Quantity be of only one Dimension; you may seek for the greatest common Divisor of the Terms in which that Letter is found, and of the remaining Terms in which it is not found; for that Divisor will divide the whole. And if there is no such common Divisor, there will be no Divisor of the whole.* For Example, if there be proposed the Quantity $x^4 - 3ax^3 - 8aaxx + 18a^3x + cx^3 - acxx - 8aacx + 6a^3c - 8a^4$; let there be sought the common Divisor of the Terms $+cx^3 - acxx - 8aacx + 6a^3c$, in which c is only of one Dimension, and of the remaining Terms $x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4$, and that Divisor, viz. $xx + 2ax - 2aa$, will divide the whole Quantity.

LV. *But*

LV. *But the greatest common Divisor of two Numbers, if it is not known, or does not appear at first Sight, it is found by a perpetual Subtraction of the less from the greater, and of the Remainder from the last Quantity subtracted. For that will be the sought Divisor, which at length leaves nothing.* Thus, to find the greatest common Divisor of the Numbers 203 and 667, subtract thrice 203 from 667, and the Remainder 58 thrice from 203, and the Remainder 29 twice from 58, and there will remain nothing; which shews, that 29 is the Divisor sought (*e*).

LVI. *After the same Manner the common Divisor in Species, when it is compounded, is found, by subtracting either Quantity, or its Multiple, from the other; provided both those Quantities and the Remainder be ranged according to the Dimensions of any Letter, as is shewn in Division, and be each Time managed by dividing them all by their Divisors, which are either simple, or divide each of its Terms as if it were a simple one.* Thus, to find the greatest common Divisor of the Numerator and Denominator of this Fraction $\frac{x^4 - 3ax^3 - 8aaxx + 18a^3x - 8a^4}{x^3 - axx - 8aax + 6a^3}$ multiply the Denominator by x, that its first Term may become the same with the first Term of the Numerator. Then subtract it, and there will remain $-2ax^3 + 12a^3x - 8a^4$, which being rightly ordered by dividing by $-2a$, it becomes $x^3 - 6a^2x + 4a^3$. Subtract this from the Denominator, and there will remain $-axx - 2aax + 2a^3$; which again divided by $-a$, becomes $xx + 2ax - 2aa$. Multiply this by x, that its first Term may become the same with the first Term of the last subtracted Quantity $x^3 - 6aax + 4a^3$, from which it is to be likewise subtracted, and there will remain $-2axx - 4aax + 4a^3$, which divided by $-2a$, becomes also $xx + 2ax - 2aa$. And since this is the same with the former Remainder, and consequently being subtracted from it, will leave nothing, it will be the Divisor sought; by

LV. (*e*) Eucl. VII. 2.

by which the proposed Fraction, by dividing both the Numerator and Denominator by it, may be reduced to a more Simple one, viz. to $\frac{xx-5ax+4aa}{x-3a}$. And so, if you have the Fraction $\frac{6a^5+15a^4b-4a^3cc-10aabcc}{9a^3b-27aabc-6abcc+18bc^3}$, its Terms must be first abbreviated, by dividing the Numerator by aa, and the Denominator by $3b$: Then subtracting twice $3a^3-9aac-2acc+6c^3$ from $6a^3+15aab-4acc-10bcc$, there will remain $\begin{matrix}15b\\+18c\end{matrix}aa\begin{matrix}-10bcc\\-12c^3\end{matrix}$. Which being ordered, by dividing each Term by $5b+6c$ after the same Way as if $5b+6c$ was a simple Quantity, it becomes $3aa-2cc$. This being multiplied by a, subtract it from $3a^3-9aac-2acc+6c^3$, and there will remain $-9aac+6c^3$, which being again ordered by a Division by $-3c$, becomes also $3aa-2cc$, as before. Wherefore $3aa-2cc$ is the Divisor sought. Which being found, divide by it the Parts of the proposed Fraction, and you will have $\frac{2a^3+5aab}{3ab-9bc}$ (*f*).

LVII. *Now, if a common Divisor cannot be found after this Way, it is certain there is none at all; unless, perhaps, it may arise out of the Terms that abbreviate the Numerator and Denominator of the Fraction.* As, if you have the Fraction $\frac{aadd-ccdd-aacc+c^4}{4aad-4acd-2acc+2c^3}$, and so dispose its

LVI. (*f*) If all the Terms of both Quantities have a common Divisor, divide them by it, and by it multiply the common Divisor, when found; so shall the Product be the greatest common Divisor: But no Quantity, by which the Terms of one Quantity only have been abbreviated, can enter the Composition of a Divisor, which is to be common to both.

its Terms, according to the Dimensions of the Letter d, that the Numerator may become $\begin{matrix} aa \\ -cc \end{matrix} dd \begin{matrix} -aacc \\ +c^4 \end{matrix}$, and the Denominator $\begin{matrix} 4aa \\ -4ac \end{matrix} d \begin{matrix} -2acc \\ +2c^3 \end{matrix}$. These must first be abbreviated by dividing each Term of the Numerator by $aa - cc$, and each of the Denominator by $2a - 2c$, just as if $aa - cc$ and $2a - 2c$ were simple Quantities. And so, in Room of the Numerator there will come out $dd - cc$, and in Room of the Denominator $2ad - cc$, from which, thus prepared, no common Divisor can be obtained. But, out of the Terms $aa - cc$ and $2a - 2c$, by which both the Numerator and Denominator are abbreviated, there comes out a Divisor, viz. $a - c$, by which the Fraction may be reduced to this, viz. $\frac{add + cdd - acc - c^3}{4ad - 2cc}$. Now, if neither the Terms $aa - cc$ and $2a - 2c$ had not had a common Divisor, the proposed Fraction would have been irreducible.

LVIII. And this is a general Method of finding common Divisors: *But most commonly they are more expeditiously found by seeking all the prime Divisors of either of the Quantities, that is, such as cannot be divided by others, and then by trying if any of them will divide the other without a Remainder.* Thus, to reduce $\frac{a^3 - aab + abb - b^3}{aa - ab}$ to the least Terms, you must find the Divisors of the Quantity $aa - ab$, viz. a and $a - b$; then you must try whether either a, or $a - b$, will also divide $a^3 - aab + abb - b^3$ without any Remainder.

Of

Of the REDUCTION *of* FRACTIONS *to a common Denominator.*

LIX. *FRACTIONS are reduced to a common Denominator by multiplying the Terms of each by the Denominator of the other* (*a*).

Thus, having $\frac{a}{b}$ and $\frac{c}{d}$ multiply the Terms of one $\frac{a}{b}$ by *d*, and alſo the Terms of the other $\frac{c}{b}$ by *b*, and they will become $\frac{ad}{bd}$ and $\frac{bc}{bd}$, whereof the common Denominator is *bd*. And thus *a* and $\frac{ab}{c}$, or $\frac{a}{1}$ and $\frac{ab}{c}$ become $\frac{ac}{c}$ and $\frac{ab}{c}$. *But where the Denominators have a common Diviſor, it is ſufficient to multiply them alternately by the Quotients.* Thus the Fraction $\frac{a^3}{bc}$ and $\frac{a^3}{bd}$ are reduced to theſe $\frac{a^3d}{bcd}$, and $\frac{a^3c}{bcd}$, by multiplying alternately by the Quotients *c* and *d*, ariſing by the Diviſion of the Denominators by the common Diviſor *b*.

This Reduction is moſtly of Uſe in the Addition and Subtraction of Fractions, which, if they have different Denominators, muſt be firſt reduced to the ſame Denominator before they can be added (*b*).

K Thus

LIX. (*a*) Eucl. VII. 17. (*b*) Art. XVIII. Note (*b*).

Thus $\frac{a}{b} + \frac{c}{d}$ by Reduction becomes $\frac{ad}{bd} + \frac{bc}{bd}$, or $\frac{ad + bc}{bd}$, and $a + \frac{ab}{c}$ becomes $\frac{ac + ab}{c}$. And $\frac{a^3}{bc} - \frac{a^3}{bd}$ becomes $\frac{a^3d - a^3c}{bcd}$, or $\frac{d - c}{bcd}a^3$. And $\frac{c^4 + x^4}{cc - xx} - cc - xx$ becomes $\frac{2x^4}{cc - xx}$. And so $\frac{2}{3} + \frac{5}{7}$ becomes $\frac{14}{21} + \frac{15}{21}$, or $\frac{14 + 15}{21}$, that is, $\frac{29}{21}$. And $\frac{11}{6} - \frac{3}{4}$ becomes $\frac{22}{12} - \frac{9}{12}$, or $\frac{13}{12}$. And $\frac{3}{4} - \frac{5}{12}$ becomes $\frac{9}{12} - \frac{5}{12}$, or $\frac{4}{12}$, that is, $\frac{1}{3}$. And $3\frac{4}{7}$, or $\frac{3}{1} + \frac{4}{7}$ becomes $\frac{21}{7} + \frac{4}{7}$, or $\frac{25}{7}$. And $25\frac{1}{2}$ becomes $\frac{51}{2}$.

Where there are more Fractions than two, they are to be added gradually.

Thus, having $\frac{aa}{x} - a + \frac{2xx}{3a} - \frac{ax}{a - x}$; from $\frac{aa}{x}$, take a, and there will remain $\frac{aa - ax}{x}$, to this add $\frac{2xx}{3a}$, and there will come out $\frac{3a^3 - 3aax + 2x^3}{3ax}$, from whence, lastly, take away $\frac{ax}{a - x}$, and there will remain $\frac{3a^4 - 6a^3x + 2ax^3 - 2x^4}{3aax - 3axx}$. And so if you have $3\frac{4}{7} - \frac{2}{3}$, first, you are to find the Aggregate of

$3\frac{4}{7}$,

$3\frac{4}{7}$, viz. $\frac{25}{7}$, and then to take from it $\frac{2}{3}$ and there will remain $\frac{61}{21}$ (*c*).

Of the REDUCTION *of* RADICAL QUANTITIES *to their least Terms.*

LX. *A Radical Quantity, where the Root of the whole cannot be extracted, is reduced by extracting the Root of some Divisor of it* (*a*).

Thus $\sqrt{aabc}$, by extracting the Root of the Divisor aa, becomes $a\sqrt{bc}$. And $\sqrt{48}$, by extracting the Root of the Divisor 16, becomes $4\sqrt{3}$. And $\sqrt{48aabc}$, by extracting the Root of the Divisor $16aa$, becomes $4a\sqrt{3bc}$,

(*c*) *A Fraction is reduced to its Equivalent which shall have a given Denominator, by multiplying the given Denominator by the Numerator, and by dividing the Product by the Denominator of the Fraction; for this Quote will be the Numerator of the equivalent Fraction*; for it will have the same Ratio to the given Denominator, as the Numerator of the Fraction had to its Denominator. And this is an Abbreviation of the Reduction to a common Denominator, and of a subsequent Reduction into the lowest Terms, by dividing them by the former Denominator. Also *a Fraction is reduced to its Equivalent which shall have a given Numerator, by multiplying the given Numerator into the Denominator, and by dividing the Product by the Numerator of the Fraction; for this Quote will be the Denominator of the equivalent Fraction.* This also is an Abbreviation of the two Reductions, into a common Denominator, and into the lowest Terms.

LX. (*a*) For the Radical is the same, being still the Product of the Divisor into the Quote; but the Divisor is now rational.

$4a\sqrt{3bc}$. And $\sqrt{\frac{a^3b - 4aabb + 4ab^3}{cc}}$, by extracting the Root of its Divisor $\frac{aa - 4ab + 4bb}{cc}$, becomes $\frac{a-2b}{c}\sqrt{ab}$. And $\sqrt{\frac{aaoomm}{ppzz} + \frac{4aammm}{pzz}}$, by extracting the Root of the Divisor $\frac{aamm}{ppzz}$, becomes $\frac{am}{pz}\sqrt{oo+4mp}$. And $6\sqrt{\frac{75}{98}}$, by extracting the Root of the Divisor $\frac{25}{49}$, becomes $\frac{30}{7}\sqrt{\frac{3}{2}}$, or $\frac{30}{7}\sqrt{\frac{6}{4}}$, and by farther extracting the Root of the Denominator, it becomes $\frac{15}{7}\sqrt{6}$ (*b*). And so $a\sqrt{\frac{b}{a}}$, or $a\sqrt{\frac{ab}{aa}}$, by extracting the Root of the Denominator, becomes $\sqrt{ab}$ (*c*). And $\sqrt[3]{8a^3b + 16a^4}$, by extracting the Cube Root of its Divisor $8a^3$, becomes $2a\sqrt[3]{b+2a}$. After the same manner $\sqrt[4]{a^3x}$, by extracting the Square Root of its Divisor aa, becomes $\sqrt{a}$ into $\sqrt[4]{ax}$ (*d*), or by extracting the Biquadratic Root of the Divisor a^4, it becomes $a\sqrt[4]{\frac{x}{a}}$ (*e*). And so $\sqrt[6]{a^7x^5}$ is changed into $a\sqrt[6]{ax^5}$ (*f*), or

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(*b*) For $\frac{30}{7}\sqrt{\frac{6}{4}} = \frac{30}{7} \times \frac{1}{2}\sqrt{6} = \frac{15}{7}\sqrt{6}$.

(*c*) For $a\sqrt{\frac{b}{a}} = \sqrt{\frac{a^2b}{a}} = \sqrt{ab}$.

(*d*) For $\sqrt[4]{aa} = \sqrt{a}$.

(*e*) For $\sqrt[4]{a^3x} = \sqrt[4]{\frac{a^4}{a}x} = a\sqrt[4]{\frac{x}{a}}$.

(*f*), or into $ax\sqrt[6]{\frac{a}{x}}$ (*g*), or into $\sqrt{ax}\times\sqrt[3]{aax}$ (*h*).

Moreover, *this Reduction is not only of Use for abbreviating of Radical Quantities, but also for their Addition and Subtraction, if they agree in their Roots when they are reduced to the most simple Form; for then they may be added, which otherwise they cannot* (*i*).

Thus, $\sqrt{48}+\sqrt{75}$ by Reduction becomes $4\sqrt{3}+5\sqrt{3}$, that is $9\sqrt{3}$. And $\sqrt{48}-\sqrt{\frac{16}{27}}$ by Reduction becomes $4\sqrt{3}-\frac{4}{9}\sqrt{3}$, that is, $\frac{32}{9}\sqrt{3}$. And thus, $\sqrt{\frac{4ab^3}{cc}}+\sqrt{\frac{a^3b-4aabb+4ab^3}{cc}}$, by extracting what is Rational in it, becomes $\frac{2b}{c}\sqrt{ab}+\frac{a-2b}{c}\sqrt{ab}$, that is, $\frac{a}{c}\sqrt{ab}$. And $\sqrt[3]{8a^3b+16a^4}-\sqrt[3]{b^4+2ab^3}$ becomes $2a\sqrt[3]{b+2a}-b\sqrt[3]{b+2a}$, that is, $\overline{2a-b}\sqrt[3]{b+2a}$.

 Of

(*f*) For $\sqrt[6]{a^6}=a$, therefore $\sqrt[6]{a^7}=a\sqrt[6]{a}$.

(*g*) For $\sqrt[6]{a^7x^5}=\sqrt[6]{a^7\frac{x^6}{x}}=ax\sqrt[6]{\frac{a}{x}}$.

(*h*) For $\sqrt[6]{a^7x^5}=\sqrt[6]{a^3x^3}\times\sqrt[6]{a^4x^2}$; but $\sqrt[6]{a^3x^3}=\sqrt{ax}$; and $\sqrt[6]{a^4x^2}=\sqrt[3]{aax}$; therefore $\sqrt[6]{a^7x^5}=\sqrt{ax}\times\sqrt[3]{aax}$.

(*i*) Art. XVIII. Note (*b*).

Of the REDUCTION *of* RADICAL QUANTITIES *to the same Denomination.*

LXI. *WHEN you are to multiply or divide Radicals of a different Denomination, you must first reduce them to* (a) *the same Denomination, by prefixing that radical Sign whose Index is the least Number, which their Indices divide without a Remainder, and by multiplying the Quantities under the Signs so many Times, excepting one, as that Index is become greater* (b).

For so $\sqrt{ax}$ into $\sqrt[3]{aax}$ becomes $\sqrt[6]{a^3x^3}$ into $\sqrt[6]{a^4xx}$, that is, $\sqrt[6]{a^7x^5}$. And $\sqrt{a}$ into $\sqrt[4]{ax}$ becomes $\sqrt[4]{aa}$ into $\sqrt[4]{ax}$, that is, $\sqrt[4]{a^3x}$. And $\sqrt{6}$ into $\sqrt[4]{\frac{5}{6}}$ becomes $\sqrt[4]{36}$ into $\sqrt[4]{\frac{5}{6}}$, that is, $\sqrt[4]{30}$. By the same Reason, $a\sqrt{bc}$ becomes $\sqrt{aa}$ into $\sqrt{bc}$, that is, $\sqrt{aabc}$. And $4a\sqrt{3bc}$ becomes $\sqrt{16aa}$ into $\sqrt{3bc}$, that is, $\sqrt{48aabc}$. And $2a\sqrt[3]{b+2a}$ becomes $\sqrt[3]{8a^3}$ into $\sqrt[3]{b+2a}$, that is, $\sqrt[3]{8a^3b+16a^4}$

LXI. (*a*) For in all Comparisons and Relations the Quantities are understood to be homogeneous, or of the same Kind. Art. XXXVI.

(*b*) For to reduce Radicals to the same Denomination, is to reduce their Indices to the same Denomination, and at the same time to involve the Quantities according to the Number of Units by which the Indices are respectively increased; but this Involution is made always by one Multiplication less than that Number (Art. XIV). That is, the Radicals are to be involved by the alternate Indices, and the Indices reduced to the same Denomination; whence their Values are not altered.

$\sqrt[3]{8a^3b+16a^4}$. And so $\frac{\sqrt{ac}}{b}$ becomes $\frac{\sqrt{ac}}{\sqrt{bb}}$, or $\sqrt{\frac{ac}{bb}}$. And $\frac{6ab}{\sqrt{18ab^3}}$ becomes $\frac{\sqrt{36aab^4}}{\sqrt{18ab^3}}$, or $\sqrt{2ab}$. And so in others (*c*).

Of the REDUCTION *of* RADICALS *to more simple Radicals, by the Extraction of Roots* (*a*).

LXII. THE Roots of Quantities, which are composed of Integers and Radical Quadraticks, extract thus:

Let A *denote the greater Part of any Quantity, and* B *the lesser Part; and* $\frac{A+\sqrt{AA-BB}}{2}$ *will be the Square of the greater Part of the Root; and* $\frac{A-\sqrt{AA-BB}}{2}$ *will be the Square of the lesser Part, which is to be joined to the greater Part with the Sign of* B (*b*).

(*c*) *Rationals are reduced to the Form of Irrationals of a given Index, by involving them according to that Index, and affecting the Power with the Sign of Irrationality.*

LXII. (*a*) For this Reduction of *compound Radicals*, see Number 159.

(*b*) *Simple Surds are commensurable in Power* (162). When *the Square Root of a Radical* is required, it *may be found nearly by extracting the Root of a rational Quantity, which approximates to its Value* (Art. XLI. Numb. 163). Thus to find the Square Root of $3+\sqrt{8}=3+2\sqrt{2}$; we first calculate the Value of $\sqrt{2}=1,41421$ (Art. XLI); whence $\overline{3+2\sqrt{2}}=5,82842$; and $\sqrt{5,82842}=2,41$, &c.

As if the Quantity be $3 + \sqrt{8}$, by writing 3 for A, and $\sqrt{8}$ for B, $\sqrt{AA - BB} = 1$, and thence the Square of the greater Part of the Root $\frac{3+1}{2}$, that is, 2, and the Square of the less $\frac{3-1}{2}$, that is, 1. Therefore the Root is, $1 + \sqrt{2}$ (*b*). Again, if you are to extract the Root

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165. *Sometimes we are able to express the Roots of Radicals exactly by other Radicals*, as in this Example, $\sqrt{:3 + 2\sqrt{2}} = 1 + \sqrt{2}$: for $\overline{1 + \sqrt{2}} \times \overline{1 + \sqrt{2}} = 1 + 2\sqrt{2} + 2 = 3 + 2\sqrt{2}$. Now *this can be done*, first, *when the Quantity proposed, being partly rational, and called* A, *and partly irrational, and called* B, *the Square Root of* $A^2 - B^2$ *can be extracted.* For then the Members of the Root are quadratic Radicals, the Sum of whose Squares being rational are united, and greater than their double irrational Product (115, 116); accordingly A being the Sum of their Squares, and B the Sum of their Products, AA will contain the Sum of their Biquadrates, together with double the Product of their Squares; and BB will be quadruple the Product of their Squares (Eucl. II. 4.); whence $A^2 - B^2$ will be the Sum of their Biquadrates less by double the Product of their Squares; consequently $\sqrt{AA - BB}$ will be the Difference of their Squares (Eucl. II. 7.); and therefore $A + \sqrt{AA - BB}$, to wit, the Sum of their Squares added to the Difference of their Squares is double the Square of the greater (22); and $A - \sqrt{AA - BB}$, is double the Square of the less (36); whence $\frac{A + \sqrt{AA - BB}}{2}$ is the Square of the greater; and $\frac{A - \sqrt{AA - BB}}{2}$ is the Square of the less; and the Radicals themselves are $\sqrt{\frac{A \pm \sqrt{AA - BB}}{2}}$.

Root of $\sqrt{32} - \sqrt{24}$, by putting $\sqrt{32}$ for A, and $\sqrt{24}$ for B; $\sqrt{AA - BB}$ will be $= \sqrt{8}$, and thence $\frac{\sqrt{32} + \sqrt{8}}{2}$, and $\frac{\sqrt{32} - \sqrt{8}}{2}$, that is, $3\sqrt{2}$ and $\sqrt{2}$ will be the Squares of the Parts of the Root. The Root therefore is $\sqrt[4]{18} - \sqrt[4]{2}$. After the same Manner, if out of $aa + 2x\sqrt{aa - xx}$ you are to extract the Root, for A write aa, and for B write $2x\sqrt{aa - xx}$, and $AA - BB$ will be $= a^4 - 4aaxx + 4x^4$, the Root whereof is $aa - 2xx$. Whence the Square of one Part of the Root will be $aa - xx$, and that of the other xx; and so the Root will be $x + \sqrt{aa - xx}$ (c). Again, if you have $aa + 5ax - 2a\sqrt{ax + 4xx}$, by writing $aa + 5ax$ for A, and $2a\sqrt{ax + 4xx}$ for B, $AA - BB$ will be $= a^4 + 6a^3x + 9aaxx$, whose Root is $aa + 3ax$.

(c) 166. Again, *Radicals will be reducible to more simple Radicals, if the Members of the Root, though not quadratic Surds, or Roots of Integers, are yet the Roots of like Surds*; which is known by extracting the Roots of some Divisors of the given Radicals; if they agree in their irrational Part (Art. LX.): For then the greater A is the Sum of the rational Coefficients of the Roots multiplied into their irrational Part. Thus $\sqrt{32} - \sqrt{24}$ being given, $\sqrt{32} = 4\sqrt{2}$, and $\sqrt{24} = 2\sqrt{3} \times \sqrt{2}$; whence $\sqrt{32} = 4\sqrt{2} = A$; and $A^2 - B^2 = 32 - 24 = 8$; whence $\sqrt{A^2 - B^2} = \sqrt{8} = 2\sqrt{2}$; and $\frac{A + \sqrt{A^2 - B^2}}{2} = \frac{4\sqrt{2} + 2\sqrt{2}}{2} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$; and $\frac{A - \sqrt{A^2 - B^2}}{2} = \frac{4\sqrt{2} - 2\sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}$; whence the Root $\sqrt{:3\sqrt{2} - \sqrt{2}} = \sqrt{:\sqrt{18} - \sqrt{2}} = \sqrt[4]{18} - \sqrt[4]{2}$.

$3ax$. Whence the Square of the greater Part of the Root will $aa + 4ax$, and that of the lesser Part ax, and the Root $\sqrt{aa + 4ax} - \sqrt{ax}$ (*d*). Lastly, If you have $6 + \sqrt{8} - \sqrt{12} - \sqrt{24}$, putting $6 + \sqrt{8} = A$, and $-\sqrt{12} - \sqrt{24} = B$, $AA - BB$ will be $= 8$; whence the greater Part of the Root is $\sqrt{3 + \sqrt{8}}$, that is, as above $1 + \sqrt{2}$, and the lesser Part $\sqrt{3}$, and consequently the Root itself $1 + \sqrt{2} - \sqrt{3}$ (*e*).

LXIII. *But where there are more of this Sort of radical Terms, the Parts of the Root may be sooner found, by dividing the Product of any two of the Radicals by some third Radical, which shall produce a rational and integer Quotient. For the Root of twice that Quotient will be double of the Part of the Root sought.* As in the last Example, $\frac{\sqrt{8} \times \sqrt{12}}{\sqrt{24}} = 2$, $\frac{\sqrt{8} \times \sqrt{24}}{\sqrt{12}} = 4$, and $\frac{\sqrt{12} \times \sqrt{24}}{\sqrt{8}} = 6$. Therefore

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167. (*d*) Again, *If, after extracting the Square Root of a Divisor of each Part of the given Radical, the irrational Parts are different, and are not Multiples one of the other, or of some Number which measures them both by a square Number; then the Quantity given must be a Trinome, and to proceed by this Rule, A itself must be made a Binome.*

168. (*e*) Lastly, *If a Quadrinome be given, having one rational Term, the Root must be a Trinome* (124, 125), and the Sum of the Squares of all the Terms of the Root is the rational Term; wherefore this Term added to any other Term of the given Quantity, will contain the compleat Square of two Terms of the Root taken as a Binome, together with the Square of the third Term (Eucl. II. 4.), and are the greater Part of the Quadrinome; and being denoted by A, the Root may be extracted by this Rule; though it will lead into perplexed Calculations.

fore the Parts of the Root are 1, $\sqrt{2}$, $\sqrt{3}$, as above (*f*).

There is also a Rule of extracting higher Roots out of numeral Quantities consisting of two Parts, whose Squares are commensurable.

LXIV. *Let there be the Quantity* A ± B. *And its greater Part* A. *And the Index of the Root to be extracted* c. *Seek the least Number* n, *whose Power* n^c *may be divided by* AA — BB *without any Remainder, and let the Quotient be* Q. *Compute* $\sqrt[c]{A + B} \times \sqrt{Q}$ *in the nearest integer Numbers. Let it be* r. *Divide* A $\sqrt{Q}$ *by the greatest rational Divisor. Let the Quotient be* s, *and let* $\frac{r + \frac{n}{r}}{2s}$ *in the next greatest Integers be called* t. *And* $\frac{ts \pm \sqrt{ttss - n}}{\sqrt[2c]{Q}}$ *will be the Root sought if the Root can be extracted* (*g*).

As

LXIII. 169. (*f*) For each Term of the proposed, which is not entirely rational, is the double Product of two Terms of the Root; the Product therefore of two Terms of the proposed is quadruple the Product of four Terms of the Root, or else, if the same Term of the Root be in both Factors, double the Square of one multiplied into double the Product of two other Terms of the Root: *If*, therefore, *the Product of two Terms of the proposed divided by a third gives a rational and integer Quote*, that Divisor must be the double Product of two Terms of the Root; and that Quote must be the double Square of one Term of the Root; which were all contained in the Product: Therefore double of the Quote will be quadruple the Square of that Term of the Root; and therefore *the Root of the double Quote will be double that Term of the Root.*

LXIV. (*g*) Because by Construction $\frac{n^c}{AA - BB} =$ Q, therefore AAQ — BBQ = n^c: Let $x\sqrt{y} \pm \sqrt{z}$ be

As if the Cube Root be to be extracted out of $\sqrt{968} + 25$; $AA - BB$ will be $= 343$; and 7, 7, 7, will

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be the Root c of $A\sqrt{Q} \pm B\sqrt{Q}$, and let $x\sqrt{y}$ be the greater Member; then because $\overline{xxy - z}^{c} = AAQ - BBQ$ (117), and $AAQ - BBQ = n^{c}$, therefore $\overline{xxy - z}^{c} = n^{c}$; and $xxy - z = n$: But $xxy - z = \overline{x\sqrt{y} + \sqrt{z}} \times \overline{x\sqrt{y} - \sqrt{z}}$ (Eucl. II. 5.); therefore $\overline{x\sqrt{y} + \sqrt{z}} \times \overline{x\sqrt{y} - \sqrt{z}} = n$. Hence $x\sqrt{y} + \sqrt{z} : \sqrt{n} :: \sqrt{n} : x\sqrt{y} - \sqrt{z}$ (Eucl. VII. 19.); but also $r : \sqrt{n} :: \sqrt{n} : \frac{n}{r}$ (Eucl. VII. 19.): Now by Construction $x\sqrt{y} + \sqrt{z}$ is greater than $\sqrt{n}$, whence $\sqrt{n}$ is greater than $x\sqrt{y} - \sqrt{z}$; also r is greater than $\sqrt{n}$, whence $\sqrt{n}$ is greater than $\frac{n}{r}$: Wherefore if r is less than $x\sqrt{y} + \sqrt{z}$, then $\frac{n}{r}$ is greater than $x\sqrt{y} - \sqrt{z}$ (74); and consequently $x\sqrt{y} - \sqrt{z} - \frac{n}{r}$ is negative: Also if $x\sqrt{y} + \sqrt{z}$ is less than r, then $x\sqrt{y} - \sqrt{z}$ will be greater than $\frac{n}{r}$ (74); and $x\sqrt{y} - \sqrt{z} - \frac{n}{r}$ is positive. But the Difference between $x\sqrt{y} + \sqrt{z}$ and r must be less than an Unit (163), and the Difference between $x\sqrt{y} - \sqrt{z}$ and $\frac{n}{r}$ is less than the Difference between $x\sqrt{y} + \sqrt{z}$ and r (75), and therefore the Difference between $x\sqrt{y} - \sqrt{z}$ and $\frac{n}{r}$ is still much less than an Unit; consequently, adding them together, $2x\sqrt{y} - r - \frac{n}{r}$, that is, the Difference between $2x\sqrt{y}$ and will be its Divisors; therefore $n = 7$, and $Q = 1$. Moreover, $\overline{A+B} \times \sqrt{Q}$, or $\sqrt{968} + 25$, having extracted the former Part of the Root, is a little greater than 56; and its Cube Root in the nearest Numbers is 4; therefore $r = 4$. Moreover, $A\sqrt{Q}$ or $\sqrt{968}$, by taking out whatever is rational, becomes $22\sqrt{2}$. There-

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and $r + \frac{n}{r}$ is less again than an Unit; that is, $2x\sqrt{y} - \overline{r + \frac{n}{r}} \sqsupset 1$; whence $x\sqrt{y} - \overline{r + \frac{n}{r}} \sqsupset \frac{1}{2}$; and $x - \frac{\overline{r + \frac{n}{r}}}{2\sqrt{y}} \sqsupset \frac{1}{2}$; that is, much less than Half an Unit: Now let $r + \frac{n}{r} = t$, and $\sqrt{y} = s$, then $ts = x\sqrt{y}$, and $xxy = ttss$; but $xxy - z = n$; therefore by Subduction $z = ttss - n$; and therefore $\sqrt{z} = \sqrt{ttss - n}$. Whence $x\sqrt{y} \pm \sqrt{z} = \sqrt[c]{A\sqrt{Q} \pm B\sqrt{Q}} = ts \pm \sqrt{ttss - n}$; and consequently $\frac{ts \pm \sqrt{ttss - n}}{\sqrt[c]{:\sqrt{Q}}}$, that is, $\frac{ts \pm \sqrt{ttss - n}}{\sqrt[2c]{Q}} = \sqrt[c]{A \pm B}$; if the Root c can be extracted from $A \pm B$. Now as c may be odd or even, and the greater Term A either irrational or rational, so a Variety of Cases are here to be considered.

170. *If* A *is irrational,* B *rational, and* c *even; the Root* c *cannot be extracted.* Because to extract a Root, whose Index c is even, from $A \pm B$; it is always necessary that the greater Member A should be rational; to the End that the Excess of the Sum of the Powers of the Parts of the Root above the Sum of their double Product may be a real Quantity.

Therefore $\sqrt{2}$ its radical Part is s, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{5\frac{1}{4}}{2\sqrt{2}}$ in the nearest integer Numbers is 2. Therefore $t = 2$. Lastly, ts is $2\sqrt{2}$, $\sqrt{ttss - n}$ is 1, and $\sqrt[2c]{Q}$, or $\sqrt[6]{1}$, is 1. Therefore $2\sqrt{2} + 1$ is the Root sought, if it can be extracted. I try therefore by Multiplication if the Cube of $2\sqrt{2} + 1$ be $\sqrt{968} + 25$, and it succeeds (*b*).

Again, if the Cube Root is to be extracted out of $68 - \sqrt{4374}$, $AA - BB$ will be $= 250$, whose Divisors are 5, 5, 5, 2. Therefore $n = 5 \times 2 = 10$, and $Q = 4$ (*i*). And $\sqrt[c]{\overline{A+B} \times \sqrt{Q}}$, or $\sqrt[3]{\overline{68 + \sqrt{4374}} \times 2}$ in the nearest integer Numbers is $7 = r$. Moreover, $A\sqrt{Q}$, or $68\sqrt{4}$, by extracting or taking out what is rational, becomes $136\sqrt{1}$. Therefore $s = 1$, and $\frac{r+\frac{n}{r}}{2s}$, or $\frac{7+\frac{10}{7}}{2}$ in the nearest integer Numbers is $4 = t$. Therefore $ts = 4$, $\sqrt{ttss - n} = \sqrt{6}$, and $\sqrt[2c]{Q} =$

(*b*) 171. Again, *If* A *is irrational,* B *rational, and* c *odd, then* $A\sqrt{Q}$ is irrational; and being reduced to its least Terms (LX), will have its radical Part the same with $\overline{x\sqrt{y^c}}$ (116): Whence $s = \sqrt{y}$, and $\frac{r+\frac{n}{r}}{2s}$ will differ less than ½ an Unit from x; therefore *the Root can be rightly determined.*

(*i*) For that the Power n^c should be a Multiple of $AA - BB$, Q must be an Integer: If, therefore, no prime Divisor of $A^2 - B^2$ involved to c will give Q an Integer, some compound Divisor, and the least which will serve, must be taken.

$\sqrt[2c]{Q} = \sqrt[6]{4}$, or $\sqrt[3]{2}$; and so the Root to be tried is $\frac{4 - \sqrt{6}}{\sqrt[3]{2}}$ (*k*).

Again,

(*k*). 172. Again, *If* A *is rational*, B *irrational*, *and* c *odd*, *then* A $\sqrt{Q}$ is rational and already in its lowest Terms, and $x \sqrt{y}$ is rational (116), and $Q = 1$, and $s = \sqrt{y} = 1$, and $x \sqrt{y} = \frac{r + \frac{n}{r}}{2}$ accurately, and *the Root accurately determined.*

173. Again, *If* A *is rational*, B *irrational*, *and* c *even*, *then* A $\sqrt{Q}$ is rational, and $s = 1$; but $x \sqrt{y}$ may be either rational, or irrational (116): *If* $x \sqrt{y}$ *is rational*, then $\sqrt{y} = s = 1$; and *the Root is accurately determined*, as in Number 172. *But if* $x \sqrt{y}$ *is irrational*, *the Root is generally indeterminable*; *however*, B $\sqrt{Q}$ *may be used in the Stead of* A $\sqrt{Q}$, *because* $\sqrt{y}$ *must be equal to* s (116).

174. Again, *If* A *and* B *are both irrational*, *and* c *odd*, *then* the irrational Part of each will be different (116); but A $\sqrt{Q}$, when reduced to its least Terms, will have its irrational Part the same with $x \sqrt{y}$; that is, $\sqrt{y} = s$; whence *the Root can be determined*, as in the Case of Number 171.

175. Lastly, *If* A *and* B *being both irrational*, c *is even*, *then the Root cannot be determined.* For every Power, whose Index is even, implies the Terms to be alternately rational and irrational; and consequently one Member of the Binome, arising from the Union of the alternate Terms, to be also rational; whether the Terms of the Root were both, or either of them, irrational (116).

176. Hence it appears, that *when* c *is odd*, *the Root can be determined*, *whether* A *the greater Member be rational or irrational*; *but that when* c *is even*, *and the greater Member* A, *or both Members are irrational*, *no Root can be extracted*;

Again, if the fifth Root be to be extracted out of $29\sqrt{6}+41\sqrt{3}$; $AA-BB$ will be $=3$, and consequently $n=3$, $Q=81$, $r=5$, $s=\sqrt{6}$, $t=1$, $ts=\sqrt{6}$, $\sqrt{ttss-n}=\sqrt{3}$, and $\sqrt[20]{Q}=\sqrt[10]{81}$, or $\sqrt[5]{9}$; and so the Root to be tried is $\frac{\sqrt{6}+\sqrt{3}}{\sqrt[5]{9}}$.

But

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tracted; and that *when* c *is even, and the greater Member* A *is rational, it is ambiguous whether the greater Part of the Root is rational or irrational.*

177. *In this ambiguous Case, if a Root cannot be found whose greater Part is rational*, by proceeding by the Rule, and as in Number 172; yet a Root, whose greater Part is irrational, and less Part is rational, may be found, as was hinted in Number 173, *by subducting* $\frac{n}{r}$ *from* r, *so that* $t=\frac{r-\frac{n}{r}}{2s}$, *and* $x\sqrt{y}=ts\pm\sqrt{ttss+n}$; *the Expression being the same as when* c *is odd, with the Sign of* n *changed*; *and if this does not succeed, and a prime Number stands under the radical Sign, no further Trial need be made.*

178. *But if a composite Number stands under the radical Sign, the Root may possibly have both its Members irrational, and that composite Number being the Product of their irrational Parts* (116), *the rational Parts may be sought for in the nearest Integers, and Trial made with a Root, whose Members consist of these Integers combined with the radical Factors. In this Manner they are sought immediately*; *but to avoid Ambiguity, and needless Trouble, it is better to depress them by extracting first the Square Root* (LXII); for the Square Root can be extracted still; that is, the Index may be halved (85) until it becomes odd, which will bring the Affair to the Extraction of a Root whose Index is odd.

But *if in these Sorts of Operations, the Quantity be a Fraction, or its Parts have a common Divisor, extract separately the Roots of the Terms, and of the Factors.* As if the Cube Root be to be extracted out of $\sqrt{242} - 12\frac{1}{2}$, this, having reduced its Parts to a common Denominator, will become $\frac{\sqrt{968} - 25}{2}$. Then having extracted separately the Cube Root of the Numerator and the Denominator, there will come out $\frac{2\sqrt{2} - 1}{\sqrt[3]{2}}$. Again, if you are to extract any Root out of $\sqrt[3]{3993} + \sqrt[6]{17578125}$; divide the Parts by the common Divisor $\sqrt[3]{3}$, and there will come out $11 + \sqrt{125}$. Whence the proposed Quantity is $\sqrt[3]{3}$ into $\overline{11 + \sqrt{125}}$, whose Root will be found by extracting separately the Root of each Factor $\sqrt[3]{3}$, and $11 + \sqrt{125}$ (*l*).

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179. (*l*) *In the Resolution of Cubic Equations, by Carden's Rule, we have Binomes of this Form* $A \pm B\sqrt{-q}$, *whose Cube Roots must be found.* Let p be a Divisor of A, and l a Divisor of B: *Because* $\sqrt[c]{A^2 - B^2} = x^2 - z^2$ (117), *in this Case* $\sqrt[3]{A^2 + B^2 q} = (x^2 - z^2) = p^2 + l^2 \times q$: *If we divide the Part under the radical Sign by its greatest rational Divisor, the Quote is* $\sqrt{-q}$, *and subducting* p^2 *from* $\sqrt[3]{A^2 + B^2 q}$, *the Remainder is* $l^2 \times q$, *a known Multiple of* l, *the Divisor of* B: *And* p *and* l *must be affected with such Signs, as that* $p \times \overline{p^2 - 3l^2 q} = A$, *may have the Sign of* A, *and that* $l \times \overline{3p^2 - l^2 q} = B$, *may*

 have

have the Sign of B. Thus to find $\sqrt[3]{81+\sqrt{-2700}}$; $81+\sqrt{-2700}=81+30\sqrt{-3}$, whence A $=81$, B $=30$, $q=3$, $\sqrt[3]{A^2+B^2q}=\sqrt[3]{81\times 81+2700}$ $=21=p^2+l^2q$; assume $p=\pm 3$, then $\sqrt[3]{A^2+B^2q}$ $-p^2=21-9=l^2\times q=8=2\times 2\times 3$: Therefore $l=2$, a Divisor of B $=30$. Now because we have $+81$, and $p^2-3l^2q=9-36=-27$, therefore it is $-3=p$; and because we have $+30$, and $3p^2$ $-l^2q=27-12=+15$, therefore it is $l=2$, whence the Root is $-3+2\sqrt{-3}$. Now because the Cube Roots of 1, are 1, $\frac{-1+\sqrt{-3}}{2}$, and $\frac{-1-\sqrt{-3}}{2}$ (299), therefore by Multiplication the other Cube Roots are $-\frac{3}{2}-\frac{5}{2}\sqrt{-3}$, and $\frac{9}{2}+\frac{1}{2}\sqrt{-3}$. Or because (by dividing the given Binomial by the greatest Cube it contains, and multiplying the Root of the Quote by the Root of that Cube) $81+\sqrt{-2700}=27\times$ $\overline{3+\sqrt{-\frac{100}{27}}}$; and because the Roots of $3+\sqrt{-\frac{100}{27}}$, are $-1+2\sqrt{-\frac{1}{3}}$, $-\frac{1}{2}-\frac{5}{2}\sqrt{-\frac{1}{3}}$, and $\frac{3}{2}$ $+\frac{1}{2}\sqrt{-\frac{1}{3}}$; multiplying therefore these by 3 the Root of 27, we have the Roots required, the same as above.

180. *If the Coefficient of the imaginary Member of the Binome has a contrary Sign, the Root will be the same with the Signs of the imaginary Parts changed:* Thus the Cube Roots of $81-\sqrt{-2700}=81-30\sqrt{-3}$, will be

$-3-2$

Of the FORM *of an* EQUATION.

LXV. *EQUATIONS are Ranks of Quantities either equal to one another, or, taken together, equal to nothing* (a). *These are to be considered* chiefly after two Ways;

$-3-2\sqrt{-3}$, $-\frac{3}{2}+\frac{5}{2}\sqrt{-3}$, and $\frac{9}{2}-\frac{1}{2}\sqrt{-3}$. Wherefore $\sqrt[3]{81+\sqrt{-2700}}+\sqrt[3]{81-\sqrt{-2700}}$ $=-3\times 2=-6$, or $=\frac{-3}{2}\times 2=-3$, or $\frac{9}{2}\times 2=9$, the imaginary Parts vanishing, by the Contrariety of their Signs.

But such Roots, whether expressible in rational Numbers or not, are found by evolving by the Theorem of Number 107, *and summing the alternate Terms:* Thus $\overline{81+30\sqrt{-3}}|^{\frac{1}{3}}$, or rather $\overline{81}|^{\frac{1}{3}}\times\overline{1+\frac{10}{27}\sqrt{-3}}|^{\frac{1}{3}}$, being expanded into a Series; the Sum of the odd Terms will continually approach to $4{,}5=\frac{9}{2}$; and the Sum of the Coefficients of the even Terms to $\frac{1}{2}$, which is the Coefficient of the imaginary Part. See De Moivre's Appendix to Saunderson's Algebra, and Transf. Philos. N° 451.

LXV. (a) In each Form the Quantity, or the Aggregate of the Quantities on each Side of the Sign of Equality $=$ is called *a Member of the Equation*. Thus in the Equation $x=p+q$, x is one Member, and $p+q$ the other; and in the Equation $x-p-q=0$, $x-p-q$

Ways; *either as the last Conclusions to which you come in the Resolution of Problems*; *or as Means*, by the Help whereof you are to obtain *final Equations*. *An Equation of the former Kind is composed only out of one unknown Quantity involved with known ones, if the Problem be determined, and proposes something certain to be found out* (*b*). *But those of the latter Kind involve several unknown Quantities, which, for that Reason, must be compared among one another, and so connected, that out of all there may emerge a new Equation, in which there is only one unknown Quantity which we seek mixed with known Quantities.* Which Quantity, that it may be the more easily discovered, *that Equation must be transformed* most commonly various Ways, *until it becomes the most Simple that it can, and also like some of the following Degrees* of them, in which *x denotes the Quantity sought, according to whose Dimensions the Terms, as you see, are ordered, and* p, q, r, s, *denote any other Quantities from which, being known and determined,* x *is also determined*, and may be investigated by Methods hereafter to be explained.

LXVI.
$$\begin{aligned} x &= p \\ xx &= px + q. \\ x^3 &= px^2 + qx + r. \\ x^4 &= px^3 + qx^2 + rx + s. \quad \&c. \end{aligned}$$

Or,
$$\begin{aligned} x - p &= 0. \\ xx - px - q &= 0. \\ x^3 - pxx - qx - r &= 0. \\ x^4 - px^3 - qx^2 - rx - s &= 0. \quad \&c. \ (c) \end{aligned}$$

After

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is one, and Cypher the other Member; and an Equation is easily transmuted from one into the other Form, by Art. LXVII. &c.

(*b*) See Art. LXXV.* Numb. 194.

LXVI. (*c*) *In the Resolution of a final Equation the first Form*, viz. $x=p$, $x^2=px+q$, &c. *is preferable*, and to be used as in Art. LXXIV, because when the Value of the unknown is sought, it ought alone to make one Member

After this Manner therefore the Terms of Equations are to be ordered according to the Dimensions of the unknown

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Member of the Equation: *But in all other Cases the Form, in which Cypher is one Member, is most eligible*; because the Aggregate of the Terms in the other Member (the Terms being ranged by the Dimensions of the unknown) is then the Product of so many Binomes as the Equation has Dimensions, as is shewn in Art. CXIII. Whence, as the Rules for Evolution are found by observing and tracing back the Steps of Involution, as in Numb. 107; so we may discover Rules for the Resolution of Equations, by observing their Generation from the Multiplication of Binomes. And as compound Equations are Products equal to nothing, so the Factors or Binomes are equal to nothing; that is, the Binomes are simple Equations of the same Form.

181. *The second Term of every generating Binome, or simple Equation, must have its Sign contrary to its real* Sign; that is, if it be an affirmative Quantity, it is to be affected with a negative Sign; but if a negative one, with an affirmative Sign. For if $x = p$, then by subducting p from Equals, the Residues will be equal; that is, $x - p = 0$; and if $x = -p$ by adding p to Equals, the Sums will be equal, viz. $x + p = 0$.

182. Again, the whole Aggregate of the Terms being equal to 0, each Term, which is also an Aggregate of Terms (94, 95) is $= 0$; that is, *a new Equation can be deduced from every Term after the first, by which the Coefficients of the Terms may be determined*; and this is one of the most fruitful Principles of this Art, for the Resolution of compound Problems; the given Quantities being represented as undetermined. Thus,

If $x^3 \begin{matrix}\pm a\\ \pm b\\ \pm c\end{matrix}\, x^2 \begin{matrix}\pm ab\\ \pm ac\\ \pm bc\end{matrix}\, x \pm abc = 0$, then $a \pm b \pm c = 0$, also $ab \pm ac \pm bc = 0$, and $\pm abc = 0$.

 183. As

known Quantity, so that those may be in the first Place, in which the unknown Quantity is of the most Dimensions, as x, xx, x^3, x^4, &c. and those in the second Place, in which x is of the next greatest Dimension, as p, px, px^2, px^3, and so on. As to what regards the Signs, they may stand any how; and one or more of the intermediate Terms may be sometimes wanting. Thus, $x^3 * - bbx + b^3 \mp\!= 0$, or $x^3 = bbx - b^3$, is an Equation of the third Degree, and $Z^4 \begin{array}{l} +a \\ -b \end{array} Z^3 * \begin{array}{l} +ab^3 \\ -b^4 \end{array} = 0$, is

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183. As Ideas are associated to Words by constant Use, so *let constant Use associate the Idea of the Coefficient of the second Term of an Equation to the Letter* p, *that of the third Term to* q, *that of the fourth to* r, *&c.* Thus shall $x \pm p = 0$, $x^2 \pm px \pm q = 0$, $x^3 \pm px^2 \pm qx \pm r \mp\!= 0$, &c. represent constantly an Equation of the 1st, 2d, 3d, &c. Degree, or of 1, 2, 3, &c. Dimensions, respectively; and are called the Formulas of each Degree; in which the Coefficients are determined, as those in Numb. 182 are undetermined: Thus if the final Equation deduced from a Problem is of three Dimensions, let it be $x^3 \pm px^2 \pm qx \pm r = 0$, and supposed $= x^3 \begin{array}{l} \pm a \\ \pm b \\ \pm c \end{array} x^2 \begin{array}{l} \pm ab \\ \pm ac \\ \pm bc \end{array} x \pm abc = 0$; whence (LXVII) $x^3 \begin{array}{l} \pm p \\ \mp a \\ \mp b \\ \mp c \end{array} x^2 \begin{array}{l} \pm q \\ \mp ab \\ \mp ac \\ \mp bc \end{array} x \begin{array}{l} \pm r \\ \mp abc \end{array} = 0$; whence (182) $\pm p \mp a \mp b \mp c = 0$, $\pm q \mp qb \mp ac \mp bc = 0$, and $\pm r \mp abc \mp\!= 0$; whence the Values of $\pm p$, $\pm q$, and $\pm r$, may be found; which Values being substituted in the Formula, become the true Expressions of those Coefficients. After this Manner a Formula being an universal Expression for all Equations of the same Degree, the Resolutions of the Formulas become also general Formulas for the Resolutions of all particular Equations of the same Degree.

is an Equation of the fourth Degree: For *the Degree of an Equation is always estimated by the greatest Dimension of the unknown Quantity, without any Regard to the known ones, or to the intermediate Terms. But by the Defect of the intermediate Terms, the Equation is most commonly rendered much more simple, and may be sometimes depressed to a lower Degree.* For thus, $x^4 = qxx + s$ is to be reckoned an Equation of the second Degree, because it may be resolved into two Equations of the second Degree. For, supposing $xx = y$, and y being accordingly writ for xx in that Equation, there will come out in its stead $yy = qy + s$, an Equation of the second Degree; by the Help whereof when y is found, the Equation $xx = y$ also of the second Degree, will give x (*d*).

And these are the Conclusions to which Problems are to be brought. But before I go upon their Resolution, it will be necessary to shew the Methods of transforming and reducing Equations into Order, and the Methods of finding the final Equations. I shall comprize the Reduction of a single Equation in the following Rules.

(*d*) Any Equation of this Form $x^{2n} = qx^n + s$, where the greatest Index of the unknown Quantity x is double of the Index of x in the other Term, may be reduced to a Quadratic $y^n = qy + s$, by putting $x^n = y$, and consequently $x^{2n} = y^2$; and this Quadratic being resolved by Art. LXXIV. gives y, and therefore also x, by the Equation $x^n = y$, universally.

184. *The Terms being all in one Member, if the Index of the unknown in the Penultimate is a Divisor of its Indices in all the Terms, the Dimensions of the Equation will be reduced to those indicated by the Quote of the greatest Index divided by the least.* Thus $x^6 \pm qx^4 \pm sx^2 \pm v = 0$, can be reduced to the cubic $y^3 \pm qy^2 \pm sy \pm v = 0$; and $x^6 \pm rx^3 \pm v = 0$, to the Quadratic $y^2 \pm ry \pm v = 0$; but $x^6 \pm px^5 \pm rx^2 \pm v = 0$ is irreducible, because 2 will not divide the Index 5. See Numb. 254.

Of ordering a Single or Final EQUATION.

LXVII. RULE I. *If there are any Quantities that may destroy one another, or may be joined into one by Addition or Subtraction, the Terms are that Way to be diminished.*

As if you have $5b - 3a + 2x = 5a + 3x$, take from each Side $2x$, and add $3a$, and there will come out $5b = 8a + x$. And thus, $\frac{2ab + bx}{a} - b = a + b$, by striking out the equivalent Quantities $\frac{2ab}{a} - b = b$, becomes $\frac{bx}{a} = a$.

To this Rule may also be referred the ordering of the Terms of an Equation, which is usually performed by the Transposition of the Members to the contrary Sides under the contrary Sign (*a*). As if you had the Equation $5b = 8a + x$, you are to find x; take from each Side $8a$, or, which is the same Thing, transpose $8a$ to the contrary Side with its Sign changed, and there will come out $5b - 8a = x$. After the same Way, if you have $aa - 3ay = ab - bb + by$, and you are to find y; transpose $-3ay$ and $ab - bb$, so that there may be the Terms multiplied by y on the one Side, and the other Terms on the other Side, and there will come out $aa - ab + bb = 3ay + by$, whence you will have y by the fifth Rule.

LXVII. (*a*) For to take away any Quantity from one Member, and to place it with a contrary Sign in the other, is to add, or to subduct it from both (Art. XXVII): And it is certain, that when to, or from equal Quantities you add, or subduct the same Quantity, the Sums, or Residues, must be respectively equal (Eucl. Axiom 3.)

Rule following, viz. by dividing each Part by $3a+b$, for there will come out $\frac{aa-ab+bb}{3a+b}=y$. And thus the Equation $abx+a^3-aax=abb-2abx-x^3$, by due ordering and Transposition becomes $x^3 = \begin{matrix} aa \\ -3ab \end{matrix} x \begin{matrix} -a^3 \\ +abb \end{matrix}$ or $x^3 \begin{matrix} -aa \\ +3ab \end{matrix} x \begin{matrix} +a^3 \\ -abb \end{matrix} = 0$ (*b*).

LXVIII. RULE II. *If there is any Quantity by which all the Terms of the Equation are multiplied, all of them must be divided by that Quantity; or, if all are divided by the same Quantity, all must be multiplied by it too.*

Thus, having $15bb=24ab+3bx$, divide all the Terms by b, and you will have $15b=24a+3x$; then by 3, and you will have $5b=8a+x$. Or having $\frac{b^3}{ac}-\frac{bbx}{cc}=\frac{xx}{c}$, multiply all by c, and there comes out $\frac{b^3}{a}-\frac{bbx}{c}=xx$.

LXIX. RULE III. *If there be any irreducible Fraction, in whose Denominator there is found the Letter, according to whose Dimensions the Equation is to be ordered, all the Terms of the Equation must be multiplied by that Denominator, or by some Divisor of it.*

As if the Equation $\frac{ax}{a-x}+b=x$ be to be ordered according

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(*b*). The Uses of Transposition are, 1st. To exterminate Quantities which are found in both Members. 2d. To bring all the Quantities into one Member. 3d. To disengage the known and the unknown Quantities from each other, by placing them in the opposite Members. And, lastly, to change negative Quantities into affirmative, by transferring them to the opposite Member; thus $-x^3+px^2-qx+r=0$ becomes, by Transposition, $x^3-px^2+qx-r=-0$.

according to x, multiply all its Terms by $a-x$ the Denominator of the Fraction $\frac{ax}{a-x}$ seeing x is contained therein, and there comes out $ax+ab-bx=ax-xx$, or $ab-bx=-xx$, and transposing each Part you will have $xx=bx-ab$. And so if you have $\frac{a^3-aab}{2cy-cc}=y-c$, and the Terms are to be ranged according to the Dimensions of y, multiply them by the Denominator $2cy-cc$, or, at least, by its Divisor $2y-c$, that y may vanish in the Denominator, and there will come out $\frac{a^3-abb}{c}=2yy-3cy+cc$, and by farther ordering $\frac{a^3-abb}{c}-cc+3cy=2yy$. After the same manner $\frac{aa}{x}-a=x$, by being multiplied by x, becomes $aa-ax=xx$, and $\frac{aabb}{cxx}=\frac{xx}{a+b-x}$, by multiplying first by xx, and then by $a+b-x$, becomes $\frac{a^3bb+aab^3-aabbx}{c}=x^4$ (*c*).

LXX. Rule IV. *If that particular Letter, according to whose Dimensions the Equation is to be ordered, be involved with an irreducible Surd, all the other Terms are to be transposed to the other Side, their Signs being changed, and each Part*

LXIX. (*c*). If there are many irreducible Fractions, whose Denominators contain the unknown x by which the Equation is ordered, the shorter Method is to reduce them all to a common Denominator (LIX.), and to multiply all the Terms by it, or by some Divisor of it, *Part of the Equation must be once multiplied by itself, if the Root be a* Square one, *or twice, if it be a* Cubick one, &c.

Thus, to order the Equation $\sqrt{aa - ax} + a = x$ according to the Letter x, transpose a to the other Side, and you have $\sqrt{aa - ax} = x - a$; and having squared the Parts, $aa - ax = xx - 2ax + aa$, or $0 = xx - ax$, that is, $x = a$. So also $\sqrt[3]{aax + 2axx - x^3} - a + x = 0$, by transposing $-a + x$, becomes $\sqrt[3]{aax + 2axx - x^3} = a - x$, and multiplying the Parts cubically, $aax + 2axx - x^3 = a^3 - 3aax + 3axx - x^3$, or $xx = 4ax - aa$. And so $y = \sqrt{ay + yy - a\sqrt{ay - yy}}$, having squared the Parts, becomes $yy = ay + yy - a\sqrt{ay - yy}$, and the Terms being rightly transposed, it becomes $ay = a\sqrt{ay - yy}$, or $y = \sqrt{ay - yy}$, and the Parts being again squared $yy = ay - yy$, and lastly by transposing $2yy = ay$, or $2y = a$ (*d*).

LXXI. Rule V. *The Terms, by Help of the preceding Rules, being disposed according to the Dimensions of some one of the Letters, if the highest Dimension of that Letter be multiplied by any known Quantity, the whole Equation must be divided by that Quantity.*

Thus,

LXX. (*d*). For if any other Quantity was suffered to remain in the same Member with the Surd, that Member being a Binome, irrational Terms would remain in any Power of it (116); wherefore the irrational Quantity would not be exterminated. Now there is a Necessity for exterminating the irreducible Surd with which the unknown x is involved, that the Dimensions of x, that is, of the Equation may be known.

Thus, $2y = a$, by dividing by 2, becomes $y = \frac{1}{2}a$.

And $\frac{bx}{a} = a$, by dividing by $\frac{b}{a}$, becomes $x = \frac{aa}{b}$.

And $\begin{matrix}2ac\\-cc\end{matrix} x^3 \begin{matrix}+a^3\\+aac\end{matrix} xx \begin{matrix}-2a^3c\\+aacc\end{matrix} x - a^3cc = 0$, by dividing by $2ac - cc$, becomes $x^3 + \frac{\begin{matrix}a^3\\aac\end{matrix} x^2 \begin{matrix}-2a^3c\\+aacc\end{matrix} x - a^3cc}{2ac - cc}$ $= 0$, or $x^3 + \frac{a^3 + aac}{2ac - cc} xx - aax - \frac{a^3c}{2a - c} = 0$ (*e*).

LXXII. Rule VI. *Sometimes the Reduction may be performed by dividing the Equation by some compounded Quantity.*

For thus $y^3 = \begin{matrix}-2c\\+b\end{matrix} yy + 3bcy - bbc$, is reduced to this, viz. $yy = -2cy + bc$, by transferring all the Terms to the same Side thus, $y^3 \begin{matrix}+2c\\-b\end{matrix} yy - 3bcy + bbc = 0$, and dividing by $y - b$, as is shewn in the Chapter of Division, for there will come out $yy + 2cy - bc = 0$ (*f*). But the Invention of this Sort of Divisors is Difficult, and we have taught it already (*g*).

LXXIII.

LXXI. (*e*). By comparing the 2d, 3d, and 4th Rules, it appears that Multiplication is of Use to exterminate Surds and Fractions, and any Quantity by which the unknown was divided; now this Multiplication cannot destroy the Equality of the Members of the Equation (Eucl. Axiom 6.)

LXXII. (*f*). By comparing the 2d, 5th, and 6th Rules, we see that Division exterminates the common Divisors of the Terms, and the Coefficient of the highest Term, and it cannot destroy the Equality of the Members of the Equation; neither can the Evolution destroy it, which is prescribed in the 7th Rule (Eucl. Axiom 7.).

(*g*) Art. L. LI.

LXXIII. Rule VII. *Sometimes also the Reduction is performed by extraction of the Root out of each Part of the Equation.*

As if you have $xx = \frac{1}{4}aa - bb$, having extracted the Root on both Sides, there comes out $x = \sqrt{\frac{1}{4}aa - bb}$. If you have $xx + aa = 2ax + bb$, transpose $2ax$, and there will arise $xx - 2ax + aa = bb$; and extracting the Roots of the Parts $x - a = +$ or $-b$, or $x = a \pm b$. So also having $xx = ax - bb$, add on each Side $-ax + \frac{1}{4}aa$, and there comes out $xx - ax + \frac{1}{4}aa = \frac{1}{4}aa - bb$, and extracting the Root on each Side $x - \frac{1}{2}a = \pm\sqrt{\frac{1}{4}aa - bb}$, or $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa - bb}$.

LXXIV. *And thus universally if you have* $xx = .\,px\,.\,q$; x *will be* $= .\,\frac{1}{2}p \pm \sqrt{\frac{1}{4}pp\,.\,q}$. *Where* $\frac{1}{2}p$ *and* q *are to be affected with the same Signs as* p *and* q *in the former Equation; but* $\frac{1}{4}pp$ *must be always made Affirmative. And this Example is a Rule according to which all Quadratick Equations may be reduced to the Form of Simple ones.*

For Example, having proposed the Equation $yy = \frac{2xxy}{a} + xx$, to extract the Root y, compare $\frac{2xx}{a}$ with p, and xx with q, that is, write $\frac{xx}{a}$ for $\frac{1}{2}p$, and $\frac{x^4}{aa} + xx$ for $\frac{1}{4}pp\,.\,q$, and there will arise $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx}$ or $y = \frac{xx}{a} - \sqrt{\frac{x^4}{aa} + xx}$. After the same way, the Equation $yy = ay - 2cy + aa - cc$, by comparing $a - 2c$ with p, and $aa - cc$ with q, will give $y = \frac{1}{2}a - c \pm \sqrt{\frac{5}{4}aa - ac}$.

Moreover,

Moreover, the Biquadratick Equation $x^4 = -aaxx + ab^3$, whose odd Terms are wanting, by Help of this Rule becomes $xx = -\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}$, and extracting again the Root $x = \sqrt{-\frac{1}{2}aa \pm \sqrt{\frac{1}{4}a^4 + ab^3}}$. And so in others (*b*).

And

LXXIV. (*b*) *Compound Equations are said to be affected when they involve different Powers of the unknown*, as $x^n \pm px^{n-1} \pm qx^{n-2} \pm r = 0$, consequently no compound Equation can be unaffected, but such as wants all its intermediate Terms, as $x^n \pm r = 0$, and all Quadraticks, which want the second Term, are unaffected, as $x^2 \pm q = 0$, the Equations are every where supposed affected where the contrary is not mentioned. The Formule $x^2 . px . q = 0$ represents Quadratics in the most general Manner; for as the Letters p, q, r, &c. represent all Variety of the Coefficients, so the Points represent the Variety of the Signs + and —. In like Manner $x^3 . px^2 . qx . r = 0$, $x^4 . px^3 . qx^2 . rx . s = 0$, &c. will be the most general Expressions for cubic, biquadratic, &c. Equations. Now

185. The first or highest Term of every Equation being always supposed affirmative, the Variation of the Signs has Place only in the following Terms, whose Number is n, their whole Number being $n + 1$ (211); whence the Number of Signs being 2, putting n for the Dimensions of the Equation, *all the Variety of Signs in any Degree is* 2n; whence, in Quadratics, the general Formule $x^2 . px . q = 0$, is four-fold; that is, *all Quadratics, with Regard to the Signs of their Terms, are reduced to one of these four particular Forms.*

	I	II	III
1.	$x^2 + px - q = 0$. or	$x^2 = -px + q$ or	$x^2 + px = q$.
2.	$x^2 - px - q = 0$.	$x^2 = px + q$	$x^2 - px = q$.
3.	$x^2 - px + q = 0$.	$x^2 = px - q$	$x^2 - px = -q$.
4.	$x^2 + px + q = 0$.	$x^2 = -px - q$	$x^2 + px = -q$.

Now it is evident, that every Quadratic, whether given in the first or second Manner, must be reduced to the third

And *these are the Rules for ordering one only Equation*, the Use whereof, when the Analyst is sufficiently acquainted

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third Manner of Expression, in order to be solved; that the Terms containing the unknown x may make one Member of the Equation LXXVII. and this Member, viz. $x^2 \pm px$, which contains the unknown, having two Terms, cannot be the Square of a simple Quantity; and having but two Terms, must be defective of a perfect Square; and adding the Square of $\frac{1}{2}p$ to it, it will be a compleat Square (Eucl. II. 4.). Consequently, adding $\frac{1}{4}p^2$ to each Member of the Equation, and extracting the Square Root, and transposing the known Quantities, so as to leave x the unknown alone in one Member, its Value will be in the other, viz. $x = \mp \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 \pm q}$; in which $\frac{1}{4}p^2$ is always affirmative (89); and $\frac{1}{2}p$, and q, retain the Signs which p, and q, had in the second Manner of expressing the Forms; that is, in the first Form, $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + q}$; in the second, $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 + q}$; in the third, $x = \frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$; and in the fourth, $x = -\frac{1}{2}p \pm \sqrt{\frac{1}{4}p^2 - q}$.

In the Problems which follow before the Resolution of Equations, and the Nature of their Roots is taught, Quadratics occur frequently to be resolved: It will, therefore, be here necessary to be more explicit concerning their Resolution.

186. Because the Square Root of any Quantity may be either affirmative or negative [(88) for $a^2 = a \times a$, and $a^2 = -a \times -a$]; therefore, *all Quadratics admit of two Solutions, or have two Roots.* Thus, having found, by Art. LXXIV. that $x^2 \pm px \pm \frac{1}{4}p^2 = \frac{1}{4}p^2 \pm q$; it may be inferred, that $x \pm \frac{1}{2}p = + \sqrt{\frac{1}{4}p^2 \pm q}$, or to $- \sqrt{\frac{1}{4}p^2 \pm q}$; since $-\sqrt{\frac{1}{4}p^2 \pm q} \times -\sqrt{\frac{1}{4}p^2 \pm q}$ gives $\frac{1}{4}p^2 \pm q$, as well as $+\sqrt{\frac{1}{4}p^2 \pm q} \times +\sqrt{\frac{1}{4}p^2 \pm q}$

quainted with, so that he knows how to dispose any proposed Equation, according to any of the Letters contained

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(88): There are therefore two Values of x in every one of the four Formulas. In the first, $x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$, or to $-\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$: In the second, $x = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 + q}$, or to $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 + q}$: In the third, $x = \frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, or to $\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$: And in the fourth, $x = -\frac{1}{2}p + \sqrt{\frac{1}{4}p^2 - q}$, or to $-\frac{1}{2}p - \sqrt{\frac{1}{4}p^2 - q}$. Thus the Equation $x^2 + 5x - 6 = 0$ gives $x = +1$, and to -6. $x^2 - 5x - 6 = 0$ gives $x = -1$, and to $+6$. $x^2 - 5x + 6 = 0$ gives $x = +3$, and to $+2$. And $x^2 + 5x + 6 = 0$ gives $x = -3$, and to -2.

187. Hence, *when* q *is negative, as in the first and second Forms, one Value of* x *is affirmative, and the other negative; and when* q *is affirmative, both Values of* x *are of the same Affection; being both affirmative, when* p *is negative, as in the third Form; and both negative, when* p *is affirmative, as in the fourth Form.*

188. Hence, *when* q *is negative, as in the first and second Forms, the Quantity* $\frac{1}{4}p^2 + q$ *is affirmative, and the Quantity* $\sqrt{\frac{1}{4}p^2 + q}$ *under the radical Sign is the Root of a positive Square, and can be assigned, and both Roots are real.*

189. Hence also, *when* q *is affirmative, as in the third and fourth Forms, if* q *is greater than* $\frac{1}{4}p^2$, *the Quantity* $\frac{1}{4}p^2 - q$ *is negative, and the Quantity* $\sqrt{\frac{1}{4}p^2 - q}$ *under the radical Sign is the Root of a negative Square, and being impossible* (89), *cannot be assigned.* Consequently

190. *In a Quadratic, when one Root is affirmative and the other negative, both Roots must be real* (187, 188). And

191. *When in a Quadratic both Roots have the same Affection, they are either both real, or both impossible; and never the one real and the other impossible.*

192. *In*

tained in it, and to obtain the Value of that Letter if it be of one Dimension, or of its greatest Power if it be of more:

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192. *In an unaffected Quadratic, viz. which wants the second Term, as* $x^2 + q = 0$, *if the third Term* q *be affirmative, both Roots are impossible*; for the p being wanting, the Quantity $\frac{1}{4}p^2 - q$ becomes $-q$, and $\sqrt{\frac{1}{4}p^2 - q}$ becomes $\sqrt{-q}$ (89).

The impossible Roots of unaffected Quadratics are called pure, as having no real Quantity joined with them; such are $x - \sqrt{-q} = 0$, $x + \sqrt{-q} = 0$, which generate the unaffected Quadratic $x^2 + q = 0$, and which may be called both affirmative or both negative: But the *impossible Roots of affected Quadratics, are called mixed, as having a real Quantity under the same Sign, connected to the impossible and radical Part under contrary Signs; and if the real Quantity is affected with the Sign* $-$ *they are both affirmative* (181), as $x - \frac{1}{2}b - \sqrt{-c}$, and $x - \frac{1}{2}b + \sqrt{-c}$; whose Product is $x^2 - bx \begin{matrix}+b\\+c\end{matrix} = 0$; that is, putting $-b = -p$, and $\overline{+b+c} = +q$, $x^2 - px + q = 0$, a Quadratic of the third Form: *But if the real Quantity is affected with the Sign* $+$, *they are both* (181) *negative*, as $x + \frac{1}{2}b + \sqrt{-c} = 0$, and $x + \frac{1}{2}b - \sqrt{-c} = 0$, which generate the Quadratic $x^2 + bx \begin{matrix}+b\\+c\end{matrix} = 0$; or, as before, $x^2 + px + q = 0$, of the fourth Form: But in all Cases, as the Terms of the Quadratic are supposed rational, the radical Part of the Root is affected with contrary Signs (120, 153.).

193. *A Quadratic, whose Roots are imaginary, contains in* q *its last Term; a positive Quantity, exclusive of the positive Square of the Root; so that its last Term will always exceed the last of a Quadratic, whose Roots are the same but real Radicals, by double the Square of the radical Parts.* Thus if we subduct $x^2 \pm bx - c = 0$ (whose Roots are $x \pm \frac{1}{2}b - \sqrt{c} = 0$, and $x \pm \frac{1}{2}b + \sqrt{c} = 0$), from $x^2 \pm bx + c = 0$ (whose Roots are $x \pm \frac{1}{2}b - \sqrt{-c} = 0$, and $x \pm \frac{1}{2}b + \sqrt{-c} = 0$), the Excess is $2c$.

more: The Comparison of several Equations among one another, will not be difficult to him; which I am now going to shew.

Of the Transformation of two or more EQUATIONS *into one, in order to exterminate the unknown Quantities* (*a*).

LXXV. *WHEN, in the Solution of any Problem, there are more Equations than one to comprehend the State of the Question, in each of which there are several unknown Quantities; those Equations (two by two, if there are more than two) are to be so connected, that one of the unknown Quantities may be made to vanish at each of the Operations, and*

LXXV. (*a*) *By the Reduction of medial Equations is understood the gradual Transformation of the Equations, which, by containing the Conditions of the Problem, contain different unknown Quantities, into one final Equation, which shall comprehend all the Conditions of the Problem, and contain but one unknown Quantity*; consequently *in every Transformation, one unknown Quantity ought to be exterminated.* This Extermination and Transformation is accomplished, when the unknown Quantity to be exterminated is of one Dimension in both the Medials, either by connecting them together, as in Art. LXXV; or by equating the Values of the unknown, found in each, as in Art. LXXVI; but if the unknown is of one Dimension in one only, then by substituting the Value of the unknown found in that one, into its Place in the other, as in Article LXXVII. And when the unknown Quantity to be exterminated is of different Dimensions, and above one, it is to be made of equal Dimensions in both, as in Numb. LXXVIII; and then it may be exterminated by one of the above Methods of connecting or equating, whichever shall seem most conducive to keep down the Dimensions of the final Equation. But in exterminating by any Method, Care is to be taken that the Value of the unknown may emerge positive.

and so produce a new Equation. Thus, having the Equations $2x = y + 5$, and $x = y + 2$, by taking equal Things out of equal Things, there will come out $x = 3$ (*b*). *And you are to know, that by each Equation one unknown Quantity may be taken away; and, consequently, when there are as many Equations as unknown Quantities, all may at length be reduced into one, in which there shall be only one Quantity unknown. But if there be more unknown Quantities by one than there are Equations, then there will remain in the Equation last resulting two unknown Quantities; and if there are more unknown Quantities by two than there are Equations, then in the last resulting Equation there will remain three, and so on* (*c*)

There

(*b*) *This connecting consists in adding the Equations, when the unknown in each has contrary Signs, and in subducting the one from the other, when it has the same Sign; and the Equations are always supposed to be ordered by the Dimensions of the unknown to be exterminated.*

(*c*) 194. A Problem is a Proposition requiring the Investigation of Quantities from their given Properties; it *is determined, when the Number of Quantities, which answer what is required, is determined and certain, otherwise indetermined; the Properties given are the Laws or Conditions of the Problem, each Property gives an Equation between the Quantities to which it is peculiar*, and which are represented by Letters and Symbols signifying the unknown and their Properties: Hence *there are as many Equations as Properties given.* Each additional Property also, if it be not superfluous, limits the Extent, and Number of Quantities, to which the foregoing Properties agreed, and excludes those to which it does not itself agree; whence each Equation connected to another, exterminates one Letter signifying an unknown; whence in the final Equation, all Quantities are excluded to which the Aggregate of the Properties given does not agree; that is, so many Letters denoting unknown Quantities are exterminated, as there are Properties given, and Equations to be collected from them: Therefore, *if the Number of Equations, or charac-*

 terising

There may also, perhaps, two or more unknown Quantities be made to vanish, by only two Equations. As if you have $ax - by = ab - az$, and $bx + by = bb + az$: Then adding Equals to Equals, there will come out $ax + bx = ab + bb$, both y and z being exterminated. But such Cases either argue some Fault to lie hid in the State of the Question, or that the Calculation is erroneous, or not artificial enough. The Method by which one unknown Quantity may be exterminated or taken away by each of the Equations, will appear by what follows.

The Extermination of an unknown Quantity by an Equality of its Values.

LXXVI. *WHEN the Quantity to be exterminated is only of one Dimension in both Equations, both*

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terising Properties, is equal to the Number of Quantities sought, every Letter for an unknown Quantity will be exterminated, except the one representing that to which the Aggregate of Properties agrees; and therefore its Value may certainly be found; whence *the Problem will be determined. Now if the Number of Quantities sought is one, two, &c. more than the Number of Equations,* the Limitations, Exclusions, and Exterminations, will be one, two, &c. less, and consequently two, three, &c. unknown Quantities must remain in the final Equation, whose Values cannot be certainly determined, because one unknown must enter into the Value of another unknown; and the Value not being certainly determined, *the Problem is undetermined. Now if the Equations are more than the Number of Quantities required, either the Properties are superfluous,* as not limiting the Extent or Number of Quantities to which other Properties agree, *or else they may be inconsistent with each other, and make the Resolution impossible; or lastly, the Translation of the Problem* into Algebra, i. e. *into Equations, is erroneous or inartificial.*

both its Values are to be sought by the Rules already delivered, and the one made equal to the other (d).

Thus, putting $a + x = b + y$, and $2x + y = 3b$, that y may be exterminated, the first Equation will give $a + x - b = y$, and the second will give $3b - 2x = y$. Therefore $a + x - b$ is $= 3b - 2x$, or by due ordering $x = \frac{4b - a}{3}$.

And thus, $2x = y$ and $5 + x = y$ give $2x = 5 + x$ or $x = 5$.

And $ax - 2by = ab$ and $xy = bb$ give $\frac{ax - ab}{2b}$ $(= y) = \frac{bb}{x}$; or by due ordering the Terms $xx - bx - \frac{2b^3}{a} = 0$.

Also $\frac{bbx - aby}{a} = ab + xy$, and $bx + \frac{ayy}{c} = 2aa$, by taking away x, give $\frac{aby + aab}{bb - ay}$ $(= x) = \frac{2aac - ayy}{bc}$; and by Reduction $y^3 - \frac{bb}{a}yy - \frac{2aac + bbc}{a}y + bbc = 0$.

Lastly, $x + y - z = 0$ and $ay = xz$ by taking away z give $x + y\ (= z) = \frac{ay}{x}$ or $xx + xy = ay$.

 The

LXXVI. (*d*) The Values may be equated although the unknown is of more than one Dimension in each, as is seen in LXXVIII; but then, the unknown will not be exterminated by one Operation, but only lowered by one Dimension.

The same is also performed by subtracting either of the Values of the unknown Quantities from the other, and making the Remainder equal to nothing. Thus, in the first of the Examples, take away $3b - 2x$ from $a + x - b$, and there will remain $a + 3x - 4b = 0$, or $x = \frac{4b - a}{3}$.

The Extermination of an unknown Quantity, by substituting its Value for it.

LXXVII. *WHEN, at least, in one of the Equations the Quantity to be exterminated is only of one Dimension, its Value is to be sought in that Equation, and then to be substituted in its room in the other Equation.* Thus, having proposed $xyy = b^3$, and $xx + yy = by - ax$; to exterminate x, the first will give $\frac{b^3}{yy} = x$; wherefore I substitute in the second $\frac{b^3}{yy}$ in the room of x, and there comes out $\frac{b^6}{y^4} + yy = by - \frac{ab^3}{yy}$, and by Reduction $y^6 - by^5 + ab^3yy + b^6 = 0$ (*e*).

But having proposed $ayy + aay = z^3$, and $yz - ay = az$, to take away y, the second will give $y = \frac{az}{z - a}$. Wherefore for y I substitute $\frac{az}{z - a}$ into the first, and

LXXVII. (*e*) *In all Substitutions, the Value substituted must have its own Sign, if that of the Quantity into whose Place it is put, is affirmative; but the contrary Sign, if negative.* And this Value must be multiplied, or divided, or involved, or evolved, &c. in the same manner as was the unknown, into the Place of which it is substituted.

and there comes out $\frac{a^3zz}{zz-2az+aa}+\frac{a^3z}{z-a}=z^3$. And by Reduction, $z^4-2az^3+aazz-2a^3z+a^4=0$.

In the like manner, having proposed $\frac{xy}{c}=z$ and $cy+zx=cc$, to take away z, I substitute in its room $\frac{xy}{c}$ in the second Equation, and there comes out $cy+\frac{xxy}{c}=cc$.

But a Person used to these Sorts of Computations, will oftentimes find shorter Methods than these by which the unknown Quantity may be exterminated. Thus, having $ax=\frac{bbx-b^3}{z}$ and $x=\frac{az}{x-b}$, if equal Quantities are multiplied by Equals, there will come out equal Quantities, viz. $axx=abb$, or $x=b$.

But I leave particular Cases of this kind to be found out by the Students, as Occasion shall offer.

The Extermination of an unknown Quantity of several Dimensions in each Equation.

LXXVIII. *WHEN the Quantity to be taken away is of more than one Dimension in both the Equations, the Value of its greatest Power must be sought in both; then if those Powers are not the same, the Equation that involves the lesser Power must be multiplied by the Quantity to be taken away, or by its Square, or Cube, &c. that it may become of the same Power with the other Equation. Then the Values of those Powers are to be made equal, and there will come out a new Equation, where the greatest Power or Dimension of the Quantity to be taken away is diminished. And*

 by

by repeating this Operation, the Quantity will at length be taken away (*f*).

As if you have $xx + 5x = 3yy$ and $2xy - 3xx = 4$; to take away x, the first Equation will give $xx = -5x + 3yy$, and the second $xx = \frac{2xy - 4}{3}$. I put therefore $3yy - 5x = \frac{2xy - 4}{3}$, and so x is reduced to only one Dimension, and so may be taken away by what I have before shewn, viz. by a due Reduction of the last Equation there comes out $9yy - 15x = 2xy - 4$, or $x = \frac{9yy + 4}{2y + 15}$. I therefore substitute this Value for x in one of the Equations first proposed (as in $xx + 5x = 3yy$) and there arises $\frac{81y^4 + 72yy + 16}{4yy + 60y + 225} + \frac{45yy + 20}{2y + 15} = 3yy$. To reduce which into Order, I multiply by $4yy + 60yy + 225$, and there comes out $81y^4 + 72yy + 16 + 90y^3 + 40y + 675yy + 300 = 12y^4 + 180y^3 + 675yy$, or $69y^4 - 90y^3 + 72yy + 40y + 316 = 0$.

Moreover, if you have $y^3 = xyy + 3x$, and $yy = xx - xy - 3$; to take away y, I multiply the latter Equation by y, and you have $y^3 = xxy - xyy - 3y$, of as many Dimensions as the former. Now, by making the Values of y^3 equal to one another, I have $xyy + 3x = xxy - xyy - 3y$, where y is depressed to two Dimensions. By this therefore, and the most simple one of the Equations first proposed $yy = xx - xy - 3$, the Quantity

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LXXVIII. (*f*) Or, raise the Value, found in the Equations of lowest Dimensions, to the Dimensions of the other Equation, and equate this raised Value with the Value found in the other Equation.

Quantity y may be wholly taken away by the same Method as in the former Example.

There are moreover other Methods by which this may be done, and that oftentimes more concisely. As if there be given $yy = \frac{2x^2y}{a} + xx$, and $yy = 2xy + \frac{x^4}{aa}$; that y may be extirpated, extract the Root y in each, as is shewn in the seventh Rule, and there will come out $y = \frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx}$, and $y = x + \sqrt{\frac{x^4}{aa} + xx}$. Now, by making these two Values of y equal you will have $\frac{xx}{a} + \sqrt{\frac{x^4}{aa} + xx} = x + \sqrt{\frac{x^4}{aa} + xx}$, and by rejecting the equal Quantities $\sqrt{\frac{x^4}{aa} + xx}$, there will remain $\frac{xx}{a} = x$, or $xx = ax$, and $x = a$.

Moreover, to take x out of the Equations $x + y + \frac{yy}{x} = 20$, and $xx + yy + \frac{y^4}{xx} = 140$, take away y from the Parts of the first Equation, and there remains $x + \frac{yy}{x} = 20 - y$, and squaring the Parts, it becomes $xx + 2yy + \frac{y^4}{xx} = 400 - 40y + yy$, and taking away yy on both Sides, there remains $xx + yy + \frac{yyyy}{xx} = 400 - 40y$. Wherefore, since $400 - 40y$ and 140 are equal to the same Quantities, $400 - 40y$ will be equal to 140, or $y = 6\frac{1}{2}$; and so you may contract the Matter in most other Equations.

LXXIX. But

LXXIX. *But when the Quantity to be exterminated is of several Dimensions, sometimes there is required a very laborious Calculus to exterminate it out of the Equations; but then the Labour will be much diminished by the following Examples made use of as Rules.*

Rule I.

From $dxx+bx+c=0$, and $fxx+gx+h=0$, x being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ah:+\overline{bh-cg}\times bf:\times\overline{agg+cff}\times c=0.$$

Rule II.

From $ax^3+bxx+cx+d=0$, and $fxx+gx+h=0$, x being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ahh:+\overline{bh-cg-2df}\times bfh:+\overline{ch-dg}\times\overline{agg+cff}:+\overline{3agh+bgg+dff}\times df=0.$$

Rule III.

From $ax^4+bx^3+cxx+dx+e=0$, and $fxx+gx+h=0$, x being exterminated, there comes out

$$\overline{ah-bg-2cf}\times ah^3:+\overline{bh-cg-2df}\times bfhh:+\overline{agg+cff}\times\overline{chh+dgh+egg-2efh+3agh+bgg+dff}\times dfh:+\overline{2ahh+3bgh-dfg+eff}\times eff:-\overline{bg-2ah}\times efgg=0.$$

Rule IV.

From $ax^3+bxx+cx+d=0$, and $fx^3+gx^2+hx+k=0$, x being exterminated, there comes out

$$\overline{ah-bg-2cf}\times\overline{adhh-achk}:+\overline{ak+bh-cg-2df}\times bdfh:-\overline{ak+bh+2cg+3df}\times aakk+\overline{cdh-ddg-cck+2bdk}\times\overline{agg+cff}:+$$

+ $3agh$

$+\overline{3agh+bgg+dff-3afk}\times ddf : -\overline{3ak-bh+cg+df}\times bcfk : +\overline{bk-2dg}\times bbfk-\overline{bbk-3adh-cdf}\times agk=0.$

For Example, to exterminate x out of the Equations $xx+5x-3yy=0$, and $3xx-2xy+4=0$: I respectively substitute in the first Rule for a, b, c; f, g, and h, these Quantities, 1, 5, $-3yy$; 3, $-2y$ and 4; and duly observing the Signs $+$ and $-$, there arises $\overline{4+10y+18yy}\times 4 : +\overline{20-6y^3}\times 15 : +\overline{4yy-27yy}\times -3yy=0$, or $16+40y+72yy+300-90y^3+69y^4=0$.

By the like Reason that y may be expunged out of the Equations $y^3-xyy-3x=0$, and $yy+xy-xx+3=0$, I substitute into the second Rule for a, b, c, d; f, g, h, and x, these Quantities 1, $-x$, 0, $-3x$; 1, x, $-xx+3$, and y respectively, and there comes out $\overline{3-xx+xx}\times\overline{9-6xx+x^4} : -\overline{3x+x^3+6x}\times\overline{-3x+x^3} : +3xx\times xx : +\overline{9x-3x^3-x^3-3x}\times -3x=0$: Then blotting out the superfluous Quantities and multiplying, you have $27-18xx+3x^4$, $-9xx+x^6$, $+3x^4-18x^2+12x^4=0$. And ordering, $x^6+18x^4-45xx+27=0$.

LXXX. Hitherto we have discoursed of taking away one unknown Quantity out of two Equations. Now, *if several are to be taken out of several, the Business must be done by degrees:* Out of the Equations $ax=yz$, $x+y=z$, and $5x=y+3z$; if the Quantity y is to be found, first, take out one of the Quantities x or z, suppose x, by substituting for it its Value $\frac{yz}{a}$ (found by the

first

first Equation) in the second and third Equations; and then you will have $\frac{yz}{a} + y = z$, and $\frac{5yz}{a} = y + 3z$, out of which take away z as above (*g*).

Of

LXXX. (*g*) 195. *When two Equations are given involving two unknown Quantities, as* $\begin{matrix} ax + by = c \\ dx + ey = f \end{matrix}$ *then shall* $y = \frac{af - dc}{ae - db}$. Where the Numerator is the Difference of the Products of the opposite Coefficients in the Order in which y is not found, and the Denominator is the Difference of the Products of the opposite Coefficients taken from the Orders that involve the two unknown Quantities. (Coefficients are of the same Order which either affect no unknown Quantity, as c and f; or the same unknown Quantity in the different Equations, as a and d. Coefficients are opposite when they affect the different unknown Quantities in the different Equations, as a and e; d and b). For from the first Equation, $ax = c - by$; and $x = \frac{c - by}{a}$ (R. 1, 5.); also from the second, $dx = f - ey$; and $x = \frac{f - ey}{d}$: Whence $\frac{c - by}{a} = \frac{f - ey}{d}$ (LXXVI.); and $cd - dby = af - aey$ (R. 3.); and $aey - dby = af - cd$ (R. 1.); and $y = \frac{af - cd}{ae - db}$ (R. 5.): After the same Manner, $x = \frac{ce - bf}{ae - db}$.

196. When

Of the Method of taking away any Number of Surd Quantities out of Equations.

LXXXI. *HITHERTO may be referred the Extermination of Surd Quantities, by making them equal to any Letters.* As if you have $\sqrt{ay} - \sqrt{aa - ay} = 2a + \sqrt[3]{ayy}$, by writing t for $\sqrt{ay}$, and v for $\sqrt{aa - ay}$, and x for $\sqrt[3]{ayy}$, you will have the Equations

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196. *When three Equations involve three unknown Quantities,* x, y, *and* z; *thus,* $\begin{matrix} ax + by + cz = m \\ dx + ey + fz = n \\ gx + hy + kz = p \end{matrix}$ *then*

$$z = \frac{aep - ahn + dhm - dbp + ghn - gem}{aek - ahf + dhc - dbk + ghf - gec}.$$

Where the Numerator consists of all the different Products, which can be made of three opposite Coefficients taken from the Orders in which z is not found; and the Denominator consists of all the Products that can be made of the three opposite Coefficients taken from the Orders which involve the three unknown Quantities.

For $y = \dfrac{an - afz - dm + dcz}{ae - db}$, and $y = \dfrac{ap - akz - gm + gcz}{ab - gb}$ (195); therefore $\dfrac{an - afz - dm + dcz}{ae - db} = \dfrac{ap - akz - gm + gcz}{ab - gb}$ (LXXVI), and $\overline{an - afz - dm + dcz} \times ab - gb \times \overline{an - afz} + gbdm - gbdcz = \overline{ap - gm - akz + gcz} \times ae - db \times \overline{ap - akz} + gbdm - gbdcz$ (R. 5.): Take $gbdm - gbdcz$ from both tions $t - v = 2a + x$, $tt = ay$, $vv = aa - ay$, and $x^3 = ayy$, out of which taking away by degrees t, v, and x, there will result an Equation entirely free from Surdity.

How a Question may be brought to an Equation.

LXXXII. AFTER the Learner has been some Time exercised in managing and transforming Equations, Order requires that he should try his Skill in bringing Questions to an Equation. And *any Question being proposed, his Skill is particularly required to denote all its Conditions by so many Equations. To do which he must first consider whether the Propositions or Sentences in which it is expressed, be all of them fit to be denoted in algebraick Terms,* just as we express our Conceptions in Latin or Greek Characters. *And if so,* (as will happen in Questions conversant about Numbers or abstract Qantities) *then let him give Names to both known and unknown Quantities, as far as Occasion requires; and express the Sense of the Question in the Analytick Language,* if I may so speak. *And the Conditions thus translated to algebraick Terms will give as many Equations as are necessary to solve it.*

As

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both Members, and divide by ba, so shall $\overline{an - dm - afz + dcz} \times b - ghn + gbfz = \overline{ap - gm - akz + gcz} \times e - dbp + dhkz$; transpose and divide, and so shall you find

$$z = \frac{aep - ahn + dhm - dbp + gbn - gem}{aek - abf + dhc - dbk + gbf - gec}:$$

After the same Manner,

$$y = \frac{afp - akn + dkm - dcp + gen - gfm}{aek - ahf + dhc - dbk + gbf - gec};$$

And,

$$x = \frac{bfp - bkn + ekm - ecp + hcn - hfm}{aek - ahf + dhc - dbk + gbf - gec}.$$

As if there are required three Numbers in continual Proportion, whose Sum is 20, and the Sum of their Squares 140; putting x, y, and z for the Names of the three Numbers sought, the Question will be translated out of the Verbal to the Symbolical Expression, as follows:

The Question in Words.	*The same in Symbols.*
There are sought three Numbers on these Conditions:	x, y, z?
That they shall be continually proportional.	$x : y :: y : z$, or $xz = yy$.
That the Sum shall be 20.	$x + y + z = 20$.
And the Sum of their Squares 140.	$xx + yy + zz = 140$.

And so the Question is brought to these Equations, viz. $xz = yy$, $x + y + z = 20$, and $xx + yy + zz = 140$, by the Help whereof x, y, and z, are to be found by the Rules delivered above.

But you must note, That the Solutions of Questions are (for the most Part) so much the more expedite and artificial, by how fewer unknown Quantities you have at first. Thus, in the Question proposed, putting x for the first Number, and y for the second, $\frac{yy}{x}$ will be the third Proportional; which then being put for the third Number, I bring the Question into Equations, as follows:

The Question in Words.	*Symbolically.*
There are sought three Numbers in continual Proportion.	$x, y, \frac{yy}{x}$?
Whose Sum is 20.	$x + y + \frac{yy}{x} = 20$.
And the Sum of their Squares 140.	$xx + yy + \frac{y^4}{xx} = 140$.

You

You have therefore the Equations $x+y+\frac{yy}{x}=20$, and $xx+yy+\frac{y^4}{xx}=140$, by the Reduction whereof x and y are to be determined.

Take another Example. A certain Merchant encreases his Estate yearly by a third Part, abating 100 *l.* which he spends yearly in his Family; and after three Years he finds his Estate doubled. *Query*, What was he worth?

To resolve this, you must know there are or lie hid several Propositions, which are all thus found out and laid down.

In English.	*Algebraically.*
A Merchant has an Estate - - -	x.
Out of which the first Year he expends 100 *l.*	$x-100$.
And augments the rest by one third -	$x-100+\frac{x-100}{3}$ or $\frac{4x-400}{3}$.
And the second Year expends 100 *l.* -	$\frac{4x-400}{3}-100$ or $\frac{4x-700}{3}$.
And augments the rest by a third - -	$\frac{4x-700}{3}+\frac{4x-700}{9}$ or $\frac{16x-2800}{9}$.
And so the third Year expends 100 *l.* -	$\frac{16x-2800}{9}-100$ or $\frac{16x-3700}{9}$.
And by the rest gains likewise one third Part - - - -	$\frac{16x-3700}{9}+\frac{16x-3700}{27}$, or $\frac{64x-14800}{27}$.
And he becomes at length twice as rich as at first - -	$\frac{64x-14800}{27}=2x$.

Therefore

Therefore the Question is brought to this Equation, $\frac{64x - 14800}{27} = 2x$, by the Reduction whereof you are to find x; viz. multiply it by 27, and you have $64x - 14800 = 54x$; subtract $54x$, and there remains $10x - 14800 = 0$, or $10x = 14800$, and dividing by 10, you have $x = 1480$. Wherefore, 1480 *l.* was his Estate at first, as also his Profit or Gain since.

You see therefore, that *to the Solution of Questions which only regard Numbers, or the abstracted Relations of Quantities, there is scarce any Thing else required, than that the Problem be translated* out of the English, or any other Tongue it is proposed in, *into the algebraical Language, that is, into Characters fit to denote our Conceptions of the Relations of Quantities. But it may sometimes happen, that the Language or the Words wherein the State of the Question is expressed, may seem unfit to be turned into the algebraical Language; but making Use of a few Changes, and attending to the Sense, rather than the Sound of the Words, the Version will become easy* (*a*). Thus, the Forms of Speech among different Nations have their proper Idioms; which, where they happen, the Translation out of one into another is not to be made literally, but to be determined by the Sense. But that I may illustrate these Sorts of Problems, and make familiar the Method of reducing them to Equations; and since Arts are more easily learned by Examples

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LXXXII. (*a*) If such Equations cannot be derived without some previous Operations (which frequently happens to be the case), let the Learner consider what Method or Process he would use, to prove the Truth of the Solution, were the Numbers that answer the Conditions of the Question to be given; and then by following the very same Steps, only using Symbols instead of Numbers, the Question will be brought to an Equation.

amples than Precepts, I have thought fit to adjoin the Solutions of the following Problems.

PROBLEM I.

Having given the Sum of two Numbers, a, *and the Difference of their Squares* b, *to find the Numbers.*

Let the least of them be x, the other will be $a - x$, and their Squares xx, and $aa - 2ax + xx$: The Difference whereof $aa - 2ax$ is supposed b. Therefore $aa - 2ax = b$, and then by Reduction $aa - b = 2ax$, or $\frac{aa - b}{2a} \left(= \frac{1}{2}a - \frac{b}{2a}\right) = x$.

For Example, if the Sum of the Numbers or a be 8, and the Difference of the Squares or b be 16; $\frac{1}{2}a - \frac{b}{2a}$ $(= 4 - 1)$ will be $= 3 = x$, and $a - x = 5$. Wherefore the Numbers are 3 and 5.

PROBLEM II.

To find three Quantities, x, y, *and* z, *the Sum of any two of which shall be given.*

If the Sum of two of them, viz. x and y be a; of x and z, b; and of y and z, c; there will be had three Equations to determine the three Quantities sought, x, y, and z, viz. $x + y = a$, $x + z = b$, and $y + z = c$. Now, that two of the unknown Quantities, viz. y and z may be exterminated, take away x on both Sides in the first and second Equation, and you will have $y = a - x$, and $z = b - x$; which Values substitute for y and z in the third Equation, and there will come out $a - x + b - x = c$, and by Reduction $x = \frac{a + b - c}{2}$; and having found

found x, the Equations above $y = a - x$ and $z = b - x$ will give y and z.

EXAMPLE. If the Sum of x and y be 9, of x and z, 10, and y and z, 13; then, in the Values of x, y, and z, write 9 for a, 10 for b, and 13 for c; and you will have $a + b - c = 6$, and consequently $x \left(= \frac{a+b-c}{2}\right) = 3$, $y\ (= a - x) = 6$, and $z\ (= b - x) = 7$.

PROBLEM III.

To divide a given Quantity into as many Parts as you please; so that the greater Parts may exceed the least by any given Differences.

Let a be a Quantity to be divided into four such Parts, and its first or least Part call x, and the Excess of the second Part above this call b, and of the third Part c, and of the fourth d; and $x + b$ will be the second Part, $x + c$ the third, and $x + d$ the fourth, the Aggregate of all which $4x + b + c + d$ is equal to the whole Line a. Take away on both Sides $b + c + d$, and there remains $4x = a - b - c - d$, or $x = \frac{a - b - c - d}{4}$.

EXAMPLE. Let there be proposed a Line of 20 Feet, so to be divided into four Parts, that the Excess of the second above the first Part shall be 2 Feet, of the third 3 Feet, and of the fourth 7 Feet; and the four Parts will be $x \left(= \frac{a - b - c - d}{4} \text{ or } \frac{20 - 2 - 3 - 7}{4}\right) = 2$, $x + b = 4$, $x + c = 5$, and $x + d = 9$.

After

After the ſame Manner a Quantity is divided into more Parts on the ſame Conditions (*b*).

PROBLEM IV.

A Perſon being willing to diſtribute ſome Money among Beggars, wanted eight Pence to give three Pence a piece to them; he therefore gave to each two Pence, and had three Pence remaining over and above. To find the Number of the Beggars.

Let the Number of the Beggars be x, and there will be wanting eight Pence to give all $3x$ Pence; he has therefore $\overline{3x - 8}$ Pence. Out of theſe he gives $2x$ Pence, and the remaining Pence $\overline{x - 8}$ are three. That is, $x - 8 = 3$, or $x = 11$.

PROBLEM V.

If two Poſt-Boys A *and* B, *at* 59 *Miles Diſtance from one another, ſet out in the Morning in order to meet. And* A *rides* 7 *Miles in two Hours, and* B 8 *Miles in three Hours, and* B *begins his Journey one Hour later than* A; *to find what Number of Miles* A *will ride before he meets* B.

Call that Length x, and you will have $59 - x$, the Length of *B*'s Journey. And ſince *A* travels 7 Miles in two Hours, he will make the Space x in $\frac{2x}{7}$ Hours, becauſe 7 Miles : 2 Hours :: x Miles : $\frac{2x}{7}$ Hours. And ſo,

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Prob. III. (*b*) For let the Dividend be a, the Number of the Parts be n, and the Sum of all the Differences be d; and we ſhall have $x = \frac{a - d}{n}$.

so, ſince *B* rides 8 Miles in 3 Hours, he will deſcribe his Space or ride his Journey $59-x$ in $\frac{177-3x}{8}$ Hours. Now, ſince the Difference of theſe Times is one Hour; to the End they may become equal, add that Difference to the ſhorter Time $\frac{177-3x}{8}$, and you will have $1+\frac{177-3x}{8}=\frac{2x}{7}$, and by Reduction $35=x$. For, multiplying by 8 you have $185-3x=\frac{16x}{7}$. Then multiplying alſo by 7 you have $1295-21x=16x$, or $1295=37x$. And, laſtly, dividing by 37, there ariſes $35=x$. Therefore, 35 Miles is the Diſtance that *A* muſt ride before he meets *B*.

The ſame more generally.

Having given the Velocities of two moveable Bodies, A *and* B, *tending to the ſame Place, together with the Interval or Diſtance of the Places and Times from, and in which they begin to move; to determine the Place they ſhall meet in.*

Suppoſe, the Velocity of the Body *A* to be ſuch, that it ſhall paſs over the Space c in the Time f; and of the Body *B* to be ſuch as it ſhall paſs over the Space d in the Time g; and that the Interval of the Places is e, and h the Interval of the Times in which they begin to move.

CASE I. Then if both tend to the ſame Place [or the ſame Way], and *A* be the Body that, at the Beginning of the Motion, is fartheſt diſtant from the Place they tend to: Call that Diſtance x, and ſubtract from it the Diſtance e, and there will remain $x-e$ for the Diſtance of *B* from the Place it tends to. And ſince *A* paſſes through the Space c in the Time f, the Time in which it will paſs over the Space x will be $\frac{fx}{c}$, becauſe the

Space c is to the Time f, as the Space x to the Time $\frac{fx}{c}$. And so, since B passes the Space d in the Time g, the Time in which it will pass the Space $x - e$ will be $\frac{gx - ge}{d}$. Now since the Difference of these Times is supposed h, that they may become equal, add h to the shorter Time, *viz.* to the Time $\frac{fx}{c}$ if B begins to move first, and you will have $\frac{fx}{c} + h = \frac{gx - ge}{d}$, and by Reduction $\frac{cge + cdh}{cg - df}$ or $\frac{ge + dh}{g - \frac{d}{c}f} = x$. But if A begins to move first, add h to the Time $\frac{gx - ge}{d}$, and you will have $\frac{fx}{c} = h + \frac{gx - ge}{d}$, and by Reduction $\frac{cge - cdh}{cg - df} = x$.

Case II. If the moveable Bodies proceed towards one another, and x, as before, be made the initial Distance of the moveable Body A, from the Place it is to move to; then $e - x$ will be the initial Distance of the Body B from the same Place; and $\frac{fx}{c}$ the Time in which A will describe the Distance x, and $\frac{ge - gx}{d}$ the Time in which B will describe its Distance $e - x$. To the lesser of which Times, as above, add the Difference h, *viz.* to the Time $\frac{fx}{c}$ if B begin first to move, and so you will have $\frac{fx}{c} + h = \frac{ge - gx}{d}$, and by Reduction

$\frac{cge - cdh}{cg + df}$

$\frac{cge - cdb}{cg + df} = x$. But if A begins first to move, add b to the Time $\frac{ge - gx}{d}$ and it will become $\frac{fx}{c} = b + \frac{ge - gx}{d}$, and by Reduction $\frac{cge + cdb}{cg + df} = x$.

Example I. If the Sun moves every Day one Degree, and the Moon thirteen, and at a certain Time the Sun be at the Beginning of Cancer, and, in three Days after, the Moon in the Beginning of Aries, the Place of their next following Conjunction is demanded. Answer, in $10\frac{3}{4}$ Degrees of Cancer: For since they both are going towards the same Parts, and the Motion of the Moon, which is farther distant from the Conjunction, hath a later Epocha, the Moon will be A, the Sun B, and $\frac{cge + cdb}{cg - df}$ the Length of the Moon's Way, which, if you write 13 for c, 1 for f, d, and g, 90 for e, and 3 for b, will become $\frac{13 \times 1 \times 90 + 13 \times 1 \times 3}{13 \times 1 - 1 \times 1}$, that is, $\frac{1209}{12}$, or $100\frac{3}{4}$ Degrees; and then add these Degrees to the Beginning of Aries, and there will come out $10\frac{1}{4}$ Degrees of Cancer.

Example II. If two Post-Boys, A and B, being in the Morning 59 Miles asunder, set out to meet each other, and A goes 7 Miles in 2 Hours, and B 8 Miles in 3 Hours, and B begins his Journey 1 Hour later than A, it is demanded how far A will have gone before he meets B? Answer, 35 Miles. For since they go towards each other, and A sets out first, $\frac{cge + cdb}{cg + df}$ will be the Length of his Journey; and writing 7 for c, 2 for f, 8 for d, 3 for g, 59 for e, and 1 for b, this will become $\frac{7\times 3\times 59+7\times 8\times 1}{7\times 3+8\times 2}$, that is, $\frac{1295}{37}$ or 35.

Problem VI.

Giving the Power of any Agent, to find how many such Agents will perform a given Effect a, *in a given Time* b.

Let the Power of the Agent be such that it can produce the Effect c in the Time d, and it will be as the Time d to the Time b, so the Effect c, which that Agent can produce in the Time d, to the Effect which he can produce in the Time b, which therefore will be $\frac{bc}{d}$. Again, as the Effect of one Agent $\frac{bc}{d}$ to the Effect of all a; so that single Agent to all the Agents; and thus the Number of the Agents will be $\frac{ad}{bc}$.

Example. If a Scribe can in 8 Days write 15 Sheets, how many such Scribes must there be to write 405 Sheets in 9 Days? Answer, 24. For if 8 be substituted for d, 15 for c, 405 for a, and 9 for b, the Number $\frac{ad}{bc}$ will become $\frac{405\times 8}{9\times 15}$, that is, $\frac{3240}{135}$, or 24.

Problem VII.

The Forces of several Agents being given, to determine x *the Time, wherein they will jointly perform a given Effect* d.

Let the Forces of the Agents A, B, C, be supposed, which in the Times e, f, g can produce the Effects a, b, c respectively; and these in the Time x will produce

duce the Effects $\frac{ax}{e}$, $\frac{bx}{f}$, $\frac{cx}{g}$; wherefore is $\frac{ax}{e} + \frac{bx}{f} + \frac{cx}{g} = d$, and by Reduction $x = \frac{d}{\frac{a}{e} + \frac{b}{f} + \frac{c}{g}}$.

EXAMPLE. Three Workmen can do a Piece of Work in certain Times, viz. *A* once in 3 Weeks, *B* thrice in 8 Weeks, and *C* five Times in 12 Weeks. It is desired to know in what Time they can finish it jointly? Here then are the Forces of the Agents *A*, *B*, *C*, which in the Times 3, 8, 12, can produce the Effects 1, 3, 5, respectively, and the Time is sought wherein they can do one Effect. Wherefore, for *a*, *b*, *c*; *d*; *e*, *f*, *g*, write 1, 3, 5, 1, 3, 8, 12, and there will arise $x = \frac{1}{\frac{1}{3} + \frac{3}{8} + \frac{5}{12}}$, or $\frac{8}{9}$ of a Week, that is, [allowing 6 working Days to a Week, and 12 Hours to each Day] 5 Days and 4 Hours, the Time wherein they will jointly finish it.

PROBLEM VIII.

So, to compound unlike Mixtures of two or more Things, that the Things mixed together may have a given Ratio to one another.

Let the given Quantity of one Mixture be $d\mathrm{A} + e\mathrm{B} + f\mathrm{C}$, the same Quantity of another Mixture $g\mathrm{A} + h\mathrm{B} + k\mathrm{C}$, and the same of a third $l\mathrm{A} + m\mathrm{B} + n\mathrm{C}$, where A, B, C, denote the Things mixed; and *d*, *e*, *f*, *g*, *h*, &c. the Proportions of the same in the Mixtures. And let $p\mathrm{A} + q\mathrm{B} + r\mathrm{C}$ be the Mixture which must be composed of these three Mixtures; and suppose *x*, *y*, and *z*, to be the Numbers, by which if the three given Mixtures be respectively multiplied, their Sum will become $p\mathrm{A} + q\mathrm{B} + r\mathrm{C}$. Therefore is

$$\left\{\begin{array}{l} dx\,\mathrm{A} + ex\,\mathrm{B} + fx\,\mathrm{C} \\ + gy\,\mathrm{A} + hy\,\mathrm{B} + ky\,\mathrm{C} \\ + lz\,\mathrm{A} + mz\,\mathrm{B} + nz\,\mathrm{C} \end{array}\right\} = p\mathrm{A} + q\mathrm{B} + r\mathrm{C}.$$

And then comparing the Terms by making $dx + gy + lz = p$,

$+ lz = p$, $ex + hy + mz = q$, and $fx + ky + nz = r$ (c), and by Reduction $x = \frac{p - gy - lz}{d} = \frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$. And again, the Equations $\frac{p - gy - lz}{d} = \frac{q - hy - mz}{e}$, and $\frac{q - hy - mz}{e} = \frac{r - ky - nz}{f}$ by Reduction give $\frac{ep - dq + dmz - elz}{eg - dh} (= y) = \frac{fq - er + enz - fmz}{fh - ek}$: Which, if abbreviated by writing α for $ep - dq$, β for $dm - el$, γ for $eg - dh$, δ for $fq - er$, ζ for $en - fm$, and θ for $fh - ek$, will become $\frac{\alpha + \beta z}{\gamma} = \frac{\delta + \zeta z}{\theta}$, and by Reduction $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$. Having found z, put $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

EXAMPLE. If there were three Mixtures of Metals melted down together; of the first of which a Pound [Averdupois] contains of Silver ℥ 12, of Brass ℥ 1, and of Tin ℥ 3; of the second, a Pound contains of Silver ℥ 1, of Brass ℥ 12, and of Tin ℥ 3; and a Pound of the third contains of Brass ℥ 14, of Tin ℥ 2, and no Silver; and let these Mixtures be so to be compounded, that a Pound of the Composition may contain of Silver ℥ 4, of Brass ℥ 9, and of Tin ℥ 3: For d, e, f; g, h, k; l, m, n; p, q, r; write 12, 1, 3; 1, 12, 3; 0, 14, 2; 4, 9, 3, respectively, and α will be ($= ep - dq = 1 \times 4 - 12 \times 9$) $= -104$, and β ($= dm - el = 12 \times 14 - 1 \times 0$) $= 168$, and so $\gamma = -143$; $\delta = 24$, $\zeta = -40$,

Prob. VIII. (c) Number 182, 183.

$= -40$, and $\theta = 33$. And therefore $z \left(= \frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = \frac{-3432 + 3432}{5720 - 5544}\right) = 0$; $y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-104 + 0}{-143}\right) = \frac{8}{11}$, and $x \left(= \frac{p - gy - lz}{d} = \frac{4 - \frac{8}{11}}{12}\right) = \frac{3}{11}$. Wherefore, if there be mixed $\frac{8}{11}$ Parts of a Pound of the second Mixture, $\frac{3}{11}$ Parts of a Pound of the first, and nothing of the third, the Aggregate will be a Pound, containing four Ounces of Silver, nine of Brass, and three of Tin.

PROBLEM IX.

The Prices of several Mixtures of the same Things, and the Proportions of the Things mixed together being given, to determine the Price of each of the Things mixed.

Of each of the Things A, B, C, let the Price of the Mixture $dA + gB + lC$ be p, of the Mixture $eA + hB + mC$ the Price q, and of the Mixture $fA + kB + nC$ the Price r; and of these Things A, B, C, let the Prices x, y, z, be demanded. For the Things A, B, C, substitute their Prices x, y, z, and there will arise the Equations $dx + gy + lz = p$, $ex + hy + mz = q$, and $fx + ky + nz = r$; from which, by proceeding as in the foregoing Problem, there will in like manner be got $\frac{\theta\alpha - \gamma\delta}{\gamma\zeta - \beta\theta} = z$, $\frac{\alpha + \beta z}{\gamma} = y$, and $\frac{p - gy - lz}{d} = x$.

EXAMPLE. One bought 40 Bushels of Wheat, 24 Bushels of Barley, and 20 Bushels of Oats together for 15 Pounds 12 Shillings. Again, he bought of the same Grain 26 Bushels of Wheat, 30 Bushels of Barley, and 50 Bushels of Oats together, for 16 Pounds. And thirdly, he bought of the like Kind of Grain, 24 Bushels of Wheat, 120 Bushels of Barley, and 100 Bushels of Oats together, for 34 Pounds. It is demanded at what

what Rate a Buſhel of each of the Grains ought to be valued. Anſwer, a Buſhel of Wheat at 5 Shillings, of Barley at 3 Shillings, and of Oats at 2 Shillings. For inſtead of d, g, l; e, h, m; f, k, n; p, q, r; by writing reſpectively 40, 24, 20; 26, 30, 50; 24, 120, 100; $15\frac{3}{5}$, 16, and 34, there ariſes α $(= ep - dq = 26 \times 15\frac{3}{5} - 40 \times 16) = -234\frac{2}{5}$; and β $(= dm - el = 40 \times 50 - 26 \times 20) = 1480$; and thus $\gamma = -576$, $\delta = -500$, $\zeta = 1400$, and $\theta = -2400$. Then z

$$\left(= \frac{\theta\alpha - \delta\gamma}{\gamma\zeta - \beta\theta} = \frac{562560 - 288000}{-806400 + 3552000} = \frac{274560}{2745600}\right)$$

$$= \tfrac{1}{10};\ y \left(= \frac{\alpha + \beta z}{\gamma} = \frac{-234\frac{2}{5} + 148}{-576}\right) = \tfrac{3}{20};$$

$$\text{and } x \left(= \frac{p - gy - lz}{d} = \frac{15\frac{3}{5} - \frac{18}{5} - 2}{40}\right) = \tfrac{1}{4}.$$

Therefore a Buſhel of Wheat coſt $\frac{1}{4}$ ℔, or 5 Shillings; a Buſhel of Barley $\frac{3}{20}$ ℔, or 3 Shillings; and a Buſhel of Oats $\frac{1}{10}$ ℔, or 2 Shillings.

Problem X.

There being given the ſpecifick Gravity both of the Mixture and the Things mixed, to find the Proportion of the mixed Things to one another.

Let e be the ſpecifick Gravity of the Mixture $A + B$, a the ſpecific Gravity of A, and b the ſpecifick Gravity of B; and ſince the abſolute Gravity, or the Weight, is compoſed of the Bulk of the Body and the ſpecifick Gravity, aA will be the Weight of A; bB of B; and $eA + eB$ the Weight of the Mixture $A + B$; and therefore $aA + bB = eA + eB$; and from thence $aA - eA = eB - bB$ or $e - b : a - e :: A : B$.

Example. Suppoſe the Gravity or Specifick Weight of Gold to be as 19, and of Silver as $10\frac{1}{3}$, and King Hiero's Crown as 17; and it will be $10 : 3$ $(e - b : a - e :: A : B)$ :: Bulk of Gold in the Crown : Bulk of

 Silver,

Silver, or 190 : 31 (:: 19 × 10 : $10\frac{1}{3}$ × 3 :: $a \times \overline{e-b}$: $b \times \overline{a-e}$) :: the Weight of Gold in the Crown, to the Weight of Silver, and 221 : 31 :: the Weight of the Crown, to the Weight of the Silver.

Problem XI.

If the Number of Oxen a *eat up the Meadow* b *in the Time* c; *and the Number of Oxen* d *eat up as good a Piece of Pasture* e *in the Time* f, *and the Grass grows uniformly; to find how many Oxen will eat up the like Pasture* g *in the Time* h.

If the Oxen a in the Time c eat up the Pasture b; then, by Proportion, the Oxen $\frac{e}{b}a$ in the same Time c, or the Oxen $\frac{ec}{bf}a$ in the Time f, or the Oxen $\frac{ec}{bh}a$ in the Time h will eat up the Pasture e; supposing the Grass did not grow at all after the Time c (d). But since, by reason of the Growth of the Grass, all the Oxen d in the Time f can eat up only the Meadow e, therefore that Growth of the Grass in the Meadow e in the Time $f-c$ will be so much as alone would be sufficient to feed the Oxen $d-\frac{eca}{bf}$ the Time f, that is, as

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Prob. XI. (d) For in equal Times c, as Field is to Field, so Number of Oxen to Number of Oxen; that is, $b : e :: a : \frac{ea}{b}$ in the same Time c; and in equal Fields, the Numbers of Oxen are reciprocally as the Times; that is, $f : c :: \frac{ea}{b} : \frac{eca}{bf}$, in the Time f; also $h : c :: \frac{ea}{b} : \frac{eca}{bh}$, in the Time h.

PROBLEM XII.

Having given the Magnitudes and Motions of Spherical Bodies perfectly elastick, moving in the same Right-Line, and striking against one another, to determine their Motions after Reflexion.

The Resolution of this Question depends on these Conditions, that each Body will suffer as much by Reaction as the Action of each is upon the other, and that they must recede from each other after Reflexion with the same Velocity or Swiftness as they met before it. These Things being supposed, let the Velocity of the Bodies A and B, be a and b respectively; and their Motions (as being composed of their Bulk and Velocity together) will be aA and bB. And if the Bodies tend the same Way, and A moving more swiftly, follows B, make x the Decrement of the Motion aA, and the Increment of the Motion bB arising by the Percussion; and the Motions after Reflexion will be $aA - x$ and $bB + x$; and the Celerities $\frac{aA - x}{A}$ and $\frac{bB + x}{B}$, whose Difference is $= a - b$ the Difference of the Celerities before Reflection. Therefore there arises this Equation $\frac{bB + x}{B} - \frac{aA - x}{A} = a - b$, and thence by Reduction x becomes $= \frac{2aAB - 2bAB}{A + B}$, which being substituted for x in the Celerities $\frac{aA - x}{A}$, and $\frac{bB + x}{B}$, there comes out $\frac{aA - aB + 2bB}{A + B}$ for the Celerity of A, and $\frac{2aA - bA + bB}{A + B}$ for the Celerity of B after Reflexion.

But

But if the Bodies move towards one another, then changing every where the Sign of b, the Velocities after Reflexion will be $\frac{aA - aB - 2bB}{A + B}$ and $\frac{2aA + bA - bB}{A + B}$; either of which, if they come out, by Chance, negative, it argues that Motion, after Reflexion, to tend a contrary Way to that which A tended to before Reflexion. Which is also to be understood of A's Motion in the former Case.

Example. If the homogeneous Bodies [or Bodies of the same Sort] A of 3 Pounds with 8 Degrees of Velocity, and B a Body of 9 Pounds with 2 Degrees of Velocity, tend the same Way; then for A, a, B and b, write 3, 8, 9, and 2; and $\left(\frac{aA - aB + 2bB}{A + B}\right)$ becomes -1, and $\left(\frac{2aA - bA + bB}{A + B}\right)$ becomes 5. Therefore A will return back with one Degree of Velocity after Reflexion, and B will go on with 5 Degrees.

Problem XIII.

To find three Numbers in continual Proportion, whose Sum shall be 20, and the Sum of their Squares 140?

Make the first of the Numbers $= x$, and the second $= y$, and the third will be $\frac{yy}{x}$, and consequently $x + y + \frac{yy}{x} = 20$; and $xx + yy + \frac{y^4}{xx} = 140$. And by Reduction $xx \begin{matrix} +y \\ -20 \end{matrix} x + yy = 0$, and $x^4 \begin{matrix} +yy \\ -140 \end{matrix} xx + y$ $= 0$. Now to exterminate x, for a, b, c, d, e, f, g, h, in the third Rule, substitute respectively 1, 0, $yy -$

$\overline{140, 0, y^4}; 1, y - 20$, and yy; and there will come out $\overline{-yy+280}\times y^6 : +\overline{2yy-40y+260}\times\overline{260y^4-40y^5}$ $: +3y^4\times y^4 : -2yy\times\overline{y^6-40y^5+400y} := 0$; and by Multiplication $1600y^6 - 20800y^5 - 67600y^4 = 0$. And by Reduction $4yy - 52y + 169 = 0$. Or (the Root being extracted) $2y - 13 = 0$, or $y = 6\frac{1}{2}$. Which is found more short by another Method before, but not so obvious as this. Moreover, to find x, substitute $6\frac{1}{2}$ for y in the Equation $xx\frac{+y}{-20}x + yy = 0$, and there will arise $xx - 13\frac{1}{2}x + 42\frac{1}{4} = 0$, or $xx = 13\frac{1}{2}x + 42\frac{1}{4}$, and having extracted the Root $x = 6\frac{3}{4}$ + or $-\sqrt{3\frac{5}{16}}$; viz. $6\frac{3}{4}+\sqrt{3\frac{5}{16}}$ is the greatest of the three Numbers sought, and $6\frac{3}{4}-\sqrt{3\frac{5}{16}}$ the least. For x denotes ambiguously either of the extreme Numbers, and thence there will come out two Values, either of which may be x, the other being $\frac{yy}{x}$.

The same otherwise.

Putting the Numbers x, y, and $\frac{yy}{x}$ as before, you will have $x + y + \frac{yy}{x} = 20$, or $xx = {}^{\;20}_{-y}\,x - yy$, and extracting the Root $x = 10 - \frac{1}{2}y + \sqrt{100 - 10y - \frac{1}{4}yy}$ for the first Number: Take away this and y from 20, and there remains $\frac{yy}{x} = 10 - \frac{1}{2}y - \sqrt{100 - 10y - \frac{3}{4}yy}$ the third Number. And the Sum of the Squares arising from these three Numbers is $400 - 40y$, and so $400 - 40y = 140$, or $y = 6\frac{1}{2}$. And having found the mean Number $6\frac{1}{2}$, substitute it for y in the first and third Number above found; and the first will become $6\frac{3}{4} + \sqrt{3\frac{5}{16}}$, and the third $6\frac{3}{4} - \sqrt{3\frac{5}{16}}$, as before.

PROBLEM

Problem XIV.

To find four Numbers in continual Proportion, the two Means whereof together make 12, *and the two Extremes* 20.

Let x be the second Number; and $12 - x$ will be the third; $\frac{xx}{12 - x}$ the first; and $\frac{144 - 24x + xx}{x}$ the fourth; and consequently $\frac{xx}{12 - x} + \frac{144 - 24x + xx}{x} = 20$. And by Reduction $xx = 12x - 30\frac{6}{7}$, or $x = 6 + \sqrt{5\frac{1}{7}}$. Which being found, the other Numbers are given from those above.

Problem XV.

To find four Numbers continually proportional, whereof the Sum a *is given, and also the Sum of their Squares* b.

Although we ought for the most Part to seek the Quantities required immediately, yet if there are two that are ambiguous, that is, that involve both the same Conditions (as here the two Means and two Extremes of the four Proportionals), the best Way is to seek other Quantities that are not ambiguous, by which these may be determined, as suppose their Sum, or Difference, or Rectangle. Let us therefore make the Sum of the two mean Numbers to be s, and the Rectangle r; and the Sum of the Extremes will be $a - s$, and the Rectangle also r, because of the Proportionality. Now that from hence these four Numbers may be found, make x the first, and y the second; and $s - y$ will be the third; and $a - s - x$ the fourth; and the Rectangle under the Means $sy - yy = r$, and thence one Mean $y = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - r}$, the other $s - y = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - r}$,

$\sqrt{\frac{1}{4}ss - r}$. Also, the Rectangle under the Extremes $ax - sx - xx = r$, and thence one Extreme $x = \frac{a-s}{2} + \sqrt{\frac{ss - 2as + aa}{4} - r}$, and the other $a - s - x = \frac{a-s}{2} - \sqrt{\frac{ss - 2as + aa}{4} - r}$.

The Sum of the Squares of these four Numbers is $2ss - 2as + aa - 4r$ which is $= b$. Therefore $r = \frac{1}{2}ss - \frac{1}{2}as + \frac{1}{4}aa - \frac{1}{4}b$, which being substituted for r, there come out the four Numbers as follows:

The two Means $\begin{cases} \frac{1}{2}s + \sqrt{\frac{1}{4}b - \frac{1}{4}ss + \frac{1}{2}as - \frac{1}{4}aa} \\ \frac{1}{2}s - \sqrt{\frac{1}{4}b - \frac{1}{4}ss + \frac{1}{2}as - \frac{1}{4}aa}. \end{cases}$

The two Extremes $\begin{cases} \frac{a-s}{2} + \sqrt{\frac{1}{4}b - \frac{1}{4}ss} \\ \frac{a-s}{2} - \sqrt{\frac{1}{4}b - \frac{1}{4}ss}. \end{cases}$

Yet there remains the Value of s to be found. Wherefore to abbreviate the Terms, for these Quantities substitute

$$\frac{1}{2}s + p. \qquad \frac{a-s}{2} + q$$

and

$$\frac{1}{2}s - p. \qquad \frac{a-s}{2} - q$$

And make the Rectangle under the second and fourth equal to the Square of the third, since this Condition of the Question is not yet satisfied, and you will have $\frac{as - ss}{4} - \frac{1}{2}qs + \frac{pa - ps}{2} - pq = \frac{1}{4}ss - ps + pp$. Make also the Rectangle under the first and third equal to to the Square of the second, and you will have $\frac{as - ss}{4}$ $+ \frac{1}{2} qs + \frac{-pa + ps}{2} - pq = \frac{1}{4} ss + ps + pp.$ Take the first of these Equations from the latter, and there will remain $qs - pa + ps = 2ps$, or $qs = pa + ps$. Restore now $\sqrt{\frac{1}{4}b - \frac{1}{4}ss + \frac{1}{2}as - \frac{1}{4}aa}$ in the Place of p, and $\sqrt{\frac{1}{4}b - \frac{1}{4}ss}$ in the Place of q, and you will have $\sqrt{\frac{1}{4}b - \frac{1}{4}ss} = \overline{a + s}$ $\sqrt{\frac{1}{4}b - \frac{1}{4}ss + \frac{1}{2}as - \frac{1}{4}aa}$, and by squaring $ss = -\frac{b}{a}s + \frac{1}{2}aa - \frac{1}{2}b$, or $s = -\frac{b}{2a} + \sqrt{\frac{bb}{4aa} + \frac{1}{2}aa - \frac{1}{2}b}$; which being found, the four Numbers sought are given from what has been shewn above.

Problem XVI.

If an annual Pension of the Number of Pounds a, *to be paid in the five next following Years, be bought for the ready Money* c, *to find what the Compound Interest of* 100 l. Annum *will amount to?*

Make $1 - x$ the Compound Interest of the Money x for a Year, that is, that the Money 1 to be paid after one Year is worth x in ready Money: and, by Proportion, the Money a to be paid after one Year, will be worth ax in ready Money; and after two Years, it will be worth axx; and after three Years, ax^3; and after four Years, ax^4; and after five Years, ax^5. Add these five Terms, and you will have $ax^5 + ax^4 + ax^3 + axx + ax = c$, or $x^5 + x^4 + x^3 + x^2 + x = \frac{c}{a}$ an Equation of five Dimensions, by Help of which,

 when

when x is found by the * Rules to be taught hereafter, put $x : 1 :: 100 : y$, and $y - 100$ will be the Compound Interest of 100 *l. per Annum*.

It is sufficient to have given these Instances in Questions where only the Proportions of Quantities are to be considered, without the Positions of Lines: Let us now proceed to the Solutions of Geometrical Problems.

* Viz. by finding the first Figures of the Root by any mechanical Construction, and the remaining Figures by the Method of Vietz.

How GEOMETRICAL QUESTIONS *may be reduced to* EQUATIONS.

LXXXIII. *Geometrical Questions may be reduced sometimes to Equations with as much Ease, and by the same Laws, as those we have proposed concerning abstracted Quantities.* As if the Right-Line [See Fig. 6.] AB be to be divided in mean and extreme Proportion in C, that is, so that BE the Square of the greatest Part shall be equal to the Rectangle BD contained under the whole and the least Part; having put $AB = a$, and $BC = x$, then will AC be $= a - x$, and $xx = a$ into $a - x$ (*a*); an Equation which by Reduction gives $x = -\frac{1}{2}a + \sqrt{\frac{5}{4}aa}$ (*b*).

LXXXIV. *But in Geometrical Affairs, which more frequently occur, they so much depend on the various Positions and complex Relations of Lines, that they require some farther Invention and Artifice to bring them into Algebraick Terms.* And though it is difficult to prescribe any Thing in these Sorts of Cases, and every Person's own Genius ought to be his Guide in these Operations; yet I will endeavour to shew the Way to Learners. You are to know therefore, that *Questions about the same Lines, related*

LXXXIII. (*a*) Euclid II. 2. (*b*) Art. LXXIV.

lated after any definite Manner to one another, may be variously proposed, by making different Quantities the Quæsita *or Things sought, from different* Data *or Things given. But of what* Data *or* Quæsita *soever the Question be proposed, its Solution will follow the same Way by an Analytick Series, without any other Variation of Circumstance besides the feigned Species of Lines, or the Names by which we are used to distinguish the given Quantities from those sought.*

As if the Question be of an Isosceles Triangle CBD [See Fig. 7.] inscribed in a Circle, whose Sides BC, BD, and Base CD, are to be compared with the Diameter of the Circle AB. This may either be proposed of the Investigation of the Diameter from the given Sides and Base, or of the Investigation of the Basis from the given Sides and Diameter; or lastly, of the Investigation of the Sides from the given Base and Diameter; but however it be proposed, it will be reduced to an Equation by the same Series of an Analysis, viz. If the Diameter be sought, I put $AB = x$, $CD = a$, and BC or $BD = b$. Then (having drawn AC) by reason of the similar Triangles ABC, and CBE, it will be $AB : BC :: BC : BE$, or $x : b :: b : BE$ (*c*). Wherefore $BE = \frac{bb}{x}$. Moreover CE is $= \frac{1}{2} CD$, or $\frac{1}{2} a$; and by reason of the Right-Angle CEB, $CEq + BEq = BCq$, that is $\frac{1}{4} aa + \frac{b^4}{xx} = bb$ (*d*). Which Equation, by Reduction, will give the Quantity x sought (*e*).

 But

(*c*) Euclid VI. 8. (*d*) Euclid I. 47.

(*e*) For multiplying by $4x^2$ (R. 3. of Equations), and transposing (R. 1.), we have $4b^4 = 4b^2x^2 - a^2x^2$; and dividing by $4b^2 - a^2$ (R. 5.), and evolving (R. 7. LXXIV.) $x = \frac{2b^2}{\sqrt{4b^2 - a^2}}$.

But if the Base be sought, put AB $= c$, CD $= x$, and BC or BD $= b$. Then (AC being drawn) because of the similar Triangles ABC and CBE, there is AB : BC :: BC : BE, or $c : b :: b :$ BE (c). Wherefore BE $= \frac{bb}{c}$; and also CE $= \frac{1}{2}$ CD, or $\frac{1}{2}x$. And because the Angle CEB is right, CEq + BEq = BCq, that is, $\frac{1}{4}xx + \frac{b^4}{cc} = bb$ (d); an Equation which will give by Reduction the sought Quantity x (f).

But if the Side BC or BD be sought, put AB $= c$, CD $= a$, and BC or BD $= x$. And (AC being drawn as before) by reason of the similar Triangles ABC and CBE, it is AB : BC :: BC : BE, or $c : x :: x :$ BE (c). Wherefore BE $= \frac{xx}{c}$. Moreover CE is $= \frac{1}{2}$ CD, or $\frac{1}{2}a$; and by reason of the right Angle CEB, CEq + BEq is = BCq, that is, $\frac{1}{4}aa + \frac{x^4}{cc} = xx$ (d); and the Equation, by Reduction, will give the Quantity sought, viz. x (g).

LXXXV. You

(f) After the same Manner, $x^2 = \frac{4b^2c^2 - 4b^4}{c^2}$; whence $x = \sqrt{\frac{4b^2c^2 - 4b^4}{c^2}} = \frac{2b}{c}\sqrt{c^2 - b^2}$.

(g) In like manner, $x^4 = c^2x^2 - \frac{1}{4}a^2c^2$; whence $x^2 = \frac{c^2}{2} \pm \sqrt{\frac{c^4 - a^2c^2}{4}}$ (R. 7.) $= \frac{c^2 \pm c}{2}\sqrt{c^2 - a^2}$; whence $x = \sqrt{\frac{c^2 \pm c}{2}\sqrt{c^2 - a^2}} = \sqrt{\frac{1}{4}c^2 \pm \frac{1}{2}c\sqrt{c^2 - a^2}}$.

LXXXV. *You see therefore that in every Case, the Calculus, by which you come to the Equation, is the same every where, and brings out the same Equation, excepting only that I have denoted the Lines by different Letters, according as I made the* Data *and* Quæsita *different. And from different* Data *and* Quæsita *there arises a Diversity in the Reduction of the Equation found:* For the Reduction of the Equation $\frac{1}{4}aa + \frac{b^4}{xx} = bb$, in order to obtain $x = \frac{2bb}{\sqrt{4bb - aa}}$ the Value of AB, is different from the Reduction of the Equation $\frac{1}{4}xx + \frac{b^4}{cc} = bb$, in order to obtain $x = \frac{2b}{c}\sqrt{cc - bb}$, the Value of CD; and the Reduction of the Equation $\frac{1}{4}aa + \frac{x^4}{cc} = xx$ very different to obtain $x = \sqrt{\frac{1}{4}cc \pm \frac{1}{2}c\sqrt{cc - aa}}$ the Value of BC or BD: (as well as this also, $\frac{1}{4}aa + \frac{b^4}{cc} = bb$, to bring out c, a, or b, ought to be reduced after different Methods) but there was no Difference in the Investigation of these Equations (*b*). *And hence it is that Analysts order us to make no Difference between the given and sought Quantities. For since the same Computation agrees to any Case of the given and sought Quantities, it is convenient* that they should be conceived

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LXXXV. (*b*) As there is no Difference in the Investigation, let the Base CD be a, the Diameter AB $= c$, and BC or BD $= b$; then $\frac{aa}{4} + \frac{b^4}{cc} = bb$ will be an universal Equation, in which substituting x for c, if the Diameter is sought; for a, if the Base; and for b, if a Side is sought; and the former Equations emerge.

conceived and compared without any Difference, that we may the more rightly judge of the Methods of computing them; or rather it is convenient *that you should imagine, that the Question is proposed of those* Data *and* Quæsita, *given and sought Quantities, by which you think it is most easy for you to make out your Equation.*

LXXXVI. *Having therefore any Problem proposed, compare the Quantities which it involves, and making no Difference between the given and sought ones, consider how they depend one upon another, that you may know what Quantities if they are assumed, will, by proceeding synthetically, give the rest.* To do which, there is no need that you should at first of all consider how they may be deduced from one another algebraically; but this general Consideration will suffice, that they may be some how or other deduced by a direct Connexion with one another. For Example; If the Question be put of the Diameter of the Circle AD [See Fig. 8.], and the three Lines AB, BC, and CD, inscribed in a Semi-circle, and from the rest given you are to find BC; at first Sight it is manifest, that the Diameter AD determines the Semi-circle, and then that the Lines AB and CD by Inscription determine the Points B and C, and consequently the Quantity sought BC, and that by a direct Connexion; and yet after what Manner BC is to be had from these Data or given Quantities, is not so evident to be found by an Analysis. The same Thing is also to be understood of AB or CD, if they were to be sought from the other Data. Now, if AD were to be found from the given Quantities AB, BC, and CD, it is equally evident it could not be done synthetically; for the Distance of the Points A and D depends on the Angles B and C, and those Angles on the Circle in which the given Lines are to be inscribed, and that Circle is not given without knowing the Diameter AD. The Nature of the Thing therefore requires, that AD be sought, not synthetically, but by assuming it as given to make thence a Regression to the Quantities given.

LXXXVII.

LXXXVII. When you shall have thoroughly perceived the different Orderings of the Process by which the Terms of the Question may be explained, *make Use of any of the synthetical Methods by assuming Lines as given, from which the Process to others seems very easy, and the Regression to them very difficult.* For the Computation, tho' it may proceed through various Mediums, yet will begin from those Lines; and will be sooner performed by supposing the Question to be such, as if it was proposed of those Data, and some Quantity sought that would easily come out from them, than by thinking of the Question as it is really proposed. Thus, in the proposed Example, if from the rest of the Quantities given you were to find AD. Since I perceive that it cannot be done synthetically, but yet provided it was given, I could proceed in my Ratiocination in a direct Connexion from that to other Things, I assume AD as given, and then I begin to compute as if it was given indeed, and some of the other Quantities, viz. some of the given ones, as AB, BC, or CD, were sought. *And by this Method, by carrying on the Computation from the Quantities assumed after this Way to the others, as the Relations of the Lines to one another direct, there will always be obtained an Equation between two Values of some one Quantity,* whether one of those Values be a Letter set down as a Representation or Name at the Beginning of the Work for that Quantity, and the other a Value of it found out by Computation, or whether both be found by a Computation made after different Ways (*i*).

LXXXVIII.

LXXXVII. (*i*) Beside *being directed* in the Choice of proper Quantities to substitute for, *by the Relations of the Lines; it will be of Use to choose those which lie nearest the known Parts of the Figure, by the Help of which the next adjacent Parts may be expressed without the Intervention of Surds*, by Addition and Subtraction only. Thus, if the Perpendicular of a plane Triangle is sought, from the three Sides given; 'tis better to substitute for the Segments

LXXXVIII. But when you have compared the Terms of the Queſtion thus generally, there is more Art and Invention required to find out the particular Connexions or Relations of the Lines that ſhall accommodate them to Computation. For thoſe Things, which to a Perſon that does not ſo thoroughly conſider them, may ſeem to be immediately, and by a very near Relation connected together, when we have a Mind to expreſs that Relation algebraically, require a great deal more round-about Proceeding, and oblige you to begin your Schemes anew, and carry on your Computation Step by Step; as may appear by finding BC from AD, AB, and CD. For *you are only to proceed by ſuch Propoſitions or Enunciations that can fitly be repreſented in algebraick Terms*, whereof in particular you have ſome from Euclid, Ax. 19. Prop. 4. Book 6. and Prop. 47. of the firſt.

LXXXIX. *In the firſt Place, therefore, the Calculus may be aſſiſted by the Addition and Subtraction of Lines*, ſo that from the Values of the Parts you may find the Values of the Whole, or from the Value of the Whole and one

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ments of the Baſe (by which the final Equation will be ſimple, and the Perpendicular will be found from the Segments) than to ſubſtitute for the Perpendicular (whereby the Segments would be ſurd Quantities, and the final Equation a Quadratic). *Again, if two Lines or Quantities have the ſame Relation to other Parts of the Figure or Problem, the beſt way is to make uſe of neither, but of their Sum, or of their Rectangle, or of the Sum of their alternate Quotes, or of ſome Line, or Lines, to which they both have the ſame Relation, as in Art.* CIX. *And laſtly, if the Area or Periphery of a Figure be given, or ſuch Parts thereof, as have but a remote Relation to the Parts required, it will ſometimes be of uſe to aſſume another Figure, ſimilar to the propoſed, whereof one Side is Unity, or ſome other known Quantity*, from whence the other Parts of this Figure, by the known Proportions of the homologous Sides or Parts, may be found, and an Equation obtained.

one of the Parts, you may obtain the Value of the other Part.

XC. *In the ſecond Place, the Calculus is promoted by the Proportionality of Lines*; for we ſuppoſe (as above) that the Rectangle of the mean Terms, divided by either of the Extremes, gives the Value of the other; or, which is the ſame Thing, if the Values of all four of the Proportionals are firſt had, we make an Equality between the Rectangles of the Extremes and Means. *But the Proportionality of Lines is beſt found out by the Similarity of Triangles*, which, as it is known by the Equality of their Angles, the Analyſt ought in particular to be converſant in comparing them, and conſequently not to be ignorant of Euclid, Prop. 5, 13, 15, 29, and 32, of the firſt Book; and of Prop. 4, 5, 6, 7, and 8, of the ſixth Book; and of the 20, 21, 22, 27, and 31, of the third Book of his Elements. To which alſo may be added the 3d Prop. of the ſixth Book, wherein, from the Proportion of the Sides, is inferred the Equality of the Angles, and *e contra*. Sometimes, likewiſe, the 36th and 37th Prop. of the third Book will do the ſame Thing.

XCI. *In the third Place, the Calculus is promoted by the Addition or Subtraction of Squares*, viz. In right-angled Triangles we add the Squares of the leſſer Sides to obtain the Square of the greateſt, or from the Square of the greateſt Side we ſubtract the Square of one of the leſſer, to obtain the Square of the other.

XCII. *And on theſe few Foundations (if we add to them Prop. 1. of the 6th Element when the Buſineſs relates to Superficies, as alſo ſome Propoſitions taken out of the 11th and 12th of Euclid, when Solids come in Queſtion) the whole Analytick Art, as to right-lined Geometry, depends.* Moreover, all the Difficulties of Problems may be reduced to the ſole Compoſition of Lines out of Parts, and the Similarity of Triangles; ſo that there is no Occaſion to make uſe of other Theorems; becauſe they may all be reſolved into theſe two, and conſequently into the Solutions that may be drawn from them. And, for an Inſtance of this, I have

I have subjoined a Problem about letting fall a Perpendicular upon the Base of an oblique-angled Triangle; which is solved without the Help of the 47th Prop. of the first Book of Euclid. But although it may be of Use not to be ignorant of the most simple Principles on which the Solutions of Problems depend, and though by only their Help any Problems may be solved; yet, *for Expedition sake*, it will be convenient not only that the 47th Prop. of the first Book of Euclid, whose use is most frequent, but also that *other Theorems should sometimes be made Use of.*

XCIII. As if, for Example, a Perpendicular being let fall upon the Base of an oblique-angled Triangle, the Question were (for the sake of promoting algebraick Calculus) to find the Segments of the Base; here it would be of Use to know, that *the Difference of the Squares of the Sides is equal to the double Rectangle under the Base, and the Distance of the Perpendicular from the Middle of the Basis* (*k*).

XCIV. *If the vertical Angle of any Triangle be bisected*, it will not only be of Use to know, *that the Base is divided in Proportion to the Sides* (*l*), but also that *the Difference of the Rectangles made by the Sides, and the Segments of the Base is equal to the Square of the Line that bisects the Angle* (*m*).

XCV. If the Problem relate to Figures inscribed in a Circle, this Theorem will frequently be of Use, viz. That *in any quadrilateral Figure inscribed in a Circle, the Rectangle of the Diagonals is equal to the Sum of the Rectangles of the opposite Sides* (*n*).

XCVI. *The*

XCIII. (*k*) See Problem XII. Number 199.
XCIV. (*l*) Eucl. VI. 3.
(*m*) Whiston's 1st. Schol. Eucl. VI. 17.
XCV. (*n*) Whiston's Schol. Eucl. VI. 16.

XCVI. *The Analyst may observe several Theorems of this Nature in his Practice, and reserve them for his Use; but* let him use them sparingly, if he can, with equal Facility, or not much more Difficulty, deduce the Solution from more simple Principles of Computation. Wherefore *let him take especial Notice of the three Principles first proposed, as being more known, more simple, more general, but a few, and yet sufficient for all Problems, and let him endeavour to reduce all Difficulties to them before others.*

XCVII. But that these Theorems may be accommodated to the Solution of Problems, *the Schemes are oft times to be farther constructed, and that most frequently, by producing out some of the Lines till they cut others, or become of an assigned Length; or by drawing from some remarkable Point, Lines parallel or perpendicular to others, or by conjoining some remarkable Points; as also sometimes by constructing after other Methods, according as the State of the Problem, and the Theorems which are made Use of to solve it, shall require.* As for Example, If *two Lines that do not meet each other, make given Angles with a certain third Line,* perhaps we *produce them so, that when they concur, or meet, they shall form a Triangle, whose Angles, and consequently the Ratio's of their Sides, shall be given;* or, *if any Angle is given,* or be equal to any one, we often *compleat it into a Triangle given in Specie, or similar to some other, and that by producing some of the Lines in the Scheme, or by drawing a Line subtending an Angle. If the Triangle be an oblique angled one,* we often *resolve it into two right-angled ones, by letting fall a Perpendicular.* If the *Business concerns multilateral or many-sided Figures, we resolve them into Triangles, by drawing diagonal Lines;* and so in others; always aiming at this End, viz. *that the Scheme may be resolved either into given, or similar, or right-angled Triangles* (*o*). Thus, in the Example proposed [See Fig. 9.] I draw

XCVII. (*o*) *In this Preparation of the Figure, if an Angle be given, let the Perpendicular be opposite to that Angle, and also fall from the End of a given Line if possible.*

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I draw the Diagonal BD, and the Trapezium ABCD may be resolved into the two Triangles, ABD a right angled one, and BDC an oblique angled one. Then I resolve the oblique angled one into two right angled Triangles, by letting fall a Perpendicular from any of its Angles, B, C, or D, upon the opposite Side; as from B upon CD produced to E, that BE may meet it perpendicularly. But since the Angles BAD and BCD make in the mean while two right ones (by 22 Prop. 3. Elem.) as well as BCE and BCD, I perceive the Angles BAD and BCE to be equal; consequently the Triangles BCE and DAB to be similar. And so I see that the Computation (by assuming AD, AB, and BC, as if CD were sought) may be thus carried on, viz. AD and AB (by reason of the right angled Triangle ABD) give you BD. AD, AB, BD, and BC (by reason of the similar Triangles ABD and CEB) give BE and CE. BD and BE (by reason of the right angled Triangle BED) give ED; and ED — EC gives CD. Whence there will be obtained an Equation between the Value of CD so found out, and the algebraick Letter that was put for it. *We may also (and for the greatest Part it is better so to do, than to follow the Work too far in one continued Series) begin the Computation from different Principles, or at least promote it by divers Methods to any one and the same Conclusion, that at length there may be obtained two Values of any the same Quantity, which may be made equal to one another.* Thus, AD, AB, and BC, give BD, BE, and CE, as before; then CD + CE, gives ED; and lastly, BD, and ED (by reason of the right angled Triangle BED) give BE. *You might also very well form the Computation thus, that the Values of those Quantities should be sought between which any other known Relation intercedes, and then that Relation will bring it to an Equation.* Thus, since the Relation between the Lines BD, DC, BC, and CE, is manifest from the 12th Prop. of the second Book of the Elements, viz. that $BDq - BCq - CDq$ is $= 2CD \times CE$: I seek BDq from the assumed AD and AB; and CE from the assumed AD, AB, and BC. And, lastly, assuming CD, I make $BDq - BCq -$

$CDq =$

CD q = 2 CD × CE. *After such Ways, and led by these Sorts of Consultations, you ought always to take care of the Series of the Analysis, and of the Scheme to be constructed in order to it, at once.*

XCVIII. *Hence, I believe, it will be manifest what Geometricians mean, when they bid you imagine that to be already done which is sought. For making no Difference between the known and unknown Quantities, you may assume any of them to begin your Computation from, as much as if all had indeed been known by a previous Solution, and you were no longer to consult the Solution of the Problem, but only the Proof of that Solution.* Thus, in the first of the three Ways of computing already described, although perhaps AD be really sought, yet I imagine CD to be the Quantity sought, as if I had a mind to try whether its Value derived from AD will coincide with its Quantity before known. So also in the two last Methods, I do not propose, as my Aim, any Quantity to be sought, but only some how or other to bring out an Equation from the Relations of the Lines. And, for sake of that Business, I assume all the Lines AD, AB, BC, and CD, as known, as much as if (the Question being before solved) the Business was to enquire whether such and such Lines would satisfy the Conditions of it, by agreeing with any Equations which the Relations of the Lines can exhibit. I entered upon the Business at first Sight after this Way, and with such Sort of Consultations; but when I arrive at an Equation, I change my Method, and endeavour to find the Quantity sought, by the Reduction and Solution of that Equation. *Thus, lastly, we assume often more Quantities as known, than what are expressed in the State of the Question.* Of this you may see an eminent Example in the 55th of the following Problems, where I have assumed a, b, and c, in the Equation $aa + bx + cx^2 = yy$, for determining the conick Section; as also the other Lines r, s, t, v, of which the Problem, as it is proposed, hints nothing. *For you may assume any Quantities by the Help whereof it is possible to come to Equations; only taking this Care, that you obtain as many Equations from them as you assume Quantities really unknown.*

XCIX. *After you have consulted your Method of Computation, and drawn up your Scheme, give Names to the Quantities that enter into the Computation, (that is, from which being assumed, the Values of others are to be derived, until at last you come to an Equation) chusing such as involve all the Conditions of the Problem, and seem accommodated before others to the Business, and that shall render the Conclusion (as far as you can guess) more simple, but yet not more than what shall be sufficient for your Purpose* (p). *Wherefore, do not give proper Names to Quantities which may be denominated from Names already given.* Thus, from a whole Line and its Parts, from the three Sides of a right-angled Triangle, and from three or four Proportionals, some one of the least considerable we leave without a Name, because its Value may be derived from the Names of the rest. As in the Example already brought, if I make AD $= x$, and AB $= a$, I denote BD by no Letter, because it is the third Side of a right-angled Triangle ABD, and consequently its Value is $\sqrt{xx - aa}$. Then if I call BC $= b$, since the Triangles DAB and BCE are similar, and thence the Lines AD : AB :: BC : CE proportional, to three whereof, viz. to AD, AB, and BC, there are already Names given; for that Reason I leave the fourth CE without a Name, and in its room I make Use of $\frac{ab}{x}$, discovered from the foregoing Proportionality. And so if DC be called c, I give no Name to DE, because from its Parts, DC and CE, or c and $\frac{ab}{x}$, its Value $c + \frac{ab}{x}$ comes out. [See Figure 10.]

C. But

XCIX. (p) When nothing is required but an algebraical Expression, the more simple it is the better; but if we are to proceed to a geometrical Construction also, it is not so much the Simplicity of the Expression, as the Ease and Simplicity of the Construction, to which we are to attend; and *we are always to aim at Simplicity in the Equation, and Ease in the Construction.*

C. But while I am talking of these Things, the Problem is almost reduced to an Equation. For, *after the aforesaid Letters are set down for the Species of the principal Lines, there remains nothing else to be done, but that out of those Species the Values of other Lines be made out according to a preconceived Method, until after some foreseen Way they come out to an Equation.* And I see nothing wanting in this Case, except that by means of the right-angled Triangles BCE and BDE, I can bring out a double Value of BE, viz. $BCq - CEq$ $\left(\text{or } bb - \frac{aabb}{xx}\right) = BEq$; as also $BDq - DEq$ $\left(\text{or } xx - aa - cc - \frac{2abc}{x} - \frac{aabb}{xx}\right) = BEq$. And hence $\left(\text{blotting out on both Sides } \frac{aabb}{xx}\right)$ I shall have the Equation $bb = xx - aa - cc - \frac{2abc}{x}$; which, being reduced, becomes

$$x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc.$$

CI. But since I have reckoned up several Methods for the Solution of this Problem, and those not much unlike one another in the precedent Paragraphs, of which that taken from Prop. 12. of the second Book of the Elements being something more elegant than the rest, we will here subjoin it. Make therefore $AD = x$, $AB = a$, $BC = b$, and $CD = c$, and you will have $BDq = xx - aa$, and $CE = \frac{ab}{x}$, as before. These Species, therefore, being substituted in the Theorem $BDq - BCq - CDq = 2CD \times CE$, there will arise $xx - aa - bb - cc = \frac{2abc}{x}$; and, after Reduction,

$$x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc, \text{ as before.}$$

 But

But that it may appear how great a Variety there is in the Invention of Solutions, and that it is not very difficult for a prudent Geometrician to light upon them; I have thought fit to shew other Ways of doing the same Thing. And having drawn the Diagonal BD, if in room of the Perpendicular BE, which before was let fall from the Point B upon the Side DC, you now let fall a Perpendicular from the Point D upon the Side BC, or from the Point C upon the Side BD, by which the oblique angled Triangle BCD may any how be resolved into two right angled Triangles, you may come almost by the same Methods I have already described to an Equation. And there are other Methods very different from these.

CII. As if there are drawn two Diagonals, AC and BD [See Fig. 11.], BD will be given by assuming AD and AB; as also AC, by assuming AD and CD; then by the known Theorem of quadrilateral Figures inscribed in a Circle, viz. That $AD \times BC + AB \times CD$ is $= AC \times BD$, you will obtain an Equation [See Fig. 11.]. The Names, therefore, of the Lines AD, AB, BC, CD, remaining, viz. x, a, b, c; BD will be $= \sqrt{xx - aa}$, and $AC = \sqrt{xx - cc}$, by the 47th Prop. of the first Element, and these Species of the Lines being substituted in the Theorem we just now mentioned, there will come out $xb + ac = \sqrt{xx - cc} \times \sqrt{xx - aa}$. The Parts of which Equation being squared and reduced, you will again have

$$x^3 = \begin{matrix} +aa \\ +bb \\ +cc \end{matrix} x + 2abc.$$

CIII. But, moreover, that it may be manifest after what Manner the Solutions drawn from that Theorem may be thence reduced to only the Similarity of Triangles; erect BH perpendicular to BC, and meeting AC in H, and there will be formed the Triangles BCH, BDA, similar, by reason of the right Angles at B, and equal Angles at C and D (by the 21. 3. Elem.); as as also the Triangles BCD, BHA, similar, by reason of the equal Angles both at B (as may appear by taking away the common Angle DBH from the two right ones), as also at D and A (by 21. 3. Elem.). You may see, therefore, that from the Proportionality BD : AD :: BC : HC, there is given the Line HC; as also AH from the Proportionality BD : CD :: AB : AH. Whence, since AH + HC = AC, you have an Equation. The Names therefore, aforesaid, of the Lines remaining, viz. x, a, b, c, as also the Values of the Lines AC and BD, viz. $\sqrt{xx - cc}$ and $\sqrt{xx - aa}$, the first Proportionality will give $HC = \frac{bx}{\sqrt{xx - aa}}$, and the second will give $AH = \frac{ac}{\sqrt{xx - aa}}$. Whence, by reason of $AH + HC = AC$, you will have $\frac{bx + ac}{\sqrt{xx - aa}} = \sqrt{xx - cc}$; an Equation which (by multiplying by $\sqrt{xx - aa}$, and by squaring) will be reduced to a Form often described in the preceding Pages.

CIV. But that it may yet farther appear what a Plenty of Solutions may be found, produce BC and AD [See Fig. 12.] till they meet in F, and the Triangles ABF and CDF will be similar, because the Angle at F is common, and the Angles ABF and CDF (while they compleat the Angle CDA to two right ones, by 13. 1. and 22. 3. Elem.) are equal. Wherefore, if besides the four Terms which compose the Question, there was given AF, the Proportion AB : AF :: CD : CF would give CF. Also AF — AD would give DF, and the Proportion CD : DF :: AB : BF would give BF; whence (since BF — CF is = BC) there would arise an Equation. But since there are assumed two unknown Quantities AD and DF as if they were given, there remains another Equation to be found. I let fall, therefore, BG at right Angles upon AF, and the Proportion AD : AB :: AB : AG, will give AG; which

 being

being had, the Theorem borrowed from the 13. 2. Euclid, viz. that $BFq + 2FAG$ is $= ABq + AFq$ will give another Equation. a, b, c, x, remaining therefore as before, and making $AF = y$, you will have (by insisting on the Steps already laid down) $\frac{cy}{a} = CF$. $y - x = DF$. $\frac{\overline{y-x} \times a}{c} = BF$. And thence $\frac{\overline{y-x} \times a}{c} - \frac{cy}{a} = b$, the first Equation. Also $\frac{aa}{x}$ will be $= AG$, and consequently $\frac{aayy - 2a^2xy + a^2x^2}{cc} + \frac{2aay}{x} = aa + yy$ for the second Equation. Which two, by Reduction, will give the Equation sought, viz. The Value of y found by the first Equation is $\frac{abc + aax}{aa - cc}$, which being substituted in the second, will give an Equation, from which, rightly ordered, will come out

$$x^3 = \begin{matrix} + aa \\ + bb \\ + cc \end{matrix} x + 2abc, \text{ as before } (q).$$

CV. And

CIV. (*q*) For $\frac{a^2y^2 - 2a^2xy + a^2x^2}{c^2} + \frac{2a^2y}{x} = a^2 + y^2$; which, by multiplying by c^2x, by transposing, and by ranging the Terms according to the Dimensions of y, becomes $y^2 \times \overline{a^2x - c^2x} - y \times \overline{2a^2x^2 - 2a^2c^2} + a^2x^3 - a^2c^2x = 0$; which, by substituting $\frac{a^2b^2c^2 + 2a^3bcx + a^4x^2}{\overline{a^2 - c^2} \times \overline{a^2 - c^2}}$ (the Value of y^2) for y^2, and for $-y$ (its Value) $\frac{-abc - a^2x}{a^2 - c^2}$, by abbreviating the Terms (R. 1.), and by extermi-

nating

CV. And so, if AB and CD are produced till they meet one another, the Solution will be much the same, unless perhaps it be something easier. Wherefore I will rather subjoin another Specimen of this Problem, drawn from a Fountain very unlike the former, viz. by seeking the Area of the quadrilateral Figure proposed, and that doubly. I draw therefore the Diagonal BD, and the quadrilateral Figure may be resolved into two Triangles. Then using the Names of the Lines x, a, b, c, as before, I find $BD = \sqrt{xx - aa}$, and thence $\frac{1}{2} a \sqrt{xx - aa}$ $(= \frac{1}{2} AB \times BD)$ the Area of the Triangle ABD. Moreover, having let fall BE perpendicularly upon CD, you will have (by reason of the similar Triangles ABD, BCE) AD : BD :: BC : BE, and consequently $BE = \frac{b}{x} \sqrt{xx - aa}$. Wherefore also $\frac{bc}{2x} \sqrt{xx - aa}$ $(= \frac{1}{2} CD \times BE)$ will be the Area of the Triangle BCD. Now, by adding these Areas, there will arise $\frac{ax + bc}{2x} \sqrt{xx - aa}$, the Area of the whole Quadrilateral. After the same Way, by drawing the Diagonal AC, and seeking the Areas of the Triangles ACD and ACB, and adding them, there will again be obtained the Area of the Quadrilateral Figure $\frac{cx + ba}{2x} \sqrt{xx - cc}$. Wherefore, by making these Areas equal, and multiplying both by $2x$, you will have $\overline{ax + bc} \sqrt{xx - aa} = \overline{cx + ba} \sqrt{xx - cc}$, an Equation which, by squaring

P 4

nating the Fraction (R. 3.), becomes $a^2 b^2 c^2 x - a^4 x^3 + 2a^3 bc^3 + 2a^4 c^2 x + a^4 x^3 - a^2 c^2 x^3 - a^4 c^2 x + a^2 c^4 x = 0$; which being again abbreviated, and the Terms divided by $a^2 c^2$, and by transposing $-x^3$, becomes $x^3 = \overline{a^2 + b^2 + c^2} \times x + 2abc$.

ing and dividing by $aax - ccx$, will be reduced to the Form already often found out,

$$x^3 = \begin{matrix} + aa \\ + bb \\ + cc \end{matrix} x + 2abc.$$

CVI. Hence it may appear, how great a Plenty of Solutions may be had, and that some Ways are much more neat than others. Wherefore, *if the Method you take from your first Thoughts, for solving a Problem, be but ill accommodated to computation, you must again consider the Relations of the Lines, until you shall have hit on a Way as fit and elegant as possible. For those Ways that offer themselves at first Sight, may often create sufficient Trouble if they are made use of.* Thus, in the Problem we have been upon, it would not have been more difficult to have fallen upon the following Method, than upon one of the precedent ones [See Fig. 13.]. Having let fall BR and CS perpendicular to AD, as also CT to BR, the Figure will be resolved into right angled Triangles. And it may be seen, that AD and AB give AR, AD and CD give SD, AD — AR — SD gives RS or TC. Also AB and AR give BR, CD and SD give CS or TR, and BR — TR gives BT. Lastly, BT and TC give BC, whence an Equation will be obtained. But if any one should go to compute after this Rate, he would fall into larger and more perplexed algebraick Terms than are any of the former, and more difficult to be brought to a final Equation (r).

So

CVI. (r) For supposing AC, and BD to be drawn, the Triangles ABD, ACD, are rectangular (Eucl. VI. 8.) whence $x : a :: a : \frac{a^2}{x}$, and $x : c :: c : \frac{c^2}{x}$, and $RS = DA - AR - SD = x - \frac{a^2 - c^2}{x} = \frac{x^2 - a^2 - c^2}{x}$

So much for the Solution of Problems in right-lined Geometry; unless it may perhaps be worth while to *note moreover, that when Angles, or Positions of Lines, expressed by Angles, enter the State of the Question, Lines, or the Proportions of Lines, ought to be used instead of Angles, viz. such as may be derived from given Angles by a Trigonometrical Calculation; or from which being found, the Angles sought will come out by the same Calculus* Several Instances of which may be seen in the following Pages.

CVII. *As*

$\frac{x^2 - a^2 - c^2}{x} = TC$. But $BR = \sqrt{BA^2 - AR^2} = \sqrt{a^2 - \frac{a^4}{x^2}}$; and $CS = \sqrt{CD^2 - DS^2} = \sqrt{c^2 - \frac{c^4}{x^2}} = TR$; and $BT = BR - RT = \sqrt{a^2 - \frac{a^4}{x^2}} - \sqrt{c^2 - \frac{c^4}{x^2}}$; and $BC^2 = CT^2 + TB^2$: Whence $b^2 = \frac{x^4 - 2a^2x^2 - 2c^2x^2 + a^4 + 2a^2c^2 + c^4}{x^2} + a^2 - \frac{a^4}{x^2} - 2\sqrt{a^2c^2 - \frac{a^2c^4 - c^2a^4}{x^2} + \frac{a^4c^4}{x^4}} + c^2 - \frac{c^4}{x^2}$; which, by multiplying by x^2, by abbreviating, and by squaring, becomes $4a^2c^2x^4 \overline{- 4a^2c^4 - 4a^4c^2} \times x^2 + a^4c^4 = x^8 - \overline{2a^2 - 2c^2 - 2b^2} \times x^6 + \overline{6a^2c^2 + 2a^2b^2 + a^4 + b^4 + 2b^2c^2 + c^4} \times x^4 \overline{- 4a^4c^2 - 4a^2c^4 - 4a^2b^2c^2} \times x^2 + 4a^4c^4$; which again abbreviated, and divided by x^2, becomes $x^6 \overline{- a^2 - b^2 - c^2} \times x^4 + \overline{a^4 + b^4 + c^4 + 2a^2b^2 + 2a^2c^2 + 2b^2c^2} \times x^2 - 4a^2b^2c^2 = 0$; that is (184), $x^3 - \overline{a^2 - b^2 - c^2} \times x - 2abc = 0$; viz. by dividing by $x^3 - \overline{a^2 - b^2 - c^2} \times x + 2abc = 0$.

CVII. *As for what belongs to the Geometry of curve Lines, we use to denote them, either by describing them by the local Motion of right Lines, or by using Equations indefinitely expressing the Relation of right Lines disposed according to some certain Law, and ending at the curve Lines. The Antients did the same by the Sections of Solids, but less commodiously. But the Computations that regard Curves described after the first Way, are no otherwise performed than in the precedent Pages* [See Fig. 14.] As if AKC be a curve Line described by K the vertical Point of the Square AKϕ, whereof one Leg AK freely slides through the Point A given by Position, while the other Kϕ of a determinate Length is carried along the right Line AD also given by Position, and you are to find the Point C in which any right Line CD given also by Position shall cut this Curve: I draw the right Lines AC, CF, which may represent the Square in the Position sought, and the Relation of the Lines (without any Difference or Regard of what is given or sought, or any Respect had to the Curve) being considered, I perceive the Dependency of the others upon CF and any of these four, viz. BC, BF, AF, and AC, to be synthetical; two whereof I therefore assume, as $CF = a$, and $CB = x$, and beginning the Computation from thence, I presently obtain $BF = \sqrt{aa - xx}$, and $AB = \frac{xx}{\sqrt{aa - xx}}$, by reason of the right Angle CBF, and that the Lines BF : BC :: BC : AB are continual Proportionals. Moreover, from the given Position of CD, AD is given, which I therefore call b; there is also given the Ratio of BC to BD, which I make as d to e, and you have $BD = \frac{ex}{d}$, and $AB = b - \frac{ex}{d}$. Therefore $b - \frac{ex}{d}$ is $= \frac{xx}{\sqrt{aa - xx}}$, an Equation which (by squaring its Parts, and multiplying

ing by $aa - xx$, &c.) will be reduced to this Form,

$$x^4 = \frac{2bdex^3 \begin{smallmatrix} -bbdd \\ +aaee \end{smallmatrix} xx - 2aabdex + aabbdd}{dd + ee}.$$

Whence, lastly, from the given Quantities a, b, d, and e, there may be found x, by Rules hereafter to be given, and at that Interval or Distance x or BC, a right Line drawn parallel to AD will cut CD in the Point sought C.

CVIII. *But if we do not use Geometrical Descriptions but Equations to denote the Curve Lines by, the Computations will thereby become as much shorter and easier, as the gaining of those Equations can make them* [See Fig. 15.]. As if the Intersection C of the given Ellipsis ACE with the right Line CD given by Position, be sought. To denote the Ellipsis, I take some known Equation proper to it, as $rx - \frac{r}{q}xx = yy$, where x is indefinitely put for any Part of the Axis Ab or AB, and y for the Perpendicular bc or BC terminated at the Curve; and r and q are given from the given Species of the Ellipsis. Since therefore CD is given by Position, AD will be also given, which call a; and BD will be $a - x$; also the Angle ABC will be given, and thence the Ratio of BD to BC, which call 1 to e, and BC (y) will be $= ea - ex$, whose Square $eeaa - 2eeax + eexx$ will be equal to $rx - \frac{r}{q}xx$. And thence by Reduction there will arise $xx = \frac{2aeex + rx - aaee}{ee + \frac{r}{q}}$, or

$$x = \frac{aee + \frac{1}{2}r \pm e\sqrt{ar + \frac{rr}{4ee} - \frac{aar}{q}}}{ee + \frac{r}{q}}.$$

Moreover,

Moreover, although a Curve be denoted by a geometrical Description, or by a Section of a Solid, yet thence an Equation may be obtained, which shall define the Nature of the Curve, and consequently all the Difficulties of Problems proposed about it may be reduced hither.

Thus, in the former Example [See Fig. 14.], if AB be called x, and BC y, the third Proportional BF will be $\frac{yy}{x}$, whose Square, together with the Square of BC, is equal to CFq, that is, $\frac{y^4}{xx} + yy = aa$; or $y^4 + xxyy = aaxx$. And this is an Equation by which every Point C of the Curve AKC, agreeing or corresponding to any Length AB of the Base (and consequently the Curve itself) is defined, and from whence therefore you may obtain the Solutions of Problems proposed concerning this Curve.

After the same Manner almost, when a Curve is not given in Specie, but proposed to be determined, you may feign an Equation at Pleasure, that may generally contain its Nature; and assume this to denote it as if it was given, that from its Assumption you can any Way come to Equations by which the Assumptions may at length be determined: Examples whereof you have in some of the following Problems, which I have collected for a more full Illumination of this Doctrine, and for the Exercise of Learners, and which I now proceed to deliver (*s*).

PROBLEM

CVIII. (*s*) Sometimes the finding the relative algebraic Expression of the Quantities sought, resolves the Problem; but most commonly a geometrical Effection, or Construction, is also necessary; and the Problem is said to be constructed, when the Point or Line which resolves it, is found. And this is done, in Problems of one

Problem I.

Having a finite right Line BC *given, from whose Ends the two right Lines* BA, CA, *are drawn in the given Angles* ABC, ACB; *to find* AD *the Height of their Concourse* A, *above the given Line* BC [See Fig. 16.].

Make BC $= a$, and AD $= y$; and since the Angle ABD is given, there will be given (from the Table of Signs or Tangents) the Ratio between the Lines AD and BD, which make as d to e. Therefore $d : e ::$ AD (y) : BD. Wherefore BD $= \frac{ey}{d}$. In like manner

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one or two Dimensions, by finding the Right-Lines whose Sum or Difference is the Value of the Line sought, and connecting them, by Addition or Subtraction, each under its proper Sign. In this

197. We are to observe, that *a Fraction shews that a fourth Proportional is to be found to the three given, the first Extreme being the Denominator, and the two mean Terms the Factors of the Numerator:* This Proportional is had by Eucl. VI. 11. and if the Numerator is of more than two, and the Denominator of more than one Dimension, this Operation is to be repeated. And

198. *A Square Root shews that a mean Proportional between the Factors under the Sign is to be found.* This, if it be a simple Term, is found by Eucl. VI. 13. if the Sum of two Terms, it will be the Hypothenuse of a right-angled Triangle, whose Sides are mean Proportionals between the Factors of the Terms, Eucl. I. 47. or, if the Terms have a common Factor, it will be a Mean between it and the Sum of the other Factors, Eucl. VI. 17. if the Difference of two Terms, it will be the Side of a right-angled Triangle, whose Hypothenuse is the Root of the positive Term, and the other Side is the Root of the negative one, and is found by Eucl. I. 47. or it is a Mean between the Sum and Difference of the Terms, and found by Eucl. II. 5.

ner, by reason of the given Angle ACD, there will be given the Ratio between AD and DC, which make as d to f, and you will have DC $= \frac{fy}{d}$. But BD + DC = BC, that is, $\frac{ey}{d} + \frac{fy}{d} = a$. Which reduced, by multiplying both Parts of the Equation by d, and dividing by $e + f$ becomes $y = \frac{ad}{e+f}$.

Problem II.

The Sides AB, AC, *of the Triangle* ABC *being given, and also the Base* BC, *which the Perpendicular* AD *let fall from the vertical Angle cuts in* D, *to find the Segments* BD *and* DC [See Fig. 17.]

Let AB $= a$, AC $= b$, BC $= c$, and BD $= x$, and DC will $= c - x$. Now since ABq − BDq $(aa - xx)$ = ADq; and ACq − DCq $(bb - cc + 2cx - xx)$ = ADq; you will have $aa - xx = bb - cc + 2cx - xx$; which by Reduction, becomes $\frac{aa - bb + cc}{2c} = x$.

But that it may appear that all the Difficulties of all Problems may be resolved by only the Proportionality of Lines, without the Help of the 47th of 1. Euclid, although not without round-about Methods, I thought fit to subjoin the following Solution of this Problem over and above. From the Point D let fall the Perpendicular DE upon the Side AB, and the Names of the Lines, already given, remaining, you will have AB : BD :: BD : BE. $a : x :: x : \frac{xx}{a}$ (*a*). And BA − BE $\left(a - \frac{xx}{a}\right)$ = EA.

Prob. II. (*a*) Eucl. VI. 8.

$= EA$. Also $EA : AD :: AD : AB$, and consequently $EA \times AB$ $(aa - xa) = ADq$. And so, by reasoning about the Triangle ACD (*b*), there will be found again $ADq = bb - cc + 2cx - xx$. Whence you will obtain, as before, $x = \frac{aa - bb + cc}{2c}$.

Problem III.

The Area and Perimeter of the right-angled Triangle ABC *being given, to find the Hypothenuse* BC [See Fig. 18.]

Let the Perimeter be called *a*, the Area *bb*, make $BC = x$, and $AC = y$; then will be $AB = \sqrt{xx - yy}$ (*a*); whence again the Perimeter $(BC + AC + AB)$ is $x + y + \sqrt{xx - yy}$, and the Area $(\frac{1}{2} AC \times AB)$ is $\frac{1}{2} y \sqrt{xx - yy}$ (*b*). Therefore $x + y + \sqrt{xx - yy} = a$, and $\frac{1}{2} y \sqrt{xx - yy} = bb$.

The latter of these Equations gives $\sqrt{xx - yy} = \frac{2bb}{y}$; wherefore I write $\frac{2bb}{y}$ for $\sqrt{xx - yy}$ in the former Equation, that the Asymmetry may be taken away; and there comes out $x + y + \frac{2bb}{y} = a$, or multiplying by *y*, and ordering the Equation $yy = ay - xy - 2bb$. Moreover, from the Parts of the former Equation, I take away $x + y$, and there remains $\sqrt{xx - yy} = a - x - y$, and squaring the Parts to take away again the Asymmetry, there comes out $xx - yy = aa - 2ax - 2ay + xx + 2xy + yy$, which, ordered and divided

Prob. II. (*b*) i. e. By drawing from D, a Perpendicular to the Side AC, &c.

Prob. III. (*a*) Eucl. I. 47. (*b*) Eucl. I. 41.

vided by 2, becomes $yy = ay - xy + ax - \frac{1}{2}aa$. Laftly, making an Equality between the two Values of yy, I have $ay - xy - 2bb = ay - xy + ax - \frac{1}{2}aa$, which reduced becomes $\frac{1}{2}a - \frac{2bb}{a} = x$.

The fame otherwife.

Let $\frac{1}{2}$ the Perimeter be $= a$, the Area $= bb$, and BC $= x$, and it will be AC + AB $= 2a - x$. Now fince xx (BCq) is = ACq + ABq, and $4bb =$ 2AC × AB, $xx + 4bb$ will be = ACq + ABq + 2AC × AB = to the Square of $\overline{AC + AB}$ = to the Square of $\overline{2a - x} = 4aa - 4ax + xx$. That is, $xx + 4bb = 4aa - 4ax + xx$, which reduced becomes $a - \frac{bb}{a} = x$.

Problem IV.

Having given the Perimeter and Perpendicular of a right-angled Triangle, to find the Triangle [See Fig. 67.].

Let C be the right Angle of the Triangle ABC, and CD a Perpendicular let fall thence to the Bafe AB. Let there be given AB + BC + AC $= a$, and CD $= b$. Make the Bafe AB $= x$, and the Sum of the Sides will be $a - x$. Put y for the Difference of the Legs, and the greater Leg AC will be $= \frac{a - x + y}{2}$ (*a*); the lefs BC $= \frac{a - x - y}{2}$ (*b*). Now, from the Nature of a right-angled Triangle, you have ACq + BCq = ABq, that is $\frac{aa - 2ax + xx + yy}{2} = xx$ (*c*). And alfo AB : AC :: BC : DC (*d*); therefore AB × DC = AC × BC.

Prob. IV. (*a*) Numb. 22. (*b*) Numb. 36. (*c*) Eucl. I. 47. (*d*) Eucl. VI. 8.

AC x BC, that is $bx = \frac{aa - 2ax + xx - yy}{4}$. By the former Equation $yy = xx + 2ax - aa$. By the latter $yy = xx - 2ax + aa - 4bx$. And consequently $xx + 2ax - aa = xx - 2ax + aa - 4bx$. And, by Reduction, $4ax + 4bx = 2aa$, or $x = \frac{aa}{2a + 2b}$.

Geometrically, thus:

In every right angled Triangle, as the Sum of the Perimeter and Perpendicular is to the Perimeter, so is Half the Perimeter to the Base.

Subtract $2x$ from a, and there will remain $\frac{ab}{a+b}$, the Excess of the Sides above the Base. Whence, again, as *in every right angled Triangle, the Sum of the Perimeter and Perpendicular is to the Perimeter, so is the Perpendicular to the Excess of the Sides above the Base.*

PROBLEM V.

Having given the Base AB *of a right angled Triangle, and the Sum of the Perpendicular and the Legs,* CA + CB + CD; *to find the Triangle.*

Let CA + CB + CD $= a$, AB $= b$, CD $= x$, and AC + CB will be $= a - x$. Put AC − CB $= y$, and AC will $= \frac{a - x + y}{2}$ (*a*), and CB $= \frac{a - x - y}{2}$ (*b*). But ACq + CBq is = ABq (*c*); that

Prob. V. (*a*) Numb. 22. (*b*) Numb. 36. (*c*) Euclid I. 47.

that is, $\frac{aa - 2ax + xx + yy}{2} = bb$. Moreover it is AC × CB = AB × CD (*d*); that is, $\frac{aa - 2ax + xx - yy}{4} = bx$. Which being compared, you have $2bb - aa + 2ax - xx = yy = aa - 2ax + xx - 4bx$. And by Reduction, $xx = 2ax + 2bx - aa + bb$, and $x = a + b - \sqrt{2ab + 2bb}$.

Geometrically, thus:

In any right-angled Triangle, from the Sum of the Legs and Perpendicular, subtract the mean Proportional between the said Sum and the Double of the Base, and there will remain the Perpendicular.

The same otherwise.

Make CA + CB + CD = a, AB = b, and AC = x, and BC will be $= \sqrt{bb - xx}$, CD $= \frac{x\sqrt{bb - xx}}{b}$. And x + CB + CD = a, or CB + CD = $a - x$. And therefore $\frac{b + x}{b}\sqrt{bb - xx} = a - x$ (*e*). And the Parts being squared and multiplied by

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Prob. V. (*d*) Eucl. VI. 8. AB : AC :: BC : CD, i.e. $b : x :: \sqrt{b^2 - x^2} : \frac{2\sqrt{bb - xx}}{b}$; whence

(*e*) BC : AB :: CD : AC (Eucl. V. 40.), i.e. $\sqrt{bb - xx} : b :: \frac{x\sqrt{b^2 - x^2}}{b} : x$; and AC + AB : AB :: DC + CB : BC (Eucl. V. 18.), i.e. $x + b : b :: a$

by bb, there will be made $-x^4-2bx^3+2b^3x+b^4 = aabb-2abbx+bbxx$. Which Equation being ordered, by Transposition of Parts, after this Manner,

$$x^4+2bx^3\begin{matrix}+3bb\\+2ba\end{matrix}xx\begin{matrix}+2b^3\\+2abb\end{matrix}x\begin{matrix}+b^4\\+2ab^3\\+aabb\end{matrix}=$$

$$\begin{matrix}2bb\\2ab\end{matrix}xx\begin{matrix}+4b^3\\+4abb\end{matrix}x\begin{matrix}+2b^4\\+2ab^3\end{matrix}$$ (*f*) and extracting the Roots on both Sides, there will arise $xx+bx+bb+ab=\overline{x+b}\sqrt{2ab+2bb}$. And the Root being again extracted, $x=-\frac{1}{2}b+\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}\pm\sqrt{b\sqrt{\frac{1}{2}bb+\frac{1}{2}ab}-\frac{1}{4}bb-\frac{1}{2}ab}$.

The Geometrical Construction [See Fig. 68.].

Take therefore AB $=\frac{1}{2}b$, BC $=\frac{1}{2}a$, CD $=\frac{1}{2}$ AB, AE *a mean Proportional between* b *and* AC, *and* EF *on both*

$:b::a-x:\sqrt{bb-xx}$; whence $a-x=$ DC $+$ CB $=\frac{x+b}{b}\sqrt{bb-xx}$ (Eucl. VI. 11.)

(*f*) By Transposition it becomes $x^4+2bx^3+b^2x^2-b^4+a^2b^2=\overline{2b^3+2ab^2}\times x$; and adding to each Member [in order to compleat the Square (CXLII)] $\overline{2ab+2b^2}\times\overline{x^2+bx+b^2}=\overline{2b^2+2ab}\times x^2+\overline{2b^3+2ab^2}\times x+2b^4+2ab^3$, we shall have $x^4+2bx^3+\overline{3b^2+2ab}\times x^2+\overline{2b^3+2ab^2}\times x+b^4+2ab^3+2a^2b^2=\overline{2b^2+2ab}\times x^2+\overline{4b^3+4ab^2}\times x+2b^4+2ab^3$.

both Sides a mean Proportional between b *and* DE, *and* BF; BF *will be the two Legs of the Triangle* (*g*).

Problem VI.

Having given, in the right angled Triangle ABC, *the Sum of the Sides* AC + BC, *and the Perpendicular* CD, *to find the Triangle.*

Let AC + BC $= a$, CD $= b$, AC $= x$, and BC will be $= a - x$, AB $= \sqrt{aa - 2ax + 2xx}$ (*a*). Moreover, CD : AC :: BC : AB (*b*). Therefore, again,

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Prob. V. (*g*) For the Side $x = \sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} \pm \sqrt{b\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b^2 - \frac{1}{2}ab} - \frac{1}{2}b$; but (because AB $= \frac{1}{2}b$, and BC $= \frac{1}{2}a$, therefore AC $=$ AB $+$ BC $= \frac{1}{2}b + \frac{1}{2}a$; and the mean Proportional between $\frac{1}{2}b + \frac{1}{2}a$ and b (198) is $\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab}$), $\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} =$ AE; and (because BD $=$ BC $-$ CD $= \frac{1}{2}a - \frac{1}{4}b$; and AD $=$ AC $-$ CD $= \frac{1}{2}b + \frac{1}{2}a - \frac{1}{4}b = \frac{1}{4}b + \frac{1}{2}a$; and because DE $=$ AE $-$ AD; thence DE $= \sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b - \frac{1}{2}a$; and the mean Proportional between b, and $\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b - \frac{1}{2}a$ (198), is $\sqrt{b\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b^2 - \frac{1}{2}ab}$), $\sqrt{b\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b^2 - \frac{1}{2}ab} =$ EF; and (because BF $=$ AE $\pm$ EF $-$ AB) $\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} \pm \sqrt{b\sqrt{\frac{1}{2}b^2 + \frac{1}{2}ab} - \frac{1}{4}b^2 - \frac{1}{2}ab} - \frac{1}{2}b =$ BF: therefore, $x =$ BF.

Prob. VI. (*a*) Eucl. I. 47. (*b*) Eucl. VI. 8.

again, $AB = \frac{ax - xx}{b}$. Wherefore, $ax - xx = b\sqrt{aa - 2ax + 2xx}$; and the Parts being squared and ordered, $x^4 - 2ax^3 \begin{matrix} + aa \\ - 2bb \end{matrix} xx + 2abbx - aabb = 0$. Add to both Parts $aabb + b^4$ (*c*), and there will be made $x^4 - 2ax^3 \begin{matrix} + aa \\ - 2bb \end{matrix} xx + 2abbx + b^4 = aabb + b^4$. And the Root being extracted on both Sides, $xx - ax - bb = -b\sqrt{aa + bb}$, and the Root being again extracted, $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}aa + bb - b\sqrt{aa + bb}}$.

The Geometrical Construction [See Fig. 69.].

Take $AB = BC = \frac{1}{2}a$. At C erect the Perpendicular $CD = b$. Produce DC to E, so that DE shall be $= DA$. And between CD and CE take a mean Proportional CF. And let a Circle GH described from the Center F and the Radius BC, cut the right Line BC in G and H, and BG and BH will be the two Sides of the Triangle (*d*).

The

Problem VI. (*c*) Article CXLII.

(*d*) For GH is bisected in C (Eucl. III. 3.), and $GC = CH = \sqrt{BC^2 - CF^2}$; now the Side $x = \frac{1}{2}a \pm \sqrt{\frac{1}{4}a^2 + bb - b\sqrt{aa + bb}}$; but $\frac{1}{2}a = BC$, and (because $DE = \sqrt{a^2 + b^2}$, whence CF (198) $= \sqrt{-b^2 + b\sqrt{a^2 + b^2}}$) $\sqrt{\frac{1}{4}a^2 + b^2 - b\sqrt{a^2 + b^2}} =$ CH (198); whence $\frac{1}{2}a + \sqrt{\frac{1}{4}a^2 + b^2 - b\sqrt{a^2 + b^2}} = BH = BC + CH$; and $\frac{1}{2}a - \sqrt{\frac{1}{4}a^2 + b^2 - b\sqrt{a^2 + b^2}} = BG = BC - CH$.

The same otherwise.

Let $AC + BC = a$, $AC - BC = y$, $AB = x$, and $DC = b$, and $\frac{a+y}{2}$ will be $= AC$ (*a*), $\frac{a-y}{2}$ $= BC$ (*b*), $\frac{aa+yy}{2}$ (*c*) $= ACq + BCq = ABq$ (*d*) $= xx$. $\frac{aa-yy}{4b} = \frac{AC \times BC}{DC} = AB$ (*e*) $= x$. Therefore, $2xx - aa = yy = aa - 4bx$, and $xx = aa - 2bx$ (*f*), and the Root being extracted, $x = -b + \sqrt{bb+aa}$. Whence, in the Construction above, CE is the Hypothenuse of the Triangle sought. But the Base and Perpendicular, as well in this as the Problem above being given, the Triangle is thus expeditiously constructed [See Fig. 70.]. Make a Parallelogram CG, whose Side CE shall be the Basis of the Triangle, and the other Side CF the Perpendicular. And upon CE describe

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Problem VI. (*a*) Number 22.

(*b*) Number 36.

(*c*) For $\frac{a^2+2ay+y^2}{4} + \frac{a^2-2ay+y^2}{4} = \frac{2a^2+2y^2}{4} = \frac{a^2+y^2}{2}$.

(*d*) Euclid I. 47.

(*e*) For AB : AC :: BC : CD. Eucl. VI. 8.

(*f*) $aa+yy = 2xx$; whence $2xx - aa = yy$; and $aa - yy = 4bx$; whence $aa - 4bx = yy$: Therefore $2xx - aa = aa - 4bx$; and $2xx = 2aa - 4bx$, and $xx = aa - 2bx$.

describe a Semicircle, cutting the opposite Side FG in H. Draw CH, EH, and CHE will be the Triangle sought (*g*).

PROBLEM VII.

In a right angled Triangle, having given the Sum of the Legs, and the Sum of the Perpendicular and Base, to find the Triangle.

Let the Sum of the Legs AC and BC be a, the Sum of the Base AB and of the Perpendicular CD be b, the Leg AC $= x$, the Base AB $= y$, and BC will be $= a - x$, CD $= b - y$, $aa - 2ax + 2xx =$ ACq $+$ BCq $=$ ABq $= yy$, $ax - xx =$ AC $\times$ BC $=$ AB $\times$ CD (*a*) $= by - yy = by - aa + 2ax - 2xx$, and $by = aa - ax + xx$. Make its Square $a^4 - 2a^3x + 3aaxx - 2ax^3 + x^4$ equal to $yy \times bb$, that is, equal to $aabb - 2abbx + 2bbxx$ (*a*). And ordering the Equation, there will come out $x^4 - 2ax^3 \begin{matrix} +3aa \\ -2bb \end{matrix} xx \begin{matrix} -2a^3 \\ +2abb \end{matrix} x \begin{matrix} +a^4 \\ -aabb \end{matrix} = 0$. Add to each Side of the Equation $b^4 - aabb$ (*b*), and there will come out $x^4 - 2ax^3 \begin{matrix} +3aa \\ -2bb \end{matrix} xx \begin{matrix} -2a^3 \\ +2abb \end{matrix} x \begin{matrix} +a^4 \\ -2aabb \\ +b^4 \end{matrix} = b^4 - aabb$. And the Root being extracted on both Sides $xx - ax + aa - bb = -b\sqrt{bb - aa}$, and the Root being again extracted, $x = \frac{1}{2}a \pm \sqrt{bb - \frac{3}{4}aa - b\sqrt{bb - aa}}$ (*c*).

The

Problem VI. (*g*) Euclid III. 31.

Prob. VII. (*a*) Euclid VI. 8, and 16.

(*b*) Article CXIII.

(*c*) For $aa - bb$ transposed, is $bb - aa$; to which adding

The Geometrical Construction.

Take R a mean Proportional between $b+a$ and $b-a$, and S a mean Proportional between R and $b-R$, and T a mean Proportional between $\frac{1}{2}a+S$ and $\frac{1}{2}a-S$; and $\frac{1}{2}a+T$, and $\frac{1}{2}a-T$, will be the Sides of the Triangle (*e*).

Problem VIII.

Having given the Area, Perimeter, and one of the Angles A *of any Triangle* A B C, *to determine the rest* [See Figure 19.].

Let the Perimeter be $=a$, and the Area $=bb$, and from either of the unknown Angles, as C, let fall the Perpendicular C D to the opposite Side A B; and, by reason of the given Angle A, A C will be to C D in a given Ratio, suppose as d to e. Call, therefore, A C $=x$,

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adding $\frac{1}{4}aa$, it is $bb-aa+\frac{1}{4}aa=bb-\frac{3}{4}aa$; whence the Root is $\sqrt{bb-\frac{3}{4}aa-b\sqrt{bb-aa}}$.

Problem VII. (*e*) For the Side $x=\frac{1}{2}a\pm\sqrt{b^2-\frac{3}{4}a^2-b\sqrt{b^2-a^2}}$; and $\sqrt{b^2-a^2}$ is the mean Proportional between $b+a$, and $b-a$ (198) $=$ R; and $\sqrt{b\sqrt{b^2-a^2}-b^2+a^2}$ is the mean Proportional between $\sqrt{b^2-a^2}$, and $b-\sqrt{b^2-a^2}=$ S; and $\sqrt{b^2-\frac{3}{4}a^2-b\sqrt{b^2-a^2}}$ is the mean Proportional between $\frac{1}{2}a+\sqrt{b\sqrt{b^2-a^2}-b^2+a^2}$ and $\frac{1}{2}a-\sqrt{b\sqrt{b^2-a^2}-b^2+a^2}=$ T; wherefore $x=\frac{1}{2}a\pm$ T.

$= x$, and CD will be $= \frac{ex}{d}$, by which divide the Double of the Area, and there will come out $\frac{2bbd}{ex} =$ AB. Add AD (viz. $\sqrt{ACq - CDq}$, or $\frac{x}{d} \times \sqrt{dd - ee}$) (*a*) and there will come out $BD = \frac{2bbd}{ex} + \frac{x}{d} \times \sqrt{dd - ee}$; to the Square whereof add CDq, and there will arise $\frac{4b^4dd}{eexx} + xx + \frac{4bb}{e}\sqrt{dd - ee} = BCq$. Moreover, from the Perimeter take away AC and AB, and there will remain $a - x - \frac{2bbd}{ex} = BC$, the Square whereof $aa - 2ax + xx - \frac{4abbd}{ex} + \frac{4bbd}{e} + \frac{4b^4dd}{eexx}$, make equal to the Square before found; and neglecting the Equivalents, you will have $\frac{4bb}{e}\sqrt{dd - ee} = aa - 2ax - \frac{4abbd}{ex} + \frac{4bbd}{e}$. And this, by assuming $4af$ for the given Terms $aa + \frac{4bbd}{e} - \frac{4bb}{e}\sqrt{dd - ee}$, and by reducing,

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Prob. VIII. (*a*) $\sqrt{ACq - CDq} = \sqrt{x^2 - \frac{e^2x^2}{d^2}} = \sqrt{\frac{d^2x^2}{d^2} - \frac{e^2x^2}{d^2}} = \frac{x}{d}\sqrt{d^2 - e^2}$.

ducing, becomes $xx = 2fx - \frac{2bbd}{e}$, or $x = f \pm \sqrt{ff - \frac{2bbd}{e}}$ (b).

The ſame Equation would have come out alſo by ſeeking the Leg AB; for the Sides AB and AC are indifferently alike to all the Conditions of the Problem. Wherefore, if AC be made $= f - \sqrt{ff - \frac{2bbd}{e}}$, AB will be $= f + \sqrt{ff - \frac{2bbd}{e}}$, and reciprocally; and the Sum of theſe $2f$ ſubtracted from the Perimeter, leaves the third Side BC $= a - 2f$.

Problem IX.

Having given the Altitude, Baſe, and Sum of the Sides, to find the Triangle.

Let the Altitude CD be $= a$, half the Baſis AB $= b$, half the Sum of the Sides $= c$, and their Semi-difference

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Prob. VIII. (b) For $a^2 + \frac{4b^2d}{e} - \frac{4b^2}{e}\sqrt{d^2 - e^2} = 2ax + \frac{4ab^2d}{ex} = 4af$; which, multiplied by ex, and dividing by $2ae$, and tranſpoſing $\frac{2b^2d}{e}$; becomes $x^2 = 2fx - \frac{2b^2d}{e}$; and tranſpoſing $2fx$, is $x^2 - 2fx = -\frac{2b^2d}{e}$; whence $x = f \pm \sqrt{ff - \frac{2b^2d}{e}}$.

difference $= z$; and the greater Side as BC will be $= c + z$ (*a*), and the leſſer AC $= c - z$ (*b*). Subtract CD*q* from CB*q*, and alſo from AC*q*, and hence will BD be $= \sqrt{cc + 2cz + zz - aa}$, and thence AD $= \sqrt{cc - 2cz + zz - aa}$. Subtract alſo AB from BD, and AD will again be $= \sqrt{cc + 2cz + zz - aa} - 2b$. Having now ſquared the Values of AD, and ordered the Terms, there will ariſe $bb + cz = b\sqrt{cc + 2cz + zz - aa}$. Again, by ſquaring and reducing into Order, you will obtain $cczz - bbzz = bbcc - bbaa - b^4$. And $z = b \times \sqrt{1 - \frac{aa}{cc - bb}}$ (*c*). Whence the Sides are given.

Problem X.

Having given the Baſe AB, *and the Sum of the Sides* AC + BC, *and alſo the vertical Angle* C, *to determine the Sides.* [See Fig. 20.]

Make the Baſe $= a$, Half the Sum of the Sides $= b$, and Half the Difference $= x$, and the greater Side BC will be $= b + x$, and the leſſer AC $= b - x$. From either of the unknown Angles A, let fall the Perpendicular AD to the oppoſite Side BC; and, by reaſon of

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Problem IX. (*a*) Number 22.

(*b*) Number 36.

(*c*) For $z^2 = \frac{b^2c^2 - b^2a^2 - b^4}{c^2 - b^2}$; whence $z = \sqrt{\frac{b^2c^2 - b^2a^2 - b^4}{c^2 - b^2}} = b\sqrt{1 - \frac{a^2}{c^2 - b^2}}$.

of the given Angle C, there will be given the Ratio of AC to CD, ſuppoſe as d to e, and then CD will be $= \frac{eb - ex}{d}$. Alſo, by 13. 2 Elem. $\frac{ACq - ABq + BCq}{2BC}$, that is $\frac{2bb + 2xx - aa}{2b + 2x} = CD$; and ſo you have an Equation between the Values of CD. And this reduced, x becomes $= \sqrt{\frac{daa + 2ebb - 2dbb}{2d + 2e}}$; whence the Sides are given.

If the Angles at the Baſe were ſought, the Concluſion would be more neat; as draw EC biſecting the given Angle, and meeting the Baſe in E; and it will be AB : AC + BC (: : AE : AC) : : Sine Angle ACE : Sine Angle AEC. And if from the Angle AEC, and alſo from its Complement BEC, you ſubtract $\frac{1}{2}$ the Angle C, there will be left the Angles ABC and BAC.

Problem XI.

Having the Sides of a Triangle given, to find the Angles.
[See Figure 72.]

Let the given Sides be $AB = a$, $AC = b$, $BC = c$, to find the Angle A. Having let fall to AB the Perpendicular CD, which is oppoſite to that Angle, you will have in the firſt Place, $bb - cc = ACq - BCq = ADq - BDq$ (a) $= \overline{AD + BD} \times \overline{AD - BD}$ (b) $= AB$

Problem XI. (a) For $AC^2 = CD^2 - DA^2$, and $BC^2 = BC^2 - DC^2$; whence $AC^2 - BC^2 = CD^2 + DA^2 - BD^2 - CD^2 = DA^2 - BD^2$.

(b) Euclid, II. 5.

$= AB \times \overline{2AD - AB}$ (c) $= 2AD \times a - aa$. And consequently $\frac{1}{2}a + \frac{bb - cc}{2a} = AD$ (d). Whence comes out this *first Theorem.*

I. As AB to AC + BC, so AB — BC to a fourth Proportional N. $\frac{AB + N}{2} = AD$. As AC to AD, so Radius to the Co-sine of the Angle A.

Moreover, $DCq = ACq - ADq =$

$$\frac{2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4}{4aa} =$$

$$\frac{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}{4aa}.$$

Whence, having multiplied the Roots of the Numerator and Denominator by b, there is made this *second Theorem.*

II. As $2ab$ to a mean Proportional between $\overline{a+b+c} \times \overline{a+b-c}$ and $\overline{a-b+c} \times \overline{-a+b+c}$, so is Radius to the Sine of the Angle A (e).

Moreover,

Prob. XI. (c) For AD + DB = AB; whence subducting 2AD, then DB — AD = AB — 2AD; or if 2AD is greater than AB, then BD — AD = 2AD — AB; whence $\overline{AD + BD} \times \overline{AD - BD} = AB \times \overline{2AD - AB}$.

(d) For $2ADa = aa + bb - cc$, therefore $AD = \frac{aa + bb - cc}{2a} = \frac{1}{2}a + \frac{bb - cc}{2a}$.

(e) For putting AC the Radius, DC is the Sine of A, and $DC = \frac{\text{Mean Proportional}}{2a}$ (198) $= \frac{\text{Mean Proportional} \times b}{2a \times b}$.

Moreover, on AB take AE = AC, and draw CE, and the Angle ECD will be equal to Half the Angle A (*f*). Take AD from AE, and there will remain $DE = b - \frac{1}{2}a - \frac{bb + cc}{2a} = \frac{cc - aa + 2ab - bb}{2a}$ $= \frac{\overline{c + a - b} \times \overline{c - a + b}}{2a}$. Whence $DEq =$ $\frac{\overline{c + a - b} \times \overline{c + a - b} \times \overline{c - a + b} \times \overline{c - a + b}}{4aa}$.

And hence is made the *third* and *fourth Theorem*, viz.

III. As $2ab$ to $\overline{c + a - b} \times \overline{c - a + b}$ (so AC to DE) so Radius to the versed Sine of the Angle A (*g*).

IV. And, as a mean Proportional between $a + b + c$ and $a + b - c$, to a mean Proportional between $c + a - b$ and $c - a + b$, (so CD to DE) so Radius to the Tangent of Half the Angle A, or the Co-tangent of Half the Angle to Radius (*h*).

Besides,

Prob. XI. (*f*) For the Angles at D being right, then (Eucl. I. 32.) DAC + DCA = DEC + DCE = ECA (Eucl. I. 5.) = DCA + DCE; whence DAC + DCA = DCA + 2DCE, and DAC = 2DCE.

(*g*) For putting AC the Radius, DE is the versed Sine of A, and $DE = \frac{\overline{c + a - b} \times \overline{c - a + b}}{2a} =$ $\frac{\overline{c + a - b} \times \overline{c - a + b} \times b}{2a \times b}$.

(*h*) For $DC^2 = \frac{\overline{a + b + c} \times \overline{a + b - c} \times \overline{a - b + c} \times \overline{-a + b + c}}{4a^2}$.

and

Besides, CEq is $= CDq + DEq = \frac{2abb + bcc - baa - b^3}{a}$ $= \frac{b}{a} \times \overline{c + a - b} \times \overline{c - a + b}$. Whence the *fifth* and *sixth Theorem.*

V. As a mean Proportional between $2a$ and $2b$, to a mean Proportional between $c + a - b$ and $c - a + b$, or as 1 to a mean Proportional between $\frac{c + a - b}{2a}$ and $\frac{c - a + b}{2b}$, (*i*) (so AC to $\frac{1}{2}$ CE, or

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and $DE^2 = \frac{\overline{a-b+c} \times \overline{a-b+c} \times \overline{-a+b+c} \times \overline{-a+b+c}}{4a^2}$;
and therefore $CD^2 : DE^2 ::$
$\frac{\overline{a+b+c} \times \overline{a+b-c} \times \overline{a-b+c} \times \overline{-a+b+c}}{4aa}$:
$\frac{\overline{a-b+c} \times \overline{a-b+c} \times \overline{-a+b+c} \times \overline{-a+b+c}}{4aa}$;
i. e. $\left(\text{by dividing by } \frac{\overline{a-b+c} \times \overline{-a+b+c}}{4aa}\right) ::$
$\overline{a+b+c} \times \overline{a+b-c} : \overline{a-b+c} \times \overline{-a+b+c}$;
therefore $CD : DE :: \sqrt{\overline{a+b+c} \times \overline{a+b-c}} :$
$\sqrt{\overline{a-b+c} \times \overline{-a+b+c}}$.

Prob. XI. (*i*) For $AC^2 : CE^2 :: bb : \frac{b}{a} \overline{c+a-b} \times \overline{-a+b+c}$, i. e. by dividing by $\frac{b}{a}$; $:: ab : \overline{a-b+c} \times \overline{-a+b+c}$; whence $AC : CE :: \sqrt{ab} : \sqrt{\overline{a-b+c} \times \overline{-a+b+c}} :: 2\sqrt{ab} :$

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or CE to DE) so Radius to the Sine of $\frac{1}{2}$ the Angle A (*k*).

VI. And as a mean Proportional between $2a$ and $2b$, to a mean Proportional between $a+b+c$ and $a+b-c$ (so CE to CD), so Radius to the Co-sine of Half the Angle A (*l*).

But if, besides the Angles, the Area of the Triangle be also sought, multiply CDq by $\frac{1}{4}ABq$, and the Root, viz. $\frac{1}{4}\sqrt{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{-a+b+c}}$ will be the Area sought.

$2\sqrt{\overline{a-b+c}\times\overline{-a+b+c}}$; whence AC : $2\sqrt{ab}$ $(=\sqrt{4ab})$ :: CE : $2\sqrt{\overline{a-b+c}\times\overline{-a+b+c}}$ (Eucl. V. 16.) :: $\frac{CE}{2}$: $\sqrt{\overline{a-b+c}\times\overline{-a+b+c}}$.

Prob. XI. (*k*) For the Angles at D being right, and CEA and CED being equal (Eucl. I. 5.), AC : $\frac{1}{2}$CE :: CE : ED.

(*l*) For $CE^2 : CD^2 :: \frac{b}{a}\overline{c+a-b}\times\overline{c-a+b}$: $\frac{\overline{a+b+c}\times\overline{a+b-c}\times\overline{a-b+c}\times\overline{b-a+c}}{4aa}$; i. e. dividing by $\frac{\overline{c+a-b}\times\overline{c-a+b}}{a}$, :: b : $\frac{\overline{a+b+c}\times\overline{a+b-c}}{4a}$ :: $4ab$: $\overline{a+b+c}\times\overline{a+b-c}$; whence CE : CD :: $\sqrt{4ab}$: $\sqrt{\overline{a+b+c}\times\overline{a+b-c}}$.

PROBLEM XII.

Having the Sides and Base of any right lined Triangle given, to find the Segments of the Base, the Perpendicular, the Area, and the Angles [See Fig. 40.].

Let there be given the Sides AC, BC, and the Base AB of the Triangle ABC. Bisect AB in I, and take on it (being produced on both Sides) AF and AE equal to AC, and BG and BH equal to BC. Join CE, CF; and from C to the Base, let fall the Perpendicular CD. And $ACq - BCq$ will be $= ADq + CDq - CDq - BDq = ADq - BDq = \overline{AD + BD} \times \overline{AD - BD} = AB \times 2DI$ (*a*). Therefore $\frac{ACq - BCq}{2AB} = DI$. And $2AB : AC + BC :: AC - BC : DI$. Which is a Theorem for determining the Segments of the Base (*b*).

From IE, that is, from $AC - \frac{1}{2}AB$, take away DI, and there will remain $DE = \frac{BCq - ACq + 2AC \times AB - ABq}{2AB}$ (*c*); that is, $= BC$

Prob. XII. (*a*) 199. For $AD = AI + ID = BI + ID = BD + ID + ID$; whence $AD - DB = 2ID$; Therefore, because $AB = AD + BD$, $ACq - BCq = ADq - BDq = AB \times 2ID$: i. e. The Difference of the Squares of the Segments of the Base; i. e. the Rectangle under the Sum and Difference of the Segments of the Base is equal to the Rectangle under the whole Base, and double the Distance of the Perpendicular from the Middle of the Base. (*b*) See Theorem I. of Prob. XI.

(*c*) For $DE = AC - \frac{1}{2}AB - (DI =) \frac{AC^2 - BC^2}{2AB}$; whence $DE = \frac{\overline{AC - \frac{1}{2}AB} \times 2AB - AC^2 + BC^2}{2AB}$

$= \frac{BC^2 - AC^2 + 2AC \times AB - AB^2}{2AB}$

$= \frac{\overline{BC + AC - AB} \times \overline{BC - AC + AB}}{2AB}$, or $= \frac{HE \times EG}{2AB}$ (*d*). Take away DE from FE, or 2AC, and there will remain FD = $\frac{ACq + 2AC \times AB + ABq - BCq}{2AB}$; that is, $= \frac{\overline{AC + AB + BC} \times \overline{AC + AB - BC}}{2AB}$, or $= \frac{FG \times FH}{2AB}$. And since CD is a mean Proportional between DE and DF, and CE a mean Proportional between DE and EF, and CF a mean Proportional between DF and EF (*e*), CD will be $= \frac{\sqrt{FG \times FH \times HE \times EG}}{2AB}$, CE $= \sqrt{\frac{AC \times HE \times EG}{AB}}$, and CF $= \sqrt{\frac{AC \times FG \times FH}{AB}}$. Multiply CD into $\frac{1}{2}$ AB, and you will have the Area $= \frac{1}{4}\sqrt{}$

(*d*) For BC + CA = BH + AE = BE + EH + AH + HE, but BE + EH + HA = BA; whence BC + CA = BA + HE; and BC + CA − AB = HE; and BC − CA = BG − AE; and AE + EB = AB: Whence BC − CA + AB = BG − AE + AE + EB = EG.

(*e*) For EA, AC, AF, being equal, a Semi-circle from the Center A, with the Radius AF, will pass through C, and E; whence FCE is a right Angle, Euclid III. 31; whence the Proportions follow, by Euclid VI. 8.

$= \frac{1}{4}\sqrt{FG \times FH \times HE \times EG}$. But for determining the Angle A, there come out several Theorems:

1. As $2AB \times AC : HE \times EG$ (:: $AC : DE$) :: Radius : versed Sine of the Angle A.
2. $2AB \times AC : FG \times FH$ (:: $AC : FD$) :: Radius : versed Co-sine of A.
3. $2AB \times AC : \sqrt{FG \times FH \times HE \times EG}$ (:: $AC : CD$) :: Radius : Sine of A.
4. $\sqrt{FG \times FH} : \sqrt{HE \times EG}$ (:: $CF : CE$) :: Radius : Tangent of $\frac{1}{2}$ A.
5. $\sqrt{HE \times EG} : \sqrt{FG \times FH}$ (:: $CE : FC$) :: Radius : Co-tangent of $\frac{1}{2}$ A.
6. $2\sqrt{AB \times AC} : \sqrt{HE \times EG}$ (:: $FE : CE$) :: Radius : Sine of $\frac{1}{2}$ A.
7. $2\sqrt{AB \times AC} : \sqrt{FG \times FH}$ (:: $FE : FC$) :: Radius : Co-sine of $\frac{1}{2}$ A (*f*).

Problem XIII.

To subtend the given Angle CBD *with the given right Line* CD; *so that if* AD *be drawn from the End of that right Line* D *to the Point* A, *given on the right Line* CB *produced, the Angle* ADC *shall be equal to the Angle* ABD [See Fig. 71.].

Make $CD = a$, $AB = b$, $BD = x$, and it will be $BD : BA :: CD : DA = \frac{ab}{x}$ (*a*). Let fall the Perpendicular

(*f*) This Problem is the same with XI. and the I. III. IV. and V. VI. VII. Theorems of this, fall in with the III. II. IV. V. VI. of the XI, respectively.

Prob. XIII. (*a*) For the Triangles ABD, ACD, have dicular DE, and BE will be $= \frac{BDq - ADq + BAq}{2BA}$ (*b*)

$= \frac{xx - \frac{aabb}{xx} + bb}{2b}$. By reason of the given Triangle DBA, make BD : BE :: $b : e$, and you will have again BE $= \frac{ex}{b}$, therefore $xx - \frac{aabb}{xx} + bb = 2ex$. And $x^4 - 2ex^3 + bbxx - aabb = 0$.

Problem XIV.

To find the Triangle ABC, *whose three Sides* AB, AC, BC, *and its Perpendicular* DC *are in arithmetical Progression* [See Fig. 46.].

Make AC $= a$, BC $= x$, and DC will be $= 2x - a$, and AB $= 2a - x$ (*a*). Also AD will be ($= \sqrt{ACq - DCq}$) $= \sqrt{4ax - 4xx}$, and BD ($= \sqrt{BCq}$

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have the Angle at A common, and the Angles ABD, ADC, equal; and are similar.

(*b*) For AE $=$ EB — BA, and $AE^2 = EB^2 - 2AB \times BE + BA^2$ (Eucl. II. 7.) $= AD^2 - DE^2$, the Triangle DAE being rectangular; but $DE^2 = DB^2 - BE^2$, the Triangle DBE being also rectangular; whence $EB^2 - 2AB \times BE + BA^2 = AD^2 - DB^2 + EB^2$; whence abbreviating, transposing, and dividing by 2AB, BE $= \frac{BD^2 - AD^2 + AB^2}{2AB}$.

Prob. XIV. (*a*) For AB, AC, BC, DC, being in arithmetical Progression, if AC $= a$, and BC $= x$, then AB $= 2a - x$, and DC $= 2x - a$; because $2a - x$, a, x, and $2x - a$, are in a Progression, whose common Difference is $a - x$.

$\sqrt{BCq - DCq}) = \sqrt{4ax - 3xx - aa}$. And fo again $AB = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$. Wherefore $2a - x = \sqrt{4ax - 4xx} + \sqrt{4ax - 3xx - aa}$, or $2a - x - \sqrt{4ax - xx} = \sqrt{4ax - 3xx - aa}$. And the Parts being fquared, $4aa - 3xx - \overline{4a + 2x} \times \sqrt{4ax - 4xx} = 4ax - 3xx - aa$, or $5aa - 4ax = \overline{4a - 2x} \times \sqrt{4ax - 4xx}$. And the Parts being again fquared, and the Terms rightly difpofed, $16x^4 - 80ax^3 + 144aaxx - 104a^3x + 25a^4 = 0$. Divide this Equation by $2x - a$, and there will arife $8x^3 - 36axx + 54aax - 25a^3 = 0$; an Equation, by the Solution whereof x is given from a, being any how affumed. a and x being had, make a Triangle, whofe Sides fhall be $2a - x$, a and x, and a Perpendicular let fall upon the Side $2a - x$ will be $2x - a$.

If I had made the Difference of the Sides of the Triangle to be d, and the Perpendicular to be x, the Work would have been fomething neater; this Equation at laft coming out, viz. $x^3 = 24ddx + 48d^3$ (*b*).

Prob. XIV. (*b*) For DC, BC, AC, AB, being the Terms, and $DC = x$, and the Difference $= d$, the Terms will be refpectively x, $x + d$, $x + 2d$, $x + 3d$; and $AD = \sqrt{4dx + 4dd}$, and $BD\ \sqrt{2dx + dd}$; whence $AB = x + 3d = \sqrt{4dx + 4dd} + \sqrt{2dx + dd}$; and $x + 3d - \sqrt{4dx + 4dd} = \sqrt{2dx + dd}$; and fquaring the Parts, and reducing, and tranfpofing, $x^2 + 8dx + 12dd = \overline{2x + 6d}\sqrt{4dx + 4dd}$; and fquaring again, $x^4 + 16dx^3 + 88ddx^2 + 192d^3x + 144d^4 = 240d^3x + 112d^2x^2 + 16dx^3 + 144d^4$; that is, by tranfpofing, and dividing by x, $x^3 = 24d^2x + 48d^3$.

PROBLEM XV.

To find a Triangle ABC, *whose three Sides* AB, AC, BC, *and the Perpendicular* CD *shall be in a Geometrical Progression.*

Make AC $= x$, BC $= a$; and AB will be $= \frac{xx}{a}$. And CD $= \frac{aa}{x}$ (*a*). And AD $(= \sqrt{ACq - CDq})$ $= \sqrt{xx - \frac{a^4}{xx}}$; and BD $(= \sqrt{BCq - DCq}) = \sqrt{aa - \frac{a^4}{xx}}$ and consequently $\frac{xx}{a}$ ($=$ AB) $= \sqrt{xx - \frac{a^4}{xx}} + \sqrt{aa - \frac{a^4}{xx}}$, or $\frac{xx}{a} - \sqrt{aa - \frac{a^4}{xx}} = \sqrt{xx - \frac{a^4}{xx}}$; and the Parts of the Equation being squared, $\frac{x^4}{aa} - \frac{2xx}{a} \times \sqrt{aa - \frac{a^4}{xx}} + aa - \frac{a^4}{xx} = xx - \frac{a^4}{xx}$, that is, $x^4 - aaxx + a^4 = 2aax\sqrt{xx - aa}$

Prob. XV. (*a*) For AB, AC, BC, DC, being in geometrical Progression, if AC $= x$, and BC $= a$, then AB $= \frac{xx}{a}$, and CD $= \frac{aa}{x}$; for $\frac{xx}{a}$, x, a, $\frac{aa}{x}$, are in a Progression, whose common Divisor is $\frac{x}{a}$.

$\sqrt{xx - aa}$ (*b*). And the Parts being again squared, $x^8 - 2aax^6 + 3a^4x^4 - 2a^6xx + a^8 = 4a^4x^4 - 4a^6xx$. That is, $x^8 - 2aax^6 - a^4x^4 + 2a^6xx + a^8 = 0$. Divide this Equation by $x^4 - aaxx - a^4$, and there will arise $x^4 - aaxx - a^4$. Wherefore x^4 is $= aaxx + a^4$. And extracting the Root $xx = \frac{1}{2}aa + \sqrt{\frac{5}{4}a^4}$, or $x = a\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$ (*c*). Take therefore a or BC, of any Length, and make BC : AC :: AC : AB :: 1 : $\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$, and the Perpendicular DC of a Triangle ABC made of these Sides, will be to the Side BC in the same Ratio.

Prob. XV. (*b*) For $\frac{x^4}{a^2} - \frac{2x^2}{a}\sqrt{a^2 - \frac{a^4}{x^2}} + a^2 - \frac{a^4}{x^2} = x^2 - \frac{a^4}{x^2}$, which, by abbreviating, and multiplying by a^2, becomes $x^4 - 2ax^2\sqrt{a^2 - \frac{a^4}{x^2}} + a^4 = a^2x^2$; and by transposing, $x^4 - a^2x^2 + a^4 = 2ax^2\sqrt{a^2 - \frac{a^4}{x^2}}$; or putting x^2 under the Sign, we have $2a\sqrt{a^2x^4 - a^4x^2}$; and the radical Part divided by a^2x^2, gives $x^2 - a^2$; which being kept under the Sign, and the Root of the Divisor, viz. ax, being drawn into $2a$, we have $2a^2x\sqrt{x^2 - a^2}$.

(*c*) For $x^2 = \frac{aa}{2} + \sqrt{5\frac{a^4}{4}}$; that is, by dividing $\sqrt{5\frac{a^4}{4}}$ by $\sqrt{a^4}$; $x^2 = \frac{aa}{2} + aa\sqrt{\frac{5}{4}}$: Whence $x = \sqrt{\frac{aa}{2} + aa\sqrt{\frac{5}{4}}}$; that is, by dividing by $\sqrt{a^2}$, $x = a\sqrt{\frac{1}{2} + \sqrt{\frac{5}{4}}}$.

The same otherwise [See Fig. 47].

Since AB : AC :: BC : DC. I say the Angle ACB is a right one. For if you deny it, draw CE, making the Angle ECB a right one. Therefore the Triangles BCE, DBC, are similar, by 8. 6. Elem. and consequently EB : EC :: BC : DC; that is, EB : EC :: AB : AC. Draw AF perpendicular to CE, and by reason of the parallel Lines AF, BC, EB will be : EC :: AE : FE :: AB : FC (*d*). Therefore by 9. 5 Elem. AC is = FC; that is, the Hypothenuse of a right-angled Triangle is equal to the Side, contrary to the 19. 1 Elem. Therefore the Angle ECB is not a right one; wherefore it is necessary ACB should be a right one. Therefore $ACq + BCq$ is $= ABq$. But $ACq = AB \times BC$, therefore $AB \times BC + BCq = ABq$, and extracting the Root $AB = \frac{1}{2} BC + \sqrt{\frac{5}{4} BCq}$. Wherefore take $BC : AB :: 1 : \frac{1+\sqrt{5}}{2}$, and AC a mean Proportional between BC and AB, and a Triangle being made of these Sides, AB, AC, BC, DC, will be continually Proportionals.

Problem XVI.

To make the Triangle ABC *upon the given Base* AB, *whose Vertex* C *shall be in the right Line* EC *given in Position, and the Base an arithmetical Mean between the Sides* [See Fig. 48.]

Let the Base AB be bisected in F, and produced until it meet the right Line given in Position EC in E, and let fall to it the Perpendicular CD; and making AB = *a*,

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Prob. XV. (*d*) For EB : EC :: BE + EA : CE + EF (Eucl. V. 12.), i. e. BA : CF :: BA : AC.

AB $= a$, FE $= b$, and BC $-$ AB $= x$, BC will be $= a + x$, AC $= a - x$; and by the 13. 2 Elem. BD $\left(= \frac{BCq - ACq + ABq}{2AB}\right) = 2x + \frac{1}{2}a$. And consequently, FD $= 2x$ (*a*), DE $= b + 2x$, and CD $(= \sqrt{CBq - BDq}) = \sqrt{\frac{3}{4}aa - 3xx}$. But by reason of the given Positions of the right Lines CE and AB, the Angle CED is given; and consequently the Ratio of DE to CD, which, if it be put as d to e, will give the Proportion $d : e :: b + 2x : \sqrt{\frac{3}{4}aa - 3xx}$. Whence the Means and Extremes being multiplied by each other, there arises the Equation $eb + 2ex = d\sqrt{\frac{3}{4}aa - 3xx}$, the Parts whereof being squared and rightly ordered, you have $xx = \frac{\frac{3}{4}d^2a^2 - eebb - 4eebx}{4ee + 3dd}$, and the Root being extracted $x = \frac{-2eeb + d\sqrt{3eeaa - 3eebb + \frac{9}{4}ddaa}}{4ee + 3dd}$. But x being given, there is given BC $= a + x$, and AC $= a - x$.

Prob. XVI. (*a*) For BD $= \frac{\text{Sum} + \text{Difference}}{2} = \frac{\text{Sum}}{2} + \frac{2\ \text{Difference}}{2}$; now BD $=$ BF $+$ FD $= \frac{1}{2}a + 2x$; and BF $= \frac{1}{2}a$; whence FD $= 2x$.

PROBLEM

Problem XVII.

Having given the Sides of any Parallelogram AB, BD, DC, *and* AC, *and one of the Diagonals* BC, *to find the other Diagonal* AD [See Fig. 21.].

Let E be the Concourse of the Diagonals, and to the Diagonal BC let fall the Perpendicular AF, and by the 13. 2 Elem. $\frac{ACq - ABq + BCq}{2BC} = CF$, and also $\frac{ACq - AEq + ECq}{2EC} = CF$. Wherefore, since EC is $= \frac{1}{2}BC$ (*a*), and $AE = \frac{1}{2}AD$, it will be $\frac{ACq - ABq + BCq}{2BC} = \frac{ACq - \frac{1}{4}ADq + \frac{1}{4}BCq}{BC}$, and having reduced, $AD = \sqrt{2ACq + 2ABq - BCq}$ (*b*).

Whence, by the bye, *in any Parallelogram, the Sum of the Squares of the Sides is equal to the Sum of the Squares of the Diagonals* (*c*).

Problem

Prob. XVII. (*a*) 200. *In every Parallelogram the Diagonals bisect each other*; for the Triangles CED, AEB, having the Angles CED, AEB, vertically opposite, equal (Eucl. I. 15), and the alternate Angles, CDA, DAB, equal (Eucl. I. 29.), will be equiangular; but also the Bases CD, AB, are equal (Eucl. I. 34.), therefore the Triangles are equal, and the Side CE = EB, and DE = EA (Eucl. I. 26.)

(*b*) For multiplying to exterminate the Fractions, and dividing the Terms by BC, we have $AD^2 = 2AC^2 + 2AB^2 - BC^2$; whence $AD = \sqrt{2AC^2 + 2AB^2 - BC^2}$.

(*c*) 201. In every Parallelogram, *the Sum of the Squares of the Sides is equal to the Sum of the Squares of the Diagonals*, for

PROBLEM XVIII.

Having given the Angles of the Trapezium ABCD *also its Perimeter and Area, to determine the Sides* [See Fig. 22.]

Produce any two of the Sides AB and DC till they meet in E, and let AB be $= x$, and BC $= y$, and because all the Angles are given, there are given the Ratio's of BC to CE and BE, which make d to e and f; and CE will be $= \frac{ey}{d}$ and BE $= \frac{fy}{d}$, and consequently AE $= x + \frac{fy}{d}$. There are also given the Ratio's of AE to AD and to DE; which make as g and as b to d; and AD will be $= \frac{dx + fy}{g}$ and ED $= \frac{dx + fy}{b}$, and consequently CD $= \frac{dx + fy}{b} - \frac{ey}{d}$, and the Sum of all the Sides $x + y + \frac{dx + fy}{g} + \frac{dx + fy}{b} - \frac{ey}{d}$; which, since it is given, call it a, and the Terms will be abbreviated by writing $\frac{p}{r}$ for the given

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for if the Parallelogram is rectangular, it follows from the 47. of I. of Euclid; and if not, the Angles at the same Side are equal to two right ones (Eucl. 27. I.); whence if one is obtuse the other is acute, and raising Perpendiculars from the Ends of the opposite Sides, the Square of the Diagonal opposite the obtuse Angle will exceed the Sum of the Squares of the Sides of that Angle (Eucl. II. 12.), by the same double Rectangle, by which the Square of the Diagonal opposite the acute Angle is exceeded by the Sum of the Squares of the Sides containing the acute Angle (Eucl. II. 13.).

given Quantity $1+\frac{d}{g}+\frac{d}{h}$, and $\frac{q}{r}$ for the given $1+\frac{f}{g}+\frac{f}{h}-\frac{e}{d}$, and you will have the Equation $\frac{px+qy}{r}=a$.

Moreover, by reaſon of all the Angles being given, there is given the Ratio of BCq to the Triangle BCE, which make as m to n, and the Triangle BCE will be $=\frac{n}{m}yy$. There is alſo given the Ratio of AEq to the Triangle ADE; which make as m to d; and the Triangle ADE will be $=\frac{ddxx+2dfxy+ffyy}{dm}$. Wherefore, ſince the Area AC, which is the Difference of theſe Triangles, is given, let it be bb, and $\frac{ddxx+2dfxy+ffyy-dnyy}{dm}$ will be $=bb$. And ſo you have two Equations, from the Reduction whereof all is determined, viz. The former Equation gives $\frac{ra-qy}{p}=x$, and by writing $\frac{ra-qy}{p}$ for x in the laſt, there comes out $\frac{drraa-2dqray+dqqyy}{ppm}+\frac{2afry-2fqyy}{pm}+\frac{ffyy-dnyy}{dm}=bb$. And the Terms being abbreviated by writing s for the given Quantity $\frac{dqq}{pp}-\frac{2fq}{p}+\frac{ff}{d}-n$, and st for the given $+\frac{adqr}{pp}-\frac{afr}{p}$, and stv for the given $bbm-\frac{drraa}{pp}$, there ariſes $yy=2ty+tv$, or $y=t+\sqrt{tt+tv}$.

PROBLEM

PROBLEM XIX.

To surround the Fish-pond ABCD *with a Walk* ABCD EFGH *of a given Area, and of the same Breadth every where* [See Fig. 23.].

Let the Breadth of the Walk be x, and its Area aa. And, letting fall the Perpendiculars AK, BL, BM, CN, CO, DP, DQ, AI, from the Points A, B, C, D, to the Lines EF, FG, GH, and HE, to divide the Walk into the four Trapezia IK, LM, NO, PQ, and into the four Parallelograms AL, BN, CP, DI, of the Latitude x, and of the same Length with the Sides of the given Trapezium. Let therefore the Sum of the Sides $(AB + BC + CD + DA)$ be $= b$, and the Sum of the Parallelograms will be $= bx$.

Moreover, having drawn AE, BF, CG, DH; since AI is = AK, the Angle AEI will be = Angle AEK $= \frac{1}{2}$ IEK, or $\frac{1}{2}$ DAB. Therefore the Angle AEI is given, and consequently the Ratio of AI to IE, which make as d to e, and IE will be $= \frac{ex}{d}$. Multiply this into $\frac{1}{2}$ AI, or $\frac{1}{2}x$, and the Area of the Triangle AEI will be $= \frac{exx}{2d}$. But by reason of equal Angles and Sides, the Triangles AEI and AEK are equal, and consequently the Trapezium IK (= 2 Triangles AEI) $= \frac{exx}{d}$. In like manner, by putting BL : LF :: $d : f$, and CN : NG :: $d : g$, and DP : PH :: $d : h$ (for those Ratio's are also given from the given Angles B, C, and D), you will have the Trapezium LM $= \frac{fxx}{d}$, NO $= \frac{gxx}{d}$, and PQ $= \frac{hxx}{d}$.

Wherefore

Wherefore $\frac{exx}{d}+\frac{fxx}{d}+\frac{gxx}{d}+\frac{hxx}{d}$, or $\frac{pxx}{d}$, by writing p for $e+f+g+h$ will be equal to the four Trapeziums IK + LM + NO + PQ; and consequently $\frac{pxx}{d}+hx$ will be equal to the whole Walk aa. Which Equation, by dividing all the Terms by $\frac{p}{d}$, and extracting its Root, x will become $=$ $\frac{-dh+\sqrt{hhdd+4aapd}}{2p}$. And the Breadth of the Walk being thus found, it is easy to describe it.

Problem XX.

From the given Point C, *to draw the right Line* CF, *which together with two other right Lines* AE *and* AF *given by Position, shall comprehend or constitute the Triangle* AEF *of a given Magnitude* [See Fig. 24.]

Draw CD parallel to AE, and CB and EG perpendicular to AF, and let AD $=a$, CB $=b$, and AF $=x$, and the Area of the Triangle AEF be cc, and by reason of the proportional Quantities DF : AF (:: DC : AE) :: CB : EG; that is, $a+x:x::b:\frac{bx}{a+x}$; it will be $\frac{bx}{a+x}=$ EG. Multiply this into $\frac{1}{2}$AF, and there will come out $\frac{bxx}{2a+2x}$, the Quantity of the Area AEF, which is $=cc$. And so the Equation being ordered xx will be $=\frac{2ccx+2cca}{b}$ or $x=\frac{cc+\sqrt{c^4+2ccab}}{b}$.

After

After the same manner a right Line may be drawn through a given Point, which shall divide any Triangle or Trapezium in a given Ratio (a).

Problem XXI.

To determine the Point C in the given right Line DF, from which the right Lines AC and BC drawn to two other Points A and B given by Position, shall have a given Difference. [See Fig. 25.] [See Prob. xiv.]

From the given Points let fall the Perpendiculars AD and BF to the given right Line, and make AD $= a$, BF $= b$, DF $= c$, DC $= x$, and AC will be $= \sqrt{aa + xx}$, FC $= x - c$, and BC $= \sqrt{bb + xx - 2cx + cc}$. Now let their given Difference be d, AC being greater than BC, then $\sqrt{aa + xx} - d$ will be $= \sqrt{bb + xx - 2cx + cc}$. And squaring the Parts $aa + xx + dd - 2d\sqrt{aa + xx} = bb + xx - 2cx + cc$. And reducing, and (for Abbreviation sake) writing $2ee$ instead of the given Quantities $aa + dd - bb - cc$, there will come out $ee + cx = d \times \sqrt{aa + xx}$. And again, having squared the Parts, $e^4 + 2ceex + ccxx = ddaa + ddxx$. And the Equation

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Prob. XX. (a) For let the Area of the given Triangle AEF be to the Area of the sought Triangle, as m to n; then the Area of the sought Triangle must be equal to the Area of the given one multiplied into $\frac{n}{m}$ and will therefore be given; and the Problem will be the same as this twentieth.

Equation being reduced $xx = \frac{2eecx + e^4 - aadd}{dd - cc}$, or $x = \frac{eec + \sqrt{e^4dd - aad^4 + aaddcc}}{dd - cc}$.

The Problem will be resolved after the same Way, if the Sum of the Lines AC and BC, or the Sum or the Difference of their Squares, or the Proportion or Rectangle, or the Angle comprehended by them be given: Or also, if instead of the right Line DC, you make use of the Circumference of a Circle, or any other curve Line, so the Calculation (in this last Case especially) relates to the Line that joins the Points A and B.

Problem XXII.

Having the three right Lines AD, AE, BF, *given by Position, to draw a fourth* DF, *whose Parts* DE *and* EF, *intercepted by the former, shall be of given Lengths* [See Fig. 49.]

Let fall EG perpendicular to BF, and draw EC parallel to AD; and, the three right Lines given by Position meeting in A, B, and H, make AB $= a$, BH $= b$, AH $= c$, ED $= d$, EF $= e$, and HE $= x$. Now, by reason of the similar Triangles ABH, ECH; it is AH : AB :: HE : EC $= \frac{ax}{c}$, and AH : HB :: HE : CH $= \frac{bx}{c}$. Add HB, and there comes CB $= \frac{bx + bc}{c}$. Moreover, by reason of the similar Triangles FEC, FDB, it is ED : CB :: EF : CF $= \frac{ebx + ebc}{dc}$. Lastly, by the 12 and 13. 2 Elem. you have

EC^q

$\frac{ECq - EFq}{2FC} + \frac{1}{2}FC = (CG) = \frac{HEq - ECq}{2CH} -$ $\frac{1}{2}$ CH (*a*); that is, $\frac{\frac{aaxx}{cc} - ee}{\frac{2ebx + 2ebc}{dc}} + \frac{ebx + ebc}{2dc}$ $= \frac{xx - \frac{aaxx}{cc}}{\frac{2bx}{c}} - \frac{bx}{2c}$. Or $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebx}{d}$ $+ \frac{ebc}{d} = \frac{ccx - aax - bbx}{b}$. Here, for Abbreviation fake, for $\frac{cc - aa - bb}{b} - \frac{eb}{d}$ write m, and you will have $\frac{aadxx - eedcc}{ebx + ebc} + \frac{ebc}{d} = mx$; and all the Terms

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Prob. XXII. (*a*) For $EF^2 = EC^2 + FC^2 - 2FC \times CG$ (Eucl. II. 13.); whence $CG \times 2FC = EC^2 - EF^2 + FC^2$, and $CG = \frac{EC^2 - EF^2 + FC^2}{2FC}$; but $FC^2 = 2FC \times \frac{1}{2}FC$ (Eucl. II. 2.): Whence $CG = \frac{EC^2 - EF^2}{2FC} + \frac{1}{2}FC$. Alfo $EH^2 = CE^2 + CH^2 + CG \times 2CH$ (Eucl. II. 12.); whence $CG = \frac{EH^2 - CE^2 - CH^2}{2CH}$: But $CH^2 = 2CH \times \frac{1}{2}CH$ (Eucl. II. 2.); whence $CG = \frac{EH^2 - CE^2}{2CH} - \frac{1}{2}CH = \frac{EC^2 - EF^2}{2FC} + \frac{1}{2}FC$.

Terms being multiplied by $x+r$, there will come out

$$\frac{aadxx - eedc}{eb} - \frac{ebcx}{d} + \frac{ebcc}{d} = mxx + mcx.$$

Again, for $\frac{aad}{eb} - m$ write p, and for $mc + \frac{ebc}{d}$ write $2pq$, and for $-\frac{ebcc}{d} + \frac{eedcc}{eb}$ write prr, and xx will become $= 2qx + rr$, and $x = q \pm \sqrt{qq + rr}$. Having found x or HE, draw EC parallel to AB, and take FC : BC :: e : d, and having drawn FED, it will satisfy the Conditions of the Question.

Problem XXIII.

To determine the Point Z, from which if four right Lines ZA, ZB, ZC, *and* ZD, *are drawn at given Angles to four right Lines given by Position, viz.* FA, EB, FC, GD, *the Rectangle of two of the given Lines* ZA *and* ZB, *and the Sum of the other two* ZC *and* ZD *may be given* [See Fig. 26.]

From among the Lines chuse one, as FA, given by Position, as also another, ZA, not given by Position, and which is drawn to it, from the Lengths whereof the Point Z may be determined, and produce the other Lines given by Position until they meet these, or be produced farther out if there be Occasion, as you see here. And having made EA $= x$, and AZ $= y$, by reason of the given Angles of the Triangle AEH, there will be given the Ratio of AE to AH, which make as p to q, and AH will be $= \frac{qx}{p}$. Add AZ, and ZH will be $= y + \frac{qx}{p}$. And thence, since by reason of the given Angles of the Triangle HZB, there is given the Ratio of HZ of HZ to BZ, if that be made as n to p you will have $ZB = \frac{py + qx}{n}$.

Moreover, if the given EF be called a, AF will be $= a - x$, and thence, if by reaſon of the given Angles of the Triangle AFI, AF be made to AI in the ſame Ratio as p to r, AI will become $= \frac{ra - rx}{p}$. Take this from AZ and there will remain $IZ = y - \frac{ra - rx}{p}$. And by reaſon of the given Angles of the Triangle ICZ, if you make IZ to ZC in the ſame Ratio as m to p, ZC will become $= \frac{py - ra + rx}{m}$.

After the ſame Way, if you make $EG = b$. AG : AK :: l : s, and ZK : ZD :: p : l, there will be obtained $ZD = \frac{sb - sx - ly}{p}$.

Now, from the State of the Queſtion, if the Sum of the two Lines ZC and ZD, viz. $\frac{py - ra + rx}{m} + \frac{sb - sx - ly}{p}$, be made equal to any given Quantity f; and the Rectangle of the other two $\frac{pyy + qxy}{n}$ be made $= gg$, you will have two Equations for determining x and y. By the latter there comes out $x = \frac{ngg - pyy}{qy}$, and by writing this Value of x in the room of that in the former Equation, there will come out $\frac{py - ra}{m} + \frac{rngg - rpyy}{mqy} + \frac{sb - ly}{p} -$

$sngg -$

$\frac{sngg - spyy}{pqy} = f$; and by Reduction $yy =$ $\frac{apqry - bmqsy + fmpqy + ggmns - ggnpr}{ppq - ppr - mlq + mps}$ (a); and for Abbreviation sake, writing $2h$ for $\frac{apqr - bmqs + fmpq}{ppq - ppr - mlq + mps}$ and kk for $\frac{ggmns - ggpnr}{ppq - ppr - mlq + mps}$, you will have $yy = 2hy + kk$, or $y = h \pm \sqrt{hh + kk}$. And since y is known by means of this Equation, the Equation $\frac{ngg - pyy}{qy} = x$ will give x. Which is sufficient to determine the Point Z.

After the same Way a Point may be determined from which other right Lines may be drawn to more or fewer right Lines given by Position, so that the Sum, or Difference, or Rectangle of some of them may be given, or may be made equal to the Sum, or Difference, or Rectangle of the rest, or that they may have any other assigned Conditions.

Prob. XXIII. (a) For transposing, and abbreviating, $\frac{py}{m} - \frac{rpy}{mq} - \frac{ly}{p} + \frac{sy}{q} = \frac{ar}{m} - \frac{bs}{p} + f + \frac{ggns}{pqy} - \frac{ggnr}{mqy}$, and multiplying by $mqyp$, we have $p^2qy^2 - p^2ry^2 - mqly^2 + pmsy^2 = apqry - bmsqy + pfmqy + mg^2ns - g^2npr$; which dividing by $p^2q - p^2r - mql + pms$, is $y^2 = \frac{apqry - bmqsy + fmpqy + g^2mns - g^2npr}{ppq - ppr - mlq + mps}$.

PROBLEM

PROBLEM XXIV.

To subtend the right Angle EAF *with the right Line* EF *given in Magnitude, which shall pass through the given Point* C, *equidistant from the Lines that comprehend the right Angle (when they are produced)* [See Fig. 27.].

Complete the Square ABCD, and bisect the Line EF in G. Then call CB or CD, a; EG or FG, b; and CG, x; and CE will be $= x - b$, and CF $= x + b$. Then since $CFq - BCq = BFq$, BF will be $= \sqrt{xx + 2bx + bb - aa}$. Lastly, by reason of the similar Triangles CDE, FBC, CE : CD :: CF : BF, or $x - b : a :: x + b : \sqrt{xx + 2bx + bb - aa}$. Whence $ax + ab = \overline{x - b} \times \sqrt{xx + 2bx + bb - aa}$. Each Part of which Equation being squared, and the Terms that come out being reduced into Order, there comes out $x^4 = \begin{matrix} 2aa \\ + 2bb \end{matrix} xx \begin{matrix} + 2aabb \\ - b^4. \end{matrix}$ And extracting the Root as in Quadratic Equations, there comes out $xx = aa + bb + \sqrt{a^4 + 4aabb}$; and consequently $x = \sqrt{aa + bb + \sqrt{a^4 + 4aabb}}$. And CG being thus found, gives CE or CF, which, by determining the Point E or F, satisfies the Problem.

The same otherwise.

Let CE be $= x$, CD $= a$, and EF $= b$; and CF will be $= x + b$, and BF $= \sqrt{xx + bb + 2bx - aa}$. And then since CE : CD :: CF : BF, or $x : a :: x + b : \sqrt{xx + 2bx + bb - aa}$, $ax + ab$ will be $= x \times \sqrt{xx + 2bx + bb - aa}$. The Parts of this

 Equation

Equation being ſquared, and the Terms reduced into Order, there will come out $x^4 + 2bx^3 \begin{array}{c} + bb \\ -2aa \end{array} xx - 2aabx - aabb = 0$, a biquadratic Equation, the Inveſtigation of the Root of which is more difficult than in the former Caſe. But it may be thus inveſtigated; put $x^4 + 2bx^3 \begin{array}{c} + bb \\ -2aa \end{array} xx - 2aabx + a^4 = aabb + a^4$, and extracting the Root on both Sides $xx + bx - aa = \pm a\sqrt{aa + bb}$.

Hence I have an Opportunity of giving *a Rule for the Election of Terms for the Calculus.*

CIX. Viz. *When there happens to be ſuch an Affinity or Similitude of the Relation of two Terms to the other Terms of the Queſtion, that you ſhould be obliged in making Uſe of either of them to bring out Equations exactly alike; or that both, if they are made Uſe of together, ſhall bring out the ſame Dimenſions and the ſame Form (only excepting perhaps the Signs + and —) in the final Equation (which will be eaſily ſeen) then it will be the beſt Way to make Uſe of neither of them, but in their room to chuſe ſome third, which ſhall bear a like Relation to both; as ſuppoſe the half Sum, or half Difference, or perhaps a mean Propoportional, or any other Quantity related to both indifferently, and without a like.*

Thus, in the precedent Problem, when I ſee the Line EF alike related to both AB and AD (which will be evident if you alſo draw EF in the Angle BAH) and therefore I can by no Reaſon by perſwaded why ED ſhould be rather made Uſe of than BF, or AE rather than AF, or CE rather than CF for the Quantity ſought: Wherefore, in the room of the Points E and F, from whence this Ambiguity comes (in the former Solution), I made Uſe of the intermediate Point G, which has a like Relation to both the Lines AB and AD. Then from this Point G, I did not let fall a Perpendicular to AF for finding the Quantity ſought, becauſe

cause I might, by the same Reason, have let one fall to AD. And therefore I let it fall upon neither CB nor CD, but proposed CG for the Quantity sought, which does not admit of a like; and so I obtained a biquadratic Equation without the odd Terms.

I might also (taking Notice that the Point G lies in the Periphery of a Circle described from the Center A, by the Radius EG) have let fall the Perpendicular GK upon the Diagonal AC, and have sought AK or CK, (as which bear also a like Relation to both AB and AD) and so I should have fallen upon a quadratic Equation, viz. $yy = -\frac{1}{2}ey + \frac{1}{2}bb$, making $AK = y$, $AC = e$, and $EG = b$ (a). And AK being so found, there must have

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Prob. XXIV. (a) For drawing AG, and writing $CE = z$, $ED = x$; then $AE = a - x$, and $CG = b + z$. In the Triangle ADC, $AC^2 = 2CD^2$; whence $e^2 = 2aa$. In the Triangle CDE, $CE^2 = ED^2 + CD^2$; whence $z^2 = a^2 + x^2$. In the similar Triangles FEA, CDE, AE : ED :: FE (= 2 EG) : EC; whence $AE \times EC = 2EG \times ED$; and $az - xz = 2bx$; and $az = 2bx + xz$; whence $x = \frac{az}{2b + z}$; and $x^2 = \frac{a^2z^2}{4b^2 + 4bz + z^2}$. Lastly in the obtuse-angled Triangle CAG, $GC^2 = AG^2$ ($= EG^2$) $+ AC^2 + 2AC \times AK$; whence $b^2 + 2bz + zz = b^2 + e^2 + 2ey$, or $2bz + z^2 = e^2 + 2ey$. Now, because $a^2 + x^2 = z^2$; therefore $a^2 + \frac{a^2z^2}{4b^2 + 4bz + z^2} = z^2$; that is, $2a^2z^2 + 4a^2bz + 4a^2b^2 = z^4 + 4bz^3 + 4b^2z^2$; but $2a^2 = e^2$; and $z^4 + 4bz^3 + 4b^2z^2 = e^4 + 4e^3y + 4e^2y^2$; whence $e^2z^2 + 2e^2bz + 2e^2b^2 = e^4 + 4e^3y + 4e^2y^2$;

have been erected the Perpendicular KG meeting the aforesaid Circle in G, through which CF would pass.

Taking particular Notice of this Rule in Prob. IX. and X. where the like Sides BC and AC of the Triangle were to be determined, I rather sought the Semi-difference than either of them. But the Usefulness of this Rule will be more evident from the twenty-eighth Problem.

Problem XXV.

To a Circle described from the Center C, *and with the Radius* CD, *to draw a Tangent* DB, *the Part whereof* PB *placed between the right Lines given by Position,* AP *and* AB, *shall be of a given Length* [See Fig. 50.].

From the Center C to either of the right Lines given by Position, as suppose to AB, let fall the Perpendicular CE, and produce it till it meets the Tangent DB in H. To the same AB let fall also the Perpendicular PG, and making $EA = a$, $EC = b$, $CD = c$, $BP = d$, and $PG = x$, by reason of the similar Triangles PGB, CDH, you will have GB $(\sqrt{dd - xx})$: PB :: CD : $CH = \frac{cd}{\sqrt{dd - xx}}$. Add EC, and you will have $EH = b + \frac{cd}{\sqrt{dd - xx}}$. Moreover PG is : GB :: EH : EB

$4e^2y^2$; that is, dividing by e^2, $z^2 + 2bz + 2b^2 = e^2 + 4ey + 4y^2$; and because $z^2 + 2bz = e^2 + 2ey$, $e^2 + 2ey + 2b^2 = e^2 + 4ey + 4y^2$; that is, by Reduction and dividing by 4, $\frac{1}{2}b^2 = \frac{1}{2}ey + y^2$; or $y^2 = -\frac{1}{2}ey + \frac{1}{2}b^2$; and $y = -\frac{1}{4}e \pm \sqrt{\frac{ee}{16} + \frac{bb}{2}}$.

$EH : EB = \frac{b}{x}\sqrt{dd - xx} + \frac{cd}{x}$. Farther, because of the given Angle PAG, there is given the Ratio of PG to AG, which being made as e to f, AG will be $= \frac{fx}{e}$. Add EA and BG, and you will have, lastly, $EB = a + \frac{fx}{e} + \sqrt{dd - xx}$. Therefore it is $\frac{cd}{x} + \frac{b}{x}\sqrt{dd - xx} = a + \frac{fx}{e} + \sqrt{dd - xx}$, and by Transposition of the Terms, $a + \frac{fx}{e} - \frac{cd}{x} = \frac{b - x}{x}\sqrt{dd - xx}$. And the Parts of the Equation being squared, $aa + \frac{2afx}{e} - \frac{2acd}{x} + \frac{ffxx}{ee} - \frac{2cdf}{e} + \frac{ccdd}{xx} = \frac{bbdd}{xx} - bb - \frac{2bdd}{x} + 2bx + dd - xx$. And, by a due Reduction

$$\frac{x^4 \begin{array}{l} +2aef \\ -2bee \end{array} x^3 \begin{array}{l} +aaee \\ +bbee \\ -ddee \\ -2cdef \end{array} xx \begin{array}{l} +2bddee \\ -2acdee \end{array} x \begin{array}{l} +ccddee \\ -bbddee \end{array}}{ee + ff} = 0.$$

Problem XXVI.

To find the Point D, *from which three right Lines* DA, DB, DC, *let fall perpendicular to so many other right Lines* AE, BF, CF, *given in Position; shall obtain a given Ratio to one another* [See Fig. 44.].

Of the right Lines given in Position, let us suppose one as BF be produced, as also its Perpendicular BD, till they meet the rest AE and CF, viz. BF in E

I and F,

and F, and BD in H and G. Now let EB be $= x$, and EF $= a$; and BF will be $= a - x$. But since, by reason of the given Position of the right Lines EF, EA, and FC, the Angles E and F, and consequently the Proportions of the Sides of the Triangles EBH and FBG are given; let EB be to BH as d to e; and BH will be $= \frac{ex}{d}$, and EH $(= \sqrt{EBq + BHq})$ $= \sqrt{xx + \frac{eexx}{dd}}$, that is, $\frac{x}{d} \times \sqrt{dd+ee}$. Let also BF be to BG, as d to f; and BG will be $= \frac{fa - fx}{d}$, and FG $(= \sqrt{BFq + BGq}) =$ $\sqrt{\frac{aadd - 2axdd + xxdd + ffaa - 2ffax + ffxx}{dd}}$, that is, $= \frac{a - x}{d}\sqrt{dd+ff}$. Besides, make BD $= y$, and HD will be $= \frac{ex}{d} - y$, and GD $= \frac{fa - fx}{d} - y$; and so, since AD is : HD (: : EB : EH) : : d : $\sqrt{dd+ee}$, and DC : GD (: : BF : FG) : : d : $\sqrt{dd+ff}$, AD will be $= \frac{ex - dy}{\sqrt{dd+ee}}$, and DC $=$ $\frac{fa - fx - dy}{\sqrt{dd+ff}}$. Lastly, by reason of the given Proportions of the Lines BD, AD, DC, let BD : AD : : $\sqrt{dd+ee} : h - d$, and $\frac{hy - dy}{\sqrt{dd+ee}}$ will be $(= \text{AD})$ $= \frac{ex - dy}{\sqrt{dd+ee}}$, or $hy = ex$. Let also BD : DC : : $\sqrt{dd+ff} : k - d$, and $\frac{ky - dy}{\sqrt{dd+ff}}$ will be $(= \text{DC})$ $=$

$= \frac{fa - fx - dy}{\sqrt{dd + ff}}$, or $ky = fa - fx$. Therefore $\frac{ex}{b}$ $(= y)$ is $= \frac{fa - fx}{k}$; and by Reduction $\frac{fba}{ek + fb} = x$. Wherefore take EB : EF :: b : $\frac{ek}{f} + b$, then BD : EB :: e : b, and you will have the Point sought D.

Problem XXVII.

To find the Point D, *from which three right Lines* DA, DB, DC, *drawn to the three given Points*, A, B, C, *shall have a given Ratio among themselves*. [See Fig. 45.]

Of the given three Points, join any two of them, as suppose A and C, and let fall the Perpendicular BE from the third B, to the Line that conjoins A and C, as also the Perpendicular DF from the Point sought D; and making AE $= a$, AC $= b$, EB $= c$, AF $= x$, and FD $= y$; and ADq will be $= xx + yy$. FC $= b - x$, CDq ($=$ FCq + FDq) $= bb - 2bx + xx + yy$. EF $= x - a$, and BDq ($=$ EFq + $\overline{\text{EB} + \text{FD}}q$) (*a*) $= xx - 2ax + aa + cc + 2cy + yy$. Now, since AD is to CD in a given Ratio, let it be as d to e; and CD will be $= \frac{e}{d}\sqrt{xx + yy}$. Since also AD is to BD in a given Ratio, let that be as d to f, and BD will be $= \frac{f}{d}\sqrt{xx + yy}$. And, consequently

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Prob. XXVII. (*a*) For if DF was produced, so as its whole Length should be equal to DF + EB, it would be the Side of a right-angled Triangle, whose Base equals EF, and Hypothenuse is BD.

ſequently it is $\frac{eexx+eeyy}{dd}$ (= CDq) $= bb - 2bx + xx + yy$, and $\frac{ffxx+ffyy}{dd}$ (= BDq) $= xx - 2ax + aa + cc + 2cy + yy$. In which if, for Abbreviation ſake, you write p for $\frac{dd-ee}{d}$, and q for $\frac{dd-ff}{d}$, there will come out $bb - 2bx + \frac{p}{d}xx + \frac{p}{d}yy = 0$ (*b*), and $aa + cc - 2ax + 2cy + \frac{q}{d}xx + \frac{q}{d}yy = 0$ (*c*). And by the former you have

$\frac{2bqx - bbq}{p}$

(*b*) For by multiplying $\frac{c^2x^2+e^2y^2}{d^2} = b^2 - 2bx + x^2 + y^2$, by d^2, it becomes $c^2x^2 + e^2y^2 = b^2d^2 - 2bxd^2 + x^2d^2 + y^2d^2$; which, by tranſpoſing, and dividing by d^2, it becomes $b^2 - 2bx + \frac{d^2-e^2}{d^2}x^2 + \frac{d^2-e^2}{d^2}y^2 = 0$; and by Subſtitution, $b^2 - 2bx + \frac{p}{d}x^2 + \frac{p}{d}y^2 = 0$.

(*c*) $\frac{f^2x^2+f^2y^2}{d^2} = \overline{x^2 - 2ax + a^2 + c^2 + 2cy + y^2} \times d^2$, the Terms then being tranſpoſed into one Member, and divided by d^2; and the Subſtitution being made, it becomes $a^2 + c^2 - 2ax + 2cy + \frac{q}{d}x^2 + \frac{q}{d}y^2 = 0$,

$\frac{2bqx - bbq}{p} = \frac{q}{d}xx + \frac{q}{d}yy$ (*d*). Wherefore, in the latter, for $\frac{q}{d}xx + \frac{q}{d}yy$, write $\frac{2bqx - bbq}{p}$, and there will come out $\frac{2bqx - bbq}{p} + aa + cc - 2ax + 2cy = 0$. Again, for Abbreviation ſake, write m for $a - \frac{bq}{p}$, and $2cn$ for $\frac{bbq}{p} - aa - cc$, and you will have $2mx + 2cn = 2cy$, and the Terms being divided by $2c$, there ariſes $\frac{mx}{c} + n = y$. Wherefore, in the Equation $bb - 2bx + \frac{p}{d}xx + \frac{p}{d}yy = 0$, for yy write the Square of $\frac{mx}{c} + n$, and you will have $bb - 2bx + \frac{p}{d}xx + \frac{pmm}{dcc}xx + \frac{2pmn}{dc}x + \frac{pnn}{d} = 0$. Where, laſtly, if, for Abbreviation ſake, you write $\frac{b}{r}$ for $\frac{p}{d} + \frac{pmm}{dcc}$, and $\frac{sb}{r}$ for $b - \frac{pmn}{dc}$, and $\frac{tbb}{r}$ for $bb + \frac{pnn}{d}$, you will have $xx = 2sx - tb$. And having extracted the Root, $x = s \pm \sqrt{ss - tb}$. Having found

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(*d*) For $\frac{p}{d}x^2 + \frac{p}{d}y^2 = 2bx - b^2$; which, multiplied into $\frac{q}{p}$, becomes $\frac{q}{d}x^2 + \frac{q}{d}y^2 = \frac{2bqx - b^2q}{p}$.

found x, the Equation $\frac{mx}{c} + n = y$ will give y; and from x and y given, that is, AF and FD, the given Point D is determined.

Problem XXVIII.

So to inscribe the right Line DC *of a given Length in the given conic Section* DAC, *that it may pass through the Point* G *given by Position.* [See Fig. 28.]

Let AF be the Axis of the Curve, and from the Points D, G, and C, let fall to it the Perpendiculars DH, GE, and CB. Now to determine the Position of the right Line DC, it may be proposed to find out the Point D or C; but since these are related, and so alike, that there would be the like Operation in determining either of them, whether I were to seek CG, CB, or AB; or their likes, DG, DH, or AH; therefore I look after a third Point, that regards D and C alike, and at the same Time determines them. And I see F to be such a Point.

Now let AE be $= a$, EG $= b$, DC $= c$, and EF $= z$; and besides, since the Relation between AB and BC is had in the Equation, I suppose, given for determining the conic Section, let AB $= x$, BC $= y$, and FB will be $= x - a + z$. And because GE : EF :: CB : FB, FB will again be $= \frac{yz}{b}$. Therefore, $x - a + z = \frac{yz}{b}$.

These Things being thus laid down, take away x, by the Equation that denotes the Curve. As if the Curve be a Parabola expressed by the Equation $rx = yy$, write $\frac{yy}{r}$ for x; and there will arise $\frac{yy}{r} - a$

$+ z =$

$+ z = \frac{yz}{b}$, and extracting the Root $y = \frac{rz}{2b} \pm \sqrt{\frac{rrzz}{4bb} + ar - rz}$. Whence it is evident, that $\sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$ is the Difference of the double Value of y; that is, of the Lines $+$ BC and $-$ DH, and consequently (having let fall DK perpendicular upon CB) that Difference is equal to CK. But FG : GE :: DC : CK; that is, $\sqrt{bb + zz} : b :: c : \sqrt{\frac{rrzz}{bb} + 4ar - 4rz}$.

And by multiplying the Squares of the Means, and also the Squares of the Extreams into one another, and ordering the Products, there will arise $z^4 =$

$$\frac{4bbrz^3 \begin{matrix}-4abbr\\-bbrr\end{matrix} zz + 4b^4rz \begin{matrix}-4ab^4r\\+b^4cc\end{matrix}}{rr},$$

an Equation of four Dimensions, which would have risen to one of eight Dimensions, if I had sought either CG, or CB, or AB.

PROBLEM XXIX.

To multiply or divide a given Angle, by a given Number.

[See Figure 29.]

In any Angle FAG inscribe the Lines AB, BC, CD, DE, &c. of any the same Length, and the Triangles ABC, BCD, CDE, DEF, &c. will be Isosceles, and consequently by the 32. 1. Eucl. the Angle CBD will be = Angle A + ACB = 2 Angle A, and the Angle DCE = Angle A + ADC = 3 Angle A, and the Angle EDF = A + AED = 4 Angle A, and the Angle FEG = 5 Angle A, and so onwards. Now, making

making AB, BC, CD, &c. the Radii of equal Circles, the Perpendiculars BK, CL, DM, &c. let fall upon AC, BD, CE, &c. will be the Sines of those Angles, and AK, BL, CM, DN, &c. will be their Sines Complement to a right one; or making AB the Diameter, the Lines AK, BL, CM, &c. will be Chords. Let therefore AB $= 2r$, and AK $= x$, then work thus:

AB : AK :: AC : AL.

$$2r : x :: 2x : \frac{xx}{r}.$$

And $\left\{\begin{matrix} AL - AB \\ \frac{xx}{r} - 2r \end{matrix}\right\} =$ BL, the Duplication.

AB : AK :: AD (2AL − AB) : AM.

$$2r : x :: \frac{2xx}{r} - 2r : \frac{x^3}{rr} - x.$$

And $\left\{\begin{matrix} AM - AC \\ \frac{x^3}{rr} - 3x \end{matrix}\right\} =$ CM, the Triplication.

AB : AK :: AE (2AM − AC) : AN.

$$2r : x :: \frac{2x^3}{rr} - 4x : \frac{x^4}{r^3} - \frac{2xx}{r}.$$

And $\left\{\begin{matrix} AN - AD \\ \frac{x^4}{r^3} - \frac{4xx}{r} + 2r \end{matrix}\right\} =$ DN, the Quadruplication:

AB : AK :: AF (2AN − AD) : AO.

$$2r : x :: \frac{2x^4}{r^3} - \frac{6xx}{r} + 2r : \frac{x^5}{r^4} - \frac{3x^3}{rr} + x.$$

And $\left\{\begin{matrix} AO - AE \\ \frac{x^5}{r^4} - \frac{5x^3}{rr} + 5x \end{matrix}\right\} =$ EO, the Quintuplication.

And so onwards.

Now

Now if you would divide an Angle into any Number of Parts, put q for BL, CM, DN, &c. and you will have $xx - 2rr = qr$ for the Bisection; $xxx - 3rrx = qr^2$ for the Trisection; $xxxx - 4rrxx + 2r^4 = qr^3$ for the Quadrisection; $xxxxx - 5r^2x^3 + 5r^4x = qr^4$ for the Quinquisection, &c.

PROBLEM XXX.

To determine the Position of a Comet's Course that moves uniformly in a right Line, as BD, *from three Observations.* [See Fig. 30.]

Suppose A to be the Eye of the Spectator, B the Place of the Comet in the first Observation, C in the second, and D in the third; the Inclination of the Line BD to the Line AB is to be found. From the Observations, therefore, there are given the Angles BAC, BAD; and consequently if BH be drawn perpendicular to AB, and meeting AC and AD in E and F, assuming any how AB, there will be given BE and BF, viz. the Tangents of the Angles in respect of the Radius AB. Make therefore $AB = a$, $BE = b$, and $BF = c$. Moreover, from the given Intervals of the Observations, there will be given the Ratio of BC to BD, which, if it be made as b to e, and DG be drawn parallel to AC, since BE is to BG in the same Ratio, and BE was called b, BG will be $= e$, and consequently $GF = e - c$. Farther, if you let fall DH perpendicular to BG, by reason of the Triangles ABF and DHF being alike, and alike divided by the Lines AE and DG, FE will be : AB :: FG : HD (*a*), that is, $c - b : a :: e - c : \frac{ae - ac}{c - b} = HD$.

Prob. XXX. (*a*) Because AFE, FGD, are similar, FE : EA :: FG : GD; and because AEB, DGH, are similar, EA : AB :: GD : DH; whence FE : AB :: FG : DH (Eucl. V. 18.)

$=$ HD. Moreover, FE will be : FB :: FG : FH (*b*), that is, $c - b : c :: e - c : \frac{ce - cc}{c - b} = $ FH; to which add BF, or c, and BH will be $= \frac{ce - cb}{c - b}$. Wherefore $\frac{ce - cb}{c - b}$ is to $\frac{ae - ac}{c - b}$ (or $ce - cb$ to $ae - ac$, or $\frac{ce - cb}{e - c}$ to a) as BH to HD; that is, as the Tangent of the Angle HDB, or ABK to the Radius. Wherefore since a is supposed to be the Radius, $\frac{ce - cb}{e - c}$ will be the Tangent of the Angle ABK; and therefore, by resolving them into an Analogy, it will be as $e - c$ to $e - b$ (or GF to GE), so c (or the Tangent of the Angle BAF) to the Tangent of the Angle ABK.

Say, therefore, as the Time between the first and second Observation to the Time between the first and third, so the Tangent of the Angle BAE to a fourth Proportional. Then as the Difference between that fourth Proportional and the Tangent of the Angle BAF, to the Difference between the same fourth Proportional and the Tangent of the Angle BAE, so the Tangent of the Angle BAF to the Tangent of the Angle ABK.

(*b*) Because FE : EA :: FG : GD, and AE : EB :: DG : GH, therefore FE : EB :: FG : GH (Eucl. IV. 18.); whence FE : FB :: FG : FH (Eucl. V. 22.).

PROBLEM

PROBLEM XXXI.

Rays of Light from any shining or lucid Point diverging to a refracting spherical Surface, to find the Concourse of each of the refracted Rays with the Axis of the Sphere passing through that lucid Point. [See Fig. 31.]

Let A be that lucid Point, and BV the Sphere, the Axis whereof is AD, the Center C, and the Vertex V; and let AB be the incident Ray, and BD the refracted Ray; and having let fall to those Rays the Perpendiculars CE and CF, as also BG perpendicular to AD, and having drawn BC, make AC $= a$, VC or BC $= r$, CG $= x$, and CD $= z$, and AG will be $= a - x$, BG $= \sqrt{rr - xx}$, AB $= \sqrt{aa - 2ax + rr}$; and by reason of the similar Triangles ABG and ACE, CE $= \frac{a\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}}$. Also GD $= z + x$, BD $= \sqrt{zz + 2zx + rr}$; and by reason of the similar Triangles DBG and DCF, CF $= \frac{z\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$. Besides, since the Ratio of the Sines of Incidence and Refraction, and consequently of CE to CF, is given, suppose that Ratio to be as a to f, and $\frac{fa\sqrt{rr - xx}}{\sqrt{aa - 2ax + rr}}$ will be $= \frac{az\sqrt{rr - xx}}{\sqrt{zz + 2zx + rr}}$; and multiplying cross-ways, and dividing by $a\sqrt{rr - xx}$, it will be $f\sqrt{zz + 2zx + rr} = z\sqrt{aa - 2xa + rr}$, and by squaring and reducing the Terms into Order, $zz = \frac{2ffxz + ffrr}{aa - 2ax + rr - ff}$. Then for the given $\frac{ff}{a}$ write p, and q for the given $a + \frac{rr}{a} - p$, and zz will

 be $=$

be $= \frac{2pxz + prr}{q - 2x}$, and $z = \frac{px + \sqrt{ppxx - 2prrx + pqrr}}{q - 2x}$. Therefore z is found; that is, the Length of CD, and consequently the Point sought D, where the refracted Ray BD meets with the Axis. Q. E. F.

Here I made the incident Rays to diverge, and fall upon a thicker Medium; but changing what is requisite to be changed, the Problem may be as easily resolved when the Rays converge, or fall from a thicker Medium into a thinner one.

Problem XXXII.

If a Cone be cut by any Plane, to find the Figure of the Section. [See Fig. 32 and 33.]

Let ABC be a Cone standing on a circular Base BC, and IEM its Section sought; and let KILM be any other Section parallel to the Base, and meeting the former Section in HI; and ABC a third Section, perpendicularly bisecting the two former in EH and KL, and the Cone in the Triangle ABC. And producing EH till it meet AK in D; and having drawn EF and DG, parallel to KL, and meeting AB and AC in F and G, call EF $= a$, DG $= b$, ED $= c$, EH $= x$, and HI $= y$; and by reason of the similar Triangles EHL, EDG, ED will be : DG :: EH : HL $= \frac{bx}{c}$. Then by reason of the similar Triangles DEF, DHK, DE will be : EF :: DH : ($c - x$ in the thirty-second Figure, and $c + x$ in the thirty-third Figure) HK $= \frac{ac \mp ax}{c}$. Lastly, since the Section KIL is parallel to the Base, and consequently circular, HK × HL will be $=$ HIq, that is, $\frac{ab}{c}x \mp \frac{ab}{cc}xx = yy$, an Equation

tion which expresses the Relation between EH (x) and HI (y), that is, between the Axis and the Ordinate of the Section EIM; which Equation, since it expresses an Ellipse in the thirty-second Figure, and an Hyperbola in the thirty-third Figure, it is evident, that that Section will be Elliptical or Hyperbolical.

Now if ED no where meets AK, being parallel to it, then HK will be $=$ EF (a), and thence $\frac{ab}{c}x$ (HK $\times$ HL) $= yy$, an Equation expressing a Parabola.

PROBLEM XXXIII.

If the right Line XY *be turned about the Axis* AB, *at the Distance* CD, *with a given Inclination to the Plane* DCB, *and the Solid* PQRUTS, *generated by that Circumrotation, be cut by any Plane as* INQLK, *to find the Figure of the Section.* [See Fig. 34.]

Let BHQ, or GHO, be the Inclination of the Axis AB to the Plane of the Section; and let L be any Concourse of the right Line XY with that Plane. Draw DF parallel to AB, and let fall the Perpendiculars LG, LF, LM, to AB, DF, and HO, and join FG and MG. And having called CD $= a$, CH $= b$, HM $= x$, and ML $= y$, by reason of the given Angle GHO, making MH : HG :: $d : e$, $\frac{ex}{d}$ will be $=$ GH, and $b + \frac{ex}{d} =$ GC or FD. Moreover, by reason of the given Angle LDF (viz. the Inclination of the right Line XY to the Plane GCDF) putting FD : FL :: $g : h$, it will be $\frac{hb}{g} + \frac{hex}{dg} =$ FL, to whose Square add FGq (DCq, or aa) and there will come out GLq

 $= aa$

$= aa + \frac{bbbb}{gg} + \frac{2bbbex}{dgg} + \frac{bbeexx}{ddgg}$. Hence subtract MG$q$ (HMq — HGq, or $xx - \frac{ee}{dd}xx$) and there will remain $\frac{aagg + bbbb}{gg} + \frac{2bbbe}{dgg}x + \frac{bbee - ddgg + eegg}{ddgg} \times xx$ (= MLq) = yy: an Equation that expresses the Relation between x and y, that is, between HM the Axis of the Section, and ML its Ordinate. And therefore, since in this Equation x and y ascend only to two Dimensions, it is evident, that the Figure INQLK is a conic Section. As for Example, if the Angle MHG is greater than the Angle LDF, this Figure will be an Ellipse; but if less, an Hyperbola; and if equal, either a Parabola, or (the Points C and H moreover coinciding) a Parallelogram.

Problem XXXIV.

If you erect AD *of a given Length perpendicular to* AF; *and* ED, *one Leg of a Square* DEF, *pass continually through the Point* D, *while the other Leg* EF *equal to* AD *slide upon* AF; *to find the Curve* HIC, *which the Leg* EF *describes by its middle Point* C. [See Fig. 35.]

Let EC or CF be $= a$, the Perpendicular CB $= y$, AB $= x$; and on account of the similar Triangles FBC, FEG (*a*), it will be BF ($\sqrt{aa - yy}$) : BC + CF ($y + a$) :: EF ($2a$) : EG + GF (AG + GF) or AF

Prob. XXXIV. (*a*) For they are rectangular in E, and B, and have the Angle EFG common.

or AF (*b*). Wherefore $\frac{2ay+2aa}{\sqrt{aa-yy}}$ ($=$ AF $=$ AB $+$ BF) $= x + \sqrt{aa-yy}$. Now by multiplying by $\sqrt{aa-yy}$, there is made $2ay + 2aa = aa - yy + x\sqrt{aa-yy}$, or $2ay + aa + yy = x \times \sqrt{aa-yy}$, and by squaring the Parts, divided by $\sqrt{a+y}$ (*c*), and ordering them, there comes out $y^3 + 3ayy \begin{matrix}+3aa\\+xx\end{matrix}\, y \begin{matrix}+a^3\\-axx\end{matrix} = 0$.

The same otherwise [See Fig. 36.].

On BC take at each End, BI, and CK, equal to CF; and draw KF, HI, HC, and DF; whereof let HC and DF meet AF, and IK, in M and N, and upon HC let fall the Perpendicular IL; and the Angle K will be $= \frac{1}{2}$ BCF $= \frac{1}{2}$ EGF $=$ GFD $=$ AMH $=$ MHI $=$ CIL (*d*); and consequently the right-angled Triangles

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(*b*) The Triangles DAG, FEG, are rectangular in A, and E, and have the Angles at G vertical, that is, equal, and the Side DA equal to EF by Supposition, wherefore AG $=$ GE; and Triangle DAG $=$ FEG.

(*c*) For $a^2 + 2ay + y^2 = \overline{a+y}^2 = \overline{a+y} \times \sqrt{a+y} \times \sqrt{a+y}$; but $x\sqrt{a^2-y^2} = x\sqrt{a+y} \times \sqrt{a-y} = \sqrt{a+y} \times x\sqrt{a-y}$; whence dividing by $\sqrt{a+y}$, and squaring, $a^3 + 3a^2y + 3ay^2 + y^3 = x^2a - yx^2$, and transposing, $y^3 + 3ay^2 + \overline{3a^2+x^2} \times y + a^3 - ax^2 = 0$.

(*d*) The Triangle CKF being equicrural by Construction, the exterior BCF is double of CKF, or

 CFK;

angles KBF, FBN, HLI, and ILC will be similar. Make therefore FC $= a$, HI $= x$, and IC $= y$; and BN $(2a - y)$ will be : BK (y) :: LC : LH :: CIq (yy) : HIq (xx), and consequently $2axx - yxx = y^3$. From which Equation it is easily inferred, that this Curve is the Cissoid of the Antients, belonging to a Circle, whose Center is A, and its Radius AH.

Problem XXXV.

If a right Line ED *of a given Length subtending the given Angle* EAD, *be so moved, that its Ends* D *and* E *always touch the Sides* AD *and* AE *of that Angle; let it be proposed to determine the Curve* FCG, *which any given Point* C *in that right Line* ED *describes.* [See Fig. 37.]

From the given Point C draw CB parallel to EA; and make AB $= x$, BC $= y$, CE $= a$, and CD $= b$, and by reason of the similar Triangles DCB, DEA, it will be EC : AB :: CD : BD; that is, $a : x :: b$: BD $= \frac{bx}{a}$. Besides, having let fall the Perpendicular CH, by reason of the given Angle DAE, or DBC, and consequently of the given Ratio of the Sides of the right-

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CFK; but the Triangles BCF, and EGF, being similar, EGF is double of GFD, or GDF, (EGF and BCF being equal) therefore the Triangle DGF is equicrural; whence the Angles FDG, GFD, KFC, CKF, are equal: But the Angles BCF + CFB make one right Angle; whence NFK is also a right Angle; and a Circle, whose Center is C, and Radius CK, will pass through the Points F, and N: Whence NC = FC = DH, and HC and DF are parallel; and GFD = AMH; and AMH = MHI; and MHI = CIL, for the Triangles HIC and CLI, being rectangular at I and L, and having HCI common, are similar.

right-angled Triangle BCH, you will have $a : e :: BC : BH$, and BH will be $= \frac{ey}{a}$. Take away this from BD, and there will remain $HD = \frac{bx - ey}{a}$. Now in the Triangle BCH, becaufe of the right Angle BHC, it is $BCq - BHq = CHq$; that is $yy - \frac{eeyy}{aa} = CHq$. In like manner, in the Triangle CDH, becaufe of the right Angle CHD, it is $CDq - CHq = HDq$; that is, $bb - yy + \frac{eeyy}{aa} \left(= HDq = \frac{bx - ey}{a} q.\right) = \frac{bbxx - 2bexy + eeyy}{aa}$; and by Reduction $yy = \frac{2be}{aa} \times xy + \frac{aabb - bbxx}{aa}$. Where, fince the unknown Quantities rife but to two Dimenfions, it is evident that the Curve is a conic Section. Then extracting the Root, you will have $y = \frac{bex \pm b\sqrt{eexx - aaxx + a^4}}{aa}$ Where, in the radical Term, the Coefficient of xx is $ee - aa$. But it was $a : e :: BC : BH$; and BC is neceffarily a greater Line than BH, viz. the Hypothenufe of a right-angled Triangle is greater than the Side of it; therefore a is greater than e, and $ee - aa$ is a negative Quantity, and confequently the Curve will be an Ellipfis.

PROBLEM

PROBLEM XXXVI.

If the Ruler E B D, *forming a right Angle, be so moved, that one Leg of it,* E B, *continually subtends the right Angle* E A B, *while the End of the other Leg,* B D, *describes some curve Line, as* F D; *to find that Line* F D, *which the Point* D *describes.* [See Fig. 38.]

From the Point D let fall the Perpendicular D C to the Side A C; and making A C $= x$, and D C $= y$, and E B $= a$, and B D $= b$. In the Triangle B D C, by reason of the right Angle at C, B Cq is $=$ B Dq $-$ D Cq $= bb - yy$. Therefore B C $= \sqrt{bb - yy}$; and A B $= x - \sqrt{bb - yy}$. Besides, by reason of the similar Triangles B E A, D B C, it is B D : D C :: E B : A B; that is, $b : y :: a : x - \sqrt{bb - yy}$. Wherefore $bx - b\sqrt{bb - yy} = ay$, or $bx - ay = b\sqrt{bb - yy}$. And the Parts being squared and duly reduced $yy = \frac{2abxy + b^4 - bbxx}{aa + bb}$. And extracting the Root $y = \frac{abx \pm bb\sqrt{aa + bb - xx}}{aa + bb}$. Whence it is again evident, that the Curve is an Ellipsis.

This is so where the Angles E B D and E A B are right; but if those Angles are of any other Magnitude, as long as they are equal, you may proceed thus: [See Fig. 39.] Let fall D C perpendicular to A C as before, and draw D H, making the Angle D H A equal to the Angle H A E, suppose obtuse, and calling E B $= a$, B D $= b$, A H $= x$, and H D $= y$; by reason of the similar Triangles E A B, B H D, B D will be : D H :: E B : A B; that is, $b : y :: a :$ A B $= \frac{ay}{b}$. Take this from A H and there will remain B H $= x - \frac{ay}{b}$.

Besides,

Besides, in the Triangle DHC, by reason of all the Angles given, and consequently the Ratio of the Sides given, assume DH to HC in any given Ratio, suppose as b to e; and since DH is y, HC will be $\frac{ey}{b}$, and HB × HC $= \frac{exy}{b} - \frac{aeyy}{bb}$. Lastly, by the 12, 2 Elem. in the Triangle BHD, it is BDq = BHq + DHq + 2BH × HC; that is, $bb = xx - \frac{2axy}{b} + \frac{aayy}{bb} + yy + \frac{2exy}{b} - \frac{2aeyy}{bb}$. And extracting the Root $x = \frac{ay - ey \pm \sqrt{eeyy - bbyy + bbbb}}{b}$. Where when b is greater than e, that is, when $ee - bb$ is a negative Quantity, it is again evident, that the Curve is an Ellipse.

PROBLEM XXXVII.

In the given Angle PAB *having any how drawn the right Lines,* BD, PD, *in a given Ratio, on this Condition, that* BD *shall be parallel to* AP, *and* PD *terminated at the given Point* P *in the right Line* AP; *to find the Locus of the Point* D. [See Fig. 41.]

Draw CD parallel to AB, and DE perpendicular to AP; and make AP $= a$, CP $= x$, and CD $= y$, and let BD be to PD in the same Ratio as d to e, and AC or BD will be $= a - x$, and PD $= \frac{ea - ex}{d}$. Moreover, by reason of the given Angle DCE, let the Ratio of CD to CE be as d to f, and CE will be $= \frac{fy}{d}$, and EP $= x - \frac{fy}{d}$. But by reason of the Angles at E being

being right ones, it is $CDq - CEq$ ($= EDq$) $= PDq - EPq$; that is, $yy - \frac{ffyy}{dd} = \frac{eeaa - 2eeax + eexx}{dd} - xx + \frac{2fxy}{d} - \frac{ffyy}{dd}$; and blotting out on each Side $-\frac{ffyy}{dd}$, and the Terms being rightly dispos'd, $yy = \frac{2fxy}{d} + \frac{eeaa - 2eeax + eexx - ddxx}{dd}$, and extracting the Root $y =$

$$\frac{fx}{d} \pm \frac{\sqrt{eeaa - 2eeax \begin{matrix} + ee \\ - dd \\ + ff \end{matrix} xx}}{d}.$$

Where, since x and y in the last Equation ascends only to two Dimensions, the Place of the Point D will be a conic Section, and that either an Hyperbola, Parabola, or Ellipse, as $ee - dd + ff$, (the Co-efficient of xx in the last Equation) is greater, equal to, or less than nothing.

PROBLEM XXXVIII.

The two right Lines VE *and* VC *being given in Position, and cut any how in* C *and* E *by another right Line,* PE *turning about the Pole* P, *given also in Position; if the intercepted Line* CE *be divided into the Parts* CD, DE, *that have a given Ratio to one another, it is proposed to find the Place of the Point* D. [See Fig. 42.]

Draw VP, and parallel to it, DA and EB, meeting VC in A and B. Make $VP = a$, $VA = x$, and $AD = y$; and since the Ratio of CD to DE is given, or conversely of CD to CE, that is, the Ratio of DA to EB, let it be as d to e, and EB will be $= \frac{ey}{d}$. Besides,

sides, since the Angle EVB is given, and consequently the Ratio of EB to VB, let that Ratio be as e to f, and VB will be $= \frac{fy}{d}$. Lastly, by reason of the similar Triangles CEB, CDA, CPV, it is EB : CB :: DA : CA :: VP : VC, and by Composition EB + VP : CB + VC :: DA + VP : CA + VC; that is, $\frac{ey}{d} + a : \frac{fy}{d} :: y + a : x$, and multiplying together the Means and Extremes $eyx + dax = fyy + fay$. Where since the indefinite Quantities x and y ascend only to two Dimensions, it follows, that the Curve VD, in which the Point D is always found, is a conic Section, and that an Hyperbola, because one of the indefinite Quantities, viz. x is only of one Dimension, and in the Term exy is multiplied by the other indefinite one y.

Problem XXXIX.

If two right Lines, AC *and* BC, *in any given Ratio, are drawn from the two Points* A *and* B *given in Position, to a third Point* C, *to find the Place of* C, *the Point of Concourse.* [See Fig. 43.]

Join AB, and let fall to it the Perpendicular CD; and making AB $= a$, AD $= x$, DC $= y$, AC will be $= \sqrt{xx+yy}$, BD $= x-a$, and BC $(= \sqrt{BDq+DCq})$ $= \sqrt{xx - 2ax + aa + yy}$. Now since there is given the Ratio of AC to BC, let that be as d to e; and the Means and Extremes being multiplied together, you will have $e\sqrt{xx+yy} = d\sqrt{xx - 2ax + aa + yy}$, and by Reduction $\sqrt{\frac{ddaa - 2ddax}{ee - dd} - xx} = y$. Where since xx is negative, and affected only by Unity, and also the Angle ADC a right one, it is evident, that the Curve meeting AD in F. Moreover, let fall the Perpendicular DE to BF, as also DC perpendicular to AB meeting BF in G. And making $AB = a$, $AC = x$, and $CD = y$, BC will be $= a - x$. Now, since in the Triangle BCG there are given all the Angles, there will be given the Ratio of the Sides BC and GC, let that be as d to a, and CG will be $= \frac{aa - ax}{d}$; take away this from DC or y, and there will remain DG $= \frac{dy - aa + ax}{d}$. Besides, because of the similar Triangles BGC, and DGE, it is BG : BC :: DG : DE. But in the Triangle BGC, it is $a : d$ :: CG : BC. And consequently $aa : dd :: CGq : BCq$, and by compounding $aa + dd : dd :: BGq : BCq$, and extracting the Roots $\sqrt{aa + dd} : d$ (:: BG : BC) :: DG : DE. Therefore DE $= \frac{dy - aa + ax}{\sqrt{aa + dd}}$. Moreover, since the Angle ABF is the Difference of the Angles BAD and ABD, and consequently the Angles BAD and FBD are equal, the right-angled Triangles CAD and EBD will be similar, and therefore the Sides proportional, or DA : DC :: DB : DE. But DC is $= y$. DA $(= \sqrt{ACq + DCq}) = \sqrt{xx + yy}$. DB $(= \sqrt{BCq + DCq}) = \sqrt{aa - 2ax + xx + yy}$, and above DE was $= \frac{dy - aa + ax}{\sqrt{aa + dd}}$. Wherefore $\sqrt{xx + yy} : y :: \sqrt{aa - 2ax + xx + yy} : \frac{dy - aa + ax}{\sqrt{aa + dd}}$; and the Squares of the Means and Extremes being multiplied by each other, $aayy - 2axyy +$

$xxyy$

$$xxyy + y^4 = \frac{ddxxyy + ddy^4}{aa+dd}$$
$$\frac{-2aadxxy - 2aady^3 + 2adyx^3 + 2adxy^3 + a^4x^2 + a^4yy}{aa+dd}$$
$$\frac{-2a^3x^3 - 2a^3xyy + aax^4 + a^2x^2y^2}{aa+dd}.$$

Multiply all the Terms by $aa+dd$, and reduce those Terms that come out into due Order, and there will arise

$$x^4 \begin{matrix} -aa \\ +\frac{2d}{a}y \end{matrix} x^3 \begin{matrix} -2dy \\ +aa \end{matrix} xx \begin{matrix} +\frac{2d}{a}y^3 \\ +\frac{2dd}{a}yy \end{matrix} x \begin{matrix} -ddyy \\ -y^4 \end{matrix} -2dy^3 = 0.$$

Divide this Equation by $xx - ax \begin{matrix} +dy \\ +yy \end{matrix}$, and there will arise $xx \begin{matrix} -a \\ +\frac{2d}{a}y \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$; there come out therefore two Equations in the Solution of this Problem: The first, $xx - ax \begin{matrix} +dy \\ +yy \end{matrix} = 0$ is to a Circle, viz. the Place of the Point D, where the Angle FBD is taken on the other Side of the right Line BF than what is described in the Figure, the Angle ABF being the Sum of the Angles DAB and DBA at the Base, and so the Angle ADB at the Vertex being given. The last, viz. $xx \begin{matrix} -a \\ +\frac{2d}{a}y \end{matrix} x \begin{matrix} -yy \\ -dy \end{matrix} = 0$, is an Hyperbola, the Place of the Point D, where the Angle FBD obtains the Situation from the right Line BF, which we described in the Figure; that is, so that the Angle ABF may be the Difference of the Angles DAB, DBA, at the Base. And this is the Determination of the Hyperbola: Bisect AB in P; draw PQ, making the Angle BPQ equal to Half the Angle ABF: To this draw the Perpendicular P R, and P Q and P R will be the Asymptotes of this Hyperbola, and B a Point thro' which the Hyperbola will pass.

Hence arises this Theorem. *Any Diameter, as* A B, *of a right-angled Hyperbola, being drawn, and having drawn the right Lines* A D, B D, A H, B H, *from its Ends to any two Points* D *and* H *of the Hyperbola; these right Lines will make equal Angles* D A H, D B H, *at the Ends of the Diameter.*

The same after a shorter Way.

At Problem xxiv. I laid down a Rule about the most commodious Election of Terms to proceed with in the Calculus of Problems, where there happens any Ambiguity in the Election of such Terms. Here the Difference of the Angles at the Base is indifferent in respect to both Angles; and in the Construction of the Scheme, it might equally have been added to the lesser Angle D A B, by drawing from A a right Line parallel to B F; or subtracted from the greater Angle D B A, by drawing the right Line B F. Wherefore I neither add nor subtract it, but add Half of it to one of the Angles, and subtract Half of it from the other. Then since it is also doubtful whether A C or B C must be made use of for the indefinite Term whereon the Ordinate D C stands, I use neither of them; but I bisect A B in P, and I make use of P C; or rather, having drawn M P Q [See Figure 53.] making, on both Sides, the Angles A P Q, B P M, equal to Half the Difference of the Angles at the Base, so that it, with the right Lines A D, B D, may make the Angles D Q P, D M P, equal; I let fall to M Q the Perpendiculars A R, B N, D O, and I use D O for the Ordinate, and P O for the indefinite Line it stands on. I make therefore P O $= x$, D O $= y$, A R or B N $= b$, and P R or P N $= c$. And by reason of the similar Triangles B N M, D O M, B N will be : D O :: M N : M O. And by Division D O $-$ B N $(y - b)$: D O (y) :: M O $-$ M N (O N

(ON or $c-x$) : MO. Wherefore $MO = \frac{cy - xy}{y - b}$. In like manner on the other Side, by reason of the similar Triangles ARQ, DOQ, AR will be : DO :: RQ : QO, and by Composition DO + AR ($y+b$) : DO (y) :: QO + RQ (OR or $c+x$) : QO. Wherefore $QO = \frac{cy + xy}{y + b}$. Lastly by reason of the equal Angles DMQ, DQM, MO and QO are equal, that is, $\frac{cy - xy}{y - b} = \frac{cy + xy}{y + b}$. Divide all by y, and multiply by the Denominators, and there will arise $cy + cb - xy - xb = cy - cb + xy - xb$, or $cb = xy$, the most noted Equation that expresses the Hyperbola.

Moreover, the Locus or Place of the Point D might have been found without an algebraic Calculus; for from what we have said above, DO — BN : ON :: DO : MO (QO) :: DO + AR : OR. That is, DO — BN : DO + BN :: ON : OR. And mixtly (*a*), DO : BN :: $\frac{ON + OR}{2}$ (NP) : $\frac{OR - ON}{2}$ (OP). And consequently, DO × OP = BN × NP.

Prob. XLI. (*a*) For DO — BN : DO + BN :: ON : OR; whence DO — BN + DO + BN (= 2DO) : DO + BN :: NO + OR : OR; whence also DO + BN : DO + BN — DO + BN (= 2BN) :: OR : OR — ON; consequently ex æquo ord. 2DO : 2BN :: RO + ON : RO — ON; and dividing by 2, DO : BN :: $\frac{RO + ON}{2}$: $\frac{RO - ON}{2}$.

 PROBLEM

PROBLEM XLII.

To find the Locus or Place of the Vertex of a Triangle whose Base is given, and one of the Angles at the Base differs by a given Angle from being double of the other.

In the last Scheme of the former Problem, let ABD be that Triangle, AB its Base bisected in P, APQ or BPM the third of the given Angle, by which DBA exceeds the double of the Angle DAB; and the Angle DMQ will be double of the Angle DQM (*a*). To PM let fall the Perpendiculars AR, BN, DO, and bisect the Angle DMQ by the right Line MS meeting DO in S; and the Triangles DOQ, SOM, will be similar; and consequently OQ : OM :: OD : OS, and by dividing OQ — OM : OM :: SD : OS :: (by the 3. of the 6th Elem.) DM : OM. Wherefore (by the 9. of the 5th Elem.) OQ — OM = DM. Now making PO = x, OD = y, AR or BN = b, and PR or PN = c, you will have, as in the former Problem, OM = $\frac{cy - xy}{y - b}$, and OQ = $\frac{cy + xy}{y + b}$, and consequently OQ — OM = $\frac{-2bcy + 2xyy}{yy - bb}$. Make now DO$q$ + OMq = DMq, that is, $yy + \frac{cc - 2cx + xx}{yy - 2by + bb}yy = \frac{4bbcc - 8bcxy + 4xxyy}{y^4 - 2bbyy + b^4}yy$, and

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Prob. XLII. (*a*) Because BPM = $\frac{DBP - 2DAQ}{3}$, and because DBP = DMQ + BPM, therefore DMQ = 2DAQ + 3BPM — BPM = 2DAQ + 2BPM = 2DAQ + 2APQ; but DQM = DAQ + APQ; therefore DMQ = 2DQM.

and by due Reduction there will at length arise

$$y^4 * \begin{array}{l} +cc \\ -2bb \\ -2cx \\ -3xx \end{array} yy \begin{array}{l} +2bxx \\ +4bcx \\ +2bcc \end{array} y \begin{array}{l} +b^4 \\ -3bbcc \\ -2bbcx \\ +bbxx \end{array} = 0 \ (b).$$

Divide all by $y - b$, and it will become

$$y^3 + byy \begin{array}{l} -bb \\ +cc \\ -2cx \\ -3xx \end{array} y \begin{array}{l} -b^3 \\ +3bcc \\ +2bcx \\ -bxx \end{array} = 0.$$

Wherefore the Point D is in a Curve of three Dimensions; which, however, becomes an Hyperbola when the Angle BPM vanishes or becomes nothing; or, which is the same Thing, when one of the Angles at the Base DBA is double of the other DAB. For then BN or b vanishing, the Equation will become $yy = 3xx + 2cx - cc$.

And from the Construction of this Equation there comes this Theorem. [See Fig. 54.] *If to the Center* C, *and Asymptotes* CS, CT, *containing the Angle* SCT *of* 120 *Degrees, you describe any Hyperbola, as* DV, *whose Semi-Axis's are* CV, CA; *produce* CV to B, *so that* VB *shall be* = VC, *and from* A *and* B *you draw any how the right Lines* AD, BD, *meeting at the Hyperbola; the Angle* BAD *will be Half the Angle* ABD, *but a third Part of the Angle* ADE, *which the right Line* AD *comprehends with* BD *produced.* This is to be understood of the Hyperbola that passes through the Point V. But if the two right Lines A*d* and B*d*,

 drawn

(*b*) Because $y^4 - 2b^2y^2 + b4 = \overline{y^2 - 2by + b^2}|^2$, it is sufficient to multiply y^2, by $y^4 - 2b^2y^2 + b^4$; and $\overline{c^2 - 2cx + x^2} \times y^2$ by $y^2 - 2by + b^2$; then abbreviating, and transposing, the Biquadratic emerges.

drawn from the same Points A and B, meet in the opposite Hyperbola that passes through A, then of those two external Angles of the Triangle at the Base, that at B will be double of that at A.

Problem XLIII.

To describe a Circle through two given Points, that shall touch a right Line given in Position. [See Fig. 55.]

Let A and B be the two given Points, and EF the right Line given in Position, and let it be required to describe a Circle ABE through those Points which shall touch that right Line FE. Join AB, and bisect it in D. Upon D erect the Perpendicular DF meeting the right Line FE in F, and the Center of the Circle will fall upon this last drawn Line DF, as suppose in C. Join, therefore, CB; and on FE let fall the Perpendicular CE, and E will be the Point of Contact, and CB and CE equal, as being Radii of the Circle sought. Now since the Points A, B, D, and F, are given, let $DB = a$, and $DF = b$; and seek for DC to determine the Center of the Circle, which therefore call x. Now in the Triangle CDB, because the Angle at D is a right one, you have $\sqrt{DBq + DCq}$, that is $\sqrt{aa + xx} = CB$. Also $DF - DC$, or $b - x = CF$. And since in the right-angled Triangle CFE the Angles are given, there will be given the Ratio of the Sides CF and CE. Let that be as d to e; and CE will be $= \frac{e}{d} \times CF$, that is, $= \frac{eb - ex}{d}$. Now put CB and CE (the Radii of the Circle sought) equal to one another, and you will have the Equation $\sqrt{aa + xx} = \frac{eb - ex}{d}$. Whose Parts being squared and multiplied by dd, there arises $aadd + ddxx = eebb - 2eebx + eexx$; or $xx =$

$= \frac{-2eebx - aadd + eebb}{dd - ee}$. And extracting the Root, $x = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$.

Therefore the Length of DC, and consequently the Center C is found, from which a Circle is to be described through the Points A and B that shall touch the right Line FE.

Problem XLIV.

To describe a Circle through a given Point, that shall touch two right Lines given in Position. [See Fig. 56.]

N. B. *This Proposition is resolved as* Prop. 43. *for the Point* A *being given, there is also given the other Point* B.

Suppose the given Point to be A; and let EF, FG, be the two right Lines given by Position, and AEG the Circle sought touching the same, and passing through that Point A. Let the Angle EFG be bisected by the right Line CF, and the Center of the Circle will be found therein. Let that be C; and having let fall the Perpendiculars CE, CG, to EF and FG, E and G will be the Points of Contact. Now in the Triangles CEF, CGF, since the Angles at E and G are right ones, and the Angles at F are Halves of the Angle EFG, all the Angles are given, and consequently the Ratio of the Sides CF, and CE or CG. Let that be as d to e; and if, for determining the Center of the Circle sought C, there be assumed CF $= x$, CE or CG will be $= \frac{ex}{d}$. Besides, let fall the Perpendicular AH to FC, and since the Point A is given, the right Lines AH and FH will be given. Let them be called a and b, and taking FC or x from FH or b, there will remain CH $= b - x$. To whose Square $bb - 2bx + xx$ add the Square of AH or aa, and the Sum $aa + bb - 2bx + xx$

$2bx + xx$ will be ACq by the 47. 1. Eucl. because the Angle AHC is, by Supposition, a right one. Now make the Radii of the Circle AC and CG equal to each other; that is, make an Equality between their Values, or between their Squares, and you will have the Equation $aa + bb - 2bx + xx = \frac{eexx}{dd}$. Take away xx from both Sides, and changing all the Signs, you will have $-aa - bb + 2bx = xx - \frac{eexx}{dd}$. Multiply all by dd, and divide by $dd - ee$, and it will become

$$\frac{-aadd - bbdd + 2bddx}{dd - ee} = xx.$$

The Root of which Equation being extracted, is $x = \frac{bdd - d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$. Therefore the Length FC is found, and consequently the Point C; which is the Center of the Circle sought.

If the found Value x or FC be taken from b or HF, there will remain $HC = \frac{-eeb + d\sqrt{eebb + eeaa - ddaa}}{dd - ee}$ the same Equation which came out in the former Problem, for determining the Length of DC.

Problem XLV.

To describe a Circle through two given Points, which shall touch another Circle given in Position. [See Problem 21, and Figure 57.]

Let A, B, be the two Points given, EK the Circle given in Magnitude and Position, F its Center, ABE the Circle sought, passing through the Points A and B, and touching the other Circle in E, and let C be its Center. Let fall the Perpendiculars CD and FG to AB being produced, and draw CF cutting the Circles in the Point of Contact E, and draw also FH parallel to

DG,

DG, and meeting CD in H. These being thus constructed, make AD or DB $= a$, DG or HF $= b$, GF $= c$, and EF (the Radius of the Circle given) $= d$, and DC $= x$; and CH will be ($=$ CD $-$ FG) $= x - c$, and CFq ($=$ CHq $+$ HFq) $= xx - 2cx + cc + bb$, and CBq ($=$ CDq $+$ DBq) $= xx + aa$, and consequently CB or CE $= \sqrt{xx + aa}$. To this add EF, and you will have CF $= d + \sqrt{xx + aa}$, whose Square $dd + aa + xx + 2d\sqrt{xx + aa}$, is equal to the Value of the same CFq found before, viz. $xx - 2cx + cc + bb$. Take away from both Sides xx, and there will remain $dd + aa + 2d\sqrt{xx + aa} = cc + bb - 2cx$. Take away moreover $dd + aa$, and there will come out $2d\sqrt{xx + aa} = cc + bb - dd - aa - 2cx$. Now, for Abbreviation sake, for $cc + bb - dd - aa$, write $2gg$, and you will have $2d\sqrt{xx + aa} = 2gg - 2cx$, or $d\sqrt{xx + aa} = gg - cx$. And the Parts of the Equation being squared, there will come out $ddxx + ddaa = g^4 - 2ggcx + ccxx$. Take from both Sides $ddaa$ and $ccxx$, and there will remain $ddxx - ccxx = g^4 - ddaa - 2ggcx$. And the Parts of the Equation being divided by $dd - cc$, you will have $xx = \frac{g^4 - ddaa - 2ggcx}{dd - cc}$. And by Extraction of the affected Root, $x = \frac{-ggc + \sqrt{g^4dd - d^4aa + ddaacc}}{dd - cc}$.

Having found therefore x, or the Length of DC, bisect AB in D, and at D erect the Perpendicular DC $= \frac{-ggc + d\sqrt{g^4 - aadd + aacc}}{dd - cc}$. Then from the Center C, through the Point A or B, describe the Circle ABE; for that will touch the other Circle EK, and pass through both the Points A, B. Q. E. F.

PROBLEM

Problem XLVI.

To describe a Circle through a given Point which shall touch a given Circle, and also a right Line, both given in Position. [See Fig. 58.]

Let the Circle to be described be BD, its Center C, and B a Point through which it is to be described, and AD the right Line which it shall touch; the Point of Contact D, and the Circle which it shall touch GEM, its Center F, and its Point of Contact E. Join CB, CD, CF, and CD will be perpendicular to AD, and CF will cut the Circles in the Point of Contact E. Produce CD to Q, so that DQ shall be $=$ EF, and through Q draw QN parallel to AD. Lastly, from B and F to AD and QN, let fall the Perpendiculars BA, FN; and From C to AB and FN let fall the Perpendiculars CK, CL. And since BC is $=$ CD or AK, BK will be ($=AB-AK$) $=AB-BC$, and consequently $BKq=ABq-2AB\times BC+BCq$. Subtract this from BCq, and there will remain $2AB\times BC-ABq$ for the Square of CK. Therefore $AB\times\overline{2BC-AB}$ is $=CKq$; and for the same Reason it will be $FN\times\overline{2FC-FN}=CLq$, and consequently $\frac{CKq}{AB}+AB=2BC$, and $\frac{CLq}{FN}+FN=2FC$. Wherefore, if for AB, CK, FN, KL, and CL, you write a, y, b, c, and $c-y$, you will have $\frac{yy}{2a}+\frac{1}{2}a=BC$, and $\frac{cc-2cy+yy}{2b}+\frac{1}{2}b=FC$. From FC take away BC, and there will remain $EF=\frac{cc-2cy+yy}{2b}+\frac{1}{2}b-\frac{yy}{2a}-\frac{1}{2}a$. Now, if the Points, where FN being produced cuts the right Line AD and the Circle GEM, be marked with the Letters H, G, and M, and upon HG produced you take HR $=$ AB, since HN ($=$ DQ $=$ EF) is $=$ GF, by by adding FH on both Sides, you will have FN = GH, and consequently AB − FN (= HR − GH) = GR, and AB − FN + 2EF; that is, $a - b + 2EF = RM$, and $\frac{1}{2}a - \frac{1}{2}b + EF = \frac{1}{2}RM$. Wherefore, since above EF was $= \frac{cc - 2cy + yy}{2b} + \frac{1}{2}b - \frac{yy}{2a} - \frac{1}{2}a$, if this be written for EF you will have $\frac{1}{2}RM = \frac{cc - 2cy + yy}{2b} - \frac{yy}{2a}$. Call therefore RM, d; and d will be $= \frac{cc - 2cy + yy}{b} - \frac{yy}{a}$. Multiply all the Terms by a and b, and there will arise $abd = acc - 2acy + ayy - byy$. Take away from both Sides $acc - 2acy$, and there will remain $abd - acc + 2acy = ayy - byy$. Divide by $a - b$, and there will arise $\frac{abd - acc + 2acy}{a - b} = yy$. And extracting the Root, $y = \frac{ac}{a - b} \pm \sqrt{\frac{aabd - abbd + abcc}{aa - 2ab + bb}}$. Which Conclusions may be thus abbreviated; make $c : b :: d : e$, then $a - b : a :: c : f$; and $fe - fc + 2fy$ will be $= yy$, or $y = f \pm \sqrt{ff + fe - fc}$. Having found y or KC or AD, take $AD = f \pm \sqrt{ff + fe - fc}$, and at D erect the Perpendicular DC $(= BC) = \frac{KCq}{2AB} + \frac{1}{2}AB$; and from the Center C, at the Interval CB or CD, describe the Circle BDE, for this passing through the given Point B, will touch the right Line AD in D, and the Circle GEM in E. Q. E. F.

Hence also a Circle may be described which shall touch two given Circles, and a right Line given in Position. [See Fig. 59.] For let the given Circles RT, SV, their Centers B, F, and the right Line given in Position PQ. From the Center F, with the Radius

FS — BR, describe the Circle EM. From the Point B to the right Line PQ let fall the Perpendicular BP, and having produced it to A, so that PA shall be = BR, through A draw AH parallel to PQ, and describe a Circle which shall pass through the Point B, and touch the right Line AH and the Circle EM. Let its Center be C, join BC, cutting the Circle RT in R; and the Circle RS described from the same Center C, and the Radius CR will touch the Circles RT, SV, and the right Line PQ, as is manifest by the Construction.

Problem XLVII.

To describe a Circle that shall pass through a given Point, and touch two other Circles given in Position and Magnitude. [See Fig. 60.]

Let the given Point be A, and let the Circles given in Magnitude and Position be TIV, RHS, their Centers C and B; the Circle to be described AIH, its Center D; and the Points of Contact I and H. Join AB, AC, AD, DB, and let AB produced cut the Circle RHS in the Points R and S, and AC produced, cut the Circle TIV in T and V. And having from the Points D and C let fall the Perpendiculars DE to AB, and DF to AC meeting AB in G, and CK to AB; in the Triangle ADB, it will be $ADq - DBq + ABq = 2AE \times AB$, by the 13 of the 2 Elem. But $DB = AD + BR$, and consequently $DBq = ADq + 2AD \times BR + BRq$. Take away this from $ADq + ABq$, and there will remain $ABq - 2AD \times BR - BRq$ for $2AE \times AB$. Moreover, $ABq - BRq$ is $= \overline{AB - BR} \times \overline{AB + BR} = AR \times AS$. Wherefore, $AR \times AS - 2AD \times BR = 2AE \times AB$. And $\frac{AR \times AS - 2AB \times AE}{BR} = 2AD$.

And

And by a like reasoning in the Triangle ADC, there will come out again $2AD = \frac{TAV - 2CAF}{CT}$. Wherefore $\frac{RAS - 2BAE}{BR} = \frac{TAV - 2CAF}{CT}$. And $\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR} = \frac{2CAF}{CT}$. And $\overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AC} = AF$. Whence since it is $AK : AC :: AF : AG$, AG will be $= \overline{\frac{TAV}{CT} - \frac{RAS}{BR} + \frac{2BAE}{BR}} \times \frac{CT}{2AK}$. Take away this from AE, or $\frac{2KAE}{CT} \times \frac{CT}{2AK}$, and there will remain $GE = \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2AK}$. Whence since it is $KC : AK :: GE : DE$; DE will be $= \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2BAE}{BR} + \frac{2KAE}{CT}} \times \frac{CT}{2KC}$. Upon AB take AP, which let be to AB as CT to BR, and $\frac{2PAE}{CT}$ will be $= \frac{2BAE}{BR}$, and so $\frac{2PK \times AE}{CT} = \frac{2BAE}{BR} - \frac{2KAE}{CT}$, and so $DE = \overline{\frac{RAS}{BR} - \frac{TAV}{CT} - \frac{2PK \times AE}{CT}} \times \frac{CT}{2KC}$. Upon AB erect the Perpendicular $AQ = \overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2KC}$, and

and in it take $QO = \frac{PK \times AE}{KC}$, and AO will be $= DE$.

Join DO, DQ, and CP, and the Triangles DOQ, CKP, will be similar, becauſe their Angles at O and K are right ones, and the Sides (KC : PK :: AE, or DO : QO) proportional. Therefore the Angles OQD, KPC, are equal, and conſequently QD is perpendicular to CP. Wherefore if AN be drawn parallel to CP, and meeting QD in N, the Angle ANQ will be a right one, and the Triangles AQN, PCK, ſimilar; and conſequently PC : KC :: AQ : AN. Whence ſince AQ is $\overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2KC}$, AN will be $\overline{\frac{RAS}{BR} - \frac{TAV}{CT}} \times \frac{CT}{2PC}$. Produce AN to M, ſo that NM ſhall be $= AN$, and AD will be $= DM$, and conſequently the Circle will paſs through the Point M.

Since therefore the Point M is given, there follows this Reſolution of the Problem, without any farther Analyſis.

Upon AB take AP, which muſt be to AB as CT to BR; join CP, and draw parallel to it AM, which ſhall be to $\frac{RAS}{BR} - \frac{TAV}{CT}$, as CT to PC; and by the Help of the forty-fifth Problem, deſcribe through the Points A and M the Circle AIHM, which ſhall touch either of the Circles TIV, RHS, and the ſame Circle ſhall touch both. Q. E. F.

And hence alſo a Circle may be deſcribed, which ſhall touch three Circles given in Magnitude and Poſition. For let the Radii of the given Circles be A, B, C, and their Centers D, E, F. From the Centers E and F, with

with the Radii B $+$ A and C $+$ A describe two Circles, and let a third Circle which touches these two be also described, and let it pass through the Point D; let its Radius be G, and its Center H, and a Circle described on the same Center H, with the Radius G $+$ A, shall touch the three former Circles, as was required.

Problem XLVIII.

If at the Ends of the Thread D A E, *moving upon the fixed Tack* A, *there are hanged two Weights* D *and* E, *whereof the Weight* E *slides through the oblique Line* B G; *to find the Place of the Weight* E, *where these Weights are in Æquilibrio.* [See Fig. 63.]

Suppose the Problem done, and parallel to AD draw EF, which let be to AE as the Weight E to the Weight D. And from the Points A and F to the Line BG let fall the Perpendiculars AB, FG. Now since the Weights are, by Supposition, as the Lines AE and EF, express those Weights by those Lines, the Weight D by the Line EA, and the Weight E by the Line EF. Therefore the Body E, directed by the Force of its own Weight EF, tends towards F, and by the oblique Force EG tends towards G. And the same Body E by the direct Force AE of the Weight D is drawn towards A, and by the oblique Force BE is drawn towards B. Since therefore the Weights sustain each other in Æquilibrio, the Force by which the Weight E is drawn towards B, ought to be equal to the contrary Force by which it tends towards G, that is, BE ought to be equal to EG. But now the Ratio of AE to EF is given by the Hypothesis; and by reason of the given Angle FEG, there is also given the Ratio of FE to EG, to which BE is equal. Therefore there is given the Ratio of AE to BE. AB is also given in Length; and thence the Triangle ABE, and the Point E will easily be given, viz. make AB $= a$, BE $= x$, and AE will be $= \sqrt{aa + xx}$; moreover, let AE be to BE in the given Ratio of d to e, and

$e\sqrt{aa + xx}$

$e\sqrt{aa+xx}$ will be $=dx$. And the Parts of the Equation being squared and reduced, $eeaa=ddxx-eexx$; or $\frac{ea}{\sqrt{dd-ee}}=x$. Therefore the Length BE is found, which determines the Place of the Weight E. Q. E. F.

But if both Weights descend by oblique Lines, the Computation may be made thus. [See Fig. 64.] Let CD and BE be oblique Lines given in Position, through which those Weights D and E descend. From the fixed Tack A to these Lines let fall the Perpendiculars AC, AB, and let the Lines EG, DH, erected from the Weights perpendicularly to the Horizon, meet them in the Points G and H; and the Force by which the Weight E endeavours to descend in a perpendicular Line; that is, the whole Gravity of E, will be to the Force by which the same Weight endeavours to descend in the oblique Line BE, as GE to BE; and the Force by which it endeavours to descend in the oblique Line BE, will be to the Force by which it endeavours to descend in the Line AE, that is, to the Force by which the Thread AE is distended or stretched, as BE to AE. And consequently the Gravity of E will be to the Tension of the Thread AE, as GE to AE. And by the same reason the Gravity of D will be to the Tension of the Thread AD, as HD to AD. Let therefore the Length of the whole Thread DA + AE be c, and let its Part AE be $=x$, and its other Part AD will be $=c-x$. And because $AEq-ABq$ is $=BEq$, and $ADq-ACq=CDq$; let moreover, AB be $=a$, and $AC=b$, and BE will be $=\sqrt{xx-aa}$, and $CD=\sqrt{xx-2cx+cc-bb}$. Farther, since the Triangles BEG, CDH, are given in Specie, let BE : EG :: f : E, and CD : DH :: f : g, and EG will be $=\frac{E}{f}\sqrt{xx-aa}$, and $DH=\frac{g}{f}\sqrt{xx-2cx+cc-bb}$. Wherefore since GE : AE :: Weight

:: Weight E : Tension of AE; and HD : AD :: Weight D : Tension of AD; and those Tensions are equal, you will have $\frac{Ex}{\frac{E}{f}\sqrt{xx-aa}}$ = Tension of AE = the Tension AD $= \frac{Dc - Dx}{\frac{g}{f}\sqrt{xx-2cx+cc-bb}}$; from the Reduction of which Equation there comes out $gx\overline{x}\sqrt{xx-2cx+cc-bb} = \overline{Dc-Dx}\sqrt{xx-aa}$,

$$\text{or } -\frac{gg}{DD}x^4 \mp \frac{2ggc}{2DDc}x^3 \begin{matrix} +ggcc \\ -ggbb \\ -DDcc \\ +DDaa \end{matrix} xx - 2DDcaax + DDccaa = 0.$$

But if you desire a Case wherein this Problem may be constructed by a Rule and Compass, make the Weight D to the Weight E as the Ratio $\frac{BE}{EG}$ to the Ratio $\frac{CD}{DH}$, and g will become $= D$ (a); and so in the Room of the precedent Equation you will have this,

$$\begin{matrix} aa \\ -bb \end{matrix} xx - 2acx + aaacc = 0, \text{ or } x = \frac{ac}{a+b}.$$

Prob. XLVIII. (a) For $\frac{BE}{EG} = \frac{f}{E}$, and $\frac{CD}{DH} = \frac{f}{g}$, and $\frac{f}{E} : \frac{f}{g} :: D : E$; whence $\frac{Df}{g} = f$; whence $D = g$.

X PROBLEM

PROBLEM XLIX.

If on the String DACBF, *that slides about the two Tacks* A *and* B, *there are hung three Weights,* D, E, F; D *and* F *at the Ends of the String, and* E *at its middle Point* C, *placed between the Tacks: From the given Weights and Position of the Tacks, to find the Situation of the Point* C, *where the middle Weight hangs, and where they are in Æquilibrio.* [See Fig. 65.]

Since the Tension of the Thread AC is equal to the Tension of the Thread AD, and the Tension of the Thread BC to the Tension of the Thread BF, the Tensions of the Strings or Threads AC, BC, EC, will be as the Weights D, E, F. Then take the Parts of the Thread CG, CH, CI, in the same Ratio as the Weights. Compleat the Triangle GHI. Produce IC till it meet GH in K, and GK will be = KH, and CK = $\frac{1}{2}$ CI, and consequently C the Center of Gravity of the Triangle GHI. For, draw PQ through C, perpendicular to CE; and perpendicular to that, from the Points G and H, draw GP, HQ. And if the Force by which the Thread AC by the Force of the Weight D draws the Point C towards A, be expressed by the Line GC, the Force by which that Thread will draw the same Point towards P, will be expressed by the Line CP; and the Force by which it draws it towards K, will be expressed by the Line GP. And in like manner, the Forces by which the Thread BC, by means of the Weight F, draws the same Point C towards B, Q, and K, will be expressed by the Lines CH, CQ, and HQ; and the Force by which the Thread CE, by means of the Weight E, draws that Point C towards E, will be expressed by the Line CI. Now since the Point C is sustained in Æquilibrio by equal Forces, the Sum of the Forces by which the Threads AC and BC do together draw C towards K, will be equal to the contrary Force by which the Thread EC draws that Point towards E; that is, the Sum GP + HQ will be equal to CI; and the Force by which the Thread AC draws the Point C towards P, will be equal to the contrary Force by which the Thread BC draws the same Point C towards Q; that is, the Line PC is equal to the Line CQ. Wherefore, since PG, CK, and QH are parallel, GK will be also = KH, and CK $\left(= \frac{GP + HQ}{2}\right)$ $= \frac{1}{2}$ CI. Which was to be shewn. It remains therefore to determine the Triangle GCH, whose Sides GC and HC are given, together with the Line CK, which is drawn from the Vertex C to the Middle of the Base. Let fall therefore from the Vertex C to the Base CH the Perpendicular CL, and $\frac{GCq - CHq}{2GH}$ will be = $KL = \frac{GCq - KCq - GKq}{2GK}$. For 2GK write GH, and having rejected the common Divisor GH, and ordered the Terms, you will have $GCq - 2KCq + CHq = 2GKq$, or $\sqrt{\frac{1}{2}GCq - KCq + \frac{1}{2}CHq} = GK$. Having found GK, or KH, there are given together the Angles GCK, KCH, or DAC, FBC. Wherefore, from the Points A and B in these given Angles DAC, FBC, draw the Lines AC, BC, meeting in the Point C; and C will be the Point sought.

But it is not always necessary to solve Questions that are of the same Kind, particularly by Algebra, but from the Solution of one of them you may most commonly infer the Solution of the other. As if now there should be proposed this Question.

The Thread ACDB *being divided into the given Parts* AC, CD, DB, *and its Ends being fastened to the two Tacks given in Position,* A *and* B; *and if at the Points of Division,* C *and* D, *there are hanged the two Weights* E *and* F; *from the given Weight* F, *and the Situation of the Points* C *and* D, *to know the Weight* E. [See Figure 66.]

From the Solution of the former Problem the Solution of this may be easily enough found. Produce the

 Lines

Lines AC, BD, till they meet the Lines DF, CE, in G and H; and the Weight E will be to the Weight F as DG to CH.

And hence, by the bye, may appear a Method of making a Balance of only Threads, by which the Weight of any Body E may be known, from only one given Weight F.

Problem L.

A Stone falling down into a Well, from the Sound of the Stone striking the Bottom, to determine the Depth of the Well.

Let the Depth of the Well be x, and if the Stone descends with an uniformly accelerated Motion through any given Space a in any given Time b, and the Sound passes with an uniform Motion through the same given Space a in the given Time d, the Stone will descend through the Space x in the Time $b\sqrt{\frac{x}{a}}$; but the Sound which is caused by the Stone striking upon the Bottom of the Well, will ascend through the same Space x, in the Time $\frac{dx}{a}$. For the Spaces described by descending heavy Bodies, are as the Squares of the Times of Descent; or the Roots of the Spaces, that is, $\sqrt{x}$ and $\sqrt{a}$ are as the Times themselves. And the Spaces x and a, through which the Sound passes, are as the Times of Passage. And the Sum of these Times $b\sqrt{\frac{x}{a}}$, and $\frac{dx}{a}$, is the Time of the Stone's falling to the Return of the Sound. This Time may be known by Observation. Let it be t, and you will have $b\sqrt{\frac{x}{a}} + \frac{dx}{a} = t$. And $b\sqrt{\frac{x}{a}} = t - \frac{dx}{a}$. And the Parts being squared $\frac{bbx}{a} =$

$tt -$

$tt - \frac{2tdx}{a} + \frac{ddxx}{aa}$. And by Reduction $xx = \frac{2adt + abb}{dd}x - \frac{aatt}{dd}$. And having extracted the Root $x = \frac{adt + \frac{1}{2}abb}{dd} - \frac{ab}{2dd} \times \sqrt{bb + 4dt}$.

PROBLEM LI.

Having given the Globe A, *and the Position of the Wall* DE, *and* BD *the Distance of the Center of the Globe* B *from the Wall; to find the Bulk of the Globe* B, *on this Condition, that if the Globe* A *(whose Center is in the Line* BD, *which is perpendicular to the Wall, and produced out beyond* B), *be moved in free absolute Space, and where Gravity cannot act, with an uniform Motion towards* D, *until it strikes against the other quiescent Globe* B; *and that Globe* B, *after it is reflected from the Wall, shall meet the Globe* A *in the given Point* C. [See Fig. 81.]

Let the Velocity of the Globe A before Reflection be a, and by Problem XII. p. 192. the Velocity of the Globe A will be after Reflection $= \frac{aA - aB}{A + B}$, and the Velocity of the Globe B after Reflection will be $= \frac{2aA}{A + B}$. Therefore the Velocity of the Globe A to the Velocity of the Globe B, is as A — B to 2A. On GD take gD = GH, viz. to the Diameter of the Globe B, and those Velocities will be as GC to Gg + gC. For when the Globe A struck upon the Globe B, the Point G, which being on the Surface of the Globe B, is moved in the Line AD, will go through the Space Gg before that Globe B shall strike against the Wall, and through the Space gC after it is reflected from the Wall; that is, through the whole Space Gg + gC, in the same Time wherein the Point F of the

 Globe

Globe A shall pass through the Space GC, so that both Globes may again meet and strike one another in the given Point C. Wherefore, since the Intervals BC and CD are given, make $BC = m$, $BD + CD = n$, and $BG = x$, and GC will be $= m + x$, and $Gg + gC = GD + DC - 2gD = GB + BD + DC - 2GH = x + n - 4x$, or $= n - 3x$. Above you had $A - B$ to $2A$, as the Velocity of the Globe A to the Velocity of the Globe B, and the Velocity of the Globe A to the Velocity of the Globe B, as GC to $Gg + gC$, and consequently $A - B$ to $2A$, as GC to $Gg + gC$; therefore since GC is $= m + x$, and $Gg + gC = n - 3x$, $A - B$ will be to $2A$ as $m + x$ to $n - 3x$. Moreover, the Globe A is to the Globe B, as the Cube of its Radius AF to the Cube of the others Radius GB; that is, if you make the Radius AF to be s, as s^3 to x^3; therefore $s^3 - x^3 : 2s^3$ $(:: A - B : 2A) :: m + x : n - 3x$. And multiplying the Means and Extreams by one another, you will have this Equation, $s^3 n - 3s^3 x - nx^3 + 3x^4 = 2ms^3 + 2s^3 x$. And by Reduction, $3x^4 - nx^3 - 5s^3 x \begin{smallmatrix} + s^3 n \\ - 2s^3 m \end{smallmatrix} = 0$. From the Construction of which Equation there will be given x, the Semi-diameter of the Globe B; which being given, that Globe is also given. Q. E. F.

But note, when the Point C lies on contrary Sides of the Globe B, the Sign of the Quantity $2m$ must be changed, and written $3x^4 - nx^3 - 5s^3 x \begin{smallmatrix} + s^3 n \\ + 2s^3 m \end{smallmatrix} = 0$.

If the Globe B were given, and the Globe A sought on this Condition, that the two Globes, after Reflection, should meet in C, the Question would be easier; viz. in the last Equation found, x would be supposed to be given, and s to be sought. Whereby, by a due Reduction of that Equation, the Terms $-5s^3 x + s^3 n - 2s^3 m$ being translated to the contrary Side of the Equation, and each Part

Part divided by $5x - n + 2m$, there would come out $\frac{3x^4 - nx^3}{5x - n + 2m} = s^3$. Where s will be obtained by the bare Extraction of the Cube Root.

But if both Globes being given, you were to find the Point C, in which both would fall upon one another after Reflection: Since above it was A — B to 2 A as GC to Gg + gC, therefore by Inversion and Composition 3 A — B will be to A — B as 2 Gg to the sought Distance GC.

PROBLEM LII.

If two Globes, A and B, are joined together by a small Thread PQ, and the Globe B hanging on the Globe A; if you let fall the Globe A, so that both Globes may begin to fall together by the sole Force of Gravity in the same perpendicular Line PQ; and then the lower Globe B, after it is reflected upwards from the Bottom or horizontal Plane FG, it shall meet the upper Globe A, as falling, in a certain Point D: From the given Length of the Thread PQ, and the Distance DF of that Point D from the Bottom, to find the Height PF, from which the upper Globe A ought to be let fall to cause this Effect. [See Fig. 83.]

Let a be the Length of the Thread PQ. In the Perpendicular PQRF from F upwards take FE equal to QR the Diameter of the lower Globe, so that when the lowest Point R of that Globe falls upon the Bottom in F, its upper Point Q shall possess the Place E; and let ED be the Distance through which that Globe, after it is reflected from the Bottom, shall, by ascending, pass, before it meets the upper falling Globe in the Point D. Therefore, by reason of the given Distance DF of the Point D from the Bottom, and the Diameter EF of the inferior Globe, there will be given their Difference DE. Let that be $= b$, and let the Height RF, or QE, which that lower Globe describes by falling through it before it touches the Bottom, be $= x$, by reason it is unknown. And having found x, if to it you add EF and PQ, there will be had the Height PF, from which the upper Globe ought to fall to have the desired Effect.

Since therefore PQ is $= a$, and QE $= x$, PE will be $= a + x$. Take away DE or b, and there will remain PD $= a + x - b$. But the Time of the Descent of the Globe A is as the Root of the Space described in falling, or $\sqrt{a + x - b}$, and the Time of the Descent of the other Globe B as the Root of the Space described by its falling, or $\sqrt{x}$, and the Time of its Ascent as the Difference of that Root, and of the Root of the Space which it would describe by falling only from Q to D. For this Difference is as the Time of Descent from D to E, which is equal to the Time of Ascent from E to D. But that Difference is $\sqrt{x} - \sqrt{x - b}$. Whence the Time of Descent and Ascent together will be as $2\sqrt{x} - \sqrt{x - b}$. Wherefore, since this Time is equal to the Time of Descent of the upper Globe, it will be $\sqrt{a + x - b} = 2\sqrt{x} - \sqrt{x - b}$. the Parts of which Equation being squared, you will have $a + x - b = 5x - b - 4\sqrt{xx - bx}$, or $a = 4x - 4\sqrt{xx - bx}$; and the Equation being ordered, $4x - a = 4\sqrt{xx - bx}$; and squaring the Parts of that Equation again, there arises $16xx - 8ax + aa = 16xx - 16bx$, or $aa = 8ax - 16bx$; and dividing all by $8a - 16b$, you will have $\frac{aa}{8a - 16b} = x$. Make therefore as $8a - 16b$ to a, so a to x, and you will have x or QE. Q. E. I.

But if from the given QE you are to find the Length of the Thread PQ or a; the same Equation $aa = 8ax - 16bx$, by extracting the affected quadratic Root, would give $a = 4x - \sqrt{16xx - 16bx}$; that is, if you

you take QY a mean Proportional between QD and QE, PQ will be $= 4$ EY. For that mean Proportional will be $\sqrt{x \times \overline{x-b}}$, or $\sqrt{xx-bx}$; which subtracted from x, or QE, leaves EY, the Quadruple whereof is $4x - 4\sqrt{xx-bx}$.

But if from the given Quantities QE, or x, as also the Length of the Thread PQ, or a, there were sought the Point D in which the upper Globe falls upon the under one; the Distance DE, or b, of that Point from the given Point E, will be had from the precedent Equation $aa = 8ax - 16bx$ by transferring aa and $16bx$ to the contrary Sides of the Equation with the Signs changed, and by dividing the whole by $16x$. For there will arise $\frac{8ax - aa}{16x} = b$. Make therefore as $16x$ to $8x - a$, so a to b, and you will have b or DE.

Hitherto I have supposed the Globes tied together by a small Thread to be let fall together. Which if they are let fall at different Times connected by no Thread, so that the upper Globe A, for Example, being let fall first, shall descend through the Space PT before the other Globe begins to fall; and from the given Distances PT, PQ, and DE, you are to find the Height PF, from which the upper Globe ought to be let fall, so that it shall fall upon the inferior or lower one at the Point D; make PQ $= a$, DE $= b$, PT $= c$, and QE $= x$, and PD will be $= a + x - b$, as above. And the Times wherein the upper Globe, by falling, will describe the Spaces PT and TD, and the lower Globe by falling before, and then by re-ascending, will describe the Sum of the Spaces QE + ED, will be as $\sqrt{PT}$, $\sqrt{PD} - \sqrt{PT}$, and $2\sqrt{QE} - \sqrt{QD}$; that is, as $\sqrt{c}$, $\sqrt{a + x - b} - \sqrt{c}$, and $2\sqrt{x} - \sqrt{x-b}$. But the two last Times, because the Spaces TD and QE + ED are described together, are equal. Therefore

fore $\sqrt{a+x-b} - \sqrt{c} = 2\sqrt{x} - \sqrt{x-b}$. And the Parts being squared $a + c - 2\sqrt{ca + cx - cb} = 4x - 4\sqrt{xx - bx}$. Make $a + c = e$, and $a - b = f$, and by a due Reduction it will be $4x - e + 2\sqrt{cf + cx} = 4\sqrt{xx - bx}$, and the Parts being squared $ee - 8ex + 16xx + 4cf + 4cx + \overline{16x - 4e} \times \sqrt{cf + cx} = 16xx - 16bx$. And blotting out on both Sides $16xx$, and writing m for $ee + 4cf$, and also writing n for $8e - 16b - 4c$, you will have by due Reduction $\overline{16x - 4e} \times \sqrt{cf + cx} = nx - m$. And the Parts being squared you will have $256cfxx + 256cx^3 - 128cefx - 128cexx + 16ceef + 16ceex = nnxx - 2mnx + mm$. And having ordered the Equation

$$256cx^3 \begin{matrix} +256cf \\ -128ce \\ -nn \end{matrix} xx \begin{matrix} -128cef \\ +16cee \\ +2mn \end{matrix} x \begin{matrix} +6ceef \\ -mm \end{matrix} = 0.$$

By the Construction of which Equation x or QE will be given, to which if you add the given Distances PQ and EF, you will have the Height PF, which was to be found.

Problem LIII.

If two quiescent Globes, the upper one A *and the under one* B, *are let fall at different Times; and the lower Globe begins to fall in the same Moment that the upper one, by falling, has described the Space* PT; *to find the Places* α, β, *which these falling Globes shall occupy when their Interval or Distance* $\pi\chi$ *is given.* [See Fig. 84.]

Since the Distances PT, PQ, and $\pi\chi$ are given, call the first a, the second b, the third c, and for Pπ, or the Space that the superior Globe describes by falling before it comes to the Place sought π, put x. Now the Times wherein the upper Globe describes the Spaces PT, Pπ, Tπ, and the lower one the Space Qχ, are as $\sqrt{PT}$, $\sqrt{P\pi}$, $\sqrt{P\pi} - \sqrt{PT}$, and $\sqrt{Q\chi}$. The latter

latter two of which Times, because the Globes by falling together describe the Spaces $T\pi$ and $Q\chi$, are equal. Whence also $\sqrt{P\pi} - \sqrt{PT}$ will be equal to $\sqrt{Q\chi}$. $P\pi$ was $= x$, and $PT = a$, and by adding $\pi\chi$, or c, to $P\pi$, and subtracting PQ, or b, from the Sum, you will have $Q\chi = x + c - b$. Wherefore substituting these, you will have $\sqrt{x} - \sqrt{a} = \sqrt{x + c - b}$. And squaring both Sides of the Equation, there will arise $x + a - 2\sqrt{ax} = x + c - b$. And blotting out on both Sides x, and ordering the Equation, you will have $a + b - c = 2\sqrt{ax}$. And having squared the Parts, the Square of $a + b - c$ will be $= 4ax$, and that Square divided by $4a$ will be $= x$, or $4a$ will be to $a + b - c$ as $a + b - c$ is to x. But from x found, or $P\pi$, there is given the Place sought, viz. α of the superior Globe. And by the Distance of the Places, there is also given the Place of the lower one β.

And hence, if you were to find the Point where the upper Globe, by falling, will at length fall upon the lower one; by putting the Distance $\pi\chi = 0$, or by extirpating c, say $4a$ is to $a + b$ as $a + b$ is to x, or $P\pi$, and the Point π will be that sought.

And reciprocally, if that Point π, or χ, in which the upper Globe falls upon the under one, be given and you are to find the Place T which the lower Point P of the upper falling Globe possessed, or was then in when the lower Globe began to fall; because $4a$ is to $a + b$ as $a + b$ is to x; or multiplying the Means and Extremes together, $4ax = aa + 2ab + bb$, and by due ordering of the Equation $aa = 4ax - 2ab - bb$; extract the Square Root and you will have $a = 2x - b - 2\sqrt{xx - bx}$. Take therefore $V\pi$, a mean Proportional between $P\pi$ and $Q\pi$, and towards V take $VT = VQ$, and T will be the Point you seek. For $V\pi$ will be $= \sqrt{P\pi \times Q\pi}$, that is $= \sqrt{x \times \overline{x - b}}$, or $= \sqrt{xx - bx}$; the double whereof subtracted from $2x - b$, or from $2P\pi - PQ$, that is, from $PQ + 2Q\pi$.

$2\,Q\pi$, leaves $PQ - 2VQ$, or $PV - VQ$, that is, PT.

If, lastly, the lower of the Globes, after the upper has fallen upon the lower, and the lower, by their Shock upon one another, is accelerated, and the superior one retarded, the Places are required where, in falling, they shall acquire a Distance equal to a given right Line. In the first Place you must seek the Place where the upper one falls upon the lower one; then from the known Magnitudes of the Globes, as also from their Celerities where they fall on each other, you must find the Celerities they shall have immediately after Reflection, after the same Way as in Prob. XII. p. 192. Afterwards you must find the highest Places to which the Globes with these Celerities, if they were carried upwards, would ascend, and thence the Spaces will be known, which the Globes will describe by falling in any given Times after Reflection, as also the Difference of the Spaces; and reciprocally from that Difference assumed, you may go back analytically to the Spaces described in falling.

As if the upper Globe falls upon the lower one at the Point π, [See Fig. 85.] and after Reflection, the Celerity of the upper one downwards be so great, as if it were upwards, it would cause that Globe to ascend through the Space πN; and the Celerity of the lower one downwards was so great, as that, if it were upwards, it would cause the lower one to ascend through the Space πM; then the Times wherein the upper Globe would reciprocally descend through the Spaces $N\pi$, NG, and the inferior one through the Spaces $M\pi$, MH, would be as $\sqrt{N\pi}$, $\sqrt{NG}$, $\sqrt{M\pi}$, $\sqrt{MH}$; and consequently the Times wherein the upper Globe would run the Space πG, and the lower one πH, would be as $\sqrt{NG} - \sqrt{N\pi}$, to $\sqrt{MH} - \sqrt{M\pi}$. Make those Times equal, and $\sqrt{NG} - \sqrt{N\pi}$ will be $= \sqrt{MH} - \sqrt{M\pi}$. And, moreover, since there is given the Distance GH, put $\pi G + GH = \pi H$. And by the Reduction of these two Equations, the Problem will be solved. As if $M\pi$ is $= a$, $N\pi = b$, $GH = c$, $\pi G = x$, you will have, according

according to the latter Equation, $x + c = \pi H$. Add $M\pi$, you will have $MH = a + c + x$. To πG add $N\pi$, and you will have $NG = b + x$. Which being found, according to the former Equation, $\sqrt{b+x} - \sqrt{b}$ will be $= \sqrt{a+c+x} - \sqrt{a}$. Write e for $a + c$, and $\sqrt{f}$ for $\sqrt{a} + \sqrt{b}$, and the Equation will become $\sqrt{b+x} = \sqrt{e+x} - \sqrt{f}$. And the Parts being squared $b + x = e + x + f - 2\sqrt{ef+fx}$, or $e + f - b = 2\sqrt{ef+fx}$. For $e + f - b$ write g, and you will have $g = 2\sqrt{ef+fx}$, and the Parts being squared, $gg = 4ef + 4fx$, and by Reduction $\frac{gg}{4f} - e = x$.

Problem LIV.

If there are two Globes, A, B, *whereof the upper one* A *falling from the Height* G, *strikes upon another lower one* B *rebounding from the Ground* H *upwards; and these Globes so part from one another by Reflection, that the Globe* A *returns by Force of that Reflection to its former Altitude* G, *and that in the same Time that the lower Globe* B *returns to the Ground* H; *then the Globe* A *falls again, and strikes again upon the Globe* B, *rebounding again back from the Ground; and after this Rate the Globes always rebound from one another and return to the same Place: From the given Magnitude of the Globes, the Position of the Ground, and the Place* G *from whence the upper Globe falls, to find the Place where the Globes shall strike upon each other.* [See Fig. 86.]

Let e be the Center of the Globe A, and f the Center of the Globe B, d the Center of the Place G wherein the upper Globe is in its greatest Height, g the Center of the Place of the lower Globe where it falls on the Ground; a the Semi-diameter of the Globe A, b the Semi-diameter of the Globe B, c the Point of Contact of the Globes falling upon one another, and H the Point of Contact of the lower Globe and the Ground. And the

the Celerity of the Globe A, where it falls on the Globe B, will be the same which is generated by the Fall of the Globe from the Height de, and consequently is as $\sqrt{de}$. With this same Celerity the Globe A ought to be reflected upwards, that it may return to its former Place G. And the Globe B ought to be reflected with the same Celerity downwards wherewith it ascended, that it may return in the same Time to the Ground it took up in mounting from it. And that both these may come to pass, the Motion of the Globes in reflecting ought to be equal. But the Motions are compounded of the Celerities and Magnitudes together, and consequently the Product of the Bulk and Celerity of one Globe will be equal to the Product of the Bulk and Celerity of the other. Whence, if the Product of the Bulk and Celerity of one Globe be divided by the Bulk of the other Globe, you will have the Celerity of the other just before and after Reflection, or at the End of the Ascent, and at the Beginning of the Descent. Therefore this Celerity will be as $\frac{A\sqrt{de}}{B}$, or since the Globes are as the Cubes of the Radii as $\frac{a^3\sqrt{de}}{b^3}$. But as the Square of this Celerity to the Square of the Celerity of the Globe A just before Reflection, so is the Height to which the Globe B would ascend with this Celerity, if it was not hindered by meeting the Globe A falling upon it, to the Height ed from which the Globe B descends. That is, as $\frac{Aq}{Bq}de$ to de, or as Aq to Bq, or a^6 to b^6, so that first Height to x, if you put x for the latter Height ed. Therefore this Height, viz. to which B would ascend, if it was not hindered, is $\frac{a^6}{b^6}x$. Let that be fK. To fK add fg, or $dH - de - ef - gH$; that is, $p - x$, if for the given $dH - ef - gh$ you write p, and x for the unknown de; and you will have $Kg = \frac{a^6}{b^6}x + p - x$. Whence the Celerity of the Globe B,

when

when it falls from K to the Ground, that is, when it falls through the Space Kg, which its Center would describe in falling, will be as $\sqrt{\frac{a^6}{b^6}x+p}-x$. But that Globe falls from the Place Bcf to the Ground in the same Time that the upper Globe A ascends from the Place Ace to its greatest Height d, or on the other Hand falls from d to the Place Ace; and therefore since the Celerities of falling Bodies are equally augmented in equal Times, the Celerity of the Globe B, by falling to the Ground, will be augmented as much as is the whole Celerity which the Globe A acquires by falling in the same Time from d to e, or loses by ascending from e to d. Therefore, to the Celerity which the Globe B has in the Place Bcf, add the Celerity which the Globe A has in the Place Ace, and the Sum, which is as $\sqrt{de}+\frac{a^3\sqrt{de}}{b^3}$, or $\sqrt{x}+\frac{a^3}{b^3}\sqrt{x}$, will be the Celerity of the Globe B when it falls on the Ground. Therefore $\sqrt{x}+\frac{a^3}{b^3}\sqrt{x}$ will be equal to $\sqrt{\frac{a^6}{b^6}x+p}-x$. For $\frac{a^3+b^3}{b^3}$ write $\frac{r}{s}$, and for $\frac{a^6-b^6}{b^6}$, $\frac{rt}{ss}$, and that Equation will become $\frac{r}{s}\sqrt{x}=\sqrt{\frac{rt}{ss}x+p}$, and the Parts being squared, $\frac{rr}{ss}x=\frac{rt}{ss}x+p$. Subtract from both Sides $\frac{rt}{ss}x$, multiply all into ss, and divide by $rr-rt$, and there will arise $x=\frac{ssp}{rr-rt}$. Which Equation would have come out more simple, if I had taken $\frac{p}{s}$ for $\frac{a^3+b^3}{b^3}$, for there would have come out

$\frac{ss}{p-s}$

$\frac{ss}{p-t} = x$. Whence making, that $p - t$ shall be to s as s to x, you will have x, or ed; to which if you add ec, you will have dc; and the Point c, in which the Globes shall fall upon one another. Q. E. F.

PROBLEM LV.

Three Staves being erected, or set up an End, in some certain Part of the Earth perpendicular to the Plane of the Horizon, in the Points A, B, *and* C, *whereof that which is in* A *is six Feet long, that in* B *eighteen, and that in* C *eight, the Line* AB *being thirty Feet long; it happens on a certain Day in the Year that the End of the Shadow of the Staff* A *passes through the Points* B *and* C, *and of the Staff* B *through* A *and* C, *and of the Staff* C *through the Point* A. *To find the Sun's Declination, and the Elevation of the Pole, or the Day and Place where this shall happen.* [See Fig. 61.]

Because the Shadow of each Staff describes a conic Section, or the Section of a luminous Cone, whose Vertex is the Top of the Staff; I will feign BCDEF to be such a Curve, (whether it be an Hyperbola, Parabola, or Ellipse) as the Shadow of the Staff A describes that Day, by putting AD, AE, AF, to have been its Shadows, when BC, BA, CA, were respectively the Shadows of the Staves B and C. And besides I will suppose PAQ to be the meridional Line, or the Axis of this Curve, to which the Perpendiculars BM, CH, DK, EN, and FL, being let fall, are Ordinates. And I will denote these Ordinates indefinitely by the Letter y, and the intercepted Parts of the Axis AM, AH, AK, AN, and AL, by the Letter x. I will suppose, lastly, the Equation $aa + bx + cxx = yy$, to express the Relation of x and y, (*i. e.* the Nature of the Curve) assuming aa, b, and c, as known Quantities, as they will be found to be from the Analysis. Where I made the unknown Quantities of two Dimensions only, because the Equation is to express a conic Section: And I omitted

omitted the odd Dimensions of y, because it is an Ordinate to the Axis. And I denoted the Signs of b and c, as being indeterminate by the Note $\perp$, which I use indifferently for $+$ or $-$, and its opposite $\top$ for the contrary. But I made the Sign of the Square aa affirmative, because the concave Part of the Curve necessarily contains the Staff A, projecting its Shadows to the opposite Parts (C and F, D and E); and therefore if at the Point A you erect the Perpendicular Aβ, this will some where meet the Curve, suppose in β, that is, the Ordinate y, where x is nothing, will still be real. From thence it follows that its Square, which in that Case is aa, will be affirmative.

It is manifest therefore, that this fictitious Equation $aa \perp bx \perp cxx = yy$, as it is not filled with superfluous Terms, so neither is it more restrained than what is capable of satisfying all the Conditions of this Problem, and will denote the Hyperbola, Ellipse, or Parabola, according as the Values of aa, b, c, shall be determined, or perhaps found to be nothing. But what may be their Values, and with what Signs b and c are to be affected, and thence what Sort of a Curve this may be, will be manifest from the following Analysis.

The former Part of the Analysis.

Since the Shadows are as the Altitudes of the Staves, you will have BC : AD :: AB : AE (:: 18 : 6) :: 3 : 1. Also CA : AF (:: 8 : 6) :: 4 : 3. Wherefore naming AM $= r$, MB $= s$, AH $= t$, and HC $= \perp v$. From the Similitude of the Triangles AMB, ANE, and AHC, ALF, AN will be $= -\frac{r}{3}$. NE $= \mp\frac{s}{3}$. AL $= -\frac{3t}{4}$, and LF $= \mp\frac{3v}{4}$ (a); whose Signs I put

(a) Problem LV. AB : AM $= r$ :: AE : AN :: 3 : r :: 1 : AN, whence AN $= \frac{r}{3}$. AB : MB $= s$ :: AE put contrary to the Signs of AM, MB, AH, HC, because they tend contrary Ways with respect to the Point A from which they are drawn, or to the Axis PQ on which they stand. Now these being respectively written for x and y in the fictitious Equation $aa + bx + cxx = yy$.

r and s will give $aa + br + crr = ss$.
$-\frac{r}{3}$ and $-\frac{s}{3}$ will give $aa - \frac{br}{3} + \frac{1}{9}crr = \frac{1}{9}ss$.
t and $+v$ will give $aa + bt + ctt = vv$.
$-\frac{3}{4}t$ and $-\frac{3}{4}v$ will give $aa - \frac{3}{4}bt + \frac{9}{16}ctt = \frac{9}{16}vv$.

Now, by exterminating ss from the first and second Equations, in order to obtain r, there comes out $\frac{2aa}{+b} = r$ (*b*). Whence it is manifest, that $+b$ is affirmative. Also by exterminating vv from the third and fourth, to obtain t, there comes out $\frac{aa}{3b} = t$ (*c*). And having

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:: AE : NE :: 3 : s :: 1 : NE, whence NE $= \frac{s}{3}$. CA : AH $= t$ :: FA : AL :: 4 : t :: 3 : AL, whence AL $= \frac{3t}{4}$. AC : HC $= v$ :: FA : FL :: 4 : v :: 3 : FL, whence FL $= \frac{3v}{4}$.

(*b*) For $+br = 8aa - 3br$, i. e. $+4br - 8aa$, whence $r = \frac{2aa}{+b}$.

(*c*) For $9aa + 9bt = 16aa - 12bt$, whence $+21bt = 7aa$, i. e. $+3bt = aa$, or $t = \frac{aa}{+3b}$.

having writ $\frac{2aa}{b}$ for r in the first, and $\frac{aa}{3b}$ for t in the third, there arise $3aa + \frac{4a^4c}{bb} = ss$ (d), and $\frac{4}{3}aa + \frac{a^4c}{9bb} = vv$ (e).

Moreover, having let fall Bλ perpendicular upon CH, BC will be : AD (:: 3 : 1) :: Bλ : AK :: Cλ : DK. Wherefore, since Bλ is ($=$ AM $-$ AH $= r - t$) $= \frac{5aa}{3b}$ (f), AK will be $= \frac{5aa}{9b}$ (g), or rather $= -\frac{5aa}{9b}$ (h). Also since Cλ is ($=$ CH $+$ BM $= v + s$) $= \sqrt{\frac{4aa}{3} + \frac{a^4c}{9bb}} + \sqrt{3aa + \frac{4a^4c}{bb}}$, it will be DK ($= \frac{1}{3}$Cλ) $= \sqrt{\frac{4aa}{27} + \frac{a^4c}{81bb}} + \sqrt{\frac{1}{3}aa + \frac{4a^4c}{9bb}}$. Which being respectively written in the Equation $aa + bx + cxx = yy$, for AK and DK, or x and y, there

(d) By substituting $\frac{2aa}{+b}$ for r, $aa + br + crr = ss$ becomes $aa + \frac{2baa}{b} + \frac{4a^4c}{bb}$, i.e. $3aa + \frac{4a^4c}{bb} = ss$.

(e) But substituting $\frac{aa}{3b}$ for t, $aa + bt + ctt = vv$ becomes $aa + \frac{baa}{3b} + \frac{a^4c}{9bb}$, i.e. $\frac{4}{3}aa + \frac{a^4c}{9bb} = vv$.

(f) XXV. 1. (g) 147.

(h) As tending the contrary Way.

there comes out $\frac{4aa}{9} \perp \frac{25a^4c}{81bb} = \frac{13}{27}aa \perp \frac{37a^4c}{81bb}$ $\perp 2\sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \times \sqrt{\frac{aa}{3} \perp \frac{4a^4c}{9bb}}$ (i).

And by Reduction $-bb \mp 4aac = \perp 2\sqrt{36b^4 \perp 51aabbc + 4a^4cc}$; and the Parts being squared, and again reduced, there comes out $0 = 143b^4 \perp 196aabbc$, or $\frac{-143bb}{196aa} = \perp c$ (k).

Whence

(i) By substituting $-\frac{5aa}{9}$ for x, the Member $aa + bx \perp cxx$ becomes $aa - \frac{5aab}{9b} \perp \frac{25a^4c}{81bb}$ i. e. $\frac{4}{9}aa \perp \frac{25a^4c}{81bb}$; and by substituting $\sqrt{\frac{4}{27}aa \perp \frac{a^4c}{81bb}} \perp \sqrt{\frac{1}{3}aa \perp \frac{4a^4c}{9bb}}$ for y, the Member yy becomes $\left(\frac{4}{27}aa + \frac{9}{27}aa =\right) \frac{13}{27}aa \left(\perp \frac{a^4c}{81bb} + \frac{36a^4c}{81bb} =\right) \perp \frac{37a^4c}{81bb} \perp 2\sqrt{\frac{4aa}{27} \perp \frac{a^4c}{81bb}} \times \sqrt{\frac{aa}{3} \perp \frac{4a^4c}{9bb}}$, or $\frac{13}{27}aa \perp \frac{37a^4c}{81bb} \perp 2\sqrt{\frac{4a^4}{81} \perp \frac{17a^6c}{243bb} \perp \frac{4a^8c^2}{729bb}}$.

(k) By multiplying the above-mentioned Expression of the Equation of the Curve by $81bb = \sqrt{729b^4}$, and by dividing it by $a^2 = \sqrt{a^4}$, it becomes $36b^2 \perp 25a^2c = 39b^2 \perp 37a^2c \perp 2\sqrt{36b^4 \perp 51a^2b^2c \perp 4a^4c^2}$, i. e. by

Whence it is manifest, that $\perp c$ is negative, and consequently the fictitious Equation $aa \perp bx \perp cxx = yy$ will be of this Form, $aa + bx - cxx = yy$. And therefore the Curve, which it denotes, is an Ellipsis; whose Center and two Axis are thus found.

Making $y = 0$, as happens in the Vertex's of the Figure P and Q, you will have $aa + bx = cxx$, and having extracted the Root x, $= \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$ $= \left\{ \begin{array}{c} AQ \\ AP \end{array} \right\}$ (*l*).

And consequently, taking $AV = \frac{b}{2c}$, V will be the Center of the Ellipse, and VQ, or VP $\left(\sqrt{\frac{bb}{4cc} + \frac{aa}{c}}\right)$ the greatest Semi-Axis. If, moreover, the Value of AV, or $\frac{b}{2c}$, be put for x in the Equation $aa + bx - cxx = yy$, there will come out $aa + \frac{bb}{4c} = yy$. Wherefore

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i. e. by Reduction and Transposition, $-b^2 \mp 4a^2c = \perp \sqrt{36b^4 \perp 51a^2b^2c \perp 4a^4c^2}$, this squared becomes $b^4 \mp 8a^2b^2c - 16a^4c^2 = 144b^4 \perp 20a^2b^2c - 16a^4c^2$, and by Reduction and Transposition, $0 = 143b^4 \perp 196a^2b^2c$, or, $\perp 196a^2b^2c = -143b$, whence $\perp c = \frac{-143b^2}{196a^2}$.

(*l*) Because $y = 0$, $aa + bx - cxx = 0$, whence $xx - \frac{b}{c}x = \frac{aa}{c}$, whence $x = \frac{b}{2c} \pm \sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$.

fore $aa + \frac{bb}{4c}$ will be $= VZq$, that is, to the Square of the least Semi-Axis (*m*). Lastly, in the Values of AV, VQ, and VZ already found, writing $\frac{143\,bb}{196\,aa}$ for c, there come out $\frac{98\,aa}{143\,b} = AV$, $\frac{112\,aa\sqrt{3}}{143\,b} = VQ$, and $\frac{8\,a\sqrt{3}}{\sqrt{143}} = VZ$ (*n*).

(*m*) By substituting $\frac{b}{2c}$ for x, $aa + bx - cxx = yy$ becomes $aa + \frac{bb}{2c} - \frac{bb}{4c}$, i.e. $aa + \frac{bb}{4c} = yy$.

(*n*) By substituting $\frac{143\,bb}{196\,aa}$ for c, $\frac{b}{2c}$ becomes $\frac{98\,aab}{143\,bb} = \frac{98\,aa}{143\,b}$: Also $\sqrt{\frac{bb}{4cc} + \frac{aa}{c}}$ becomes $\frac{b}{2c} + \sqrt{\frac{aa}{c}} = \frac{98\,aa}{143\,b} + \sqrt{\frac{196\,a^4}{143\,b^2}} = \frac{98\,aa}{143\,b} + \frac{14\,aa}{b\sqrt{143}} = \frac{112\,aa \times 143\sqrt{143}}{b\,143\sqrt{143}} = \frac{112\,aa \times \sqrt{143} \times \sqrt{143} \times \sqrt{143}}{143\,b \times \sqrt{143}} = \frac{112\,aa\sqrt{3}}{143\,b}$: Also $\sqrt{aa + \frac{bb}{4c}}$ becomes $\sqrt{aa + \frac{49\,aa}{143}} = \sqrt{\frac{143\,aa + 49\,aa}{143}} = \sqrt{\frac{192\,aa}{143}} = \sqrt{\frac{64\,aa \times 3}{143}} = \frac{8\,a\sqrt{3}}{\sqrt{143}}$.

The other Part of the Analysis. [See Fig. 62.]

Suppose now the Staff AR ſtanding on the Point A, and RPQ will be the Meridional Plane, and RPZQ the luminous Cone whoſe Vertex is R. Let moreover TXZ be a Plane cutting the Horizon in VZ, and the Meridional Plane in TVX, which Section let be perpendicular to the Axis of the World, or of the Cone, and the Plane TXZ will be perpendicular to the ſame Axis, and will cut the Cone in the Periphery of the Circle TZX, which will be every where at an equal Diſtance, as RX, RZ, RT, from its Vertex. Wherefore, if PS be drawn parallel to TX, you will have RS = RP, by reaſon of the equal Quantities RX, RT; and alſo SX = XQ, by reaſon of the equal Quantities PV, VQ; whence RX or RZ = $\left(\frac{RS + RQ}{2}\right)$ is = $\frac{RP + RQ}{2}$. Laſtly, draw RV, and ſince VZ perpendicularly ſtands on the Plane RPQ, (as being the Section of the Planes perpendicularly ſtanding on the ſame Plane) the Triangle RVZ will be right-angled at V.

Now making RA = d, AV = e, VP or VQ = f, and VZ = g, you will have AP = $f - e$, and RP = $\sqrt{ff - 2ef + ee + dd}$. Alſo AQ = $f + e$, and RQ = $\sqrt{ff + 2ef + ee + dd}$; and conſequently RZ $\left(= \frac{RP + RQ}{2}\right)$ = $\frac{\sqrt{ff - 2ef + ee + dd} + \sqrt{ff + 2ef + ee + dd}}{2}$.

Whoſe Square $\frac{dd + ee + ff}{2}$ + $\frac{1}{2}\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4}$ is

 equal

equal (RVq + VZq = RAq + AVq + VZq =) $dd + ee + gg$. Now having reduced, it is $\sqrt{f^4 - 2eeff + e^4 + 2ddff + 2ddee + d^4} = dd + ee - ff + 2gg$, and the Parts being squared and reduced into Order, $ddff = ddgg + eegg - ffgg + g^4$, or $\frac{ddff}{gg} = dd + ee - ff + gg$. Lastly, 6, $\frac{98aa}{143b}$, $\frac{112aa\sqrt{3}}{143b}$, and $\frac{8a\sqrt{3}}{\sqrt{143}}$ (the Values of AR, AV, VQ, and VZ) being restored for d, e, f, and g, there arises $36 - \frac{196a^4}{143bb} + \frac{192aa}{143} = \frac{36 \times 14 \times 14aa}{143bb}$, and thence by Reduction $\frac{49a^4 + 36 \times 49aa}{48aa + 1287} = bb$ (o).

In

(o) By substituting 6 for d, $\frac{98aa}{143b}$ for e, $\frac{112aa\sqrt{3}}{143b}$ for f, and $\frac{8a\sqrt{3}}{\sqrt{143}}$ for g; $\frac{ddff}{gg} = dd + ee - ff + gg$ becomes $36 \left(+ \overline{\frac{98aa}{143b}}^2 - \overline{\frac{98aa}{143b} + \frac{14aa}{b\sqrt{143}}}^2 \right.$ i.e. $\left. \frac{-14aa}{b\sqrt{143}} \times \frac{14aa}{b\sqrt{143}} \text{ i.e.} \right) - \frac{196a^4}{143bb} \left(+ \frac{64aa \times 3}{143} \text{ i.e.} \right) + \frac{192aa}{143} = \left(\frac{36 \times 14 \times 14aa}{b\sqrt{143} \times b\sqrt{143}} \text{ i.e.} \right) \frac{36 \times 14 \times 14aa}{143bb}$, which by multiplying by $143bb$, and transposing, is $36 \times 143bb + 192aabb = 196a^4 + 36 \times 14 \times 14aa$; whence, In the first Scheme $AMq + MBq$ is $= ABq$, that is, $rr + ss = 33 \times 33$. But r was $= \frac{2aa}{b}$, and $ss = 3aa - \frac{4a^4c}{bb}$, whence $rr = \frac{4a^4}{bb}$, and (substituting $\frac{143bb}{196aa}$ for c) $ss = \frac{4aa}{49}$. Wherefore $\frac{4a^4}{bb} + \frac{4aa}{49} = 33 \times 33$, and thence by Reduction there again results $\frac{4 \times 49a^4}{53361 - 4aa} = bb$ (p). Putting therefore an Equality between the two Values of bb, and dividing each Part of the Equation by 49, you will have

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whence, $bb = \frac{196a^4 + 36 \times 14 \times 14aa}{36 \times 143 + 192aa} = \frac{196a^4 + 36 \times 196aa}{192aa + 5148}$ (i.e. dividing by 4) $= \frac{49a^4 + 36 \times 49aa}{48aa + 1287}$.

(p) $rr + ss = \frac{4a^4}{bb} + 3aa - \frac{4a^4c}{bb}$ (i.e. $\frac{4a^4}{bb} + 3aa - \overline{\frac{4a^4}{bb} \times \frac{143bb}{196aa}}$ i.e. $\frac{4a^4}{bb} + \frac{588a^2 - 572a^2}{196}$ i.e. $\frac{4a^4}{bb} + \frac{16aa}{196}$ i.e.) $= \frac{4a^4}{bb} + \frac{4aa}{49} = 33 \times 33$, and multiplying by $49bb$, and transposing, $\overline{49 \times 1089 - 4a^2} \times b^2 = 49 \times 4a^4$; whence $bb = \frac{49 \times 4a^4}{49 \times 1089 - 4aa} = \frac{4 \times 49a^4}{53361 - 4aa}$.

have $\frac{a^4 + 36aa}{48aa + 1287} = \frac{4a^4}{53361 - 4aa}$ (q); whose Parts being multiplied cross-ways, ordered and divided by 49, there comes out $4a^4 = 981aa + 39204$, whose Root aa is $\frac{981 + \sqrt{1589625}}{8} = 280,2254144$ (r).

Above was found $\frac{4 \times 49a^4}{53361 - 4aa} = bb$, or $\frac{14aa}{\sqrt{53361 - 4aa}} = b$. Whence AV $\left(\frac{98aa}{143b}\right)$ is $\frac{7\sqrt{53361 - 4aa}}{143}$, and VP or VQ $\left(\frac{112aa\sqrt{3}}{143b}\right)$ is $\frac{8}{143}\sqrt{160083 - 12aa}$ (s).

That

(q) $\frac{49a^4 + 36 \times 49aa}{48aa + 1287} = \frac{4 \times 49a^4}{53361 - 4aa}$, and dividing by 49, $\frac{a^4 + 36aa}{1287 + 48aa} = \frac{4a^4}{53361 - 4aa}$,

(r) $a^4 - \frac{981aa}{4} = 9801$, whence $a^4 - \frac{981aa}{4} + \frac{962361}{64}$ $\left(= \frac{962361}{64} + 9801\right) = \frac{1589625}{64}$, whence $a^2 = \frac{981 + \sqrt{1589625}}{8} = \frac{981 + 1260.8033149}{8} =$ 280.2254144.

(s) $bb = \frac{4 \times 49a^4}{53361 - 4aa} = \frac{14aa \times 14aa}{53361 - 4aa}$ whence $b = \frac{14aa}{\sqrt{53361 - 4aa}}$. AV $= \frac{98aa}{143b} =$

98 aa ×

That is, by substituting 280,2254144 for aa, and reducing the Terms into Decimals, AV = 11,188297, and YP or VQ = 22,147085; and consequently AP (PY − AV) = 10,958788, and AQ (AV + VQ) 33,335382 (*t*).

Lastly, if ½ AR or 1 be made Radius, ½ AQ or 5,555897 will be the Tangent of the Angle ARQ of 79 gr. 47′. 48″. and ½ AP or 1,826465 the Tangent of the Angle ARP of 61 gr. 17′. 57″. Half the Sum of which Angles 70 gr. 32′. 52″. is the Complement of the Sun's Declination; and the Semi-difference 9 gr. 14′. 56″. the Complement of the Latitude of the Place. Therefore, the Sun's Declination was 19 gr. 27′. 10″. and the Latitude of the Place 80 gr. 45′. 4″. which were to be found.

$$\frac{98\,aa \times \sqrt{53361 - 4aa}}{143 \times 14\,aa} = \frac{7\sqrt{53361 - 4aa}}{143}.\ \text{VQ} = \frac{112\,aa\sqrt{3}}{143\,b} = \frac{112\,aa\sqrt{3} \times \sqrt{53361 - 4aa}}{143 \times 14\,aa} = \frac{8\sqrt{3} \times \sqrt{53361 - 4aa}}{143} = \frac{8}{143}\sqrt{160083 - 12aa}.$$

$$(t)\ \text{AV} = \frac{7\sqrt{53361 - 4 \times 280.2254144}}{143} = \frac{7 \times \sqrt{53361 - 1120.9016576}}{143} = \frac{7 \times \sqrt{52240.9016576}}{143} = \frac{7 \times 228.582585}{143} = \frac{1600.078095}{143} = 11.188297.$$

$$\text{VQ} = \frac{8\sqrt{160083 - 12 \times 280.2254144}}{143} = \frac{8 \times 395.880404}{143} = \frac{3166.043232}{143} = 22.147085.$$

PROBLEM

PROBLEM LVI.

From the Observation of four Places of a Comet, moving with an uniform right-lined Motion through the Heaven, to determine its Distance from the Earth, and Direction and Velocity of its Motion, according to the Copernican *Hypothesis.* [See Fig. 73.]

If from the Center of the Comet in the four Places observed, you let fall so many Perpendiculars to the Plane of the Ecliptick; and A, B, C, D, be the Points in that Plane on which the Perpendiculars fall; through those Points draw the right Line AD, and this will be cut by the Perpendiculars in the same Ratio with the Line which the Comet describes by its Motion; that is, so that AB shall be to AC, as the Time between the first and second Observation to the Time between the first and third; and AB to AD, as the Time between the first and second to the Time between the first and fourth. From the Observations, therefore, there are given the Proportions of the Lines AB, AC, AD, to one another.

Moreover, let the Sun S be in the same Plane of the Ecliptick, and EH an Arch of the ecliptical Line in which the Earth moves; E, F, G, H, four Places of the Earth at the Times of the Observations, E the first Place, F the second, G the third, H the fourth. Join AE, BF, CG, DH, and let them be produced until the three latter cut the former in I, K, and L, *viz.* BF in I, CG in K, DH in L. And the Angle, AIB, AKC, ALD will be the Differences of the observed Longitudes of the Comet; AIB the Difference of the Longitudes of the first and second Place of the Comet; AKC the Difference of the Longitudes of the first and third Place, and ALD the Difference of the Longitudes of the first and fourth Place. There are given therefore from the Observations the Angles AIB, AKC, ALD.

Join

Join S E, S F, E F; and by reason of the given Points S, E, F, and the given Angle E S F, there will be given the Angle S E F. There is given also the Angle S E A, as being the Difference of Longitude of the Comet and Sun in the Time of the first Observation. Wherefore, if you add its Complement to two right Angles, viz. the Angle S E I to the Angle S E F, there will be given the Angle I E F. Therefore there are given the Angles of the Triangle I E F, together with the Side E F, and consequently there is given the Side I F. And by a like Argument there are given K E and L E. There are given therefore in Position the four Lines A I, B I, C K, D L, and consequently the Problem comes to this, that four Lines being given in Position, we may find a fifth, which shall be cut by these four in a given Ratio.

Having let fall to A I the Perpendiculars B M, C N, D O, by reason of the given Angle A I B there is given the Ratio of B M to M I. But B M to C N is in the given Ratio of B A and C A, and by reason of the given Angle C K N there is given the Ratio of C N to K N. Wherefore, there is also given the Ratio of B M to K N; and thence also the Ratio of B M to M I — K N, that is, to M N + I K. Take P to I K, as is A B to B C, and since M A is to M N in the same Ratio, P + M A will be to I K + M N in the same Ratio, that is, in a given Ratio. Wherefore, there is given the Ratio of B M to P + M A. And by a like Argument, if Q be taken to I L in the Ratio of A B to B D, there will be given the Ratio of B M to Q + M A. And therefore the Ratio of B M to the Difference of P + M A and Q + M A will be also given. But that Difference, viz. P — Q, or Q — P, is given; and therefore there will be given B M. But B M being given, there are also given P + M A and M I, and thence, M A, M E, A E, and the Angle E A B.

These being found, erect at A a Line perpendicular to the Plane of the Ecliptick, which shall be to the Line E A as the Tangent of the Comet's Latitude in the first Observation

Observation to Radius, and the End of that Perpendicular will be the Place of the Comet's Center in the first Observation. Whence the Distance of the Comet from the Earth is given in the Time of that Observation. And after the same Manner, if from the Point B, you erect a Perpendicular which shall be to the Line BF, as the Tangent of the Comet's Latitude in the second Observation to Radius, you will have the Place of the Comet's Center in that second Observation. And a Line drawn from the first Place to the second, is that in which the Comet moves through the Heaven.

PROBLEM LVII.

If the given Angle CAD *move about the angular Point* A *given in Position, and the given Angle* CBD *about the angular Point* B *given also in Position, on this Condition, that the Legs* AD, BD, *shall always cut one another in the right Line* EF *given likewise in Position; to find the Curve, which the Intersection* C *of the other Legs* AC, BC, *describes.* [See Fig. 74.]

Produce CA to d, so that Ad shall be $=$ AD, and produce CB to δ, so that Bδ shall be $=$ to BD. Make the Angle Ade equal to the Angle ADE, and the Angle Bδf equal to the Angle BDF, and produce AB on both Sides until it meet de and δf in e and f. Produce also ed to G, that dG shall be $= \delta f$; and from the Point C to the Line AB draw CH parallel to ed, and CK parallel to $f\delta$. And conceiving the Lines eG, $f\delta$, to remain immoveable while the Angles CAD, CBD, move by the aforesaid Law about the Poles A and B, Gd will always be equal to $f\delta$, and the Triangle CHK will be given in Specie. Make therefore A$e = a$, eG $= b$, B$f = c$, AB $= m$, BK $= x$, and CK $= y$. And BK will be : CK :: Bf : $f\delta$. Therefore $f\delta = \frac{cy}{x} =$ Gd. Take this from Ge, and there

will

will remain $ed = b - \frac{cy}{x}$. Since the Triangle CKH is given in Specie, make CK : CH :: $d : e$, and CH : HK :: $e : f$, and CH will be $= \frac{ey}{d}$, and HK $= \frac{fy}{d}$. And consequently AH $= m - x - \frac{fy}{d}$. But AH : HC :: Ae : ed, that is, $m - x - \frac{f}{d}y : \frac{ey}{d} :: a : b - \frac{cy}{x}$. Therefore, by multiplying the Means and Extreams together, there will be made $mb - \frac{mcy}{x} - bx + cy - \frac{bf}{d}y + \frac{cfyy}{dx} = \frac{aey}{d}$. Multiply all the Terms by dx, and reduce them into Order, and there will come out

$$fcyy \begin{matrix} +dc \\ -ae \\ -fb \end{matrix} xy - dcmy - bdxx + bdmx = 0.$$

Where, since the unknown Quantities x and y ascend only to two Dimensions, it is evident, that the Curve Line that the Point C describes is a Conick Section. Make $\frac{ae + fb - dc}{c} = 2p$, and there will come out $yy = \frac{2p}{f}xy + \frac{dm}{f}y + \frac{bd}{fc}xx - \frac{bdm}{fc}x$. And the Square Root being extracted, $y = \frac{p}{f}x + \frac{dm}{2f} \pm \sqrt{\frac{pp}{ff}xx + \frac{bd}{fc}xx + \frac{pdm}{ff}x - \frac{bdm}{fc}x + \frac{ddmm}{4ff}}$. Whence we infer, that the Curve is an Hyperbola, if $\frac{bd}{fc}$ be affirmative, or negative and less than $\frac{pp}{ff}$; and

and a Parabola, if $\frac{bd}{fc}$ be negative and equal to $\frac{pp}{ff}$; an Ellipse or a Circle, if $\frac{bd}{fc}$ be both negative and greater than $\frac{pp}{ff}$. Q. E. I.

PROBLEM LVIII.

To describe a Parabola which shall pass through four Points given. [See Fig. 75.]

Let those given Points be A, B, C, D. Join AB, and bisect it in E. And through E draw VE a right Line, which conceive to be the Diameter of a Parabola, the Point V being its Vertex. Join AC, and draw DG parallel to AB, and meeting AC in G. Make $AB = a$, $AC = b$, $AG = c$, $GD = d$. Upon AC take AP of any Length, and from P draw PQ parallel to AB, and conceiving Q to be a Point of the Parabola; make $AP = x$, $PQ = y$. And take any Equation expressive of a Parabola, which may determine the Relation between AP and PQ. As that y is $= e + fx \pm \sqrt{gg + hx}$.

Now if AP or x be put $= 0$, the Point P falling upon A, PQ or y will be $= 0$, as also $= -AB$. And by writing in the assumed Equation 0 for x, you will have $y = e \pm \sqrt{gg}$; that is, $= e \pm g$. The greater of which Values of y, $e + g$ is $= 0$, the lesser $e - g = -AB$, or to $-a$. Therefore $e = -g$, and $e - g$; that is, $-2g = -a$, or $g = \frac{1}{2}a$. And so in room of the assumed Equation you will have this $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + hx}$ (a).

Moreover,

Problem LVIII. (a) When $x = 0$, $y = e + g$, or $y = e - g$, by the assumed Equation; and $y = 0$, and $y = -AB$

Moreover, if AP or x be made $=$ AC, so that the Point P falls upon C, you will have again PQ $= 0$. For x therefore in the last Equation write AC or b, and for y write 0; and you will have $0 = -\frac{1}{2}a + fb + \sqrt{\frac{1}{4}aa + bb}$, or $\frac{1}{2}a - fb = \sqrt{\frac{1}{4}aa + bb}$; and the Parts being squared $-afb + ffbb = bb$, or $ffb - fa = b$. And so, in room of the assumed Equation, there will be had this, $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$.

Moreover, if AP or x be made $=$ AG or c, PQ or y will be $= -$ GD or $-d$. Wherefore, for x and y in the last Equation write c and $-d$, and you will have $-d = -\frac{1}{2}a + fc - \sqrt{\frac{1}{4}aa + ffbc - fac}$, or $\frac{1}{2}a - d - fc = \sqrt{\frac{1}{4}aa + ffbc - fac}$. And the Parts being squared, $-ad - fac + dd + 2dcf + ccff = ffbc - fac$. And the Equation being ordered and reduced, $ff = \frac{2d}{b-c}f + \frac{dd-ad}{bc-cc}$. For $b-c$, that is, for GC write k, and that Equation will become $ff = \frac{2d}{k}f + \frac{dd-ad}{kc}$. And the Root being extracted,

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$y = -$ AB $= -a$; from the Figure; whence putting $e + g = 0$, then $e - g = -a$. From $e + g = 0$, we have $e = -g$; whence, by Substitution, $-2g = -a$; and $g = \frac{1}{2}a$; and putting $e - g = 0$, then $e + g = -a$. From $e - g = 0$, we have $e = g$; whence $2g = -a$, and $g = -\frac{1}{2}a$; whence, by Substitution in the assumed Equation, we have $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}a^2 + bx}$ in the first Supposition, and $y = -\frac{1}{2}a + fx \pm \sqrt{-\frac{1}{2}x^2 + bx}$.

extracted, $f = \frac{d}{k} \pm \sqrt{\frac{ddc + ddk - adk}{kkc}}$. But f being found, the parabolick Equation, viz. $y = -\frac{1}{2}a + fx \pm \sqrt{\frac{1}{4}aa + ffbx - fax}$ will be fully determined; by whose Construction therefore the Parabola will also be determined. The Construction is thus: Draw CH parallel to BD, meeting DG in H. Between DG and DH take a mean Proportional DK, and draw EI parallel to CK, bisecting AB in E, and meeting DG in I. Then produce IE to V, so that EV shall be to EI :: EBq : DIq — EBq, and V will be the Vertex (*b*), VE the Diameter, and $\frac{BEq}{VE}$ the *Latus Rectum* of the Parabola sought (*c*).

Problem LIX.

To describe a conic Section through five Points given.

[See Fig. 76.]

Let those Points be A, B, C, D, E. Join AC, BE, cutting one another in H. Draw DI parallel to BE, and meeting AC in I. As also EK parallel to AC, and meeting DI produced in K. Produce ID to F, and EK to G; so that AHC shall be : BHE :: AIC : FID :: EKG : FKD, and the Points F and G will be in a conick Section, as is known.

But you ought to observe this, if the Point H falls between all the Points A, C, and B E, or without them all, the Point I must either fall between all the Points A, C, and F, D, or without them all; and the Point K be-

(*b*) Because VI : VE :: DI^2 : BE^2.
(*c*) Because BEq = VE × *Latus Rectum*.

K between all the Points D, F, and E, G, or without them all. But if the Point H falls between the two Points A, C, and without the other two B, E, or between those two B, E, and without the other two A, C, the Point I ought to fall between two of the Points A, C, and F, D, and without the other two of them; and in like Manner, the Point K ought to fall between two of the Points D, F, and E, G, and without Side of the two other of them; which will be done by taking IF, KG, on this or that Side of the Points I, K, according to the Exigency of the Problem. Having found the Points F and G, bisect AC and EG in N and O; also BE, FD, in L and M. Join NO, LM, cutting one another in R; and LM and NO will be the Diameters of the conick Section, R its Center, and BL, FM, Ordinates to the Diameter LM. Produce LM on both Sides, if there be Occasion, to P and Q, so that BLq shall be to FMq :: PLQ : PMQ, and P and Q will be the Vertex's of the conick Section, and PQ the *Latus Transversum*. Make PLQ : LBq :: PQ : T, and T will be the *Latus Rectum*. Which being known, the Figure is known.

It remains only that we may shew how LM is to be produced each Way to P and Q, so that BLq may be : FMq :: PLQ : PMQ, viz. PLQ, or PL × LQ, is $\overline{PR - LR} \times \overline{PR + LR}$; for PL is PR − LR, and LQ is RQ + LR, or PN + LR. Moreover, $\overline{PR - LR} \times \overline{PR + LR}$, by multiplying, becomes PRq − LRq. And after the same Manner, PMQ is $\overline{PR + RM} \times \overline{PR - RM}$, or PR$q$ − RMq. Therefore BLq : FMq :: PRq − LRq : PRQ − RMq; and by dividing, BLq − FMq : FMq :: RMq − LRq : PRq − RMq. Wherefore, since there are given BLq − FMq, FMq and RMq − LRq, there will be given PRq − RMq. Add the given Quantity RMq, and there will be given the Sum PRq, and consequently its Root PR, to which QR is equal.

Problem LX.

To describe a conick Section which shall pass through four given Points, and in one of those Points shall touch a right Line given in Position. [See Fig. 77.]

Let the four given Points be A, B, C, D, and the right Line given in Position be AE, which let the conick Section touch in the Point A. Join any two Points D, C, and let DC produced, if there be Occasion for it, meet the Tangent in E, Through the fourth Point B draw BF parallel to DC, which may meet the same Tangent in F. Also draw DI parallel to the Tangent, and which may meet BF in I. Upon FB, DI, produced if there be Occasion, take FG, HI, of such Length as that it may be AE*q* : CED :: AF*q* : BFG :: DIH : BIG. And the Points G and H will be in a conick Section, as is known; provided you take FG, IH, on the proper Sides of the Points F and I, according to the Rule delivered in the foregoing Problem. Bisect BG, DC, DH, in K, L, and M. Join KL, AM, cutting one another in O, and O will be the Center, A the Vertex, and HM an Ordinate to the Semi-diameter AO; which being known, the Figure is known.

Problem LXI.

To describe a conick Section which shall pass through three given Points, and touch right Lines given in Position in two of those Points. [See Fig. 78.]

Let those given Points be A, B, C, the Tangents AD, BD, to the Points A and B, and let D be the common Intersection of those Tangents. Bisect AB in E. Draw DE, and produce it till in F it meets CF drawn parallel to AB; and DF will be the Diameter, and AE and CF the Ordinates to that Diameter. Produce DF to O, and on DO take OV a mean Propor-tional

tional between DO and EO, on this Condition, that also $AEq : CFq :: VE \times \overline{VO + OE} : VF \times \overline{VO + OF}$; and V will be the Vertex, and O the Center of the Figure. Which being known, the Figure will also be known. But VE is $= VO - OE$, and consequently $VE \times \overline{VO + OE} = \overline{VO - OE} \times \overline{VO + OE} = VOq - OEq$. Besides, because VO is a mean Proportional between DO and EO, VOq will be $= DOE$; and consequently $VOq - OEq = DOE - OEq = DEO$. And by a like Argument you will have $VF \times \overline{VO + OF} = VOq - OFq = DOE - OFq$. Therefore $AEq : CFq :: DEO : DOE - OFq$. OFq is $= EOq - 2FEO + FEq$. And consequently $DOE - OFq = DOE - OEq + 2FEO - FEq = DEO + 2FEO - FEq$. And $AEq : CFq :: DEO : DEO + 2FEO - FEq ::$

$$DE : DE + 2FE - \frac{FEq}{EO}.$$

Therefore there is given $DE + 2FE - \frac{FEq}{EO}$. Take away from this given Quantity $DE + 2FE$, and there will remain $\frac{FEq}{EO}$ given. Call that N; and $\frac{FEq}{N}$ will be $= EO$, and consequently EO will be given. But EO being given, there is also given VO, the mean Proportional between DO and EO.

After this Way, by some of Apollonius's Theorems, these Problems are expeditiously enough solved; which yet may be solved by Algebra alone, without those Theorems. As if the first of the last three Problems be proposed: [See Fig. 78.] Let the five given Points be A, B, C, D, E, through which the conick Section is to pass. Join any two of them, A, C, and any other two, B, E, by right Lines intersecting one another in H. Draw DI parallel to BE, meeting AC in I; as also any other right Line KL meeting AC in K, and the conick Sec-

tion in L. And imagine the conick Section to be given, so that the Point K being known, there will at the same Time be known the Point L; and making AK = x, and KL = y, to express the Relation between x and y, assume any Equation which generally expresses the conick Sections; suppose this, $a + bx + cxx + dy + exy + yy = 0$. Wherein a, b, c, d, e, denote determinate Quantities with their Signs, but x and y indeterminate Quantities. Now if we can find the determinate Quantities a, b, c, d, e, the conick Section will be known. Let us therefore feign the Point L to fall successively upon the Points A, C, B, E, D, and let us see what will follow thence. If therefore the Point L falls upon the Point A, in that Case AK and KL, that is, x and y, will be 0. Then all the Terms of the Equation besides a will vanish, and there will remain $a = 0$. Wherefore a is to be blotted out in that Equation, and the other Terms $bx + cxx + dy + exy + yy$ will be $= 0$. Moreover if L falls upon C, AK, or x, will be = AC, and LK or $y = 0$. Put therefore AC = f, and by substituting f for x and 0 for y, the Equation for the Curve $bx + cxx + dy + exy + yy = 0$, will become $bf + cff = 0$, or $b = -cf$. And having writ in that Equation $-cf$ for b, it will become $-cfx + cxx + dy + exy + yy = 0$. Farther, if the Point L falls upon the Point B, AK or x will be = AH, and KL or y = BH. Put therefore AH = g, and BH = h, and then write g for x and h for y, and the Equation $-cfx + cxx$, &c. will become $-cfg + cgg + dh + egh + hh = 0$. But if the Point L falls upon E, AK will be = AH, or $x = g$, and KL or y = HE. For HE therefore write $-k$, with a negative Sign, because HE lies on the contrary Side of the Line AC, and by substituting g for x and $-k$ for y, the Equation $-cfx + cxx$, &c. will become $-cfg + cgg - dk - egk + kk = 0$. Take away this from the former Equation $-cfg + cgg + dh + egh + hh$, and there will remain $dh + egh + hh + dk + egk - kk = 0$. Divide this by $h + k$, and there will come out $d + eg$
$+ h - k$

$+ h - k = 0$. Take away this multiplied by h from $-cfg + cgg + dh + egh + hh = 0$, and there will remain $-cfg + cgg + hk = 0$, or $\frac{hk}{-gg + fg} = c$. Lastly, if the Point L falls upon the Point D, AK or x will be $=$ AI, and KL or y will be $=$ ID. Wherefore, for AI write m, and for ID, n; and likewise for x and y substitute m and n, and the Equation $-cfx + cxx$, &c. will become $-cfm + cmm + dn + emn + nn = 0$. Divide this by n, and there will come out $\frac{-cfm + cmm}{n} + d + em + n = 0$. Take away $d + eg + h - k = 0$, and there will remain $\frac{-cfm + cmm}{n} + em - eg + n - h + k = 0$, or $\frac{cmm - cfm}{n} + n - h + k = eg - em$. But now by reason of the given Points A, B, C, D, E, there are given AC, AH, AI, BH, EH, DI, that is, f, g, m, h, k, n. And consequently by the Equation $\frac{hk}{fg - gg} = c$, there is given c. But c being given by the Equation $\frac{cmm - cfm}{n} + n - h + k = eg - em$ there is given $eg - em$. Divide this given Quantity by the given one $g - m$, and there will come out the given e. Which being found, the Equation $d + eg + h - k = 0$, or $d = k - h - eg$, will give d. And these being known, there will at the same Time be determined the Equation expressive of the conick Section sought, viz. $cfx = cxx + dy + exy + yy$. And from that Equation, by the Method of Des Cartes, the conick Section will be determined.

Now if the four Points A, B, C, E, and the Position of the right Line AF, which touches the conick Section in one of those Points A were given, the conick Section may be thus more easily determined. Having found, as above, the Equations $cfx = cxx + dy + exy + yy$, $d = k - b - eg$, and $c = \frac{bk}{fg - gg}$, conceive the Tangent AF to meet the right Line EH in F, and then the Point L to be moved along the Perimeter of the Figure CDE till it fall upon the Point A; and the ultimate Ratio of LK to AK will be the Ratio of FH to AH, as will be evident to any one that contemplates the Figure. Make FH $= p$, and in this Case where LK, AK, are in a vanishing State, you will have $p : g :: y : x$, or $\frac{gy}{p} = x$. Wherefore for x, in the Equation $cfx = cxx + dy + exy + yy$, write $\frac{gy}{p}$, and there will arise $\frac{cfgy}{p} = \frac{cggyy}{pp} + dy + \frac{egyy}{p} + yy$. Divide all by y, and there will come out $\frac{cfg}{p} = \frac{cggy}{pp} + d + \frac{egy}{p} + y$. Now because the Point L is supposed to fall upon the Point A, and consequently KL, or y, to be infinitely small or nothing, blot out the Terms which are multiplied by y, and there will remain $\frac{cfg}{p} = d$. Wherefore make $\frac{bk}{fg - gg} = c$, then $\frac{cfg}{p} = d$. Lastly, $\frac{k - b - d}{g} = e$, and having found c, d, and e, the Equation $cfx = cxx + dy + exy + yy$ will determine the conick Section.

If,

If, lastly, there are only given the three Points A, B, C, together with the Position of the two right Lines, AT, CT, which touch the conick Section in two of those Points, A and C, there will be obtained, as above, this Equation expressive of a conick Section, $cfx = cxx + dy + exy + yy$ [See Fig. 80]. Then if you suppose the Ordinate KL to be parallel to the Tangent AT, and it be conceived to be produced, till it again meets the conick Section in M, and that Line LM to approach to the Tangent AT till it coincides with it at A; the ultimate Ratio of the Lines KL and KM to one another, will be a Ratio of Equality, as will appear to any one that contemplates the Figure. Wherefore in that Case KL and KM being equal to each other, that is, the two Values of y, (viz the affirmative one KL, and the negative one KM) being equal, those Terms of the Equation $cfx = cxx + dy + exy + yy$ in which y is of an odd Dimension, that is, the Terms $dy + exy$ in respect of the Term yy, wherein y is of an even Dimension, will vanish. For otherwise the two Values of y, viz. the affirmative and the negative, cannot be equal; and in that Case AK is infinitely less than LK, that is x than y, and consequently the Term exy than the Term yy. And consequently being infinitely less, may be reckoned for nothing. But the Term dy, in respect of the Term yy will not vanish as it ought to do, but will grow so much the greater; unless d be supposed to be nothing. Therefore the Term dy is to be blotted out, and so there will remain $cfx = cxx + exy + yy$, an Equation expressive of a conick Section. Conceive now the Tangents AT, CT, to meet one another in T, and the Point L to come to approach to the Point C, till it coincides with it. And the ultimate Ratio of KL to KC, will be that of AT to AC. KL was y; AK, x; and AC, f; and consequently KC, $f - x$; make AT $= g$, and the ultimate Ratio of y to $f - x$, will be the same as of g to f. The Equation $cfx = cxx + exy + yy$ subtracting on both Sides cxx, becomes $cfx - cxx = exy + yy$, that is, $\overline{f - x}$ into $cx = y$ into $\overline{ex + y}$.

There-

Therefore it is $y : f - x :: cx : ex + y$, and consequently $g : f :: cx : ex + y$. But the Point L falling upon C, y becomes nothing. Therefore $g : f :: cx : ex$. Divide the latter Ratio by x, and it will become $g : f :: c : e$, and $\frac{cf}{g} = e$. Wherefore, if in the Equation $cfx = cxx + exy + yy$, you write $\frac{cf}{g}$ for e, it will become $cfx = cxx + \frac{cf}{g}xy + yy$, an Equation expressive of a conick Section. Lastly, draw BH parallel to KL, or AT, from the given Point B, through which the conick Section ought to pass, and which shall meet AC in H, and conceiving KL to come towards BH, until it coincides with it, in that Case AH will be $= x$, and BH $= y$. Call therefore the given Quantity AH $= m$, and the given Quantity BH $= n$, and then for x and y, in the Equation $cfx = cxx + \frac{cf}{g}xy + yy$, write m and n, and there will arise $cfm = cmm + \frac{cf}{g}mn + nn$. Take away on both Sides $cmm + \frac{cf}{g}mn$, and there will come out $cfm - cmm - \frac{cf}{g}mn = nn$. Put $f - m - \frac{fn}{g} = s$, and csm will be $= nn$. Divide each Part of the Equation by sm, and there will arise $c = \frac{nn}{sm}$. But having found c, the Equation for the conick Section is determined $cfx = cxx + \frac{cf}{g}xy + yy$. And then, by the Method of Des Cartes, the conick Section is given, and may be described.

END of PART I.

UNIVERSAL

ARITHMETIC.

Of the Nature *of the* Roots *of* Equations.

HITHERTO I have been solving several Problems. For in learning the Sciences, Examples are of more Use than Precepts. Wherefore I have been larger on this Head. And some which occurred as I was putting down the rest, I have given their Solutions without using Algebra, that I might shew that in some Problems that at first Sight appear difficult, there is not always occasion for Algebra. But now it is time to shew the Solution of Equations. For after a Problem is brought to an Equation, you must extract the Roots of that Equation, which are the Quantities that satisfy the Problem.

How EQUATIONS *are to be solved.*

AFTER therefore in the Solution of a Question you are come to an Equation, and that Equation is duly reduced and ordered; when the Quantities which are denoted by Species, and which are supposed given, are really given in Numbers, those Numbers are to be substituted in their room in the Equation, and you will have a numeral Equation, whose Root being extracted will satisfy the Question. As if in the Division of an Angle into five equal Parts, by putting r for the Radius of the Circle, q for the Chord of the Complement of the proposed Angle to two right ones, and x for the Chord of the Complement of the fifth Part of that Angle, I had come to this Equation; $x^5 - 5rrx^3 + 5r^4x - r^4q = 0$. Where in any particular Case the Radius r is given in Numbers, and the Line q subtending the Complement of the given Angle; as if the Radius were 10, and the Chord 3; I substitute those Numbers in the Equation for r and q, and there comes

comes out the numeral Equation $x^5 - 500x^3 + 50000x - 30000 = 0$, whereof the Root being extracted will be x, or the Line subtending the Complement of the fifth Part of that given Angle.

Of the Nature of the Roots of an Equation.

CX. *But the Root is a Number which being substituted in the Equation for the Letter or Species signifying the Root, will make all the Terms vanish.*

Thus Unity is the Root of the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, because being writ for x it produces $1 - 1 - 19 + 49 - 30$; that is, nothing. But there may be more Roots of the same Equation. As if in this same Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, for x you write the Number 2, and for the Powers of x the like Powers of the Number 2, there will be produced $16 - 8 - 76 + 98 - 30$; that is, nothing. And so if for x you write the Number 3, or the negative Number -5, in both Cases there will be produced nothing, the affirmative and negative Terms in these four Cases destroying one another. Therefore since any of the Numbers written in the Equation fulfils the Condition of x, by making all the Terms of the Equation together equal to nothing, any of them will be the Root of the Equation (*a*). CXI.

CX. (*a*) *An Equation may be considered either absolutely, as* an Aggregate of Terms which involve the Powers of an unknown Quantity, and is equal to nothing; *or relatively, as* an Aggregate of Terms, which by *containing all the Conditions of a Problem,* contain the Powers of an unknown Quantity, and is equal to nothing. Art. LXXXII, LXXXIII, &c.

202. *In both Cases, any Quantity, which being substituted for the unknown, will make the Aggregate to vanish, is a Root.* For the whole Aggregate is equal to nothing (Art. LXV), and the known Quantities remain the same, and unchanged; also the unknown Quantity is the same, both before and after it is found; wherefore being

CXI. And that you may not wonder that the same Equation may have *several Roots*, you must know *that there may be more Solutions than one of the same Problem.*

As

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being substituted after it is found, in the Place of its Symbol, the Aggregate will still be equal to nothing; that is, will vanish. See Numb. 209.

203. It will be always possible to find a Root, that is, a Quantity which by Substitution shall cause all the Terms to vanish, if the extreme Terms of the Equation have contrary Signs; that is, because the first Term is always supposed affirmative, *every Equation whose last Term is negative, has a real Root.* For the first Term being affirmative, and the last by Transposition into the second Member becoming affirmative, two Numbers, whose Difference is less than any assigned Quantity, being successively substituted into the first Member, will give Sums coming nearer to an Equality than by any definite Difference; consequently a Number is given, which, by Substitution in the first Member, will make it of any assigned Magnitude, and therefore equal to the Quantity in the second Member; and that Quantity being again transposed, the Aggregate must vanish. See Numb. 205, 217.

204. *Every Equation of even Dimensions has two real Roots, when the last Term is negative; one of which is affirmative, and the other negative.* For the last Term becomes affirmative by Transposition, and every Power of even Dimensions of any Number whatsoever, whether affirmative or negative, is affirmative (87, 88); whence the last Term being transposed, and becoming affirmative, it can be equalled, as in Numb. 203, &c. See Numb. 217.

205. *Every Equation of odd Dimensions has at least one real Root, whether its last Term be negative or affirmative.* For every Power of odd Dimensions of any affirmative Number is affirmative, and of a negative Number, negative

As if there was sought the *Intersection* of two given Circles; there are *two* Intersections, and consequently the Question admits *two Answers*; and therefore the Equation determining the Intersection will have *two Roots*, whereby it determines both the Intersections, *provided there be nothing in the Data whereby the Answer is determined to only one Intersection*. [See Fig. 87.]

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gative (87, 88.); if the last Term is negative, the Root whose odd Power, being equal to it, causes the Aggregate to vanish, must be affirmative (203): And if it be affirmative, it becomes negative by Transposition, but it is still a real Power (88); the Root whose odd Power being equal to it, will cause the Terms to vanish, must be negative; and will be a real Root (87, 88.) See Numb. 217.

206. *If an Equation of even Dimensions has its last Term affirmative*, it may be impossible, by any Substitution in the first Member, to make the Aggregate to vanish; that is, *it may have impossible Roots*. Because it may be impossible to make the first Member so great, negatively, as to attain (the last Term being transposed, and then negative, and the Equation being of even Dimensions [88]) any given Magnitude: And consequently may always be deficient of the negative Quantity in the second Member, and this when transposed back being affirmative, the Aggregate will not vanish by any Substitution, but come out positive. See Numb. 189.

207. The Impossibility of finding a Number, which by Substitution would cause the Aggregate of the Terms of an Equation to vanish, arises from this, that there are no Roots of negative Powers of even Dimensions: But if such Roots are imagined to exist, they by Substitution would make the Aggregate to vanish; whence *every Equation has a Root, if not real, at least imaginary.*

And

And thus, if of the Arch APB its fifth Part AP were to be found, though perhaps you might apply your Thoughts only to the Arch APB, yet the Equation, whereby the Question will be solved, will determine the fifth Part of all the Arches which are terminated at the Points A and B; viz. the fifth Part of the Arches ASB, APBSAPB, ASBPASB, and APBSAPBSAPB, as well as the fifth Part of the Arch APB; which fifth Parts, if you divide the whole Circumference into five equal Parts, PQ, QR, RS, ST, TP, will be AT, AQ, ATS, AQR. Wherefore, by seeking the fifth Parts of the Arches which the right Line AB subtends, to determine all the Cases the whole Circumference ought to be divided in the five Points P, Q, R, S, T, therefore the Equation that will determine all the Cases will have five Roots. For the fifth Parts of all these Arches depend on the same Data, and are found by the same kind of Calculus; so that you will always fall upon the same Equation, whether you seek the fifth Part of the Arch APB, or the fifth Part of the Arch ASB, or the fifth Part of any other of the Arches. Whence, if the Equation by which the fifth Part of the Arch APB is determined, should not have more than one Root, while by seeking the fifth Part of the Arch ASB we fall upon that same Equation; it would follow, that this greater Arch would have the same fifth Part with the former, which is less, because its Subtense or Chord is expressed by the same Root of the Equation. *In every Problem therefore it is necessary, that the Equation which answers should have as many Roots as there are different Cases of the Quantity sought depending on the same Data, and to be determined by the same Method of Reasoning* (*b*).

CXII. *But an Equation may have as many Roots as it has Dimensions, and not more.*

CXI. (*b*) See Number 194, 210.

Thus

Thus the Equation $x^4 - x^3 - 19xx + 49x - 30$, $= 0$, has four Roots, 1, 2, 3, -5; but not more. For any of these Numbers writ in the Equation for x, will cause all the Terms to destroy one another, as we have said; but besides these, there is no Number by whose Substitution this will happen (*c*).

CXIII. *But the Number and Nature of the Roots will be best understood from the Generation of the Equation.*

As if we would know how an Equation is generated, whose Roots are 1, 2, 3, and -5; we are to suppose x to signify ambiguously those Numbers, or x to be $= 1$, $x = 2$, $x = 3$, and $x = -5$, or which is the same Thing, $x - 1 = 0$, $x - 2 = 0$, $x - 3 = 0$, and $x + 5 = 0$; and multiplying these together, there will come out by the Multiplication of $x - 1$ by $x - 2$ this Equation $xx - 3x + 2 = 0$, which is of two Dimensions, and has two Roots 1 and 2. And by the Multiplication of this by $x - 3$, there will come out $x^3 - 6xx + 11x - 6 = 0$, an Equation of three Dimensions and as many Roots; which again multiplied by $x + 5$ becomes $x^4 - x^3 - 19xx + 49x - 30 = 0$, as above. Since therefore this Equation is generated by four Factors, $x - 1$, $x - 2$, $x - 3$, and $x + 5$, continually multiplied by one another, where any of the Factors is nothing, that which is made by all will be nothing; but where none of them is nothing, that which is contained under them all cannot be nothing. That is, $x^4 - x^4 - 19xx + 49x - 30$ cannot be $= 0$, as ought to be, except in these four Cases, where $x - 1$ is $= 0$, or $x - 2 = 0$, or $x - 3 = 0$, or, lastly, $x + 5 = 0$, therefore only the Numbers 1, 2, 3, and -5 can exhibit x, or be the Roots of the Equation. *And you are to reason alike of all Equations.* For *we may imagine all to be generated by such a Multiplication,*

CXII. (*c*) See Number 210.

Multiplication, although it is usually very difficult to separate the Factors from one another, and is the same Thing as to resolve the Equation and extract its Roots. For the Roots being had, the Factors are had also (*d*).

CXIV. *But*

CXIII. (*d*) 208. *All Equations above one Dimension may be considered as generated by the continual Multiplication of Binomes, or simple Equations* (Article LXVI. Numb. 181.), *consisting of the same Letter denoting the unknown, and any other undetermined Quantity.* Thus the given Equation $x^3 + px^2 - qx + r = 0$, may be the same with the Equation $x^3 + \overline{a + b - c} \times x^2 + \overline{ab - ac - bc} \times x - abc = 0$, which is produced from the continual Multiplication of the Binomes, $x + a$, $x + b$, $x - c$, or the simple Equations $x + a = 0$, $x + b = 0$, $x - c = 0$. For the Number of Binomes is equal to the Index of the highest Term, that is, to the Number of Terms except the first; and consequently, by equating the Coefficients of the corresponding Terms of the given and produced Equations, there is an Equation for each assumed undeterminate Quantity (182, 183.): Therefore so many binome Factors may be separately determined, because each Equation has its Root (207), consisting of the Letter denoting the unknown, or Root, and some numeral or determinate Quantity, from whose Multiplication the Equation of those Dimensions might be generated. See Numb. 95, 97, 98, 99.

209. If in the Equation $x^3 + \overline{a + b - c} \times x^2 + \overline{ab - ac - bc} \times x - abc = 0$, produced by the Multiplication of $x + a = 0$, $x + b = 0$, $x - c = 0$, any Quantity y be substituted for the unknown x, the Equation $y^3 + \overline{a + b - c} \times y^2 + \overline{ab - ac - bc} \times y - abc = 0$ will emerge, which is the Product of $y + a = 0$, $y + b = 0$, $y - c = 0$; whence, *if in any Equation any of the second Members of the Factors be*

A a *substituted*

CXIV. *But the Roots are of two Sorts, affirmative, as in the Example brought,* 1, 2, *and* 3; *and negative, as* — 5. *And of these some are often impossible.*

Thus,

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substituted for x, the unknown, with the Sign changed, a Product will result, one of whose Factors will be that Member connected with itself under contrary Signs, that is, nothing; whence the Product is nothing; that is, *the Aggregate of the Terms vanishes.* Thus in the above Equation, $-a$ substituted, produces $-a^3 + a^3 + ba^2 - ca^2 - ba^2 + ca^2 + bca - bca = 0$; and $-b$, $-b^3 + b^3 + ab^2 - cb^2 - ab^2 + cb^2 + bca - bca = 0$; and $+c$, $c^3 - c^3 + bc^2 + ac^2 - ac^2 - bc^2 + abc - abc = 0$; there being a Part of the second Term equal to the first; the Remainder of the second Term equal to Part of the third; and so on; and Part of the Penultimate equal to the last: But the equal Parts have contrary Signs, whence being destroyed, the Aggregate vanishes.

210. *The second Members of the Factors*, that is, the Roots of the simple generating Equations, *and no other Quantities are the Roots of the Equation, their Signs being changed* (181): Because they alone can by Substitution, under contrary Signs, cause the Aggregate of the Terms to vanish (209). Hence *an Equation has so many Roots, and no more, as it has Dimensions*; and the Problem from the Sum of whose Conditions it was collected (194), admits so many Solutions, and no more, as the Equation has Dimensions, or, as there are Units in the Index of its highest Term. Hence also *the Letter denoting the unknown denotes each Root equally, and in like Manner*; because each Root substituted for the Letter causes the Aggregate in like Manner to vanish (209).

211. *If the Letter* x *denoting the Root is found, or any Power of it, in the last Term of an Equation, so many of the latter Terms of the Equation are wanting, as there are* Units

Thus, the two Roots of the Equation $xx - 2ax + bb = 0$, which are $a + \sqrt{aa - b}$ and $a - \sqrt{aa - bb}$, are

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Units in the Index of it (96), *and conversely*: Also putting n for the Dimensions of any Equation, and Asterisks for the wanting Terms, *its Number of Terms is* $n + 1$ (94).

212. *An Equation of any Dimensions is the Product of the Factors, into which it can by Division be resolved* (152), that is, may *be considered as the Product of simple or compound Equations, the Sum of whose Dimensions is equal to its Dimensions:* Thus, a Biquadratic may be considered as the Product, either of four simple; or of two quadratic; or lastly, of one simple, and one cubic Equation: Whence, Equations may be reduced to lower Dimensions, by being divided by their Factors. *If an Equation is divisible by a rational Divisor, it is said to be reducible; otherwise, irreducible:* And every Equation which is irreducible by a rational Divisor of Half its Dimensions, will also be irreducible by Divisors of greater Dimensions (164): And *all Equations are reducible to infinite Series* (155).

213. There are *two Characteristics of the Roots* of any Equation: 1. *They are Divisors of it* (212): 2. *They cause the Aggregate of the Terms to vanish, when they are substituted* (210).

214. *If any Equation is divisible by* $x - a + \sqrt{a}$, *or if* $a + \sqrt{a}$ *be its Root; then also shall* $x - a - \sqrt{a}$ *be a Divisor, and* $a - \sqrt{a}$ *a Root: Also if* $x - a + \sqrt{-a}$ *be a Divisor, and* $a + \sqrt{-a}$ *a Root; so also shall* $x - a - \sqrt{-a}$, *and* $a - \sqrt{-a}$ (153).

215. *If all the Terms of an Equation are rational, the irrational Factors,* (if any) *must have destroyed themselves* (120).

216. *If the Terms of an Equation are some rational, and some irrational; the Sum of the Rationals and Irrationals, are separately equal to nothing* (116).

are real, when aa is greater than bb: but when aa is less than bb, they become impossible; because then $aa - bb$ will be a negative Quantity, and the square Root of a negative Quantity is impossible. For every possible Root, whether it be affirmative or negative, if it be multiplied by itself, produces an affirmative Square; therefore that will be an impossible one, which is to produce a negative Square. By the same Argument you may conclude, that the Equation $x^3 - 4xx + 7x - 6 = 0$, has one real Root, which is 2, and two impossible ones, $1 + \surd - 2$ and $1 - \surd - 2$. For any of these, 2, $1 + \surd - 2$, $1 - \surd - 2$, being writ in the Equation for x, will make all its Terms destroy one another; but $1 + \surd - 2$, and $1 - \surd - 2$, are impossible Numbers, because they suppose the Extraction of the square Root out of the negative Number $- 2$ (e).

CXV. *But it is just, that the Roots of Equations should be often impossible, lest they should exhibit the Cases of Problems that are often impossible, as if they were possible.*

As if you were to determine the Intersection of a right Line and a Circle, and you should put two Letters for the Radius of the Circle and the Distance of the right Line from its Center; and when you have the Equation defining the Intersection, if for the Letter denoting the Distance of the right Line from the Center, you put a Number less than the Radius, the Intersection will be possible; but if it be greater, impossible; and the two Roots of the Equation, which determine the two Intersections, ought to be either possible or impossible, that they may truly express the Matter [See Fig. 88]. And thus, if the Circle CDEF, and the Ellipsis ACBF, cut

CXIV. (e) Every Quantity, whether real or imaginary, whether affirmative or negative, which being substituted for x, the Letter denoting the unknown, causes the Terms to vanish; or which, joined with it, divides the Equation, is looked upon as a Root or Factor.

cut one another in the Points C, D, E, F, and to any right Line given in Position, as A B, you let fall the Perpendicular C G, D H, E I, F K, and by seeking the Length of any one of the Perpendiculars, you come at length to an Equation; that Equation, when the Circle cuts the Ellipsis in four Points, will have four real Roots, which will be those four Perpendiculars. But if the Radius of the Circle, its Center remaining, be diminished until (the Points E and F meeting) the Circle at length touches the Ellipse; those two of the Roots, which expresss the Perpendiculars E I and F K now coinciding, will become equal. And if the Circle be yet diminished, so that it does not touch the Ellipse in the Points E, F, but only cuts it in the other two Points C D; then out of the four Roots, those two which expressed the Perpendiculars E I, F K, which are now become impossible, will become, together with those Perpendiculars, also impossible (*f*).

CXVI. *And after this Way in all Equations, by augmenting or diminishing their Terms, of the unequal Roots, two will become, first equal, and then impossible. And thence it is, that the Number of the impossible Roots is always even* (*g*.)

CXVII. *But*

CXV. (*f*) *In Algebra the Root of an impossible Quantity has its Expression; but in Geometry none. In Algebra* you obtain a general Solution, and there is an Expression in all Cases of the Thing required; only within certain Bounds, *that Expression* represents an imaginary Quantity, or rather is the Symbol of an Operation which in that Case cannot be performed; and *serves only to shew the Genesis of the Quantity, and the Limits within which it is possible.*

CXVI. (*g*) 217. *If an Equation has any impossible Roots, their Number is even:* for, to make the Coefficients rational, and the last Term real (215), the radical Sign must disappear (120); which it could not do, except

 their

CXVII. *But sometimes the Roots of Equations are possible, when the Schemes exhibit them as impossible. But this happens by reason of some Limitation in the Scheme, which does not belong to the Equation.* [See Fig. 89.]

As if in the Semi-circle ADB, having given the Diameter AB, and the Chord AD, and having let fall the Perpendicular DC, I was to find the Segment of the Diameter AC, you will have $\frac{ADq}{AB} = AC$. And, by this Equation, AC is exhibited a real Quantity, where the inscribed Line AD is greater than the Diameter AB; but by the Scheme, AC then becomes impossible, viz. in the Scheme the Line AD is supposed to be inscribed in the Circle, and therefore cannot be greater than the Diameter of the Circle; but in the Equation there is nothing that depends upon that Condition. From this Condition alone of the Lines the Equation comes out, that AB, AD, and AC, are continually proportional. And because the Equation does not contain all the Conditions of the Scheme, it is not necessary that it should be bound to the Limits of all Conditions. Whatever is more in the Scheme than in the Equation, may constrain that to Limits; but not this. For which Reason, when Equations are of odd Dimensions, and consequently cannot have all their Roots impossible, the Schemes often set Limits to the Quantities on which all the Roots depend, which Limits it is impossible they

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their Number was even. Hence *also their Product is positive, and does not alter the Sign of the Product of the real Roots* (57). Whence it follows, that *a Cubic can have but two impossible Roots*; and that *every Equation whose last Term is negative, has, if the Exponent be odd, one at least; and if more, an odd Number of real Roots*, as in Numb. 205; and that *if the Index be even, it has at least two; and if more, an even Number of real Roots*, as in Numb. 204.

they can exceed, keeping the ſame Conditions of the Schemes (*h*).

CXVIII. *Of thoſe Roots that are real ones, the affirmative and negative ones lie on contrary Sides, or tend contrary Ways.*

Thus, in the laſt Scheme but one, by ſeeking the Perpendicular CG, you will light upon an Equation that has two affirmative Roots CG and DH, tending from the Points C and D the ſame Way; and two negative ones, EI and FK, tending from the Points E and F the oppoſite Way. Or if in the Line AB to which the Perpendiculars are let fall, there be given any Point P, and the Part of it PG extending from that given Point to ſome of the Perpendiculars, as CG, be ſought, we ſhall light on an Equation of four Roots, PG, PH, PI, and PK, whereof the Quantity ſought PG, and thoſe that tend from the Point P the ſame Way with PG (as PK), will be affirmative, but thoſe which tend the contrary Way (as PH, PI), negative (*i*).

CXIX. *Where*

CXVII. (*h*) For not only the Magnitude, but the Poſition of Quantities, will reſtrain the Expreſſions of the Roots. See Note following.

CXVIII. (*i*) Although all Quantities which by Subſtitution make the Terms vaniſh, or which will form Diviſors of it, are in general accounted Roots of an Equation, conſidered abſolutely as an Aggregate of Terms, containing the Powers of an unknown Quantity without any Relation to the Solution of a Problem; yet, when an Equation is conſidered as containing the Relation of Quantities in order to the Solution of a Problem, *it ſeems neceſſary to diſtinguiſh between Factors and Roots*, and to reſtrain the latter Term to thoſe which anſwer to the Conditions of the Problem; in which

 Caſe,

CXIX. *Where there are none of the Roots of the Equation impossible, the Number of the affirmative and negative Roots may be known from the Signs of the Terms of the Equation. For there are so many affirmative Roots, as there are Changes of the Signs in a continual Series from + to —, and from — to +; the rest are negative.*

As in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the Signs of the Terms follow one another in this Order, + — — + —, the Variations of the second — from the first +, of the fourth + from the third —, and of the fifth — from the fourth +, shew, that there are three affirmative Roots, and consequently, that the fourth is a negative one. But where some of the Roots are *impossible*, the Rule is of no Force; unless as far as those impossible Roots, which are neither negative nor affirmative, may be taken for *ambiguous* ones. Thus in the Equation $x^3 + pxx + 3ppx - q = 0$; the Signs shew that there is one affirmative Root and two negative ones. Suppose $x = 2p$, or $x - 2p = 0$; and multiply the former Equation by this, $x - 2p = 0$, that

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Case, none but affirmative Quantities could be accounted Roots: For if the Problem is purely algebraic, where Magnitude only, and no Consideration of Position, or contrary Values, can have Place, no Roots but such as are affirmative will answer; because, that a negative Quantity should be actually less than nothing is equally impossible, as that the double Product of two Numbers should be greater than the Sum of their Squares: That is, negative and imaginary Roots are equally impossible, where Magnitude only is considered. When the Problem is geometrical, where, beside Magnitude, Position also, and contrary Values are taken into Consideration; then negative Roots do not solve the Problem, but shew the Solution of it, in the opposite Position or Value; and if the Problem was changed to that opposite Position, those negative Factors would become affirmative Roots.

that one affirmative Root more may be added to the former; and you will have this Equation,

$$x^4 - px^3 + ppxx \begin{smallmatrix} -bp^3 \\ -q \end{smallmatrix} x + 2pq = 0,$$

which ought to have two affirmative and two negative Roots; yet it has, if you regard the Change of the Signs, four affirmative ones. There are therefore *two impossible* ones, which, for their Ambiguity, in the former Case seem to be negative ones; in the latter, affirmative ones (*k*).

But

CXIX. (*k*) See Numb. 192.

218. Since Equations are the Products of Binomes, it follows, that *the Coefficients of the Terms are Unity, the Sum of the Roots, the Sums of the Products of two, of three, of four, &c.* (97). Whence

219. *If the Roots be real Quantities, the Square of the middle Term of three will be greater than the Product of the adjacent Terms; and consequently every subsequent Term divided by the next Antecedent, will decrease continually* (109): Whence *there will be a Succession for each binomial Factor, and an Alternation for each Residual* (114); *That is, a Succession for each negative, and an Alternation for each affirmative Root* (181). Now, because the last Term of a Quadratic, whose Roots are impossible, is augmented by the Square of the Quantity under the radical Sign (193); the Square of the Middle of three Terms of an Equation, into whose Composition one or more such Quadratics have entered, will not always exceed the Product of the adjacent; nor the Ratio of each subsequent Term, divided by the antecedent, continually decrease; and consequently *the Alternations and Successions will not shew the Numbers of positive and negative Roots, unless the impossible Roots be taken ambiguously for positive and negative* (192).

 This

But you may know almost, by this Rule, how many Roots are impossible.

CXX. *Make a Series of Fractions, whose Denominators are Numbers in this Progression,* 1, 2, 3, 4, 5, *&c. going on to the Number which shall be the same as that of the Dimensions*

This Rule of Harriott is otherwise thus demonstrated. Because the Roots of the Equation are by Supposition all real, the Roots in all the Equations of Limits deducible from it will be real; and consequently in all the Quadratics (271): Now the positive Roots, in every Quadratic, are equal in Number to the Permutations of its Signs (190), and this Number cannot be augmented by any negative Root; that is, by any Multiplication by a binomial Factor (57): And it can be augmented by one only, in one Multiplication; by two, &c. in two Multiplications, &c. by a residual Factor (58); that is, it can be augmented but by one positive Root in the cubic, by two in the biquadratic, &c. wherefore, universally, the Number of Permutations is equal to the Number of positive Roots: Now the Number of Terms or Signs is $n+1$ (211): Therefore the Number of Permutations and Successions together are n, that is, the Number of Roots (210): Wherefore, since the Number of Alternations is the Number of Positive, the Number of Successions is that of the negative Roots.

220. *The Variations of affirmative and negative Roots, combined together in Equations, are equal to the Number of Terms in Equations of that Degree; that is, always exceed the Number of Roots by Unity:* thus the Roots of a Quadratic may be, 1st. both affirmative; or, 2d. both negative; or 3d. one affirmative and one negative: those of a cubic may be, 1st. all affirmative; 2d. all negative; 3d. one affirmative and two negative; or 4th. two affirmative and one negative, &c. For all the Variations of the Signs are $2n$ (185), and the same recur in Number $n-1$ (35); whence $2n - \overline{n-1} = n+1$, are the Variations.

Dimensions of the Equation; and the Numerators, the same Series of Numbers in a contrary Order. Divide each of the latter Fractions by each of the former. Place the Fractions that come out, over the middle Terms of the Equation. And under any of the middle Terms, if its Square multiplied into the Fraction standing over its Head is greater than the Rectangle of the Terms on both Sides, place the Sign +; but if it be less, the Sign —. But under the first and last Term place the Sign +. And there will be as many impossible Roots, as there are Changes in the Series of the underwritten Signs from + to —, and — to + (l). As

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CXX. (*l*) For any Quadratic, whose last Term is affirmative, will have both its Roots impossible, if $\frac{1}{4}$ the Square of the middle Term does not exceed the Product of the Extremes (189); that is, if the Square of the middle Term multiplied into its Fraction $\frac{1}{4}$, does not exceed the Product of the Extremes: Now, the Roots of the proposed Equation being Limits to the Roots of the Equation of Limits, and of all Equations (and consequently of all Quadratics) deducible from it by the Multiplication of its Terms into those of arithmetical Progressions, and conversely; the Roots of the Equations of Limits being the Limits of the Roots of the proposed (267): if any Roots in the Equations of Limits are impossible, there will be as many impossible Roots in the proposed, as in all the Quadratics deducible from it (271). But the Square of the middle Term of three in the proposed, multiplied into its Fraction, has the same Ratio to the Product of the Terms adjacent to it, as the Square of the middle Term of the Quadratic deduced for those three, multiplied into $\frac{1}{4}$, has to the Product of its Extremes: Therefore, as often as the Square of the middle Term of the proposed, multiplied into its Fraction, is less than the Product of the Terms adjacent to it; so often also $\frac{1}{4}$ the Square of the middle Term of the Quadratic deduced for them, is less than the Product of the Extremes; and consequently, two

 Roots

As if you have the Equation $x^3 + pxx + 3ppx - q = 0$; I divide the second of the Fractions of this Series

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Roots of the proposed are impossible: So often therefore marking with the Sign $-$, there will be two Alternations; and consequently, the Number of impossible Roots, and of Alternations in the underwritten Signs, will be equal.

That *the Square of the middle Term of three (in any proposed Equation) multiplied into its Fraction has the same Ratio, to the Product of the Extremes adjacent to it; as* $\frac{1}{4}$ *the Square of the middle Term of the corresponding Quadratic has, to the Product of its Extremes*, appears from this: that in deducing each limiting Quadratic from the corresponding Terms of the proposed whose Index is n, there are so many Multiplications by the Series of Laterals descending or ascending to Cypher, and so many Divisions by the unknown, as there are Units in $n-2$ (CXXXVIII); and therefore, that the numeral Coefficients of the Terms of the Quadratic are generated from the continual Multiplication of the same Fractions, from whose Division the Fraction over the middle Term of the proposed emerges; and consequently, that the Square of the numeral Coefficient in the Quadratic, multiplied into $\frac{1}{4}$, produces the Fraction placed over the corresponding middle Term in the proposed Equation: that is, the Square of the middle Term in the proposed, multiplied into its Fraction, has the same Ratio to the Product of the Terms adjacent to it; as $\frac{1}{4}$ the Square of the middle Term of the Quadratic, has to the Product of its Extremes.

Thus, let the proposed be $x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$; there can be deduced Quadratics for every Term except the Extremes, by $(5 - 2 =)$ 3 Multiplications and Divisions: viz. 1st. for $x^5 - Ax^4 + Bx^3$, there is found $10x^2 - 4Ax + B$; 2d. for $-Ax^4 + Bx^3 - Cx^2$, there is found

$2Ax^2$

Series $\frac{3}{1}$. $\frac{2}{2}$. $\frac{1}{3}$, viz. $\frac{2}{2}$ by the first $\frac{3}{1}$, and the third $\frac{1}{3}$ by the second $\frac{2}{2}$, and I place the Fractions that come out (viz. $\frac{1}{3}$ and $\frac{1}{3}$) over the middle Terms of the Equation, as follows;

$$\begin{matrix} & & \frac{1}{3} & & \frac{1}{3} & & \\ x^3 & + & pxx & + & 3ppx & - & q = 0. \\ + & & - & & + & & + \end{matrix}$$

Then

$2Ax^2 - 2Bx + C$ (231); 3d. for $Bx^3 - Cx^2 + Dx$, there is found $Bx^2 - 2Cx + 2D$ (231); and lastly, for $-Cx^2 + Dx - E$, is found $Cx^2 - 4Dx + 10E$ (231). Now the Equation, with its Fractions, is

$$x^5 - \overset{\frac{2}{5}}{A}x^4 + \overset{\frac{1}{2}}{B}x^3 - \overset{\frac{1}{2}}{C}x^2 + \overset{\frac{2}{5}}{D}x - E = 0:$$

but in the first and last Quadratic, the Ratio of $4 \times 4 \times \frac{1}{4}$ to $10 = \frac{4}{10} = \frac{2}{5}$; and in the second and third Quadratic, the Ratio of $2 \times 2 \times \frac{1}{4}$ to 2 is $\frac{1}{2}$.

'Tis to be noted, 1st. that although it is a certain Criterion, that there are two impossible Roots, as often as the Square of any Term (multiplied into its Fractions) is deficient of the Product of the Terms adjacent; yet *it is no Proof that the Roots are real, if the Square of any Term (multiplied into its Fraction) exceeds the Product of the adjacent Terms*; and consequently, that nothing can be concluded from such Excess, concerning the Possibility or Reality of the Roots; that is, *the Roots may be imaginary, though there should be such an Excess.* 2*d.* That although real Roots in the proposed, give real Roots in all the Equations of Limits; yet *real Roots in all the Equations of Limits, do not give real Roots in the proposed* (271). Lastly, *every Rule, depending upon the Comparison of the Square of a Term with the Product of the adjacent Terms on either Side, must sometimes fail to discover the impossible Roots*; because the Number of such Comparisons being always less by Unity than the Number of Quantities in the Equation, they cannot include and fix the Relations, upon which the Ratios of greater or less Inequality of the Squares and Products depend.

Then, because the Square of the second Term pxx multiplied into the Fraction over its Head $\frac{1}{3}$, viz. $\frac{ppx^4}{3}$ is less than $3ppx^4$, the Rectangle of the first Term x^3 and third $3ppx$, I place the Sign — under the Term pxx. But because $9p^4xx$ (the Square of the third Term $3ppx$) multiplied into the Fraction over its Head $\frac{1}{3}$, is greater than nothing, and therefore much greater than the negative Rectangle of the second Term pxx, and the fourth $-q$, I place the Sign + under that third Term. Then, under the first Term x^3 and the last $-q$, I place the Sign +. And the two Changes of the underwritten Signs (which are in this Series $+ - + +$; the one from + into —, and the other from — into +) shew that there are two impossible Roots. And thus the Equation $x^3 - 4xx + 4x - 6 = 0$ has two impossible Roots,

$$\begin{array}{ccccccc} & & \frac{1}{3} & & \frac{1}{3} & & \\ x^3 & - & 4xx & + & 4x & - & 6 = 0. \\ + & & + & & - & & + \end{array}$$

Also the Equation $x^4 - 6xx - 3x - 2 = 0$ has two

$$\begin{array}{ccccccccc} & \frac{3}{8} & & \frac{4}{9} & & \frac{3}{8} & & & \\ x^4 & * & - & 6xx & - & 3x & - & 2 = 0. \\ + & + & & + & & - & & + \end{array}$$

For this Series of Fractions $\frac{4}{1}. \frac{3}{2}. \frac{2}{3}. \frac{1}{4}$, by dividing the second by the first, and the third by the second, and the fourth by the third, gives this Series $\frac{3}{8}. \frac{4}{9}. \frac{3}{8}$, to be placed over the middle Terms of the Equation. Then the Square of the second Term, which is here nothing, multiplied into the Fraction over Head, viz. $\frac{3}{8}$, produces nothing, which is yet greater than the negative Rectangle $-6x^6$ contained under the Terms on each Side x^4 and $-6xx$. Wherefore, under the Term that is wanting, I write +. In the rest, I go on as in the former Example; and there comes out this Series of the underwritten Signs $+ + + - +$, where two Changes shew there are two impossible Roots. And after the same Way, in the Equation $x^5 - 4x^4 + 4x^3 -$

2xx

$2xx - 5x - 4 = 0$, are difcovered two impoffible Roots, as follows;

$$\begin{array}{cccccccccccc} & & \frac{2}{5} & & \frac{1}{2} & & \frac{1}{2} & & \frac{2}{5} & & & \\ x^5 & - & 4x^4 & + & 4x^3 & - & 2xx & - & 5x & - & 4 & = 0. \\ + & & + & & - & & + & & + & & + & \end{array}$$

CXXI. *Where two or more Terms are wanting together, under the firft of the deficient Terms you muft write the Sign —, under the fecond Sign +, under the third the Sign —, and fo on, always varying the Signs; except that under the laft of fuch deficient Terms you muft always place +, when the Terms next on both Sides the deficient Terms have contrary Signs.* As in the Equations

$$\begin{array}{cccccccc} x^5 & + & ax^4 & * & * & * & + a^5 & = 0 \\ + & & + & - & + & - & + & \end{array}, \text{ and}$$

$$\begin{array}{cccccccc} x^5 & + & ax^4 & * & * & * & - a^5 & = 0 \\ + & & + & - & + & + & + & \end{array};$$

the firft whereof has four, and the latter two impoffible Roots. Thus alfo the Equation,

$$\begin{array}{cccccccccccccc} & & \frac{3}{7} & & \frac{5}{9} & & \frac{3}{5} & & \frac{3}{5} & \frac{5}{9} & \frac{3}{7} & & & \\ x^7 & - & 2x^6 & + & 3x^5 & - & 2x^4 & + & x^3 & * & * & - & 3 & = 0 \\ + & & - & & + & & - & & + & - & + & & + & \end{array}$$

has fix impoffible Roots (*m*).

CXXII. *Hence*

CXXI. (*m*) Becaufe, if the Signs on each Side the vanifhed Term be the fame, 'tis a Token that there are impoffible Roots (244); and therefore the Signs — and +, writen alternately, will by their Permutations denote their Number: But, if thefe Signs be contrary, it betokens that a Term may have vanifhed through an Equality of the fimilar Products of real Roots, with contrary Signs in that Term (112), fo that one Permutation lefs will fuffice; which is done by writing + under the latter Term, which will diminifh the Number of Alternations by Unity.

CXXII. *Hence also may be known, whether the impossible Roots are among the affirmative or negative ones. For the Signs of the Terms over Head of the subscribed changing Terms shew, that there are as many impossible affirmative Roots as there are Variations of them, and as many negative ones as there are Successions without Variations.* Thus, in the Equation

$$\begin{array}{cccccc} x^5 - & 4x^4 + & 4x^3 - & 2xx - & 5x - & 4 = 0 \\ + & + & - & + & + & + \end{array},$$

because by the Signs that are writ underneath that are changeable, viz. $+ - +$, by which it is shewn there are two impossible Roots, the Terms over Head $-4x^4 + 4x^3 - 2xx$ have the Signs $- + -$, which by two Variations shew there are two affirmative Roots; therefore there will be two impossible Roots among the affirmative ones. Since therefore the Signs of all the Terms of the Equation $+ - + - - -$ by three Variations shew that there are three affirmative Roots, and that the other two are negative, and that among the affirmative ones there are two impossible ones; it follows, that the Equation has one true affirmative Root, two negative ones, and two impossible ones. But if the Equation had been

$$\begin{array}{cccccc} x^5 - & 4x^4 - & 4x^3 - & 2xx - & 5x - & 4 = 0 \\ + & + & - & - & + & + \end{array},$$

then the Terms over Head of the subscribed former varying Terms $+ -$, viz. $-4x^4 - 4x^3$, by their Signs that do not change $-$ and $-$, shew, that one of the negative Roots is impossible; and the Terms over the latter underwritten varying Terms $- +$, viz. $-2xx - 5x$, by their Terms not varying $-$ and $-$ shew, that another of the negative Roots is impossible. Wherefore, since the Signs of the Equation $+ - - - - -$ by one Variation shew there is one affirmative Root, and that the other four are negative; it follows, there is one affirmative, two negative, and two impossible ones. And this is so, where

there

there are not more impossible Roots than what are discovered by the Rule preceding. For there may be more, although it seldom happens (*n*).

Of

CXXII. (*n*) Because the Series of Fractions, made by the Laterals, may be expounded, as in Numb. 161, to be those whose Numerators are the Indices of the Terms, or the Numbers of the Terms subsequent; and their Denominators, the Indices of the Coefficients, or the Numbers of the Terms antecedent: Therefore, *the Fraction into which the Square of any Coefficient or Term is to be multiplied*, is often enunciated to be, *that whose Numerator is the Product of the Indices of the Term and of the Coefficient, and whose Denominator is the Product of those Indices each increased by Unity: Or that, whose Numerator is the Product of the Numbers of the Terms preceding and subsequent, and Denominator the Product of them when increased each by Unity*: For 'tis manifest, that the Numerators decrease, and the Denominators increase, by Unity, in the Fractions composed of the Laterals; wherefore, in dividing each Subsequent by its Antecedent (that is, in the multiplying each Subsequent by the Reciprocal of the Antecedent (149) the Factors which compound the Denominators, exceed those which compound the Numerators by Unity each.

Other Rules were published by Mr. M'Laurin, in his second Letter to M. Foulkes, Esq; to which the Reader is referred, as well as for the Proof of the following two Rules, which are thence recited as being of more universal Use.

1st. *Let the Unciæ of the Terms be found; let the Unciæ, each diminished by Unity, be the Numerators, and the Unciæ, each doubled, be the Denominators of Fractions to be set over the middle Terms of the Equation: Then, as often as the Square of any Term, multiplied into its Fraction, does not exceed the Products of the Terms adjacent to it on each Side, taken in order and added and subducted alternately, so often there will be two impossible Roots.*

2d.

Of the TRANSMUTATIONS *of* EQUATIONS (*a*).

CXXIII. *Moreover, all the affirmative Roots of any Equation may be changed into negative ones, and the negative into affirmative ones, and that only by changing the Signs of the alternate Terms* (*b*).

Thus in the Equation $x^5 - 4x^4 + 4x^3 - 2xx - 5x - 4 = 0$, the three affirmative Roots will be changed into negative ones, and the two negative ones into Affir-

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2d. *Let the Product of the Numbers of the Terms, antecedent and subsequent to any Term, be found: To this Product, let be added the Squares of the Distances of the Pairs of Terms adjacent, in order on each Side. Lastly, let into these Sums be multiplied the Products of these Pairs: Then as often as the Square of a Term, multiplied into Half the Product of the Number of Terms which are antecedent and consequent to it, does not exceed the Products of the adjacent Terms, multiplied into the said Sums added and subducted in order alternately, so often there will be two impossible Roots.*

CXXIII. (*a*) 221. *To transform an Equation, is to change it into another of the same Dimensions, whose Roots shall have a known Relation to the Roots of the proposed. Transformation is performed, by forming an Equation from the given Relation of the Roots, by finding the Value of the Roots of the proposed in this, and by substituting this Value and its Powers, into the Place of the unknown and its Powers, in the proposed Equation. Hence the Value of the Roots of the transformed being found, the Roots of the proposed will be found, by means of the Equation which expresses their given Relation.*

(*b*) Because that in the even Places the Coefficients involve an odd Number of Roots (218), under their contrary

Affirmatives, by changing only the Signs of the second, fourth, and sixth Terms, as is done here, $x^5 + 4x^4 + 4x^3 + 2xx + 5x + 4 = 0$. This Equation has the same

contrary Signs; and that if an odd Number of Quantities is to be multiplied, all their Signs being changed before Multiplication, the Sign of the Product would be changed (88): and because that in the odd Places the Coefficients involve an even Number of Roots, under the contrary Signs (218); and that if an even Number of Quantities is to be multiplied, all their Signs being changed before Multiplication, the Sign of the Product would remain the same (88): Therefore, by changing the Signs of the Terms in the even Places only, all these Coefficients will be changed into the Sum of the Roots, the Sums of their Products by threes, fives, &c. under their proper Signs: Consequently, in this transformed Equation, all the affirmative Roots are changed into negative, and all the negative into affirmative (181). Now this transformed Equation, by transposing the Terms into the opposite Member, will be the same with the proposed, having the Signs in the odd Places only changed (LXVII); consequently, by changing the Signs of the alternate Terms, the affirmative Roots are changed into negative, and the negative Roots into affirmative. Thus, $x^5 - Ax^4 + Bx^3 - Cx^2 - Dx + E = 0$, by changing the Signs in the even Places, becomes $x^5 + Ax^4 + Bx^3 + Cx^2 - Dx - E = 0$; and this, transformed by Transposition, becomes $-x^5 - Ax^4 - Bx^3 - Cx^2 + Dx + E = 0$.

Again, let the Signs be changed in the even Places only; afterwards, let the Signs be changed in the odd Places only; these two transformed are the same in all Respects, except that the Signs of the Terms of the same Exponent are contrary: But by changing the Signs in the even Places only, the affirmative Roots were changed into negative, and the negative into affirmative; therefore, by changing the Signs in the odd Places only, the

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ſame Roots with the former, unleſs that in this, thoſe Roots are affirmative that were negative there, and negative here that were affirmative there; and the two impoſſible

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the ſame Change will be made in the Roots. The Terms alſo of this latter being tranſpoſed, will give the former; and conſequently the Change is made in the Roots, by the Change of the Signs in the alternate Places. Thus $x^5 - Ax^4 + Bx^3 - Cx^2 - Dx + E = 0$, by changing the Signs in the odd Places only, becomes $-x^5 - Ax^4 - Bx^3 - Cx^2 + Dx + E = 0$; and this, by Tranſpoſition, becomes $x^5 + Ax^4 + Bx^3 + Cx^2 - Dx - E = 0$, in which the Signs in the even Places only are changed.

Again, by changing the Signs in the alternate Places, the Alternations in the propoſed become Succeſſions in the transformed, and the Succeſſions become Alternations; and the Number of Alternations in the propoſed is the ſame with that of the Succeſſions of the tranſformed, and the Number of Succeſſions the ſame with that of the Alternations: but the propoſed had ſo many affirmative Roots as Alternations, and negative Roots as Succeſſions (CXIX); therefore the transformed has ſo many negative Roots as the propoſed has affirmative, and ſo many affirmative as the propoſed has negative; and no Change has been made, except in the Signs of the Roots: whence they are the ſame with contrary Signs.

222. *By changing the Signs of all the Terms of an Equation, no Change is made in the Signs of the Roots.* For by changing the Signs of the Terms in the even Places, the Sums of the Terms in the odd Places are not changed; and by changing the Signs of the Terms in the odd Places, the Sums of the Terms in the even Places remain unchanged: wherefore, by changing the Signs of all the Terms, the Sum of all the Terms is unchanged, and remains $= 0$ (LXV); therefore, the Roots are the ſame

impossible Roots, which lay hid there among the affirmative ones, lie hid here among the negative ones; so that these being deducted, there remains only one Root truly negative.

CXXIV. There are also other Transmutations of Equations, which are of Use in divers Cases. *For we may suppose the Root of an Equation to be composed any how out of a known and unknown Quantity, and then substitute what we suppose equivalent to it.* As if we suppose the Root to be equal to the Sum, or Difference, of any known and unknown Quantity. *For after this Rate we may augment or diminish the Roots of the Equation by that known Quantity, or subtract them from it; and thereby cause that some of them that were before negative, shall now become affirmative; or some of the affirmative ones become negative; or also that all shall become affirmative, or all negative* (*c*). Thus in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, if I have a mind to augment the Roots by Unity, I suppose $x + 1 = y$, or $x = y - 1$; and then for x I write $y - 1$ in the Equation, and for the

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same (CX): The Equation also, thus changed by transposing the Terms, is the same as at first. Thus $x^5 - Ax^4 + Bx^3 - Cx^2 - Dx - E = 0$, by changing all the Signs, becomes $-x^5 + Ax^4 - Bx^3 + Cx^2 + Dx + E = 0$; and, by Transposition, $x^5 - Ax^4 + Bx^3 - Cx^2 - Dx - E = 0$, as at first.

CXXIV. (*c*) 223. *When it is required to transform an Equation into another, whose Roots shall be less than the Roots of the proposed, by a given Difference* e, *for* x *and its Powers substitute* y + e *and its Powers.* For, by Supposition, $x - e = y$; wherefore, $x = y + e$; whence, since x denotes all the Roots of the proposed indiscriminately, and y all those of the transformed, each Root of the transformed will be deficient of each of the proposed, by the given Quantity e. Thus, $y + e$ being

 substituted

the Square, Cube, or Biquadrate of x, I write the like Power of $y - 1$, as follows:

x^4	$y^4 - 4y^3 + 6yy - 4y + 1$
$-x^3$	$-y^3 + 3yy - 3y + 1$
$-19xx$	$-19yy + 38y - 19$
$+49x$	$+49y - 49$
-30	-30
Sum	$y^4 - 5y^3 - 10yy + 80y - 96 = 0$.

And

substituted for x, in the Equation $x^3 - px^2 + qx - r = 0$, it becomes

$$\begin{array}{llll} y^3 + 3e\,y^2 & + 3e^2\,y & + e^3 & \\ \quad - p & - 2pe & - pe^2 & = 0. \\ & + q & + qe & \\ & & - r & \end{array}$$

Each of whose Roots is less by the Quantity e.

224. *When it is required to transform an Equation into another, whose Roots shall be greater than the Roots of the proposed, by a given Excess* e, *for* x *and its Powers substitute* y — e *and its Powers.* For, by Supposition, $x + e = y$; whence $x = y - e$; and all the Roots of the transformed will exceed those of the proposed, by the given Excess e. Thus $y - e$ being substituted for x in the Equation $x^3 - px^2 + qx - r = 0$, it becomes

$$\begin{array}{llll} y^3 - 3e\,y^2 & + 3e^2\,y & - e^3 & \\ \quad - p & + 2pe & - pe^2 & = 0. \\ & + q & - qe & \\ & & - r & \end{array}$$

Each of whose Roots is greater by the Excess e.

225. *When it is required to transform an Equation into another, whose Roots shall be the Excess of a given Quantity* e *above the Roots of the proposed, substitute* e — y *and its Powers for* x *and its Powers.* For, by Supposition, $x + y = e$;

And the Roots of the Equation that is produced, viz. $y^4 - 5y^3 - 10yy + 80y - 96 = 0$, will be , 2, 3, 4.

$x + y = e$; whence $x = e - y$. Thus, $e - y$ being substituted for x in $x^3 - px^2 + qx - r = 0$, it becomes

$$\left.\begin{matrix} e^3 & -3pe^2 & +3y^2e & -y^3 \\ -pe^2 & +2pye & -py^2 & \\ qe & -qy & & \\ -r & & & \end{matrix}\right\} = 0.$$

Each of whose Roots is the Excess of e above the respective Roots of the proposed.

226. *When an Equation is transformed by having its Roots diminished, or increased, by a given Quantity* e; *the last Term of the transformed is the same with the proposed, having* e *in the Place of* x; *the Signs of* e *being the same when the Roots are diminished, but contrary to those of* x, *when increased. The Coefficient of the Penultimate is found, by multiplying every Part of the Coefficient of* e *in the last Term, by the Index of* e *in each Part of the last Term, and dividing the Product by* e, viz. by the Quantity which is common to those Parts. *The Coefficient of the Antepenultimate is found, by multiplying the Parts of the Coefficient of* e *in the Penultimate, by the Index of* e *in each, and by dividing the Product by* 2e, *and so on. Whence, in general,*

227. *The Terms of the transformed may be found without Involution.* For the last is had, by substituting e and its Powers for x and its Powers; and the following Terms, viz. penultimate, ante-penultimate, &c. are found, by multiplying every Part of the last found Term in which e is, by the Index of e in that Part; and by dividing the Products by the Product of e into the Index of y in that Term which is sought: And the Indices of y are the Laterals, beginning with the penultimate Term. For the transformed Equation consists of those Powers of $y + e$, which are marked by the Indices of e in the Parts of the last Term, multiplied each by their respective

2, 3, 4, — 4, which before were 1, 2, 3, — 5, i. e. bigger by Unity. Now, if for x I had writ $y + 1$ [illegible], there

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Coefficients; wherefore, beginning from the last Term, the highest Index of e being n, the Unciæ of the Terms, reckoned from it, will be 1, $\frac{n}{1}$, $\frac{n \times n - 1}{1 \times 2}$, $\frac{n \times n - 1 \times n - 2}{1 \times 2 \times 3}$, &c. where the lateral Divisors are the Indices of y in the Term itself, as in Numb. 101, &c.

228. *When an Equation is transformed by the Diminution of its Roots by a given Quantity* e, *the affirmative Roots only are diminished, but the negative Roots are augmented by that given Quantity* e; for the Sign of e in $y + e$, is the same with the Sign of the negative, and contrary to the Sign of the affirmative Roots (181).

229. Whence *if the Quantity* e, *by which the Roots are diminished, is equal to any affirmative Root, that Root will vanish* by the Transformation; *and* consequently the Product of the Roots, that is, the last Term of the transformed will vanish; that is, *the transformed will be lowered by one Dimension: Also if two, three, &c. Roots of the proposed be equal to each other, and to* e, *the Quantity by which the Roots are diminished; the Equation will be transformed into one of two, three, &c. Dimensions lower.* Thus suppose an Equation, none of whose Roots are equal to each other, and but one of them equal to e, be proposed, viz. $x^4 - px^3 + qx^2 - rx + s = 0$, the transformed will be

$$y^4 \begin{array}{l} +4e \\ -p \end{array} y^3 \begin{array}{l} +e^2 \\ -3pe \\ +q \end{array} y^2 \begin{array}{l} +e^3 \\ -3pe^2 \\ +2qe \\ -r \end{array} y \; * = 0;$$

Whose last Term, viz. $e^4 - pe^3 + qe^2 - re + s$ is vanished, there would have come out the Equation $y^4 + 5y^3 - 10yy - \frac{5}{4}y + \frac{39}{16} = 0$, whereof there be two affirmative

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vanished, upon Account of e being equal to one Root x; just as the Aggregate of the proposed would have vanished by its Substitution. Now dividing this transformed by y, it becomes one Dimension lower, viz.

$$y^3 \begin{array}{l} +4e \\ -p \end{array} y^2 \begin{array}{l} +6e^2 \\ -3pe \\ +q \end{array} y \begin{array}{l} +e^3 \\ -3pe^2 \\ +2qe \\ -r \end{array} = 0:$$

Again, suppose two Roots x equal to e, and consequently to each other, the transformed will be

$$y^4 \begin{array}{l} +4e \\ -p \end{array} y^3 \begin{array}{l} +6e^2 \\ -3pe \\ +q \end{array} y^2 * * = 0;$$

wanting the two last Terms, and dividing by y^2, is

$$y^2 \begin{array}{l} +4e \\ -p \end{array} y \begin{array}{l} +e^2 \\ -3pe \\ +q \end{array} = 0,$$

two Dimensions lower, and so on.

230. *And conversely, if by diminishing the Roots of an Equation by any Quantity, the last Term of the transformed should vanish, that Quantity is equal to some affirmative Root of the Equation; and if the penultimate Term also should vanish, the proposed has two Roots equal, and equal to that Quantity; and if, moreover, three Terms of the transformed should vanish, three Roots of the proposed are equal, and equal to that Quantity, &c.*

231. In general, *if every Term of an Equation, having* n *Number of equal Roots, be multiplied by the Index of the unknown Quantity in the Term, and the Product divided by the unknown Quantity, the transformed will contain* n — 1 *of these equal Roots:* For this is equivalent to a Transformation,

mative Roots, $\frac{1}{2}$ and $1\frac{1}{5}$, and two negative ones, $-\frac{1}{2}$ and $-6\frac{1}{5}$. But by writing $y - 6$ for x, there would have come out an Equation whose Roots would have

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ation, by diminishing the Roots by a Quantity equal to those equal Roots; in like manner, $n - 2$ *of the equal Roots will remain, after two Multiplications and Divisions; and so on.*

232. *If* e, *the Quantity by which the Roots are diminished, be greater than the greatest affirmative Root of the proposed,* all its affirmative Roots will become negative (228); whence *all the Roots of the transformed are negative, and all its Terms affirmative* (181); *and the least Root of the transformed, answers to the greatest Affirmative of the proposed.* For the Excess of e above the Affirmatives is that, which makes them negative; and that Excess is least, which exceeds the greatest Affirmative: Hence also, *those Roots of the transformed, which are less than* e, *are those which were affirmative in the proposed; and those, which are greater than* e, *are those which were negative also in the proposed* (228).

233. *And conversely, if the Terms of the transformed become affirmative by having the Roots diminished by any Quantity, that Quantity is greater than the greatest affirmative Root of the proposed; and if the Sign — intervenes once, all the Roots of the transformed, except one, are negative; and that one is greater than* e *that Quantity.*

234. *To make all the Roots of any Equation negative, is to diminish them by a Quantity greater than the greatest affirmative Root; or to substitute an affirmative Quantity for the unknown, greater than the greatest affirmative Root:* For this Substitution is equivalent to Transformation, by Diminution of the Roots (227). 'Tis evident, that if the Quantity e is less than the least affirmative Root, the Quality of the Roots is unchanged, though the Affirmatives are diminished and the Negatives augmented by it.

235. *When*

have been 7, 8, 9, 1, viz. all affirmative; and writing for the same [x] $y + 4$, there would have come out those Roots diminished by 4, viz. $-3 -2 -1 -9$, all of them negative.

After

235. *When an Equation is transformed, by having its Roots increased by a given Quantity* e, *the affirmative Roots only are increased, but the Negatives are diminished by it*; its Sign being the same with that of the affirmative, and the contrary to that of the negative Roots.

236. Whence, *if this Quantity is equal to one, two, three, &c. of the negative Roots, so many Terms of the transformed will vanish, and it will be depressed by so many Dimensions; and conversely, if the transformed be depressed, the proposed has so many negative equal Roots, and equal to the Quantity* e *by which the Roots are increased.*

237. *If the Quantity* e, *by which the Roots are augmented, be greater (that is, more remote from nothing) than the greatest negative Root*, all the Negatives will become Affirmative by the Transformation; and *all the Roots of the transformed will be affirmative, and the Terms alternately affirmative and negative; and the least Root in the transformed answers to the greatest negative Root of the proposed*, and is the Excess of e the given Quantity above the greatest negative Root of the proposed: For the Excesses of e, above the negative Roots of the proposed, are the affirmative Roots of the transformed which are less than e; and that Excess must be least, which exceeds the greatest; *and the Roots greater than* e, *were affirmative in the proposed* (228).

238. *And conversely, if the Terms of the transformed become alternately positive and negative, the Quantity by which the Roots are augmented, is greater than the greatest negative Root; and if one Succession intervenes, there will be a negative Root greater than* e.

239. To

After this manner, by augmenting or diminishing the Roots, if any of them are impossible, they will sometimes be more easily detected than before. Thus in the Equation $x^3 - 3aax - 3a^3 = 0$, there are no Roots that appear impossible by the preceding Rule; but if you augment the Roots by the Quantity a, writing $y - a$ for x,

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239. *To make all the Roots of an Equation affirmative, is to augment them by a Quantity greater than the greatest negative Root; or to substitute a negative Quantity for the unknown, greater than the greatest negative Root: whereby the resulting Quantity will become affirmative, if the Equation is of even Dimensions; but negative, if of odd Dimensions* (88): This Substitution being equivalent to Transformation, by augmenting the Roots.

240. *When an Equation is transformed, by having its Roots subducted from a given Quantity* e, as in Numb. 225, *the affirmative Roots of the proposed become negative in the transformed, and each diminishes the Quantity* e; *likewise, the negative Roots of the proposed become affirmative in the transformed, and each increases the Quantity* e. For if a be an affirmative Root, then $a = x$, and $x = e - y$; whence $a = e - y$, and $y = e - a$: Again, if a be a negative Root, then $-a = x$, and $x = e - y$; whence $-a = e - y$, and $y = e + a$.

241. *If the Quantity* e, *from which the Roots are subducted, be equal to one, two, &c. affirmative Roots of the proposed, the transformed will be depressed so many Dimensions; and conversely, if the transformed is depressed, the proposed has so many affirmative Roots equal, and equal to* e.

242. *If the Quantity* e, *from which the Roots are subducted, is greater than the greatest affirmative Root, all the Roots of the transformed will be affirmative* (240); *and the Terms, alternately, affirmative and negative; and conversely, if the Terms be alternately affirmative and negative, the Quantity* e *is greater than the greatest affirmative Root.*

for x, you may now, by that Rule, discover two impossible Roots in the Equation resulting, $y^3 - 3ayy - a^3 = 0$.

CXXV. *By the same Operation you may also take away the second Terms of Equations. This will be done, if you subduct the known Quantity of the second Term of the Equation proposed, divided by the Number of Dimensions of the highest Term of the Equation, from the Quantity which you assume to signify the Root of the new Equation, and substitute the Remainder for the Root of the Equation proposed* (*d*). As if there was proposed the Equation $x^3 - 4xx + 4x - 6 = 0$, I subtract the known Quantity

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CXXV. (*d*) That is, *divide the Coefficient of the second Term by the Index of the first, and connect the Quote to the Letter assumed by the contrary Sign.* For, by this means, if the second Term is affirmative, being the Sum of the Roots under a contrary Sign (218); each negative Root is increased by the affirmative Sum of the Roots divided by their Number (224); that is, the negative Sum of the Roots is increased by an equal affirmative Sum, and therefore vanishes: and if the second Term is negative, each affirmative Root is diminished by the negative Sum of the Roots, divided by their Number (223); that is, the affirmative Sum of the Roots is diminished by an equal negative Sum, and therefore vanishes.

243. *The Use of exterminating the second Term, is to make the Solution of the Equation more easy:* For *if the proposed be a Quadratic, the transformed will be unaffected, and* (LXXIV. h.) *its affirmative Root is equal to the negative; if* the proposed be *cubic, in the transformed either the Sum of two affirmative Roots is equal to one negative, or the Sum of two Negatives to one Affirmative; if* the proposed be *a Biquadratic, in the transformed, either the Sum of three Negatives is equal to one Affirmative, or the Sum of two Negatives is equal to the Sum of two Affirmatives, or one Negative is equal to the Sum of three Affirmatives: The Sum of*

Quantity of the second Term, which is -4, divided by the Number of the Dimensions of the Equation, *viz.* 3, from the Species or Letter which is assumed to signify the new Root, suppose from y, and the Remainder $y+\frac{4}{3}$ I substitute for x, and there comes out

$$\begin{array}{rrrr} y^3 & +4yy & +\frac{16}{3}y & +\frac{64}{27} \\ & -4yy & -\frac{32}{3}y & -\frac{64}{9} \\ & & +4y & +\frac{16}{3} \\ & & & -6 \\ \hline y^3 & * & -\frac{4}{3}y & -\frac{146}{27}=0. \end{array}$$

CXXVI. *By the same Method, the third Term of an Equation may be also taken away.* Let there be proposed the Equation $x^4-3x^3+3xx-5x-2=0$, and make $x=y-e$, and substituting $y-e$ in the room of x, there will arise this Equation;

$$y^4 \begin{array}{l} -4e \\ -3 \end{array} y^3 \begin{array}{l} +6ee \\ +9e \\ +3 \end{array} yy \begin{array}{l} -4e^3 \\ -9ee \\ -6e \\ -5 \end{array} y \left.\begin{array}{l} +e^4 \\ +3e^3 \\ +3ee \\ +5e \\ -2 \end{array}\right\} = 0.$$

The third Term of this Equation is $6ee+9e+3$ multiplied by yy. Where, if $6ee+9e+3$ were nothing,

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of the affirmative Roots being always equal to the Sum of the negative Roots, in every Equation which wants the second Term (112). Now when the Roots of the transformed are found, those of the proposed are had from their known Relation (221).

244. *In every Equation which wants the second Term, if all its Roots are real, the third Term will be negative:* For the first Term being affirmative, the third ought to have the contrary Affection (112). If, therefore, the third Term is affirmative, when the second is wanting, there may be, and generally are, impossible Roots; but it does not follow, that when there are impossible Roots, the third Term shall be affirmative.

nothing, you would have what you desired. Let us suppose it therefore to be nothing, that we may thence find what Number ought to be substituted in this Case for e, and we shall have the quadratic Equation $6ee + 9e + 3 = 0$; which, divided by 6, will become $ee + \frac{3}{2}e + \frac{1}{2} = 0$, or $ee = -\frac{3}{2}e - \frac{1}{2}$, and extracting the Root $e = -\frac{3}{4} \pm \sqrt{\frac{9}{16} - \frac{1}{2}}$, or $= -\frac{3}{4} \pm \sqrt{\frac{1}{16}}$, that is, $= -\frac{3}{4} \pm \frac{1}{4}$, and consequently either $= -\frac{1}{2}$ or $= -1$. Whence $y - e$ will be either $y + \frac{1}{2}$, or $y + 1$. Wherefore, since $y - e$ was writ for x; in the room of $y - e$ there ought to be writ $y + \frac{1}{2}$, or $y + 1$, for x, that the third Term of the Equation that results may be taken away. And that will happen in both Cases. For if for x you write $y + \frac{1}{2}$, there will arise this Equation, $y^4 - y^3 - \frac{15}{4}y - \frac{65}{16} = 0$; but if you write $y + 1$, there will arise this Equation, $y^4 + y^3 - 4y - 6 = 0$ (*e*).

CXXVII. *More-*

CXXVI. (*e*) *The Quantity, proper to be connected with the assumed Letter* y, *in order that from its Substitution the transformed Equation shall want any assigned Term, will be found by the Solution of the Equation of the Coefficient of that Term in the transformed, into which the general Formula is changed.* Thus $x^5 - px^4 + qx^3 - rx^2 + sx - t = 0$, is transformed into

$$\begin{array}{rcllllll}
x^5 & = & y^5 & + 5ey^4 & + 10e^2y^3 & + 10e^3y^2 & + 5e^4y & + e^5 \\
-px^4 & = & & - p & - 4pe & - 6pe^2 & - 4pe^3 & - pe^4 \\
qx^3 & = & & & + q & + 3qe & + 3qe^2 & + qe^3 \\
-rx^2 & = & & & & - r & - 2re & - re^2 \\
sx & = & & & & & + s & + se \\
-t & = & & & & & & - t
\end{array}$$

Now to find the Quantity proper for the Extermination of the second Term $\overline{5e - p} \times y^4$; because $5e - p = 0$, then $5e = p$, and $e = \frac{p}{5}$: now putting $\frac{p}{5}$ universally

CXXVII. *Moreover, the Roots of Equations may be multiplied or divided by given Numbers; and after this Rate, the Terms of Equations be diminished, and Fractions and radical Quantities sometimes be taken away.*

As if the Equation were $y^3 - \frac{4}{3}y - \frac{146}{27} = 0$; in order to take away the Fractions, I suppose y to be $= \frac{1}{3}z$; and

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for the Dimensions of the Equation, the Quantity will be $y \pm \frac{p}{n}$, as in the above Rule. Again to exterminate the third Term $\overline{10e^2 - 4pe + q} \times y^3$; because $10e^2 - 4pe + q = 0$, then $e = \frac{1}{5}p + \sqrt{\frac{1}{25}p^2 - \frac{1}{5}q}$ (LXXIV); that is, putting $5 = n$, $y \pm \frac{p}{n} + \sqrt{\frac{p^2}{n^2} - \frac{q}{n}}$ will be the Quantity to be substituted: and in like manner the fourth Term may be exterminated, by a Quantity found by the Solution of the cubic Equation, which is the Coefficient of the fourth Term of the transformed, and so on. Hence it appears, that *to find the Quantity proper to be connected with the assumed Letter, to exterminate the 2d, 3d, 4th, 5th, &c. Term of an Equation, there is to be solved a simple, a quadratic, a cubic, a biquadratic Equation, &c. respectively; and consequently there is but one Quantity, which, connected with the unknown, will exterminate the 2d Term; two Quantities the 3d; three the 4th; four the 5th* (CXII); *and so on.*

In like manner an Equation which wants the 2d, 3d, &c. Term, may be transformed into one which shall have that Term, and whose Value shall be e; viz. by finding the Value of e in the Equation of the indeterminate Coefficients of that Term, and substituting $y \pm \frac{p}{n}$, $y \pm \frac{p}{n} + \sqrt{\frac{p^2}{n^2} - \frac{q}{n}}$, &c. for the 2d, 3d, &c. Term, respectively.

and then, by substituting $\frac{1}{3}z$ for y, there comes out this new Equation, $\frac{z^3}{27} - \frac{12z}{27} - \frac{146}{27} = 0$, and having rejected the common Denominator of the Terms, $z^3 - 12z - 146 = 0$, the Roots of which Equation are thrice greater than before. And again, to diminish the Terms of this Equation, if you write $2v$ for z, there will come out $8v^3 - 24v - 146 = 0$, and dividing all by 8, you will have $v^3 - 3v - 18\frac{1}{4} = 0$; the Roots of which Equation are Half of the Roots of the former. And here, if at last you find v, make $2v = z$, $\frac{1}{3}z = y$, and $y + \frac{4}{3} = x$, and you will have x the Root of the Equation $x^3 - 4xx + 4x - 6 = 0$, as first proposed.

And thus, in the Equation $x^3 - 2x + \sqrt{3} = 0$, to take away the radical Quantity $\sqrt{3}$; for x I write $y\sqrt{3}$, and there comes out the Equation $3y^3\sqrt{3} - 2y\sqrt{3} + \sqrt{3} = 0$, which, dividing all the Terms by $\sqrt{3}$, becomes $3y^3 - 2y + 1 = 0$ (*f*).

CXXVII. (*f*) 245. *An Equation may be transformed into another whose Roots shall have a given Ratio to the Roots of the proposed, viz.* y : x :: a : b; *by finding the Value of* x *in an Equation formed from the Proportion; and by substituting this Value and its Powers for* x *and its Powers in the proposed.* Thus if the Equation $x^3 - px^2 + qx - r = 0$ be proposed, because $x = \frac{yb}{a}$ the transformed will be $\frac{y^3b^3}{a^3} - \frac{pb^2y^2}{a^2} + \frac{qby}{a} - r = 0$. Now, by multiplying by a^3, and dividing by b^3, the transformed is $y^3 - \frac{pa}{b}y^2 + \frac{qa^2}{b^2} - \frac{ra^3}{b^3} = 0$. Hence it appears, that, because $x = \frac{yb}{a}$, and that $\frac{b}{a}$ is the Reciprocal of the given Ratio $\frac{a}{b}$, *the Transformation is done by substituting for* x *and its Powers, the Products of* y *and its Powers into the Reciprocal of the given Ratio and*

and its similar Powers; or, which is at length the same thing, by multiplying the Terms of the Equation by the respective Terms of a geometrical Progression, beginning from Unity, in the Ratio of Unity, to a Fraction expressing the given Ratio.

246. Hence, *if it be required to multiply the Roots of an Equation by any given Quantity* a, *for* x *and its Powers,* y *divided by the Multiplier* a, *or* y *multiplied into* $\frac{1}{a}$ *the Reciprocal of the Multiplier, and its Powers, is to be substituted*; for, from the Nature of Multiplication, the Consequent of the Ratio of Numb. 245 is Unity. *Or, which comes to the same thing, by multiplying the Terms of the Equation, by the respective Terms of a Series, (beginning from Unity) of Proportionals, in the Ratio of Unity to the Multiplier.*

247. *If it be required to divide the Roots of an Equation by any given Quantity* b, *for* x *and its Powers substitute* y *multiplied into the Divisor, viz.* yb, *or* y *divided by the Reciprocal of the Divisor, and its Powers*; because the Antecedent of the Ratio is Unity (245). *Or, which comes to the same Thing, the Terms of the Equation are to be divided respectively by the Terms of a Series, beginning from Unity, of continued Proportionals in the Ratio of Unity to the Divisor.*

248. *If the first Term of an Equation has a Coefficient different from Unity, it can be taken away by multiplying the Roots by that Coefficient*; thus $ax^3 - px^2 + qx - r = 0$, multiplied by 1, a, a^2, a^3 (246), becomes $ax^3 - pax^2 + qa^2x - a^3r = 0$. Now, by dividing by a, it becomes $x^3 - px^2 + qax - a^2r = 0$. Whence it appears, that the Operation will be performed at once, *by expunging it from the first Term, and by multiplying the Terms of the Equation, beginning with the third, by the Coefficient, its Square, Cube, &c. respectively.* Hence also

249. *If an Equation has Fractions, it may be transformed into one which shall be clear of Fractions, and whose first Term shall have Unity for its Coefficient; if the Terms being first multiplied by the Product of the Denominators, the Terms beginning with the third, are multiplied by the Coefficient of the*

first Term, and its Powers respectively; as in Numb. 248. and the Coefficient is expunged from the first Term.

250. *An Equation can be cleared of Radicals by Multiplication, when the Radicals recur in such Terms of the Equation, as that when a Series of Proportionals is formed in the Ratio of Unity to the Radical, the Product of the Numerators of the Indices of the factor Terms, in the Equation and Series, can be measured by their common Denominator:* For then the Products will be Unity or Rational (79). Thus if a Quadratic Radical recurs in the alternate Terms, it can be exterminated, as $x^5 - \sqrt{p}x^4 + qx^3 - \sqrt{p}rx^2 + sx - \sqrt{p}t = 0$ is cleared, by multiplying by the respective Terms of the Series $1, \sqrt{p}, p, p\sqrt{p}, p^2, p^2\sqrt{p}$, and becomes $x^5 - px^4 + pqx^3 - p^2rx^2 + p^2sx - p^3t = 0$. So $x^8 - \sqrt[3]{q^2}x^7 + \sqrt[3]{q}x^6 - rx^5 + \sqrt[3]{q^2}x^4 - \sqrt[3]{q}x^3 + sx^2 - \sqrt[3]{q^2}x + \sqrt[3]{q}t = 0$ into $1, \sqrt[3]{q}, \sqrt[3]{q^2}, q, q\sqrt[3]{q}, q\sqrt[3]{q^2}, q^2, q^2\sqrt[3]{q}, q^2\sqrt[3]{q^2}$, becomes $x^8 - qx^7 + qx^6 - qrx^5 + q^2x^4 - q^2x^3 + q^2sx^2 - q^3x + q^3t = 0$. Also $x^8 - \sqrt[2]{p}\sqrt[3]{q^2}\sqrt[4]{r^3}\sqrt[5]{s^4}x^7 + \sqrt[3]{q}\sqrt[4]{r^2}\sqrt[5]{s^3}x^6 - \sqrt[2]{p}\sqrt[4]{r}\sqrt[5]{s^2}x^5 + \sqrt[3]{q^2}\sqrt[5]{s}x^4 - \sqrt[2]{p}\sqrt[3]{q}\sqrt[4]{r^3}tx^3 + \sqrt[4]{r^2}\sqrt[5]{s^4}vx^2 - \sqrt[2]{p}\sqrt[3]{q^2}\sqrt[4]{r}\sqrt[5]{s^3}zx + \sqrt[3]{q}\sqrt[5]{s^2}A = 0$ into $1, \sqrt[2]{p}\sqrt[3]{q}\sqrt[4]{r}\sqrt[5]{s}, p\sqrt[3]{q^2}\sqrt[4]{r^2}\sqrt[5]{s^2}p\sqrt[2]{pq}\sqrt[4]{r^3}\sqrt[5]{s^3}, p^2q\sqrt[3]{qr}\sqrt[5]{s^4}, p^2\sqrt[2]{p}\sqrt[3]{q^2}r\sqrt[4]{rs}, p\,q^2r\sqrt[4]{r^2}s\sqrt[5]{s}, p^3\sqrt[2]{p}q^2\sqrt[3]{qr}\sqrt[4]{r^3}s\sqrt[5]{s^2}, p^4q^2\sqrt[3]{q^2}r^2s\sqrt[5]{s^3}$, becomes $x^8 - pqrsx^7 + pqrsx^6 - p^2qrsx^5 + p^2q^2rsx^4 - p\,q^2r^2stx^3 + p^3q^2r^2s^2x^2 - p^4q^3r^2s^2zx + p^4q\,r^2s^2A = 0$.

251. *The Coefficients, or the last Term of an Equation will be made divisible by any Number or Numbers, if the Roots of the Equation be multiplied by that Number, or the Product of the assigned Numbers.* For if the Equation $x^4 - px^3 + qx^2 - rx + s = 0$ has its Roots multiplied into abc, it becomes

CXXVIII. *Again, the Roots of an Equation may be changed into their Reciprocals, and after this Way the Equation may be sometimes reduced to a more commodious Form* (*g*). Thus, the last Equation $3y^3 - 2y + 1 = 0$, by writing $\frac{1}{z}$ for y, becomes $\frac{3}{z^3} - \frac{2}{z} + 1 = 0$, and all the Terms being multiplied by z^3, and the Order of the Terms changed, $z^3 - 2zz + 3 = 0$. *The last*

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becomes $x^4 - abcpx^3 + a^2b^2c^2qx^2 - a^3b^3c^3rx + a^4b^4c^4s = 0$, (246) all whose Coefficients are divisible by a, or b, or c, or abc.

CXXVIII (*g*) 252. *An Equation may be transformed into another whose Roots shall be the Reciprocals of the proposed, by substituting Unity, or rather the last Term, divided by the assumed Letter* y, *and its Powers, into the Place of* x *and its Powers*; for then the Proportion (245) $x : y :: a : b$, becomes $x : 1 :: 1 : y$, and the Value of x is $\frac{1}{y}$. If we substitute Unity or any other Quantity beside the last Term, we shall have fractional Coefficients, or the highest Term will have a Coefficient different from Unity; thus by substituting $\frac{1}{y}$ for x in the Equation $x^4 - px^3 + qx^2 - rx + s = 0$, we have $\frac{1}{y^4} - \frac{p}{y^3} + \frac{q}{y^2} - \frac{r}{y} + s = 0$; and by transposing, $s - \frac{r}{y} + \frac{q}{y^2} - \frac{p}{y^3} + \frac{1}{y^4} = 0$; and multiplying by y^4, $sy^4 - ry^3 + qy^2 - py + 1 = 0$; and if we divide by s, $y^4 - \frac{r}{s}y^3 + \frac{q}{s}y^2 - \frac{p}{s}y + \frac{1}{s} = 0$: In like manner if we substitute $\frac{a}{y}$ for x, we shall have $sy^4 - ray^3 + qa^2y^2 - pa^3y + a^4 = 0$, or $y^4 - \frac{ra}{s}y^3 + \frac{qa^2}{s}y^2 - \frac{pa^3}{s}y + \frac{a^4}{s} = 0$: Whereas if $\frac{s}{y}$ were substituted, we should have $sy^4 -$

rsy^3

Term but one of an Equation may also by this Method be taken away, provided the second was taken away before, as you see done in the precedent Example. Or if you would take away the last but two, it may be done, provided you have taken away the third before (h). Moreover, the least Root may be thus converted into the greatest, and the greatest into the least, which may be of some Use in what follows (i).

Thus, in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, whose Roots are 3, 2, 1, − 5, if you write $\frac{1}{y}$ for x, there will come out the Equation $\frac{1}{y^4} - \frac{1}{y^3} - \frac{19}{yy} + \frac{49}{y} - 30 = 0$, which, multiplying all the Terms by y^4, and dividing them by 30, the Signs being changed, becomes $y^4 - \frac{49}{30}y^3 + \frac{19}{30}y^2 + \frac{1}{30}y - \frac{1}{30} = 0$, the Roots whereof are $\frac{1}{3}$, $\frac{1}{2}$, 1, $-\frac{1}{5}$; the greatest of the affirmative Roots 3 being now changed into the least $\frac{1}{3}$, and the least 1 being now made greatest, and the negative Root − 5, which of all was the most remote from 0, now coming nearest to it.

There are also other Transmutations of Equations, but which may all be performed after that Way of transmutating we have shewn, when we took away the third Term of the Equation (k).

CXXIX.

$rsy^3 + qs^2y^2 - ps^3y + s^4 = 0$, *i. e.* dividing by s, $y^4 - ry^3 + qsy^2 - ps^2y + s^3 = 0$.

(h) Because by the Operation the Order of the Coefficients is now inverted.

(i) Because as $x = \frac{1}{y}$, and $y = \frac{1}{x}$, when x is greatest, y must be least; and conversely.

(k) 253. *An Equation is transformed into another whose Roots shall be mean Proportionals between the Roots of the proposed and any given Quantity* a, *by substituting for* x *and its Powers, the Square of the assumed* y *divided by the Quantity*

CXXIX. *From the Generation of Equations it is evident, that the known Quantity of the second Term of the Equation, if its Sign be changed, is equal to the Aggregate of all the Roots under their proper Signs; and that of the third Term equal to the Aggregate of the Rectangles of each two of the Roots; that of the fourth, if its Sign be changed, is equal to the Aggregate of the Contents under each three of the Roots; that of the fifth is equal to the Aggregate of the Contents under each four, and so on* ad infinitum (*l*).

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tity a, *and the Powers of this Quote.* For the Proportion (245) $x : y :: a : b$ becomes $x : y :: y : a$, whence the Value of x is $\frac{y^2}{a}$. Thus if in $x^3 - px^2 + qx - r = 0$, $\frac{y^2}{a}$ be substituted for x, we have $\frac{y^6}{a^3} - \frac{py^4}{a^2} + \frac{qy^2}{a} - r = 0$, that is, $y^6 - pay^4 + qa^2y^2 - ra^3 = 0$, whose Roots are mean Proportionals between a and the Roots of the proposed.

254 *An Equation may be transformed into another whose Roots shall be the Square, Cube,* &c. *Roots of the proposed, by substituting for* x *its supposed Value, that is,* y^2, y^3, &c. for since $\sqrt{x} = y$ then $x = y^2$; or if $\sqrt[3]{x} = y$, then $x = y^3$; and hence it is, that Equations can be depressed, if the Indices of all the Terms can be measured by the Index of the lowest Power of x (184): Thus $x^6 + qx^4 + sx^2 + v = 0$ is depressed to $y^3 + qy^2 + sy + v = 0$, by putting $x^2 = y$, whence $x = \sqrt{y}$; and the Roots of the transformed are the square Roots of the proposed (221,) as in N°. 184.

255. *In every kind of Transformation,* if the Quantity by which the Roots are diminished, multiplied, &c. be a real Quantity, *any imaginary Quantity* diminished, &c. by it, *must remain imaginary; wherefore all imaginary Roots remain after Transformation.* It is also manifest, that when the Roots of the transformed are found, those of the proposed may be found, from their known Relation to those of the transformed (221.)

CXXIX, (*l*) See Numb. 95, &c.

Let

Let us assume $x = a$, $x = b$, $x = -c$, $x = d$, &c. or $x - a = 0$, $x - b = 0$, $x + c = 0$, $x - d = 0$, and by the continual Multiplication of these we may generate Equations, as above. Now, by multiplying $x - a$ by $x - b$, there will be produced the Equation

$$xx \begin{array}{l} -a \\ -b \end{array} x + ab = 0;$$

where the known Quantity of the second Term, if its Signs are changed, viz. $a + b$, is the Sum of the two Roots a and b, and the known Quantity of the third Term is the only Rectangle contained under both. Again, by multiplying this Equation by $x + c$, there will be produced the cubick Equation

$$x^3 \begin{array}{l} -a \\ -b \\ +c \end{array} xx \begin{array}{l} +ab \\ -ac \\ -bc \end{array} x + abc = 0,$$

where the known Quantity of the second Term having its Signs changed, viz. $a + b - c$, is the Sum of the Roots a, and b, and $-c$; the known Quantity of the third Term $ab - ac - bc$ is the Sum of the Rectangles under each two of the Roots a and b, a and $-c$, b and $-c$; and the known Quantity of the fourth Term under its Sign changed, $-abc$, is the only Content generated by the continual Multiplication of all the Roots, a by b into $-c$. Moreover, by multiplying that cubick Equation by $x - d$, there will be produced this biquadratick one;

$$x^4 \begin{array}{l} -a \\ -b \\ +c \\ -d \end{array} x^3 \begin{array}{l} +ab \\ -ac \\ -bc \\ +ad \\ +bd \\ -cd \end{array} xx \begin{array}{l} +abc \\ -abd \\ +bcd \\ +acd \end{array} x - abcd = 0.$$

Where the known Quantity of the second Term under its Signs changed, viz. $a + b - c + d$, is the Sum of all the Roots; that of the third, $ab - ac - bc + ad + bd - cd$, is the Sum of the Rectangles under every two Roots; that of the fourth, its Signs being changed, $-abc + abd - bcd - acd$, is the Sum of the Contents under each Ternary; that of the fifth, $-abcd$, is the only Content under them all.

CXXX. *And hence we first infer, that of any Equation that involves neither Surds nor Fractions all the rational Roots, and the Rectangles of any two of the Roots, and the Contents of any three or more of them, are some of the integral Divisors of the last Term; and therefore when it is evident that no Divisor of the last Term is either a Root of the Equation, or Rectangle, or Content of two or more Roots, it will also be evident that there is no Root, or Rectangle, or Content of Roots, except what is surd* (*m*).

CXXXI. *Let us suppose now, that the known Quantities of the Terms of any Equation under their Signs changed, are* p, q, r, s, t, v, *viz. that of the second* p, *that of the third* q, *of the fourth* r, *of the fifth* s, *and so on. And the Signs of the Terms being rightly observed, make* $p = a$, $pa + 2q = b$, $pb + qa + 3r = c$, $pc + qb + ra + 4s = d$, $pd + qc + rb + sa + 5t = e$, $pe + qd + rc + sb + ta + 6v = f$, *and so on* in infinitum, *observing the Series of the Progression. And* a *will be the Sum of the Roots,* b *the Sum of the Squares of each of the Roots,* c *the Sum of the Cubes,* d *the Sum of the Biquadrates,* e *the Sum of the Quadrato-cubes,* f *the Sum of the Cubo-cubes, and so on.*

As in the Equation $x^4 - x^3 - 19xx + 49x - 30 = 0$, where the known Quantity of the second Term is -1, of the third -19, of the fourth $+49$, of the fifth -30; you must make $1 = p$, $19 = q$, $-49 = r$, $30 = s$. And there will thence arise $a = (p =)\ 1$, $b = (pa + 2q = 1 + 38 =)\ 39$, $c = (pb + qa + 3r = 39 + 19 - 147 =) - 89$, $d =$

CXXX. (*m*) It can have no fractional Root, for a fractional Root would give fractional Coefficients, and if they were removed the highest Term would have a Coefficient different from Unity, which is contrary to Supposition; now Roots which are neither Integers nor Fractions must be Surds (161). Also, if the highest Term of an Equation has a Coefficient different from Unity, one, or some, or all the generating Binomes (121) had either a Coefficient in the first Number, or their second Number fractional.

($pc + qb + ra + 4s = -89 + 741 - 49 + 120 =$) 723. Wherefore the Sum of the Roots will be 1, the Sum of the Squares of the Roots 39, the Sum of the Cubes — 89, and the Sum of the Biquadrates 723, viz. the Roots of that Equation are 1, 2, 3, and — 5, and the Sum of these $1 + 2 + 3 - 5$ is 1; the Sum of the Squares, $1 + 4 + 9 + 25$, is 39; the Sum of the Cubes, $1 + 8 + 27 - 125$, is — 89; and the Sum of the Biquadrates, $1 + 16 + 81 + 625$, is 723 (*n*).

CXXXI. (*n*) This Rule follows easily from the algebraical Expressions of the Quantities, and the binomial Theorem. For a being $= p$, and p the Sum of the Roots (CXXIX), $-q$ the Sum of the Products of 2 Roots, their Signs being changed (95), r the Sum of the Products of three Roots (CXXIX), $-s$ the Sum of the Products of 4 Roots, with their Signs changed (95), t the Sum of the Products of 5 Roots, and so on; putting $x+y+z$, &c. for the Roots: Because ($pa - 2q$) the Square of the Sum less twice the Sum of the Products of the Roots is equal to the Sum of the Squares of the Roots, therefore

from $pa = x^2 + y^2 + z^2$, &c. $+ 2xy + 2xz + 2zy$, &c.
subduct $-2q = 2xy + 2xz + 2zy$, &c.
remains $pa + 2q = x^2 + y^2 + z^2$, &c. the Sum of the Squares $= b$.

Again, because pb the Sum of the Squares multiplied into the Sum of the Roots is equal to the Sum of the Cubes, more the Sum of the Squares of each Root into the other Roots; and because $-qa$ the Sum of the Products of two Roots into the Sum of the Roots is equal to the Sum of the Squares of each Root into the other Roots, more thrice the Sum of the Products of three Roots, i. e. $3r$; therefore subducting the latter Products from the former, i. e. $pb - \overline{qa + 3r}$, the Residue, viz. $pb + qa - 3r$ will be the Sum of the Cubes less thrice the Sum of the Products of three Roots, which being therefore added, we have the Sum of the Cubes.

Thus

Thus from $pb = x^3 + y^3 + z^3$, &c. $+ \overline{2y + 2z, \&c.} \times x^2 + \overline{2x + 2z, \&c.} \times y^2 + \overline{2x + 2y} \times z^2$, &c. subduct $-aq = \overline{xyz, \&c.} \times 3 + \overline{2y + 2z, \&c.} \times x^2 + \overline{2x + 2z, \&c.} \times y^2 + \overline{2x + 2y, \&c.} \times z^2$, &c. remains $pb + aq = x^3 + y^3 + z^3$, &c. $- \overline{xyz, \&c.} \times 3$; add $3r = \overline{xyz, \&c.} \times 3$

the Sum $pb + aq + 3r = x^3 + y^3 + z^3$, &c. $= c$, the Sum of the Cubes.

Again, because pc the Sum of the Cubes into the Sum of the Roots is equal to the Sum of the Biquadrates more the Sum of the Cubes of each Root into the other Roots; and because $-qb$ the Sum of the Products of two Roots into the Sum of the Squares is equal to the Sum of the Cubes of each Root into the other Roots more the Sum of the Squares of each Root into the Sums of the Products of two Roots; therefore subducting the latter Products from the former, that is, $-qb$ from pc, the Residue $pc + qb$ is equal to the Sum of the Biquadrates less the Sum of the Squares of each Root into the Sum of the Products of two Roots: Now ra the Sum of the Products of three Roots into the Sum of the Roots is equal to the Sum of the Squares of each Root into the Sum of the Products of two Roots more $-4s$, quadruple the Sum of the Products of four Roots; adding therefore the latter Product to the former Residue, the Sum, $pc + qb + ra$, is equal to the Sum of the Biquadrates, more quadruple the Sum of the Products of four Roots; therefore subducting $-4s$ this quadruple Sum, the Residue is the Sum of the Biquadrates. Thus from $pc = x^4 + y^4$, &c. $+ \overline{y + z, \&c.} \times x^3 + \overline{x + z, \&c.} \times y^3 +$, &c. subduct $-qb = \overline{yz, \&c.} \times x^2 + \overline{xz +, \&c.} \times y^2 + \overline{y + z, \&c.} \times x^3 + \overline{x + z, \&c.} \times y^3$, &c. remains $pc + qb = x^4 + y^4$ &c. $- \overline{yz, \&c.} \times x^2 - \overline{xz + nl +, \&c.} \times y^2$, &c. add $ra = \overline{xyzn + xyzl, \&c.} \times 4 + \overline{yz +, \&c.} \times x^2 + \overline{xz +, \&c.} \times y^2$, &c.

Sum

Sum $pc+qb+ra=x^4+y^4$, &c. $+\overline{xyzn+xyzl}$, &c. $\times 4$
ſubduct $-4s=$ $\overline{xyzn+xyzl}$, &c. $\times 4$

remains $pc+qb+ra+4s=x^4+y^4+z^4$, &c. $=d$. Sum of the Biquadrates.

In like Manner, from $pd=x^5+z^5+$, &c. $+\overline{y+z+}$, &c. $\times x^4+\overline{x+z}$, &c. $\times y^4$, &c.
ſubduct $-qc=\overline{yz}$, &c. $\times x^3+\overline{xz+}$, &c. $\times y^3+\overline{y+z}\times x^4+\overline{x+z+}$, &c. $\times y^4$, &c.
remains $pd+qc=x^5+y^5$, &c. $-\overline{yz+}$, &c. $\times x^3-\overline{xz+}$ &c. $\times y^3$, &c.
add $rb=\overline{yzn+yzm}$, &c. $\times x^2$, &c. $+\overline{yz+}$, &c. $\times x^3$ &c.
the Sum $pd+qc+rb=x^5+y^5$, &c. $+\overline{yzn+yzm}$, &c. $\times x^2$, &c.
ſubduct $-sa=\overline{xyznl+yznlm}$, &c. $\times 5+\overline{yzn+yzm}$, &c. $\times x^2$, &c.
rem. $pd+qc+rb+sa=x^5+y^5$, &c. $-\overline{xyznl+yznlm}$, &c. $\times 5$
add $5t=$ $\overline{xyznl+yzlmn}$, &c. $\times 5$

Sum $pd+qc+rb+sa+5t=x^5+y^5+z^5$, &c. $=e$. Sum of Quadratocubes, and ſo on.

Now that the Sum of the Roots multiplied into the Sum of the Products of two Roots, of three Roots, of four Roots, &c. is equal reſpectively to triple, quadruple, quintuple, &c. the Sum of the Products of the Roots, by threes, by fours, by fives, &c. more the Sum of the Squares of each Root into the Sum of the other Roots, into the Sum of their Products by twos, by threes, &c. reſpectively, appears from this; that in thoſe reſpective Multiplications, each Root is multiplied either into a Product into which it had entered before, and then the Power of it, either Square, Cube, &c. is produced; or into a Product into which it had not entered before; and then this Product is produced ſo many times as there are Factors in it. Thus xyz is produced by x into yz, by y into xz and by z into xy, viz. thrice, and ſo of every other

other Product of three different Factors, when the Sum of the Roots is multiplied into the Sum of the Products of two Roots. Thus x into yzn, y into xzn, z into zyn, and n into zyz, produces $xyzn$; that is, $xyzn$ is produced four times, and so of every other Product of four different Factors, when the Sum of the Roots is multiplied into the Sum of the Products of three Roots. Thus again, x into $yznm$, y into $xznm$, z into $xynm$, n into $xyzm$, and m into $xyzn$, produces $xyznm$, that is, $xyznm$ is produced five times, and so of every other Product of five different Factors, when the Sum of the Roots is multiplied into the Sum of the Products by four Roots; and so on continually.

Of the LIMITS *of the Roots of* EQUATIONS.

CXXXII. *AND hence are collected the* Limits *between which the Roots of the Equation shall consist, if none of them is impossible. For when the Squares of all the Roots are affirmative, the Sum of the Squares will be affirmative, and therefore greater than the Square of the greatest Root. And by the same Argument, the Sum of the Biquadrates of all the Roots will be greater than the Biquadrate of the greatest Root, and the Sum of the Cubo-Cubes greater than the Cubo-Cube of the greatest Root* (*a*).

CXXXIII. *Wherefore, if you desire the Limit which no Roots can pass, seek the Sum of the Squares of the Roots, and extract its Square Root. For this Root will be* greater *than the greatest Root of the Equation.* But you will come nearer the greatest Root if you seek *the Sum of the Biquadrates, and extract its Biquadratick Root*; *and yet* nearer,

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CXXXII. (*a*) The Sums of the even Powers are affirmative, whether the Roots are affirmative or negative. (88)

if

if you seek the Sum of the Cubo-Cubes, and extract its Cubo-cubical Root; and so on in infinitum. *(b)*

CXXXIII. (*b*) For let all the Roots x, y, z, n, m, &c. be affirmative, and x the least Root, y greater than x, z greater than y, and so on continually. Then since x^2 is to y^2 as x to a third Proportional in the Ratio of x to y, and x^3 is to y^3 as x to a fourth Proportional in the same Ratio; and since x is not greater than y, the Ratio of x^2 to y^2 will not be less than the Ratio of x^3 to y^3, and (comp.) the Ratio $x^2 + y^2$ to y^2 will not be less than the Ratio of $x^3 + y^3$ to y^3 and (altern.) the Ratio of $x^2 + y^2$ to $x^3 + y^3$, will not be less than the Ratio of y^2 to y^3: But since y is not greater than z, the Ratio of 1 to y, that is, of y^2 to y^3 is not less than the Ratio of 1 to z, that is, of z^2 to z^3; therefore the Ratio of $x^2 + y^2$ to $x^3 + y^3$ is not less than the Ratio of z^2 to z^3, and (altern.) the Ratio of $x^2 + y^2$ to z^2 is not less than the Ratio of $x^3 + y^3$ to z^3; and therefore (comp.) the Ratio of $x^2 + y^2 + z^2$ to z^2 is not less than the Ratio of $x^3 + y^3 + z^3$ to z^3. And by the same reasoning, the Ratio of $x^2 + y^2 + z^2 + n^2$ to n^2 will not be less than the Ratio of $x^3 + y^3 + z^3 + n^3$ to n^3, and so on continually; *viz.* the Ratio of the Sum of the Squares to the Square of the greatest Root, or to the Square of that Root, than which there is no greater in the Equation, is not less than the Ratio of the Sum of the Cubes to the Cube of the same Root; that is, it is equal if all the Roots be equal, but greater if any of the Roots be unequal. And after the same Manner, the Ratio of the Sum of the Cubes to the Cube of the greatest Root is not less than the Ratio of the Sum of the Biquadrates to the Biquadrate of the greatest Root, and so on continually. Therefore the Ratio of the Sum of the Squares to a mean Proportional between this Sum and the Square of the greatest Root, is greater than the Ratio of the Sum of the Cubes to the first of two mean Proportionals between this Sum and the Cube of the greatest Root, and this Ratio is greater than the Ratio of the Sum of the Biquadrates to the first of three mean Proportionals between

Thus, in the precedent Equation, the Square Root of the Sum of the Squares of the Roots, or $\sqrt{39}$, is $6\frac{1}{4}$ nearly, and $6\frac{1}{4}$ is farther distant from o than any of the Roots 1, 2, 3, — 5. But the Biquadratick Root of the Sum of the Biquadrates of the Roots, viz. $\sqrt[4]{723}$, which is $5\frac{1}{4}$ nearly, comes nearer to the Root that is most remote from nothing, viz. — 5.

CXXXIV. *If, between the Sum of the Squares and the Sum of the Biquadrates of the Roots you find a mean Proportional, that will be a little greater than the Sum of the Cubes of the Roots connected under affirmative Signs* (*c*). *And hence,*

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tween this Sum and the Biquadrate of the greatest Root, and so on continually: that is, the Ratio of the square Root of the Sum of the Squares to the greatest Root is greater, or more remote from the Ratio of Equality, than the Ratio of the Cube Root of the Sum of the Cubes to the greatest Root; and this Ratio is greater than the Ratio of the Biquadrate Root of the Sum of the Biquadrates to the greatest Root, and so on continually. Now should any of those Roots be changed into negative, yet those Powers of them, whose Indices are even Numbers, will continue affirmative (88); and consequently, it follows universally, that the Ratio of the Square Root of the Sum of the Squares to the greatest Root, or most distant from nothing, is greater than the Ratio of the Biquadrate Root of the Sum of Biquadrates to the greatest Root; and this greater than the Ratio of the Cubo-Cubic Root of the Sum of the Cubo-Cubes to the greatest, and so on continually.

CXXXIV. (*c*) Since all the Roots under affirmative Signs are not equal, and since x^2, x^3, x^4, &c. are continually Proportional, also y^2, y^3, y^4, &c. and z^2, z^3, z^4, &c. &c. therefore the Products of the corresponding Terms $x^2 \times y^2$, $x^3 \times y^3$, $x^4 \times y^4$, will be continued Proportionals; that is, $x^3 \times y^3$ is a geometrical Mean between $x^2 \times y^2$ and $x^4 \times y^4$, and therefore between $x^2 \times y^4$ and $x^4 \times y^2$; therefore the Sum of the Extremes is greater than *hence, the half Sum of this mean Proportional, and of the Sum of the Cubes collected under their proper Signs, found as before, will be greater than the Sum of the Cubes of the Affirmative Roots, and the half Difference greater than the Sum of the Cubes of the Negative Roots.*

CXXXV. *And consequently, the greatest of the Affirmative Roots will be less than the Cube Root of that half Sum, and*

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than double the Mean, viz. $\overline{x^2 \times y^4} + \overline{x^4 \times y^2}$ is greater than $2 \times \overline{x^3 \times y^3}$ (Eucl. V. 25); but the Product of the Sums $\overline{x^2+y^2}$ into $\overline{x^4+y^4}$ is equal to the Products of x^2 into x^4, and of y^2 into y^4 (that is, to the Squares of x^3 and y^3 (N°. 79.) together with the Rectangles x^2 into y^4 and y^2 into x^4 (Eucl. II. 4.) now the Square of the Sum $x^3 \times y^3$ is equal to the Squares of x^3 and y^3 together, with $2 \times \overline{x^3 \times y^3}$ (Eucl. II. 4.) therefore the Product of the Sum $x^2 \times y^2$ into $x^4 \times y^4$ is greater than the Square of the Sum x^3+y^3. Whence if a third Proportional P be taken in the Ratio of x^2+y^2 to x^3+y^3, it will be less than x^4+y^4, and (as was shewn before) the Product of the Sum $x^2+y^2+z^2$ into the Sum $\overline{P+z^4}$, cannot be less than the Square of the Sum $x^3+y^3+z^3$; and consequently, the Product of the Sum $x^2+y^2+z^2$, into $x^4+y^4+z^4$, will be greater than the Square of $x^3+y^3+z^3$. In like manner, the Product of the Sum $x^2+y^2+z^2+n^2$ into $x^4+y^4+z^4+n^4$, is greater than the Square of the Sum $x^3+y^3+z^3+n^3$, and so on continually, viz. If between the Sum of the Squares and the Sum of the Biquadrates a mean Proportional be taken, it will be greater than the Sum of the Cubes under affirmative Signs. And after the same manner, if a mean Proportional be taken between the Sum of the Biquadrates and Sum of the Cubo-cubes, it will be greater than the Sum of the Quadrate-cubes, and so on *in infi- um.*

and the greatest of the Negative Roots less than the Cube Root of that Semi-difference (*d*).

Thus, in the precedent Equation, a mean Proportional between the Sum of the Squares of the Roots 39, and the Sum of the Biquadrates 723, is nearly 168. The Sum of the Cubes under their proper Signs was, as above,—89, the half Sum of this and 168 is $39\frac{1}{2}$, the Semi-difference is $128\frac{1}{2}$. The Cube Root of the former, which is about $3\frac{1}{2}$, is greater than the greatest of the Affirmative Roots 3. The Cube Root of the latter, which is $5\frac{1}{21}$ nearly, is greater than the Negative Root —5. By which Example it may be seen how near you may come this Way to the Root, where there is only one Negative Root or one Affirmative one.

CXXXVI. *And yet you might come nearer still, if you found a mean Proportional between the Sum of the Biquadrates of the Roots and the Sum of the Cubo-Cubes, and if from the Semi-Sum and Semi-Difference of this, and of the Sum of the Quadrato-Cube of the Roots, you extracted the Quadrato-Cubical Roots. For the Quadrato-Cubical Root of the Semi-Sum would be greater than the greatest Affirmative Root, and the Quadrato-Cubic Root of the Semi-Difference would be greater*

CXXXV. (*d*) For the Sum of the Cubes under their proper Signs added to the Sum of the Cubes taken affirmatively, is double the Sum of the Cubes of the affirmative Roots, (N°. 22.) and the Sum of the Cubes under their proper Signs subducted from the Sum of the Cubes taken affirmatively, is double the Sum of the Cubes of the negative Roots (36); but a mean Proportional between the Sum of the Squares and the Sum of the Biquadrates, is greater than the Sum of the Cubes taken affirmatively (CXXXIV.); therefore half the Sum of this mean Proportional, and of the Sum of the Cubes of the Roots under their proper Signs, exceeds the Sum of the Cubes of the affirmative Roots, and half their difference exceeds the Sum of the negative Roots; wherefore, by extracting the Cube Roots of the half Sum and of the half Difference, Limits are found which exceed the greatest affirmative, and the greatest negative Root.

greater than the greatest negative Root, but by a less Excess than before (e). Since therefore, any Root, by augmenting or

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CXXXVI. (*e*) By the same method of Reasoning it follows, that if a mean Proportional be taken between the Sum of the Biquadrates and Sum of the Cubo-Cubes, half the Sum of this Mean and the Sum of the Quadrato-Cubes of the Roots under their proper Signs, will be greater than the Sum of the Quadrato-Cubes of the Affirmative Roots, and half the Difference greater than the Sum of the Quadrato-Cubes of the negative Roots; and, consequently the Quadrato-Cubic Root of this half Sum, will be greater than the greatest affirmative Root; and the Quadrato-Cubic Root of the half Difference greater than the greatest negative Root, and so on continually.

256. The Cubic and Quadrato-Cubic Roots of the half Sum and half Difference being greater than the Sum of the affirmative and Sum of the negative Roots; (CXXXIV. *c.*) and the Sum of the affirmative Roots, when many, being greater than the greatest affirmative Root; also the Sum of the negative Roots, when many, being greater than the greatest negative Root; it follows, that when there is but one affirmative, or one negative Root, it is itself the Sum; and that therefore *the Cubic and Quadrato-Cubic Roots of the half Sum and half Difference, are nearer Limits to the greatest affirmative and negative Roots, when they are single in the Equation, than when there are more than one of each.*

257. *The Ratio, which the Excess of a mean Proportional between the Sum of the Squares and the Sum of the Biquadrates of the Roots above the Sum of their Cubes under affirmative Signs, has to the Cube of the greatest Root, is greater than the Ratio, which the Excess of a mean Proportional between the Sum of the Biquadrates of the Roots, and the Sum of their Cubo-Cubes above the Sum of their Quadrato-Cubes under affirmative Signs has to the Quadrato-Cube of the greatest Root; and this Ratio is greater than the Ratio which the Excess of a mean Proportional between the Sum of the* [illegible]*o-Cubes of the Roots and the Sum of their Biquadrato-*[illegible]*uadrates above the Sum of their Biquadrato-Cubes under affirmative*

or diminishing all the Roots, may be made the least, and then the

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affirmative Signs has to the Biquadrato-Cube of the greatest Root, and so on continually: for let R, S, T, denote respectively those above-mentioned mean Proportionals; r, s, t, respectively the Sums of the Cubes, Quadrato-Cubes, and Biquadrato-Cubes, under affirmative Signs; and c, e, g, respectively the Sums of the Cubes, Quadrato-Cubes, and Biquadrato-Cubes, under their own Signs: Then is $\frac{R+c}{2}$ the Sum of the Cubes of the affirmative Roots increased by the Quantity $\frac{R-r}{2}$; and $\frac{R-c}{2}$ the Sum of the Cubes of the negative Roots changed into affirmative, and increased by $\frac{R-r}{2}$: Also $\frac{S+e}{2}$ the Sum of the Quadrato-Cubes of the affirmative Roots, increased by the Quantity $\frac{S-s}{2}$; and $\frac{S-e}{2}$ the Sum of the Quadrato-Cubes of the negative Roots changed into affirmative, and increased by the Quantity $\frac{S-s}{2}$: Also $\frac{T+g}{2}$ the Sum of the Biquadrato-Cubes of the affirmative Roots, increased by the Quantity $\frac{T-t}{2}$; and $\frac{T-g}{2}$ the Sum of the Biquadrato-Cubes of the negative Roots changed into affirmative, and increased by $\frac{T-t}{2}$, and so on continually. But the Ratio of the Sum of the Cubes of the Roots, whether Affirmative or Negative under affirmative Signs, to the Cube of the greatest Root, viz. most remote from nothing under an affirmative Sign, is greater than the Ratio of the Sum of the Quadrato-Cubes of the same Roots to the Quadrato-Cube of the greatest Root; and this Ratio greater than the Ratio of the

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the least converted into the greatest, and afterwards all besides the

the Sum of the Biquadrato-Cubes to the Biquadrato-Cube of the greatest Root, and so on continually: Whence because also the Ratio of $\frac{R-r}{2}$ to the Cube of the greatest Root is greater than the Ratio of $\frac{S-s}{2}$ to the Quadrato-Cube of the greatest Root, and this Ratio greater than the Ratio of $\frac{T-t}{2}$ to the Biquadrato-Cube of the greatest Root, and so on continually; it follows, that the Ratio of $\frac{R+r}{2}$ to the Cube of the greatest Root is greater than the Ratio of $\frac{S+s}{2}$ to the Quadrato-Cube of the greatest Root, and this Ratio greater than the Ratio of $\frac{T+t}{2}$ to the Biquadrato-Cube of the greatest Root, and so on continually.

258. *If* $\frac{R\pm c}{2}$, $\frac{S\pm e}{2}$, $\frac{T\pm g}{2}$, *be respectively greater than the Cube, Quadrato-Cube, Biquadrato-Cube, of the greatest Root*; then the Ratio of $\frac{R+c}{2}$ to the first of two mean Proportionals between $\frac{R+c}{2}$ and the Cube of the greatest Root, will be greater than the Ratio of $\frac{S+e}{2}$ to the first of four mean Proportionals between $\frac{S+e}{2}$ and the Quadrato-Cube of the greatest Root; and this Ratio greater than the Ratio of $\frac{T+g}{2}$ to the first of six mean Proportionals between $\frac{T\pm g}{2}$ and the Biquadrato-Cube of the greatest

the greatest be made negative, it is manifest how any Root desired may be found nearly (*f*).

CXXXVII.

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greatest Root, and so on continually: that is, *the Cube Root of the Quantity* $\frac{R \pm c}{2}$ *will be greater, and therefore the more remote from the greatest affirmative or negative Root of the Equation, than the Quadrato-Cubic Root of the Quantity* $\frac{S \pm e}{2}$; *and this Root exceeds the greatest Root more than the Biquadrato-Cubic Root of the Quantity* $\frac{T \pm g}{2}$ *exceeds it, and so on continually. But if* $\frac{R \pm c}{2}$, $\frac{S \pm e}{2}$, $\frac{T \pm g}{2}$, *be respectively less than the Cube, Quadrato-Cube, Biquadrato-Cube, of the greatest Root, it may happen* that the Ratio of $\frac{R \pm c}{2}$ to some intermediate Cube N^3 between $\frac{R \pm c}{2}$ and the Cube of the greatest Root is less than the Ratio of $\frac{S \pm e}{2}$ to N^5, and this Ratio less than the Ratio of $\frac{T \pm g}{2}$ to N^7; and consequently, *that the Cube Root of the Quantity* $\frac{R \pm c}{2}$ *may exceed the greatest Root by a less Quantity than the Quadrato-Cubic Root of* $\frac{S \pm e}{2}$; *and this by a less Quantity than the Biquadrato-Cubic Root of* $\frac{T \pm g}{2}$, *and so on continually.*

(*f*) *When it is known that there is but one affirmative or one negative Root, and that it is the greatest*, and consequently that $\frac{R \pm c}{2}$, $\frac{S \pm e}{2}$, $\frac{T \pm g}{2}$, are greater respectively than the Cube, Quadrato-Cube, Biquadrato-Cube, &c. of the greatest

CXXXVII. *If all the Roots except two are negative, those two may be both together found this Way.*

The Sum of the Cubes of those two Roots being found according to the precedent Method, as also the Sum of the Quadrato-Cubes, and the Sum of the Quadrato-Quadrato-Cubes of all the Roots: between the two latter Sums seek a mean Proportional, and that will be the Difference between the Sum of the Cubo-Cubes of the affirmative Roots, and the Sum of the Cubo-Cubes of the negative Roots nearly; and consequently, the half Sum of this mean Proportional, and of the Sum of the Cubo-Cubes of all the Roots, will be the Sum of

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greatest Root; *the superior Limit of the greatest affirmative and greatest negative Root, will be had by extracting the Cube Root of the Quantity* $\frac{R+e}{2}$; *but more accurately, by extracting the Quadrato-Cubic Root of the Quantity* $\frac{S+e}{2}$; *and nearer still, by extracting the Biquadrato-Cubic Root of* $\frac{T+g}{2}$, *and so on continually: But if it appears that the single Root is not the greatest of the Equation, the Cube Root of* $\frac{R+c}{2}$ *may be a more accurate Limit than the Quadrato-Cubic Root* $\frac{S+e}{2}$, *and this more accurate than of the Biquadrato-Cubic Root of* $\frac{T+g}{2}$, *and so on continually* (258). *Whence it is, that the Author guards against a Misapplication of this Rule to the finding the superior Limit of any Root when it is not the greatest, to any desired Accuracy, by adding, since any Root, by augmenting or diminishing all the Roots, may be made the least,* (223) *and then the least converted into the greatest* (CXXVII.) *and afterwards all besides the greatest be made Negative, a Method is given, by which any assigned Root may be obtained to any Accuracy; viz. by making each Root in rotation the greatest, a superior Limit can be found for each, which shall no* *of the Cubo-Cubes of the affirmative Roots, and the Semi-Difference will be the Sum of the Cubo-Cubes of the negative Roots. Having therefore both the Sum of the Cubes, and also the Sum of the Cubo-Cubes of the two affirmative Roots, from the Double of the latter Sum subtract the Square of the former Sum, and the Square Root of the Remainder will be the Difference of the Cubes of the two Roots. And having both the Sum and Difference of the Cubes, you will have the Cubes themselves. Extract their Cube Roots, and you will nearly have the two affirmative Roots of the Equation. And if in higher Powers you should do the like, you will have the Roots yet more nearly* (g). But these Limitations,

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not exceed it above a given Difference; that is, the Roots themselves may be approximated as near as you please.

CXXXVII. (*g*) For having found the Sums of all the Powers as high as the Biquadrato-Cubes, the mean Proportional between the Sum of the Squares and Sum of the Biquadrates will be but little greater than the Sum of all the Cubes under affirmative Signs (CXXXIV). Whence $\frac{R+c}{2}$ the half Sum of this mean Proportional, more the Sum of the Cubes under their proper Signs, will but little exceed the Sum of the Cubes of the two affirmative Roots; and the mean Proportional between the Sum of the Biquadrato-Cubes and the Sum of the Quadrato-Cubes under their proper Signs, will but little exceed the Sum of the Cubo-Cubes under their proper Signs; that is, but little exceed the Difference of the Sums of the Cubo-Cubes of the two affirmative and of the negative Roots (XXIV): Whence the half Sum of this mean Proportional more the Sum of all the Cubo-Cubes under affirmative Signs, will but little exceed the Sum of the Cubo-Cubes of the two affirmative Roots, and half their Difference the Sum of the Cubo-Cubes of the negative Roots. Now the Square of the Sum of the Cubes of the two affirmative Roots is equal to the Sum of their Cubo-Cubes (79) more double the Product of their Cubes (Eucl. Book II. Prop. 4.) subducting therefore this Square from double the Sum of their Cubo-Cubes, the Residue tions, by reason of the Difficulty of the Calculus, are of less Use, and extend only to those Equations that have no imaginary Roots. Wherefore I will now shew how to find the Limits another Way, which is more easy, and extends to all Equations (*b*).

Residue is the Sum of their Cubo-Cubes less double the Product of their Cubes, that is, the Square of the Difference of their Cubes (Eucl. Book II. Prop. 7.); wherefore extracting the Square Root of this Residue, there is had the Difference of the Cubes of the two affirmative Roots; and the Sum added to the Difference, is double the Cube of the greater (22. 36.) and the Difference subducted from the Sum is double the Cube of the less, consequently the Cube Root of the half Sum is nearly the greater, and the Cube Root of the half Difference but little exceeds the less affirmative Root.

(*b*) This Rule for finding the superior Limits (were it not for the great Labour of the Calculation, and that it will not serve when there are impossible Roots, for the Root of an imaginary Quantity cannot be approximated to) gives nearer and therefore better Limits than *the Limits obtained by Transformations*, as is next shewn, because these latter *are true, or mean Limits, viz. equidistant from the Roots.*

259. *The greatest negative Coefficient of the Equation, made affirmative and increased by Unity, is a superior Limit to the greatest affirmative Root:* for if all the Coefficients were equal and negative, and the Equation transformed by diminishing the Roots, by any of them increased by Unity, (224) all the Terms of the Transformed must become Affirmative (the Sum of the Affirmative Parts in each being greater than the Sum of the negative Parts) a fortiori therefore, if the Roots are diminished by the greatest increased by Unity when unequal, and some of them Affirmative, the Terms must all become Affirmative, therefore the Roots of the Transformed are all Negative (232): Whence the Quantity by which they were diminished, is greater than the greatest Root

CXXXVIII. *Multiply every Term of the Equation by the Number of its Dimensions, and divide the Product by the Root of the Equation. Then again multiply every one of the Terms that come out by a Number less by Unity than before, and divide the Product by the Root of the Equation. And so go on, by always multiplying by Numbers less by Unity than before, and dividing the Product by the Root, till at length all the Terms are destroyed, whose Signs are different from the Sign of the first or highest Term, except the last. And that Number will be greater than any affirmative Root; which being writ in the Terms that come out for the Root, makes the Aggregate*

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(233). But this is commonly a very remote Limit, and useless if no Term but the last is Negative.

260. $\sqrt[4]{\frac{q^2-2pr+2s}{n}}$ *is a Theorem given by Mr* Mac Laurin *for finding an inferior Limit to the greatest Root*, that is, *from the Square of the Coefficient of the third Term, subduct double the Product of the second and fourth; to the Residue add double the Coefficient of the fifth Term, divide the Sum by the Index of the first Term, and extract the Biquadratic Root of the Quote, and it will be a little less than the greatest Root*; for from the algebraical Expressions of the Quantities and the binomial Theorem, it appears that the Coefficient of the third Term is the Sum of the Products of the Roots, taken two by two; whence its Square is equal to the Sum of the Squares of the Products of two by two, more the Sum of double the Products of the Squares of each Root into the Products of the other Roots taken two by two, and more also the Sums of the Products of the Roots taken four by four; also that the Product of the Coefficients of the second and fourth Terms is the Sum of the Products of the Squares of each Root into the Products of the Roots taken two by two, more the Sum of the Products of the Roots taken four by four; and that therefore, subducting double this Product from the Square of the third Coefficient, the Residue is the Sum of

Aggregate of those which were each Time produced by Multiplication to have always the same Sign with the first or highest Term of the Equation.

As if there was proposed the Equation $x^5 - 2x^4 - 10x^3 + 30xx + 63x - 120 = 0$. I first multiply this thus;

$$\begin{matrix} 5 & 4 & 3 & 2 & 1 & 0 \\ x^5 & -2x^4 & -10x^3 & +30xx & +63x & -120 \end{matrix}$$

Then I again multiply the Terms that come out divided by x, thus;

$$\begin{matrix} 4 & 3 & 2 & 1 & 0 \\ 5x^4 & -8x^3 & -30xx & +60x & +63 \end{matrix},$$

and dividing the Terms that come out again by x, there comes out $20x^3 - 24xx - 60x + 60$; which, to lessen them, I divide by the greatest

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of the Squares of the Products of the Roots taken two by two, less double the Sum of the Products of the Roots taken four by four: Wherefore adding this double Sum, that is, double the Coefficient of the fifth Term, the Sum is the Sum of the Products of the Square of the Roots taken two by two; that is, the Sum of the Products of the Square of each Root into the Sum of the Squares of the other Roots: Now the Sum of the Products of the Square of each Root into the Sum of the Squares of the other Roots, is less than the Product of the Sum of the Biquadrates of the Roots into half the Index of the highest Term less Unity; and this last Product is less than the Product of the Biquadrate of the greatest Root into the Index of the highest Term; therefore dividing the above Sum of double the Coefficient of the fifth Term, and the Difference of the Square of the Coefficient of the third Term, and the double Product of the second and fourth Terms, by the Index of the highest Term, the Quote is less than the Biquadrate of the greatest Root, and the biquadrate Root of the Quote is less than the greatest Root. Hence, *if the Equation be Cubic, this Limit is to be found by* $\sqrt[4]{\frac{q^2 - 2pr}{n}}$ *because* $2s = 0$.

greatest common Divisor 4, and you have $5x^3-6xx-15x+15$. These being again multiplied by the Progression 3, 2, 1, 0, and divided by x, become $15xx-12x-15$, and again divided by 3 become $5xx-4x-5$. And these multiplied by the Progression 2, 1, 0, and divided by $2x$ become $5x-2$. Now, since the highest Term of the Equation x^5 is Affirmative, I try what Number writ in these Products for x will cause them all to be Affirmative. And by trying 1, you have $5x-2=3$ Affirmative; but $5xx-4x-5$, you have -4 Negative. Wherefore the Limit will be greater than 1. I therefore try some greater Number, as 2. And substituting 2 in each for x, they become,

$$
\begin{array}{l}
5x-2=8 \\
5xx-4x-5=7 \\
5x^3-6xx-15x+15=1 \\
5x^4-8x^3-30xx+60x+63=79 \\
x^5-2x^4-10x^3+30xx+63x-120=46.
\end{array}
$$

Wherefore, since the Numbers that come out 8. 7. 1. 79. 46. are all Affirmative, the Number 2 will be greater than the greatest of the affirmative Roots. In like manner, if I would find the Limit of the negative Roots, I try negative Numbers. *Or that which is all one, I change the Signs of every other Term, and try Affirmative ones.* But having changed the Signs of every other Term, the Quantities in which the Numbers are to be substituted, will become

$$
\begin{array}{l}
5x+2 \\
5xx+4x-5 \\
5x^3+6xx-15x-15 \\
5x^4+8x^3-30xx-60x+63 \\
x^5+2x^4-10x^3-30xx+63x+120.
\end{array}
$$

Out of these I chuse some Quantity wherein the negative Terms seem most prevalent; suppose $5x^4+8x^3-30xx-60x+63$, and here substituting for x the Numbers 1 and 2, there come out the negative Numbers -14 and -33. Whence the Limit will be greater than -2. But substituting the Number 3, there comes out

out the affirmative Number 234. And in like manner in the other Quantities, by substituting the Number 3 for x, there comes out always an affirmative Number, which may be seen by bare Inspection. Wherefore the Number —3 is greater than all the negative Roots. And so you have the Limits 2 and —3, between which are all the Roots (*r*).

CXXXIX.

CXXXVIII. (*r*) Because that when the Roots are diminished by any Quantity *e*, the last Term of the transformed differs nothing from the proposed, but in the letter denoting the unknown; and that the preceding Terms of the transformed are derived according to this Rule from the last Term (226, 227); therefore by treating the proposed according to this Rule, Equations are derived which are the preceding Terms of the transformed, supposing the proposed to be its last Term. Now if the Quantity *e*, by which the Roots are diminished, is greater than the greatest affirmative Root, the Roots become all negative, and all the Terms of the transformed become Affirmative (232); and conversely (233). Therefore that affirmative Quantity, which substituted in all the Equations derived according to this Rule, makes the Results all Affirmative, that is, of the same Sign with the highest Term of the Equation, is greater than the greatest affirmative Root. Again, If the Quantity *e*, by which the Roots are increased, is greater than the greatest negative Root, the Roots become all Affirmative; and all the Terms of the transformed, if the Equation is of even Dimensions, become Affirmative and Negative alternately; but if of odd Dimensions, Negative and Affirmative alternately (238); and conversely (239. Therefore that negative Quantity, which substituted in the given Equation, and in the Expressions by this Rule derived from it, makes the Results, when the Dimensions are even, Affirmative and Negative alternately, and when the Dimensions are odd, Negative and Affirmative alternately, is greater than the greatest negative Root.

Hence

CXXXIX. *But the Invention of these Limits is of Use both in the Reduction of Equations by rational Roots, and in the*

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[1] Hence *the superior Limit of the greatest affirmative Root is found by inquiring the least integer affirmative Number, which, substituted in those Expressions, will give them all Affirmative* (234): And *it must be greater than the Coefficient of the second Term of the transformed divided by the Dimensions*, supposing all the Roots to be Affirmative (for were the Roots all equal it must be greater than this Quote); *and it must not be greater than the greatest negative Coefficient of the proposed made Affirmative, and increased by Unity* (259): And in this Inquiry *we ought always to begin with that Expression thus derived*, that is, that Term of the transformed, *where the negative Roots seem most to prevail.*

Hence also *the superior Limit of the greatest negative Root is found, by inquiring the least integer negative Number, which, substituted in these Expressions, will give them alternately Affirmative and Negative, if the Dimensions are even, but Negative and Affirmative alternately, if odd* (239): And *it must be greater than the Coefficient of the second Term of the transformed divided by the Dimensions, and not greater than the greatest affirmative Coefficient of the proposed made negative, and increased by Unity*; (for if the Signs were changed in the alternate Places, it would be the greatest negative Coefficient, and greater than the greatest affirmative Root (CXXIII); and is therefore now greater than the greatest negative Root): *and in this Inquiry we ought to begin with that Expression or Term, where the affirmative Roots seems most to prevail.*

But the superior Limit of the negative Roots is most easily found by the Substitution of affirmative Numbers, having first changed the Signs of the alternate Terms of the transformed (CXXIII); for the superior Limit of the affirmative Roots of the transformed, will be the superior Limit of the negative Roots of the proposed. CXXIII.

261.

the Extraction of Surd Roots out of them; lest we might sometimes go about to look for the Root beyond these Limits. Thus,

261. *An Equation shall be transformed into another, which shall have all its Roots affirmative, by substituting for* x *the superior Limit of the affirmative Roots* e, *diminished by* y, *the assumed Letter denoting the unknown; viz. by substituting* e — y: for the Excess being always positive, and the Terms of the highest Power of the substituted Quantity *e* — *y*, having their Signs alternately + and —, and making a Part of every Term of the transformed, will be therefore always greater than the Parts of the same Terms, which have opposite Signs; so that the Signs of this highest Power every where prevailing, the Terms of the transformed will be alternately Affirmative and Negative: whence all its Roots are Affirmative (142).

262. Again, *having found* — e *the superior Limit of the negative Roots, if for* x *we substitute* y — e, *all the Roots of the Transformed will become Affirmative; for all the Terms, as before, will be alternately Affirmative and Negative, and therefore all the Roots Affirmative.* If the Roots of the proposed are all Negative, it is plain that they will become all Affirmative, by changing the Signs of the alternate Terms (CXXIII). Having found the superior Limit of the greatest affirmative Root, the Limits of the other Roots or *the mean Limits are found in the following Manner, in which we suppose, for the sake of Brevity, that all the Roots of the proposed are Affirmative*; it being an easy Transmutation of any Equation (261, 262). Let also *the Roots be denominated 1st, 2d, 3d, &c. according to their Magnitude, the least being the first, &c.*

263. Having found *the superior Limit of the greatest Root, if it be substituted for the unknown* x, *the Result will be positive* (234, 209, 261) for no Number which is not a Root, if all the Roots are possible, can make the Result = 0; and if there are impossible Roots, their Product is Affirmative, and cannot alter the Sign of the Product of the possible Roots.

If

Thus, in the last Equation, if I would find the rational Roots, if perhaps it has any; from what we have said, it

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If 0 *be substituted for* x, by destroying all the Terms in which x was, *the Result will be the last Term with its proper Sign*, that is + if the Dimensions of the Equations are even, and — if odd; for by Supposition the Roots are all Affirmative, and therefore the Factors all Residuals (181); that is, the Signs of the Terms are alternately + and — (59); and the last +, or —, according as the Index is even, or odd, (88).

If for x *a Number less than the least Root be substituted, the Sign of the resulting Quantity will be the same as that of the last Term*; for all the Factors still retain their own Signs, and are Residuals.

If for x *a Number, mean between the first and second Roots*, (that is, greater than the least, and less than the second) *be substituted, the Sign of the resulting Number will be contrary to the Sign of the last Term of the proposed*; for the least Factor is become positive, or a Binomial, and the others remaining Residual, the Sign of the Product is contrary.

If for x *a Number, mean between the second and third Roots, be substituted, the Result will have the Sign of the last Term of the proposed*; for two Factors having changed their Signs, the Sign of the Product will remain the same as before they were changed, and therefore the Sign of the whole Product will be the same as before: *and so on continually*; that is, *if there be successively substituted Numbers less than the least Root, and mean between all the Roots, the Results will be in order the same, and the contrary alternately, with the Sign of the last Term of the proposed.*

264. *And conversely, If Numbers successively substituted for* x, *give Results whose Signs are the same and contrary alternately to the Signs of the last Term of the proposed, those Numbers* are in Succession, less than the least Root, mean between the 1st, 2d, 3d, &c. Roots of the proposed; and therefore *limit them*. Whence, *if two Numbers*

it is certain they can be no other than the Divisors of the last Term of the Equation, which here is 120. Then

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bers substituted for x, give Results with contrary Signs, one or more Roots are limited by those Numbers; for the Signs being contrary, one is contrary to the Sign of the last Term of the proposed, and therefore one, or an odd Number of Factors, must have changed their Signs.

265. *If the Roots of an Equation be successively diminished by Quantities equal to its Roots, beginning with the least or first Root, the last Term of the transformed will be always exterminated*, (229); and its Penultimate or rather (having reduced the Dimensions by dividing by x) *the last Term of the reduced is the Product of the Excesses of the other Roots of the proposed above that Root, whereby they were diminished in the Transformation* (232). Now, *when they are diminished by the least, or first Root*, those Excesses remain all positive; but the Factors, or Excesses, or Residuals, are one less in Number in the reduced, than in the proposed; *therefore the Sign of the last Terms of this reduced, and of the proposed, are contrary.* Again, *if the Roots are diminished by the second Root*, the Product of the Excesses of the other Roots above it, that is, *the last Term of this reduced will have its Sign the same with that of the last Term of the proposed*; for one Excess is become negative, i. e. one Factor a Binomial, and the residual Factors are also less by one, whence the whole Number of residual Factors is diminished by an even Number, and consequently their Product retains its Sign; that is, the Signs of the last Terms of the reduced and proposed are the same. Again, *if the Roots are diminished by the third Root, the Sign of the last Term of the reduced will be contrary to that of the last Term of the proposed*; for two Excesses will have become negative, that is, two Factors will have become binomial, and one residual Factor is wanting: whence the whole Number of Residuals is diminshed by three, an odd Number, and therefore the Sign of the Product is changed, *and so on continually.* Whence, *if the Roots of*

Then trying all its Divisors, if none of them writ in the Equation for *x* would make all the Terms vanish, it is certain

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of the proposed, beginning with the least, are successively substituted for x *in the Equation reduced by one Dimension, the Signs of the Results will be alternately the same, and contrary to the Sign of the last Term of the reduced* (for they must be alternately contrary and the same with the Sign of the last Term of the proposed) that is, *the least Root of the proposed is less than the least of the reduced, and the greatest of the proposed greater than the greatest of the reduced; and the intermediate Roots of the proposed are mean between the Roots of the reduced, and consequently are Limits to them.*

266. And *conversely, the Roots of the reduced are mean Limits to the Roots of the proposed: Hence the reduced is called the Equation of Limits; and if to its Roots be added Cipher and the superior Limit of the greatest Root of the proposed, there are given all the Limits of the Roots of the proposed, and each Equation is the Equation of Limits to the other.*

267. Hence, *if the proposed Equation be reduced by successive Multiplications and Divisions, as directed by this Rule, to a simple Equation, the Root of the simple Equation is the mean Limit of the Roots of the Quadratic, whose Roots are the mean Limits of the Roots of the Cubic, and so on to the proposed: so that there is a compleat Series of Equations from the simple Equation to the proposed, each of which determines the Limits of the following Equation: And conversely, the proposed contains all the Limits of the Roots of the first Equation of Limits, whose Roots are all the Limits of the Roots of the second Equation of Limits; and so on, descending through all the Equations deducible from the proposed by successive Multiplications of the Terms by their Indices, and Divisions by the Root; or because the Indices are in arithmetical Progression, by Multiplications by the Terms of any arithmetical Progression.*

268. If two Roots of the proposed Equation are equal, then their intermediate Limit must be equal to each of them; and this Limit is a Root of the Equation of Limits:

certain that the Equation will admit of no Root, but what is Surd. But there are many Divisors of the last Term 120, *viz.* 1. — 1. 2. — 2. 3. — 3. 4. — 4. 5. — 5. 6. — 6. 8. — 8. 10. — 10. 12. — 12. 15. — 15. 20. — 20. 24. — 24. 30. — 30. 40. — 40. 60. — 60. 120. and — 120.

To

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mits: *If* therefore *in substituting a Root of the Equation of Limits in the proposed, the Result is* = 0, *then the Limit is a Root of the proposed, and two Roots of the proposed are equal; and as often as such a Result emerges equal to nothing, so many Pair of Roots in the proposed will be equal, and each of them equal to that Limit.*

269. *No rational Number whatever substituted for* x *will give a Result* = 0, *if all the Roots of the Equation be imaginary* (206): for no rational Number can be equal to an imaginary one; and though a Number equal to the real Part of an imaginary Root should be substituted, yet the positive Product of the radical imaginary Part will always remain (193), and the Result always be affirmative.

270. *If two rational Limits are found for every Root of an Equation, the Roots are all real and unequal: If the Alternation of the Signs of the results emerging from the Substitution of rational Numbers is interrupted by a Cypher only, the Roots are all real, and there are so many pairs of equal Roots as Interruptions. If the Alternation is interrupted not by Cypher, but by the Intervention of positive Results in the place of negative, then so many pair of Roots are imaginary.*

271. *If any Roots of the Equation of Limits are impossible, or imaginary, there must be so many at least impossible in the proposed:* for the last Term of the Equation of Limits is the Product of the Excesses of the Roots of the proposed above the Quantity whereby they were Diminished (232.); if therefore there are any impossible Expressions in those Excesses, there must of consequence be impossible Expressions in the Roots of the

Pro-

To try all these Divisors would be tedious. But it being known that the Roots are between 2 and — 3, we are freed from that Labour. For now there will be no need to try the Divisors, unless those only that are within these Limits, *viz.* the Divisors 1, and — 1. and — 2. For if none of these are the Root, it is certain that the Equation has no Root but what is Surd (*k*).

The

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proposed. *But it does not follow, that if all the Roots of the Equation of Limits are Real, the Roots of the proposed shall all be Real*; because the Roots of the Equation of Limits are not all the Limits, but only the mean Limits of the Roots of the proposed: *Yet it will follow, that if all the Roots of the proposed are Real, all the Roots of all the Equations of Limits deducible from it, are also real*; for the Roots of the proposed are all the Limits of the first Equation of Limits, and so on; but of imaginary Quantities there can be no real Limits.

CXXXIX. (*k*) When any imaginary Roots are in the proposed, the superior Limit of the greatest affirmative Root will be less accurately determined, if their Product, which is always Affirmative, bears any considerable proportion to that of the affirmative Roots.

The Reduction of EQUATIONS *by Surd Divisors.*

CXL. HItherto I have treated of the Reduction of Equations which admit of rational Divisors. (a) But *before we can conclude, that an Equation of four*

CXL. (a) It may be of Use to set forth in one View the most usual Methods of finding the rational Roots of Equations, whose Dimensions ascend above the Quadratic, the Reduction of which is given in Art. LXXIV.

The first and *most general Method is by finding the Divisors of the last Term,* Art. CXXX. which being the Product of all the Roots, as many of the Roots as are Rational, must be found among the Divisors of the last Term: *Every Divisor of the last Term, which substituted for* x *causes the Aggregate to vanish, or, which being connected with* x *measures the Aggregate of the Terms,* (*Art.* CX. CXIII.) *is a Root.*

If the Equation be Cubic, it is sufficient to find the simple Divisors; if Biquadratic, the Divisors of two Dimensions are also to be sought; and in general, for Equations of superior Dimensions, Divisors are to be sought, whose Dimensions are half, or one Degree lower than half the Dimensions of the proposed. (212. 164.)

In general, *when all the Roots are found, except two, those two are most expeditiously found, by finding the Roots of the Quadratic Quote, which will emerge by dividing the proposed by the Product of the Divisors already found.*

But as the Divisors of the last Term of the proposed Equation may be numerous, *the Labour of Substitution will be abridged, by dividing the Roots by their common Divisor if they admit of one* (247.); and when the Submultiples of the Roots are found, the Roots themselves will be had by their known Relation to those of the Transformed (221).

 The

four, six, or more Dimensions, is irreducible, we must first try whether

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The Number of Divisors to be substituted is further abridged by finding the superior Limit of the greatest Root of the Proposed by Art. CXXXV. or CXXXVIII.

But this Number will be reduced to the real Roots by transforming the proposed into others successively, whose Roots shall be greater and greater, and less and less, by Unity, than those of the proposed: For the Values of x *in the proposed are some Divisors of the last Term, when* x *is supposed equal to Cypher or nothing; and the Values of* y, *in the transformed Equations, are some of the Divisors of their last Terms respectively; and these Values must be in Arithmetical Progression, whose common Difference is Unity, because* x — 2, x — 1, x, x + 1, and x + 2, *&c. are in Arithmetical Progression: But the Substitution of* 2, 1, 0, — 1 — 2, *&c. for* x *in the Proposed, is equivalent to these Transformations* (234.) *Whence the Rule of Art.* L. *abridges the simple Divisors of the last Term, and that of Art.* LI. *abridges its Divisors of two Dimensions, to the Roots.*

If the proposed involves two or more Letters, the Rules of Art. LII. *and* LIII. *are to be observed.*

The other general Method of resolving higher Equations, whose Roots are rational, is to exterminate the second Term of the proposed (Art. CXXV.) *to find the Roots of the Transformed, and from these to find the Roots of the proposed* (221). Now *the Roots of the Transformed* wanting its second Term, *are found, either by Approximation; or accurately and without Approximation.*

By Approximation, thus: Let $x^3 . * . qx . r = 0.$ be a Cubic, which wants the second Term. If all its Roots are real, it will have its third Term qx (244) Negative; but if its third Term is Affirmative, it generally has two impossible Roots (244.) And in both Cases, either one negative Root is equal to two Affirmatives, or two Negatives to one Affirmative (243).

If

whether or not it may be reduced by any Surd Divisor; or, which

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If all the Roots are real, and one Negative is equal to two Affirmatives, the last Term will be Affirmative (because $- \times - \times + = +$*); and of this form* $x^3 - qx + r = 0$*; consequently the negative Root, being single and the greatest, will be nearly (Art.* CXXXVI.) $\sqrt[3]{\frac{R-c}{2}}$*; more near* $\sqrt[5]{\frac{S-e}{2}}$*; nearer still* $\sqrt[7]{\frac{T-g}{2}}$*: &c. And if one Affirmative (the Roots being all real) is equal to the Sum of two Negatives, the last Term will be Negative (because* $+ \times + \times - = -$*); and the Equation of this Form* $x^3 - qx - r = 0$*; consequently, the affirmative Root, being single and the greatest, will be nearly* $\sqrt[3]{\frac{R+c}{2}}$*; more near* $\sqrt[5]{\frac{S+e}{2}}$*; nearer still* $\sqrt[7]{\frac{T+g}{2}}$*, &c. Now the greatest Root being found to any Accuracy (Art.* CXXXVI.) *the other Roots are had by the Reduction of the Quadratic Quote. If two Roots of* $x^3 . * . qx . r = 0$. *be imaginary, the real Root will be equal to them both; and if the Equation be of the Form* $x^3 * + qx + r = 0$, *the Rational will be Negative and greatest; and if the Equation be of the Form* $x^3 * + qx - r = 0$, *the Rational will be Affirmative and greatest: And consequently the rational Root is* $\sqrt[3]{\frac{R \pm c}{2}}$*, or* $\sqrt[5]{\frac{S \pm e}{2}}$*, or* $\sqrt[7]{\frac{T \pm g}{2}}$*; &c. as in Art.* CXXXVI.

Let a Biquadratic want (or be so Transformed as to want) the second Term, as $x^4 . * . qx^2 . rx . s = 0$. *If all its Roots are real,* the third Term qx^2 and the last s, will be Negative (244.) Now because it wants the second Term, either first, one Affirmative is equal to three Negatives; or, secondly, two Affirmatives are equal to two Negatives; or, thirdly, three Affirmatives are equal to one Negative (243.) If the first, then the fourth and fifth Terms rx and s will be Negative, because $- \times + \times + = -$ and *which is the same Thing, you must try whether the Equation can*

$=-$ and $-\times+\times+\times+=-$, *and the Equation of this form* $x^4 - qx^2 - rx - s = 0$. *the Affirmative is single, and greatest, and will therefore be* $\sqrt[3]{\frac{R+c}{2}}$, *or* $\sqrt[5]{\frac{S+e}{2}}$, *or* $\sqrt[7]{\frac{T+g}{2}}$, *&c.* If the second, then if the greatest Root is one of the two Affirmatives, the fourth Term rx will be Negative, and the last Term Affirmative, and the Equation of the Form $x^4 - qx^2 - rx + s = 0$; and if the greatest Root is one of the two Negatives, the fourth Term will be Affirmative, the last Negative, and the Equation of the Form $x^4 - qx^2 + rx - s = 0$; and if the Roots are all equal to each other, the fourth Term vanishes, and the Equation is of the Form $x^4 - qx - s = 0$: *Now in all Cases, the two Affirmatives may be approximated to, by Art.* CXXXVII. *and the two Negatives found by the Reduction of the Quadratic Quote, which will emerge by dividing the Proposed by the Product of the Affirmatives:* If the third Case, then the fourth Term will be Affirmative, the last Negative, and the Equation of the Form $x^4 - qx^2 + rx - s = 0$; *and the Negative is single and greatest; and will be* $\sqrt[3]{\frac{R-c}{2}}$, *or* $\sqrt[5]{\frac{S-e}{2}}$, *or* $\sqrt[7]{\frac{T-g}{2}}$, *&c. and in this, and the first Case, the other Roots are found by Reduction of the cubic Quote, which will emerge by dividing the Proposed by the found Root.* Now tho' the proposed Biquadratic should contain two imaginary Roots, yet if the last Term is Negative, the third being Affirmative, *the rational Roots, if any, will be Approximated to, by Art.* CXXXVI, or CXXXVII.

The Roots of the Transformed, arising from taking away the second Term, are sought directly and without Approximation, thus.

272. *In the cubic* $x^3 . * . qx . r = 0$. *If* $\frac{1}{27}q^3$ *be greater than* $\frac{1}{4}r^2$, *that is, if* $\frac{1}{27}q^3 - \frac{1}{4}r^2$ *be possible, and consequently*

can be so divided into two equal Parts, that you can extract the

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quently $\frac{1}{4}r^2-\frac{1}{27}q^3$ *impossible; then the three Roots are real: Also the two, whose Sum is equal to the third, are unequal.*

273. *If* $\frac{1}{27}q^3=\frac{1}{4}r^2$, *that is, if* $\frac{1}{27}q^3-\frac{1}{4}r^2=0$, *and* $\frac{1}{4}r^2-\frac{1}{27}q^3=0$, *then the three Roots are real; and the two, whose Sum is equal to the third, are equal.*

274. *If* $\frac{1}{27}q^3-\frac{1}{4}r^2$ *be impossible, and consequently* $\frac{1}{4}r^2-\frac{1}{27}q^3$ *possible; then the two Roots, whose Sum is equal to the third, are impossible.* Let first the Roots of $x^3-qx+r=0$ be $x-f+g$, $x-f-g$, and $x+2f$: Then by supposition, $x^3-qx+r=x^3-\overline{3f^2-g^2}\times x+\overline{2f^3-g^2f}$ (CXIII.) whence $3f^2+g^2=q$ (182.) and $2f^3-g^2f=r$: Consequently $f^2+\frac{g^2}{3}=\frac{q}{3}$, and $f^3-g^2f=\frac{r}{2}$. Let secondly the Roots of $x^3-qx-r=0$ be $x+f+g$, $x+f-g$, and $x-2f$: Then $x^3-qx-r=x^3-\overline{3f^2-g^2}\times x-\overline{2f^3+2g^2f}$; whence $3f^2+g^2=q$, and $f^2+\frac{g^2}{3}=\frac{q}{3}$: Also $-2f^3+2g^2f=-r$, and $-f^3+g^2f=-\frac{r}{2}$. Now the Cube of $f^2+\frac{g^2}{3}=\frac{q}{3}$ is in both Cases $f^6+g^2f^4+g^4f^2+\frac{g^6}{27}=\frac{q^3}{27}$; and the Square, as well of $-f^3+g^2f=-\frac{r}{2}$, as of $f^3+g^2f=\frac{r}{2}$, is $f^6-2g^2f^4+g^4f^2=\frac{r^2}{4}$: Wherefore subducting the Square from the Cube, $3g^2f^4-\frac{2g^4f^2}{3}+\frac{g^6}{27}=\frac{q^3}{27}-\frac{r^2}{4}$. In this residue, if f be greater than g, then dividing $3g^2f^4-\frac{2g^4f^2}{3}$ by g^2f^2, the Quote $3f^2$ is greater than $\frac{2}{3}g^2$, consequently the Member $3g^2f^4-\frac{2g^4f^2}{3}+\frac{g^6}{27}$ is Affirma-

 tive,

the Root out of both (*b.*) But that may be done by the following Method. CXLI.

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tive, whence $\frac{q^3}{27}-\frac{r^2}{4}$ is possible, and $\frac{r^2}{4}-\frac{q^3}{27}$ is impossible: And if in that Residue $g=0$, then $3g^2f^4-\frac{2}{3}g^4f^2+\frac{g^6}{27}=0$: whence $\frac{q^3}{27}-\frac{r^2}{4}=0$, and $\frac{q^3}{27}=\frac{r^2}{4}$: And if in the same Residue f is less than g, then $3g^2f^4-\frac{2}{3}g^4f^2+\frac{g^6}{27}$ is Negative, and consequently $\frac{q^3}{27}-\frac{r^2}{4}$ is impossible; and therefore $\frac{r^2}{4}-\frac{q^3}{27}$ is possible.

275. *When* $\frac{1}{27}q^3-\frac{1}{4}r^2$ *is possible, i. e. when* $\frac{1}{4}r^2-\frac{1}{27}q^3$ *is impossible* (272.) *the greatest Root is found thus : Subduct the Coefficient of the third Term, from the square Number next greater than itself; divide the last Term by this Residue; the Quote (which is the Root of the assumed Square) affected with the Sign contrary to that of the last Term will be the greatest Root.* For $q=3f^2+g^2$, and $\overline{2f}^2=4f^2$, and $\frac{4f^2-3f^2-g^2}{f^2-g^2}$ will divide $r=2f^3-2g^2f$, and $\frac{2f^3-2gf}{f^2-2g}=2f$. Also f^2-g^2 will divide $-r=-2f^3+2g^2f$, and $\frac{-2f^3+2gf}{f^2-2g}=-2f$; and as well $-2f\times-2f$, as $2f\times2f=4f^2$. The greatest Root being thus found, the two less are found by reducing the quadratic Quote, whose Roots are to be affected with the Sign of the last Term of the cubic. The Roots of the transformed being found, those of the proposed will be found from their known Relation (221.)

276. *When* $\frac{1}{27}q^3=\frac{1}{4}r^2$, *either of the equal Roots is* $\frac{2}{3}g^4f^2$ *found, by extracting the square Root of* $\frac{1}{3}$ *of the Coefficient of the third Term; or by extracting the cube Root of half the last Term: or, lastly, by dividing triple the last Term by double the Coefficient*

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CXLI. *Dispose the Equation according to the Dimensions of some certain Letter, so that all its Terms jointly under their proper*

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cient of the third Term; and these Roots are to be affected with the Sign of the last Term. Then double of either of these Roots, affected with the Sign contrary to that of the last Term, will be the greatest Root. For since $g=0$, then $3f^2+g^2=3f^2$; but $3f^2+g^2=q$, and $f^2-\frac{g^2}{3}=\frac{q}{3}$, whence $f^2=\frac{q}{3}$, and $f=\sqrt[2]{\frac{q}{3}}$. Likewise $2f^3-2g^2f=2f^3$ but $2f^3-2g^2f=r$, and $f^3-g^2=\frac{r}{2}$; whence $f^3=\frac{r}{2}$, and $f=\sqrt[3]{\frac{r}{2}}$: Also $f=\frac{f^3}{f^2}=\sqrt[2]{\frac{q}{3}},=\sqrt[3]{\frac{r}{2}}=\frac{3r}{2q}$ (149): *So that either of the less Roots is had, by a simple quadratic Extraction, or by a simple Division.*

277. *When* $\frac{1}{4}r^2-\frac{1}{27}q^3$ *is possible,* i. e. $\frac{1}{27}q^3-\frac{1}{4}r^2$ *impossible* (274) *the rational Root is found thus. If the Coefficient of the third Term is affirmative, add it to (if negative, subduct it from) the square Number next greater than itself; then by the Sum, or by the Difference, divide the last Term: this Quote (which if the Equation has a rational Root, will be the Root of the assumed Square) affected with the Sign contrary to that of the last Term will be the rational Root.* For putting the Roots of $x^3\pm qx+r=0$ (viz. when the rational Root is negative) $x-f+\sqrt{-3g^2}$, $\overline{x-f-\sqrt{-3g^2}}$, and $x-2f$; then $x^3\pm qx+r^3=x^3-\overline{3f^2+3g^2}\times x+\overline{2f^3+6g^2f}$; whence $-3f^2+3g^2=\mp q$, and $f^2-g^2=\frac{q}{-3}$; also $2f^3+6g^2f=r$, and $f^3+g^2=\frac{r}{2}$. Again, putting the Roots of $x^3\mp qx-r$ (viz. when the rational Root is affirmative) to be $x+f+\sqrt{-3g^2}$, $\overline{x+f-\sqrt{-3g^2}}$

proper Signs, may be equal to nothing, and let the highest Term be adfected with an affirmative Sign. Then, if the Equation

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$f - \sqrt{-3g^2}$, and $x - 2f$; then $x^3 \mp qx - r = x^3 - \overline{3f^2 + 3g^2} \times x - \overline{2f^3 - 6g^2}$; whence $-3f^2 + 3g^2 = \mp q$, and $f^2 - g^2 = \frac{q}{-3}$; alſo $-2f^3 - 6g^2 f = -r$, and $f^3 + 3g^2 f = \frac{r}{2}$. Wherefore, ſince $\mp q = -3f^2 + 3g^2$, and $2f^{2} = 4f^2$; therefore $\frac{4f^2 - 3f^2 + 3g^2}{f^2 + 3g^2}$, will divide $r = 2f^3 + 6g^2 f$; and the Quote will be equal to $2f$.

But the rational Root in this Caſe is more readily found by the Method of Diviſors, or by Cardan's Rule, Art. CLII. A Biquadratic, which wants the ſecond Term, may be ſolved by the Method of Des Cartes, Art. CLV. but more eaſily by the Method of Diviſors of the laſt Term: and, in general, Biquadratics, and all Equations of higher Dimenſions, if their Roots are all, or any of them, rational, admit thoſe Roots to be found moſt eaſily by the Method of Diviſors. Biquadratics may, however, be ſolved without taking away the ſecond Term, by a Method deduced by Mr. Thomas Simpſon, from Art. CXLIV. Numb. 298. which will be there explained, or by Theorems given by Meſſ. M'Laurin, Colſon, and others: but the moſt expeditious Method is generally that of Diviſors.

278. CXL. (*b*) *An Equation of even Dimenſions, the Coefficient of whoſe higheſt Term is Unity, and which is clear of Fractions and Surds, may be conceived to be the Difference of two compleat Squares; whence, by adding the leſs Square to it, it may be compleated into the greater; and the Root found by the Reſolution of an adfected Quadratic.*

Putting therefore $2r$ the Index of the higheſt Term of the Equation, the Index of the higheſt Term of the greater *tion be a* Quadratic, (*for we may add this Case for the Analogy*

greater Square will be alſo $2r$, and that of the higheſt Term of its Root (the Root is always ſome Power of a Binome) will be r (83); and putting p for the Coefficient of the ſecond Term of the Equation, and $\frac{1}{2}p$ for that of the ſecond Term of the Root; that of the ſecond Term of the greateſt Square will be p (122.) Let the given Equation be $x^{2r} + px^{2r-1} + qx^{2r-2} + rx^{2r-3} + sx^{2r-4} + tx^{2r-5} + vx^{2r-6} + wx^{2r-7}$, &c. &c. $ax^{2r-2r} = 0$; and let the Root of the greater Square be $x^r + \frac{1}{2}px^{r-1} + Qx^{r-2} + Rx^{r-3} + Sx^{r-4} + Tx^{r-5}$, &c, &c. Vx^{r-r}: Then the greater Square will be $x^{2r} + px^{2r-1} + \overline{2Q + \frac{1}{4}p^2} \times x^{2r-2} + \overline{2R + pQ} \times x^{2r-3} + \overline{2S + pR + Q^2} \times x^{2r-4} + \overline{2T + pS + 2QR} \times x^{2r-5} + \overline{2V + pT + 2QS + R^2} \times x^{2r-6} + \overline{2W + pV + 2QT + 2RS} \times x^{2r-7}$, &c. &c. (122).

279. Let the indeterminate Expreſſions in the Coefficients of the greater Square, viz. $2Q$, $2R$, $2S$, $2T$, $2V$, &c. be changed into the Greek Letters α, β, γ, δ, ε, ζ, η, θ, κ, λ, &c. reſpectively: Then the greater Square will be expreſſed thus $x^{2r} + px^{2r-1} + \overline{\alpha + \frac{1}{4}p^2} \times x^{2r-2} + \overline{\beta + \frac{1}{2}p\alpha} \times x^{2r-3} + \overline{\gamma + \frac{1}{2}p\beta + \frac{1}{4}\alpha\alpha} \times x^{2r-4} + \overline{\delta + \frac{1}{2}p\gamma + \frac{1}{2}\alpha\beta} \times x^{2r-5} + \overline{\varepsilon + \frac{1}{2}p\delta + \frac{1}{2}\alpha\gamma + \frac{1}{4}\beta\beta} \times x^{2r-6} + \overline{\zeta + \frac{1}{2}p\varepsilon + \frac{1}{2}\alpha\delta + \frac{1}{2}\beta\gamma} \times x^{2r-7} + \overline{\eta + \frac{1}{2}p\zeta + \frac{1}{2}\alpha\varepsilon + \frac{1}{2}\beta\delta + \frac{1}{4}\gamma\gamma} \times x^{2r-8}$, &c. &c. Wherefore *equating the correſpondent Terms of this Square with thoſe of the given Equation, we have the Values of the Greek Letters in known Quantities.*

logy of the Matter) take from both Sides the lowest Term, and

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I. $\alpha = q - \frac{1}{4}p^2$. II. $\beta = r - \frac{1}{2}p\alpha$. III. $\gamma = s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha$. IV. $\delta = t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta$. V. $\varepsilon = \frac{1}{2}p\delta - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta$. VI. $\zeta = W - \frac{1}{2}p\varepsilon - \frac{1}{2}\alpha\delta - \frac{1}{2}\beta\gamma$. VII. $\eta = Z - \frac{1}{2}p\zeta - \frac{1}{2}\alpha\varepsilon - \frac{1}{2}\beta\delta - \frac{1}{4}\gamma\gamma$. VIII. $\theta = a - \frac{1}{2}p\eta - \frac{1}{2}\alpha\zeta - \frac{1}{2}\beta\varepsilon - \frac{1}{2}\gamma\delta$. IX. $\kappa = b - \frac{1}{2}pa - \frac{1}{2}\alpha\eta - \frac{1}{2}\beta\zeta - \frac{1}{2}\gamma\varepsilon - \frac{1}{4}\delta\delta$. X. $\lambda = c - \frac{1}{2}pb - \frac{1}{2}\alpha a - \frac{1}{2}\beta\eta - \frac{1}{2}\gamma\zeta - \frac{1}{2}\delta\varepsilon$, and so on *in infinitum*.

280. Now *the Index of the highest Term of the Root of the greater Square being* r, *that of the highest Term of the Root of the less or complement Square cannot be greater than* r — 1; and because it is always supposed that the Terms of this Square have a common Divisor n, if we put $\sqrt{n} \times \overline{kx^{r-1} + lx^{r-2} + mx^{r-3} + hx^{r-4}}$, &c. for the Root, the complement Square will be (122) $nk^2 \times x^{r-2} + 2nkl \times x^{2r-3} + \overline{2nkm + nl^2} \times x^{2r-4} + \overline{2nkh + 2nlm} \times x^{2r-5} + \overline{2nk\pi + 2nkl + nm^2} \times x^{2r-6}$, &c. Whence it appears, *that the highest Index, which the first Term of the complement Square can have, is* 2r — 2; *and consequently that it can be added but to the third Term of the Equation, its second to the fourth of the Equation, and so on* in order,

281. Now supposing the above indefinite Equation compleated by the Addition of the indefinite complement Square, then by equating the Terms with the correspondent Terms of the greater Square, we shall have so many Equations for determining the Quantities sought, as there are Terms in the greater Square after the two first; that is, $2r - 1$; the whole Number being $2r + 1$ (211). Now the Number of Quantities sought

and add one fourth Part of the Square of the known Quantity of the middle Term.

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sought are $r-1$ Terms of the Root of the greater Square (the two first being Unity and $\frac{1}{2}p$ (122), and r Terms of the Root of the less Square, the Dimensions being $r-1$ (280)) and n the common Divisor; that is, $\overline{r-1}+r+1=2r$: *Whence there are* $2r-1$ *Equations to determine* $2r$ *Quantities; consequently they can only be found by Trial,* (194).

282. *These Equations,* transposing all the Terms into one Member, are I. $2Q+\frac{1}{4}p^2-q-nk^2=0$. II. $2R+pQ-r-2nkl=0$. III. $2S+pR+Q^2-s-2nkm-nl^2=0$. IV. $2T+pS+2QR-t-2nhk-2nlm=0$. V. $2V+pT+2QS+R^2-v-nk\pi-2nlh-nm^2=0$. VI. $2W+pV+2QT+2RS-w-2nk\rho-2nl\pi-2nmh=0$. VII. $2Z+pW+2QV+2RT+S^2-z-2nk\sigma-2nl\rho-2nm\pi-nh^2=0$, &c. in infinitum, *and from these there are found general Limitations for the particular Quantities* n, Q, R, S, k, l, *&c.* thus.

283. By (282, I.) $2Q=q-\frac{1}{4}p^2+nk^2$, but $q-\frac{1}{4}p^2=\alpha$ (279, I.); whence $2Q=\alpha+nk^2$, and $\alpha=2Q-nk^2$: Whence *that* n *may be a Divisor of* α, *it must also be a Divisor of* Q, *or else* Q *must vanish: also that* k *may be a Divisor of* α, *it must also divide* Q, *or else* $Q=0$. *Also we have* $Q=\frac{nk^2+\alpha}{2}=\frac{q+nk^2-\frac{1}{4}p^2}{2}$.

284. By (282, II.) $2R=r-pQ+2nkl$, $=r-\frac{1}{2}p\alpha-\frac{1}{2}pnk^2+2nkl$, (283); but $r-\frac{1}{2}p\alpha=\beta$ (279, II.) whence $2R=\beta-\frac{1}{2}pnk^2+2nkl$, and $\beta=2R-$

$\frac{1}{2}pnk^2$

out $xx - ax + \frac{1}{4}aa = b + \frac{1}{4}aa$; *and extracting on both Sides*

$$\frac{\overline{10p^3n^4 - 8\alpha p^2n^2 - 30\alpha pn^2 + 12\beta n^2} \times k^4 - \overline{48lp^2n^2}}{64}$$

$$\frac{\overline{+48mpn^2 - 4\alpha pn^2 + 32ln^4 - 32bn^2} \times k^3 - \overline{p^5n +}}{64}$$

$$\frac{\overline{6\alpha p^3n - 12p^2n\beta + 2\alpha p^2n - 12\alpha^2p + 16pn\gamma + 72pn^2l^2}}{64}$$

$$\frac{\overline{+8\alpha n^2\beta + 16\alpha n^2l - 16n\delta - 96n^2lm} \times (k^2) +}{64}$$

$$\frac{\overline{4lp^4n - 8mp^3n - 24\alpha nlp^2 + 16bp^2n + 48\beta lpn +}}{64}$$

$$\frac{\overline{16\alpha^2ln + 8\alpha\beta n + 32p\alpha mn - 32b\alpha n - 32l^3n^2 -}}{64}$$

$$\frac{\overline{32\gamma ln - 32\beta mn - 32p\pi n} \times k + 32\zeta + 4l^2p^3n + 16lmp^2n}{64}$$

$$\frac{+16\alpha pl^2n - 16\beta l^2n - 32plbn - 32\alpha lmn}{64} + \rho kn +$$

$\pi ln + mbn$. Whence, *that* n *should divide* ζ, *it must also divide* W; *or* W $= 0$. *Also that* k *should divide* ζ, *it is necessary that* $\frac{1}{16}l^2p^3n + \frac{1}{4}lmp^2n + \frac{1}{4}\alpha pl^2n - \frac{1}{4}\beta l^2n - \frac{1}{2}phln - \frac{1}{2}\alpha lmn + \pi ln + mhn$ *should vanish*; *and moreover that* k *should divide* W; *or that* W $= 0$: *and so on* in infinitum.

289. From the foregoing Determinations it follows, that n will measure so many of the antecedent Greek Letters, as there are undetermined Terms in the Root of the greater Square, if it measures the subsequent Greek Letters. Now the Number of undetermined Terms in that Root is $r - 1$ (for $\overline{r+1} - 2 = r - 1$), and the whole Number of Greek Letters or Terms after the two first of the Equation is $2r - 1$ (for $\overline{2 - r + 1} - 2 = 2$

Sides the Root, you will have $x - \frac{1}{2}a = \pm\sqrt{b + \frac{1}{4}aa}$, or $x = \frac{1}{2}a \pm \sqrt{b + \frac{1}{4}aa}$ (c).

CXLII.

$-2 = 2r - 1$; wherefore subducting $r-1$, the Number which n must divide if it divides the subsequent, from $2r-1$, the Residue r *is the Number of the last Greek Letters whose common Divisor* n *must be; and such a Divisor is always to be sought.*

290. And *because by Supposition we look for a surd Divisor, all Divisors are to be rejected which are Squares, or Multiples of Squares*; for if n is a Square, then $\sqrt{n} \times \overline{kx^{r-1} + lx^{r-2}}$, &c. would be rational; and if n is a Multiple of a Square as $\sqrt{} \times m^2$, then $m \times \overline{kx^{r-1} + lx^{r-2}}$ &c. would be rational, and n depressed to $\sqrt{}$ would be rational.

291. *If any of the Terms* Q, R S, *&c. of the Root of the greater Square is a Fraction, its Denominator must be* 2, *and the Coefficient in which its Square is, must be an odd Number*; *and* n *must be an odd Number*: for in the Square it is first multiplied into 2, (122); whence if the Denominator be 2, it will become an Integer, being a Multiple of 2; and the other Parts of the same Coefficient being double Products (122), when they are reduced to the same Denomination, will be Multiples of 4. Now the Denominator being 2, the Numerator must be odd (Euc. VII. 24.); and therefore its Square is odd (Euc. 29. IX.), whence the Coefficient and its Divisor n is odd; but if the Denominator was any other Number, the Fraction multiplied into 2 could never become an Integer; and therefore the Coefficient would still be a Fraction, or the highest Term would have a Coefficient different from Unity; both which are contrary to Hypothesis.

292. *According as any of the Terms* Q, R, S, *&c. of the Root of the greater Square is an Integer, or a Fraction*;

CXLII. *But if the Equation be of* four Dimensions, *suppose*

so must the correspondent Term of the Root of the complement Square be an Integer, or a Fraction respectively, and have 2 its Denominator. For any Coefficient of the greater Square is equal to the Sum of the correspondent Coefficients of the Equation, and of the complement Square into *n*; but *n*, and the Coefficients in the Equation, and greater Square, for the same Term being always Integers (291), the Coefficient of the complement Square must be an Integer; consequently, according as the Term in one Root is integer or fracted, the correspondent Term in the other must also be integer and fracted, and have the Denominator 2; that the Square of the Fraction into *n* may either destroy the other fractional Square, or with it make an Integer; which could not be done, if one being Integer, the other was a Fraction.

293. *If any Coefficient of the Equation, as* p, r, t, w, *&c. in a Place denominated from an even Number, be odd;* n *also must be odd:* for the Exponents of the Terms denominated from an even Number in each Square are odd, and these Coefficients consist of double Rectangles, without any Square (123), wherefore they are even Numbers, if the Roots are Integers. Consequently the Differences of those even Coefficients, that is, the Coefficients of the Equation denominated from an even Number, are even, when the Roots are Integer: If therefore any Coefficient in an even Place is odd, the correspondent Terms in the Roots of both Squares are Fractions, whose Denominators are 2; and therefore the Numerators are odd, and their Squares odd, and the Aggregate odd, and *n* the Divisor of the Aggregate odd (Eucl. 29. IX.) *and conversely.*

294. n *being odd, must, when divided by* 4, *leave Unity.* For let N represent any Number in general, and I an odd Number; then because that every odd Number is a Multiple

suppose $x^4 + p x^3 + q x x + r x + s = 0$, where p, q, r, and s denote

Multiple of 4 more or less Unity, viz. $I = 4N \pm 1$; and that the Square of an odd Number is a Multiple of 4 more Unity, viz. $I^2 = 4N + 1$ (Eucl. 29. IX.) and that if from such a Square there be taken any Multiple of 4, the Remainder, if greater than Unity, will be a Multiple of 4 more Unity, viz. $4M + 1$: And because that the Product of n, into the odd Coefficient of the complement Square, is equal to the Difference of the odd Coefficients of the greater Square, and Equation: Also because the square Roots of those Coefficients are the Halves of odd Numbers. We have the 4th Part of the Product of n into an odd Square, equal to the 4th Part of the Difference of an odd Square, and of quadruple an odd Number. Wherefore n into an odd Square is equal to the Difference of an odd Square, and of quadruple an odd Number; that is, n into a Multiple of 4 more Unity is equal to a Multiple of 4 more Unity; and consequently, n is equal to a Multiple of 4 more Unity. For it is not a Multiple of 4 less Unity, but of 4 more Unity, which can give the Product a Multiple of 4 more Unity.

295. These general Limitations being premised, let those indefinite Expressions in N°. 279 and 282, &c. become finite: That is, if there is proposed an Equation of given Dimensions, it is manifest, that the Terms involving the Letters above the given Dimensions must vanish. Let the proposed be $x^8 + p x^7 + q x^6 + r x^5 + s x^4 + t x^3 + v x^2 + w x + z = 0$. Then the following will be, as it is proposed by the Author in Art. CLI. *An Exemplar for all Reductions by surd Divisors.* Because $2r = 8$, therefore $r = 4$, and $r - 1 = 3$; and the Root of the greater Square is $x^4 + \frac{1}{2} p x^3 + Q x^2 + R x + S$; and the Root of the complement Square is $\sqrt{n} \times \overline{k x^3 + l x^2 + m x + b}$; and the seven Equations for the Greek Letters, are 1. $q - \frac{1}{4} p^2 = \alpha$; 2. $r - \frac{1}{2} \alpha p = \beta$; 3. $s - \frac{1}{2} p \beta - \frac{1}{4} \alpha \alpha = \gamma$; 4. $t - \frac{1}{2} p \gamma - \frac{1}{2} \alpha \beta = \delta$; 5. $v -$

denote the known Quantities of the Terms of the Equation affected by their proper Signs, make

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}\alpha p = \beta. \quad s - \tfrac{1}{4}\alpha\alpha = \zeta.$$

Then

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5. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = \varepsilon$; 6. $w - \frac{1}{2}\beta\gamma = \zeta$; and 7. $z - \frac{1}{2}\gamma\gamma = \eta$ (279): And the seven Equations for the Coefficients, are 1. $2Q + \frac{1}{4}p^2 - q - nk^2 = 0$; 2. $2R + pQ - r - 2nkl = 0$; 3. $2S + pR + Q^2 - s - 2mnk - nl^2 = 0$; 4. $pS + 2QR - t - 2nkb - 2nlm = 0$; 5. $2QS + R^2 - v - 2nlb - nm^2 = 0$; 6. $2RS - w - 2nmb = 0$; 7. $S^2 - z - nb^2 = 0$, (282, 288). Now r being 4, n must be a common Divisor of δ, ε, ζ, and η (289); and odd, if any of the alternate Coefficients be odd (293); and neither Square, nor Multiple of a Square (290); and divided by 4 to leave Unity (294). Then because by the 7th Equation, $S^2 = z + nb^2$, if n be even, seek a square Number b^2, to which, after it is drawn into n, the last Term of the Equation z being added by its proper Sign, $z \pm nb^2$, shall make a square Number: But if n be odd, because some Term of the Root, or reduced Equation, is a Fraction, whose Denominator being 2, the other Numbers in the same Coefficient with the Square of its Numerator, are Multiples of 4, and so $4S^2 = 4z + 4nb^2$ (291, 292, 293); connect the Product of n into a square Number to quadruple the last Term of the Equation, viz. $4z \pm 4nb^2$, until a square Number is found. Extract the square Root, and call it S, when n is even; but 2 S, when n is odd; and make $\sqrt{\frac{S^2 - z}{n}} = b$. Then if S be a Fraction, so must b; both having the Denominator 2 (291) and let all Numbers S, and b, within this Limit, be collected in a Catalogue. Having thus found n, b, and S; k is next to be found, by a successive Assumption of all Numbers, which do not make $nk \pm \frac{1}{2}p$ greater than quadruple the greatest Term of the Equation: When k is had, Q is to be found by $Q = \frac{nk^2 + \alpha}{2}$ (283). Q being found, all Numbers are

to

Then put for n *some common integral Divisor of the Terms* β *and* 2ζ, *that is not a Square, and which ought to be odd, and*

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to be tried for l, which do not make $nl \pm Q$ greater than quadruple the greatest Term of the Equation; and l being found, we have $R = \frac{-np^2k^2 + 2\beta}{4} + nkl$. (284). Lastly, to find m, all Numbers are successively to be tried, which do not make $nm \pm R$ greater than the greatest Term of the Equation: Always making k a Fraction, when p is so; l a Fraction, when Q is one; and m a Fraction, when R is one (292). From the several Values of the Letters registered in the Catalogue, those only are to be assumed, which will answer all the Conditions of the Equations; for this Coincidence is a Proof that they have been rightly assumed. Thus $S = \sqrt{z + nh^2}$ by the 7th Equation; and also $= \frac{S - pR - QQ + nll}{2} + nkl$ by the third; and its Correspondent $h = \frac{\sqrt{S^2 - z}}{n}$ by the 7th Equation; and $= \frac{pS + 2QR - t - 2nlm}{2nk}$ by the 4th; also $= \frac{2QS + R^2 - v - nm^2}{2nl}$ by the 5th; and $= \frac{2RS - w}{2nm}$ by the 6th Equation. If all these Conditions coincide, then for the proposed $x^8 + px^7 + qx^6 + rx^5 + sx^4 + tx^3 + vx^2 + wx + z = 0$, write $x^4 + \frac{1}{2}px^3 + Qx^2 + Rx + S = \sqrt{n} \times \overline{kx^3 + lx^2 + mx + h}$. But beside this general Rule, there may be particular Rules for Equations in the particular Degrees of even Dimensions, as in the following.

and divided by 4 to leave Unity, if either of the Terms p *and* r *be odd. Put also for* k *some Divisor of the Quantity* $\frac{\beta}{n}$ *if* p *be even; or half of the odd Divisor, if* p *be odd; or nothing, if the Dividual* β *be nothing. Take the Quotient from* $\frac{1}{2}$pk, *and call the half of the Remainder* l. *Then for* Q *put* $\frac{\alpha + nkk}{2}$ *and try if* n *divides* QQ — s, *and the Root of the Quotient be rational and equal to* l; *which if it happen, add to each Part of the Equation* nkkxx + 2nklx + nll, *and extract the Root on both Sides, there coming out* xx + $\frac{1}{2}$px + Q = $\sqrt{n}$ *into* $\overline{kx + l}$. (*d*)

For

CXLI. (*c*). If the Equation be a Quadratic $x^2 + px + q = 0$; here $2r = 2$, whence $r = 1$, and $r - 1 = 0$. That is, the Equation is not defective, and *n*, *k*, &c. are $= 0$. Whence to reduce it, it is to be made defective, by transposing *q*, but $x^2 + px$ is (Eucl. II. 4.) compleated into a Square, by adding $\frac{1}{4}p^2$; wherefore adding $\frac{1}{4}p^2$ to $x^2 + px$, and to *q*, there is $x^2 + px + \frac{1}{4}p^2 = \pm q + \frac{1}{4}p^2$; and extracting the Root, $x \pm \frac{1}{2}p = \sqrt{\frac{1}{4}p^2 \pm q}$, &c. (LXXIV).

CXLII. (*d*). Here $2r = 4$; whence $r = 2$, $r - 1 = 1$; whence the Root of the greater Square is $x^2 + \frac{1}{2}px + Q$; and that of the Complement $\sqrt{n} \times \overline{kx + l}$ (278); whence the three Equations for the Greek Letters are, first, $q - \frac{1}{4}p^2 = \alpha$; second, $r - \frac{1}{2}p\alpha = \beta$; third, $s - \frac{1}{4}\alpha\alpha = \gamma$ (279); but here let us use ζ, for γ; following the Notation of the Author: Because $r = 2$, thence *n* must be the common Divisor of β and ζ, (289); it must also be odd, if *p*, or *r*, be odd, (293); and divided by 4, to leave Unity, (294); and neither a Square, or Multiple of a Square, (290). The three Equations for the

Coefficients

For Example, let there be proposed the Equation $x^4 + 12x - 17 = 0$, and because p and q are both here wanting, and r is 12, and s is -17, having substituted these Numbers,

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Coefficients will be 1st, $2Q + \frac{1}{4}p^2 - q - nk^2 = 0$; 2d, $pQ - r - 2nkl = 0$; 3d, $Q^2 - s - nl^2 = 0$, (282). By the first, $Q = \frac{1}{2}a + \frac{1}{2}nk^2$, and by the third, $nl^2 = Q^2 - s$; whence $n = \frac{\overline{\frac{1}{2}a + \frac{1}{2}nk^2}|^2 - s}{l^2} = \frac{\frac{1}{4}n^2k^4 + \frac{1}{2}ank^2 + \frac{1}{4}aa - s}{l^2}$

$= \frac{\frac{1}{4}a^2 - s}{-\frac{1}{4}nk^4 - \frac{1}{2}ak^2 + l^2} = \frac{s - \frac{1}{4}a^2}{\frac{1}{4}nk^4 + \frac{1}{2}ak^2 - l^2}$

$= \frac{\zeta}{k^2 \times \overline{\frac{1}{4}nk^2 + \frac{1}{2}a} - l^2}$. But the Form will be more commodious, by writing $n = \frac{2\zeta}{k^2 \times \overline{\frac{1}{2}pk^2 + a} - 2l^2}$, that is, n is to divide 2ζ by $k^2 \times \overline{\frac{1}{2}nk^2 + a} - 2l^2$. Again, by the first Equation, $Q = \frac{1}{2}nk^2 + \frac{1}{2}a$; and by the second, $Q = \frac{2nkl + r}{p}$; whence $\frac{1}{2}pnk^2 + \frac{1}{2}pa - r = 2nkl$, that is, $-2nkl = r - \frac{1}{2}pa - \frac{1}{2}pnk^2 = \beta - \frac{1}{2}pnk^2$; whence $\frac{1}{2}pnk^2 - 2nkl = \beta$, and $n = \frac{\beta}{k \times \overline{\frac{1}{2}pk - 2l}}$. Now since n divides β, by the Quote $k \times \overline{\frac{1}{2}pk - 2l}$; k will divide $\frac{\beta}{n}$ by the Quote $\frac{1}{2}pk - 2l$; and subducting the Quote $\frac{1}{2}pk - 2l$, from $\frac{1}{2}pk$; the Residue is $2l$; and half the Remainder is l. And by the third Equation, $l = \sqrt{\frac{Q^2 - s}{n}}$: Now when p is odd, the Root $\frac{\sqrt{Q^2 - s}}{n}$ may be $2l$ (295) because of a Fraction; wherefore half the odd Divisor of $\frac{\beta}{n}$ is then to be taken for l.

Numbers, you will have $\alpha = 0$, $\beta = 12$, and $\zeta = -17$, and the only common Divisor of β and 2ζ, or 12 and -34 *viz.* 2, will be n. Moreover, $\frac{\beta}{n}$ is 6, and its Divisors 1, 2, 3, and 6, are successively to be tried for k, and -3, $-\frac{1}{2}$, -1, $-\frac{1}{2}$, for l respectively. But $\frac{\alpha + nkk}{2}$, that is, kk is equal to Q. Moreover, $\sqrt{\frac{QQ - s}{n}}$, that is, $\sqrt{\frac{QQ + 17}{2}}$ is $= l$.

Where the even Numbers 2 and 6 are writ for k, Q becomes 4 and 36, and $QQ - s$ will be an odd Number, and consequently cannot be divided by n or 2. Wherefore those Numbers 2 and 6 are to be rejected. But when 1 and 3 are writ for k, Q becomes 1 and 9, and $QQ - s$ is 18 and 98, which Numbers may be divided by n, and the Roots of the Quotients extracted. For they are ± 3 and ± 7; whereof however only -3 agrees with l. I put therefore $k = 1$, $l = -3$, and $Q = 1$, and I add the Quantity $nkkxx + 2nklx + nll$, that is, $2xx - 12x + 18$ to each Part of the Equation, and there comes out $x^4 + 2xx + 1 = 2xx$

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296. *When* $k = 0$, *then necessarily* $\beta = 0$; *because by the first Equation* $Q = \frac{1}{2}\alpha$, *and by the second*, $pQ = r$; *whence* $\beta = (r - \frac{1}{2}\alpha p) = pQ - pQ = 0$. *In this Case*, n *being some Divisor of* 2ζ, *if* $\frac{\frac{1}{4}\alpha^2 - s}{n}$ *is a Square Number, its Root is* l.

297. β *may vanish altho'* k *be a real Quantity, if* $\frac{1}{2}pk = 2l$, *or* $\frac{1}{4}pk = l$; *for* $\beta = n \times k \times \overline{\frac{1}{2}pk - 2l}$. *In this Case, of the compound Divisors of* 2ζ, *take one which is a Multiple of a Square, but itself not a Square, as* nk^2; *then if the Quote* $\frac{2\zeta}{nk^2} = \alpha + \frac{1}{2}nk^2 - \frac{1}{8}p^2$; $\frac{1}{4}pk$ *shall be* l.

$-12x + 18$, and extracting on both Sides the Root $xx + 1 = x\sqrt{2} - 3\sqrt{2}$. But if you had rather avoid the Extraction of the Root, make $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, and you will find, as before, $xx + 1 = \pm\sqrt{2} \times \overline{x - 3}$. And if again you extract the Root of this Equation, there will come out $x = \pm\frac{1}{2}\sqrt{2} \pm \sqrt{\frac{-1}{2} \mp 3\sqrt{2}}$, that is, according to the Variations of the Signs, $x = -\frac{1}{2}\sqrt{2} + \sqrt{3\sqrt{2} - \frac{1}{2}}$, and $x = -\frac{1}{2}\sqrt{2} - \sqrt{3\sqrt{2} - \frac{1}{2}}$. Also $x = \frac{1}{2}\sqrt{2} + \sqrt{-3\sqrt{2} - \frac{1}{2}}$, and $x = \frac{1}{2}\sqrt{2} - \sqrt{-3\sqrt{2} - \frac{1}{2}}$. Which are four Roots of the Equation at first proposed, $x^4 + 12x - 17 = 0$. But the two last of them are impossible.

Let us now propose the Equation $x^4 - 6x^3 - 58xx - 114x - 11 = 0$, and by writing -6, -58, -114, and -11, for p, q, r, and s respectively, there will arise $-67 = \alpha$, $-315 = \beta$, and $-1133\frac{1}{4} = \zeta$. The only common Divisor of the Numbers β and 2ζ, or of -315 and $-\frac{4533}{2}$ is 3, and consequently will be here n, and the Divisors of $\frac{\beta}{n}$ or -105, are 3, 5, 7, 15, 21, 35, and 105, which are therefore to be tried for k. Wherefore, I try first 3, and the Quotient -35, which comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 3, I subtract from $\frac{1}{2}pk$, or -3×3, and there remains 26; the half whereof, 13 ought to be l. But $\frac{\alpha + nkk}{2}$, or $\frac{-67 + 27}{2}$, that is, -20, will be Q, and $QQ - s$ will be 411, which may be divided by n, or 3, but the Root of the Quotient 137 cannot be extracted. Wherefore

fore I reject 3, and try 5 for k. The Quotient that now comes out by dividing $\frac{\beta}{n}$ by k, or -105 by 5, is -21, and subtracting this from $\frac{1}{2}pk$, or -3×5, there remains 6, the half whereof 3 will be l. Also Q or $\frac{\alpha + nkk}{2}$, that is $\frac{-67+75}{2}$, is the Number 4. And $QQ - s$, or $16 + 11$ may be divided by n; and the Root of the Quotient, which is 9, being extracted, *i. e.* 3 agrees with l. Wherefore I conclude that l is $= 3$, $k = 5$, $Q = 4$, and $n = 3$; and if $nkkxx + 2nklx + nll$, that is, $75xx + 90x + 27$ be added to each Part of the Equation, the Root may be extracted on both Sides, and there will come out $xx + \frac{1}{2}px + Q = \sqrt{n} \times \overline{kx + l}$, or $xx - 3x + 4 = \pm\sqrt{3} \times \overline{5x + 3}$; and the Root being again extracted, $x = \frac{3 \pm 5\sqrt{3}}{2} \pm \sqrt{17 \pm \frac{21 \times \sqrt{3}}{2}}$.

Thus, if there was proposed this Equation $x^4 - 9x^3 + 15xx - 27x + 9 = 0$, by writing -9, $+15$, -27, and $+9$ for p, q, r, and s respectively, there will come out $-5\frac{1}{4} = \alpha$, $-05\frac{5}{8} = \beta$, and $2\frac{7}{64} = \zeta$. The common Divisors of β and 2ζ, or $-\frac{405}{8}$ and $\frac{135}{32}$ are 3, 5, 9, 15, 27, 45, and 135; but 9 is a square Number, and 3, 15, 27, 135, divided by the Number 4, do not leave Unity, as, by reason of the odd Term p, they ought to do. These therefore being rejected, there remain only 5 and 45 to be tried for n. Let us put therefore, first $n = 5$, and the odd Divisors of $\frac{\beta}{n}$ or $-\frac{81}{8}$ being halved, viz. $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, $\frac{27}{2}$, $\frac{81}{2}$, are to be tried for k. If k be made $\frac{1}{2}$, the Quotient $-\frac{81}{4}$, which comes out by dividing $\frac{\beta}{n}$ by k, subtracted from $\frac{1}{2}pk$,

or

or $-\frac{9}{4}$, leaves 18 for $2l$, and $\frac{a+nkk}{2}$ or -2 is Q, and $QQ-s$, or -5 may be divided indeed by n or 5, but the Root of the negative Quotient -1 is impossible, which yet ought to be 9. Wherefore I conclude k not to be $\frac{1}{2}$, and then I try if it be $\frac{3}{2}$. The Quotient which arises by dividing $\frac{\beta}{n}$ by k, or $-\frac{81}{8}$ by $\frac{3}{2}$, *viz.* the Quotient $-\frac{27}{4}$ I subtract from $\frac{1}{2}pk$ or $-\frac{27}{4}$, and there remains 0. Whence now l will be nothing. But $\frac{a+nkk}{2}$ or 3 is equal to Q, and $QQ-s$ is nothing; whence again l, which is the Root of $QQ-s$, divided by n, is found to be nothing. Wherefore these things, thus agreeing, I conclude n to be $=5$, $k=\frac{3}{2}$, $l=0$, and $Q=3$, and therefore by adding to each Part of the Equation proposed the Terms $nkkxx+2nlkx+nll$, that is, $\frac{75}{4}xx$, and by extracting on both Sides the square Root, there comes out $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx+l}$, that is, $xx-4\frac{1}{2}x+3=\sqrt{5}\times\frac{3}{2}x$.

CXLIII. *By the same Method literal Equations are also reduced.* As if there was $x^4-2ax^3\begin{matrix}+2aa\\-cc\end{matrix}xx-2a^3x+a^4=0$, by substituting $-2a$, $2aa-cc$, $-2a^3$, and $+a^4$ for p, q, r, and s respectively, you will obtain $aa-cc=\alpha$, $-acc-a^3=\beta$, and $\frac{3}{4}a^4+\frac{1}{2}aacc-\frac{1}{4}c^4=\zeta$. The common Divisor of the Quantities β and 2ζ is $aa+cc$, which therefore will be n; and $\frac{\beta}{n}$ or $-a$, has the Divisors 1 and a. But because n is of two Dimensions, and $k\sqrt{n}$ ought to be of no more than one, therefore k will be of none, and consequently cannot be a. Let therefore k be 1, and $\frac{\beta}{n}$ being divided by k, take the Quotient $-a$ from $\frac{1}{2}pk$

or

or $-a$ and there will remain nothing for l. Moreover, $\frac{\alpha+nkk}{2}$ or aa is Q, and QQ $-s$ or a^4-a^4 is o; and thence again there comes out nothing for l. Which shews the Quantities n, k, l, and Q to be rightly found; and adding to each Part of the Equation proposed, the Terms $nkkxx+2nklx+nll$, that is, $aaxx+ccxx$, the Root may be extracted on both Sides; and by that Extraction there will come out $xx+\frac{1}{2}px+Q=\sqrt{n}\times\overline{kx-l}$, that is, $xx-ax+aa=\pm x\sqrt{aa+cc}$. And the Root being again extracted, you will have $x=\frac{1}{2}a\pm\frac{1}{2}\sqrt{aa+cc}\pm\sqrt{\frac{1}{4}cc-\frac{1}{2}aa\pm\frac{1}{2}a\sqrt{aa+cc}}$.

CXLIV. Hitherto I have applied *the Rule* to the Extraction of *surd Roots*; the same *may also be applied to the Extraction of rational Roots, if for the Quantity* n *you make use of Unity; and after that Manner we may examine, whether an Equation that wants fracted or surd Terms can admit of any Divisor, either rational or surd, of two Dimensions.* As if the Equation $x^4-x^3-5xx+12x-6=0$ was proposed, by substituting -1, -5, $+12$, and -6 for p, q, r, and s respectively, you will find $-5\frac{1}{4}=\alpha$, $9\frac{3}{8}=\beta$, and putting $n=1$. The Divisors of the Quantity $\frac{\beta}{n}$, or $\frac{75}{8}$, are 1, 3, 5, 15, 25, 75; the Halves whereof (if p be odd) are to be tried for k. And if for k we try $\frac{5}{2}$ you will have $\frac{1}{2}pk-\frac{\beta}{nk}=-5$, and its half $-\frac{5}{2}=l$. Also $\frac{\alpha+nkk}{2}=\frac{1}{2}=$Q, and $\frac{QQ-s}{n}=6\frac{1}{4}$, the Root whereof agrees with l.

I therefore conclude, that the Quantities n, k, l, Q, are rightly found; and having added to each Part of the Equation the Terms $nkkxx+2nklx+nll$, that is, $6\frac{1}{4}xx-12\frac{1}{2}x+6\frac{1}{4}$, the Root may be extracted on both

both Sides; and by that Extraction there will come out $xx + \frac{1}{2}px + Q = \pm\sqrt{n}\times\overline{kx+l}$, that is, $xx - \frac{1}{2}x + \frac{1}{2} = \pm 1 \times \overline{2\frac{1}{2}x - 2\frac{1}{2}}$, or $xx - 3x + 3 = 0$, and $xx + 2x - 2 = 0$, and so by these two quadratic Equations the biquadratic one proposed may be divided. (e) *But rational Divisors of this Sort may more expeditiously be found by another Method delivered above.*

CXLIV. (r). 298. *The Value of* Q *may be found in this Case, the Divisor* n *being* 1, *without Trials by the Solution of the cubic Equation* $Q^3 - \frac{1}{2}qQ^2 + \overline{\frac{1}{4}pr - s}\times Q - \frac{r^2 - 4s}{8}\times\frac{p^2 - 4q}{8} = 0$, *and thence the Values of* k *and* l *will be found*; for by the first Equation in CXLII. $2Q + \frac{1}{4}p^2 - q = k^2$; by the second, $pQ - r = 2kl$; and by the third, $Q^2 - s = l^2$; whence the Product of the first and third is equal to $\frac{1}{4}$ the Square of the second; that is, $k^2l^2 = 2Q^3 + \overline{\frac{1}{4}p^2 - q}\times Q^2 - 2sQ - s\times\overline{\frac{1}{4}p^2 - q} = \frac{1}{4}p^2Q^2 - \frac{1}{2}prQ + \frac{1}{4}r^2$; wherefore transposing the Terms into one Member, and dividing by 2, we have $Q^3 - \frac{1}{2}qQ^2 + \overline{\frac{1}{4}pr - s}\times Q - \frac{r^2 - 4s}{8}\times\frac{p^2 - 4q}{8} = 0$. and Q being found, we have $k = \sqrt{2Q + \frac{1}{4}p^2 - q}$, also $l = \frac{pQ - r}{2k}$. *Whence we shall have* x, *by extracting the square Root on both Sides, from* $x^2 + \frac{1}{2}px + Q = \pm kx \pm l$; *which solved gives* $x = \pm\frac{1}{2}k - \frac{1}{4}p \pm\sqrt{\frac{1}{4}p \mp \frac{1}{2}k^2 \pm l - Q} = \pm\frac{k}{2} - \frac{1}{4}p \pm \sqrt{\frac{1}{16}p^2 \mp pk^2 \pm l - Q}$ *exhibiting the four Roots of the given Biquadratic, according to the Variation of the Signs.*

CXLV.

CXLV. *If at any Time there are many Divisors of the Quantity* $\frac{\beta}{n}$, *so that it may be too difficult to try all of them for* k, *their Number may be soon diminished, by seeking all the Divisors of the Quantity,* $\alpha s - \frac{1}{4}rr$. *For the Quantity* Q *ought to be equal to some of these, or to the half of some odd one.* Thus, in the last Example $\alpha s - \frac{1}{4}rr$ is $-\frac{9}{2}$, some one of whose Divisors, 1, 3, 9, or of them halved $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, ought to be Q. Wherefore, by trying singly the halved Divisors of the Quantity $\frac{\beta}{n}$ *viz.* $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, and $\frac{75}{2}$ for k, I reject all that do not make $\frac{1}{2}\alpha + \frac{1}{2}n\,kk$, or $-\frac{21}{2} + \frac{1}{2}kk$; that is, Q to be one of the Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$. But by writing $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, &c. for k, there come out respectively $-\frac{5}{2}$, $-\frac{3}{2}$, $+\frac{1}{2}$, $+\frac{51}{2}$, &c. for Q; out of which only $-\frac{3}{2}$ and $\frac{1}{2}$ are found among the aforesaid Numbers 1, 3, 9, $\frac{1}{2}$, $\frac{3}{2}$, $\frac{9}{2}$, and consequently the rest being rejected, either k will be $= \frac{3}{2}$ and $Q = -\frac{3}{2}$, or $k = \frac{5}{2}$ and $Q = \frac{1}{2}$. Which two Cases let be examined. And so much of Equations of *four Dimensions.* (*f*)

CXLVI. *If an Equation of six Dimensions is to be reduced let it be* $x^6 + px^5 + qx^4 + rx^3 + sxx + tx + v = 0$, *and make*

$$q - \tfrac{1}{4}pp = \alpha. \quad r - \tfrac{1}{2}p\alpha = \beta. \quad s - \tfrac{1}{2}p\beta = \gamma.$$
$$\gamma - \tfrac{1}{4}\alpha\alpha = \zeta. \quad t - \tfrac{1}{2}\alpha\beta = \eta. \quad v - \tfrac{1}{4}\beta\beta = \theta.$$
$$\zeta\theta - \tfrac{1}{4}\eta\eta = \lambda.$$

CXLV. (*f*). For by the third Equation, $s = Q^2 - nl^2$; whence $as = aQ2 - anl^2$; and by the second Equation, $\frac{1}{4}r^2 = \frac{1}{4}p^2Q - pQnkl + n^2l^2k^2$; whence $as - \frac{1}{4}r^2 = aQ^2 - \overline{a + nk^2} \times nl^2 - \frac{1}{4}p^2Q^2 + pQnkl$; but by the first Equation, $a + nk^2 = 2Q$; wherefore $as - \frac{1}{4}r^2 = aQ^2 - 2Qnl^2 - \frac{1}{4}p^2Q^2 + pQnkl$; consequently Q being in every Term is a Divisor.

Then

Then for n *take of the Terms* 2ζ, η, 2θ, *some common integer Divisor, that is not a Square, and that likewise is not divisible by a square Number, and which also divided by the Number* 4, *shall leave Unity; provided any one of the Terms* p, r, t, *be odd. For* k *take some integer Divisor of the Quantity* $\frac{\lambda}{2nn}$ *if* p *be even, or the half of an odd Divisor if* p *be odd, or* 0 *if* λ *be* 0. *For* Q *take the Quantity* $\frac{1}{2}\alpha + \frac{1}{2}nkk$. *For* l *some Divisor of the Quantity* $\frac{Qr - QQp - t}{n}$ *if* Q *be an Integer; or the half of an odd Divisor if* Q *be a Fraction that has for its Denominator the Number* 2; *or* 0, *if that Dividual* $\frac{Qr - QQp - t}{n}$ *be nothing. And for* R *the Quantity* $\frac{1}{2}r - \frac{1}{2}Qp + nkl$. *Then try if* $RR - \upsilon$ *can be divided by* n, *and the Root of the Quotient extracted; and besides, if that Root be equal as well to the Quantity* $\frac{QR - \frac{1}{2}t}{nl}$ *as to the Quantity* $\frac{QQ + pR - nll - s}{2nk}$. *If all these happen, call that Root* m; *and in room of the Equation proposed, write this,* $x^3 + \frac{1}{2}pxx + Qx + R = \pm\sqrt{n} \times \overline{kxx + lx + m}$. *For this Equation, by squaring its Parts, and taking from both Sides the Terms on the right Hand, will produce the Equation proposed. Now if all these Things do not happen in the Case proposed, the Reduction will be impossible, provided it appears beforehand that the Equation cannot be reduced by a rational Divisor* (*g*).

For

CXLVI. (*g*). For $2r = 6$, whence $r = 3$, and $r - 1 = 2$, wherefore the Roots are $x^3 + \frac{1}{2}px^2 + Qx + R$, and $\sqrt{n} \times \overline{kx^2 + lx + m}$ (280); and the Equations for Greek Letters 1ft. $q - \frac{1}{4}p^2 = \alpha$; 2d. $r - \frac{1}{2}\alpha p = \beta$; 3d. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = \gamma$; 4th. $t - \frac{1}{2}\alpha\beta = \delta$; 5th. $v - \frac{1}{4}\beta\beta = \varepsilon$ (279).

but

For Example, let there be propofed the Equation $x^6 - 2ax^5 + 2bbx^4 + 2abbx^3 + \left.\begin{matrix} -2aabb \\ +2a^3b \\ -4ab^3 \end{matrix}\right\} xx \begin{matrix} +3aab^4 \\ -a^4bb \end{matrix}$ $= 0$, and by writing $-2a$, $+2bb$, $+2abb$, $-2aabb + 2a^3b - 4ab^3$, 0, and $3aab^4 - a^4bb$ for p, q, r, s, t, and v refpectively, there will come out $2bb - aa$

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but for γ, δ, ε, let us write ζ, η, θ, refpectively, to conform to the Author's Notation. The Equations for the Coefficients are 1ft, $2Q + \frac{1}{4}p^2 - q - nk^2 = 0$; 2d, $2R + pQ - r - 2nkl = 0$; 3d. $pR + Q^2 - s - 2nkm - nl^2 = 0$; 4th, $2QR - t - 2nlm = 0$; and 5th, $R^2 - v - nm^2 = 0$ (282). alfo $r = 3$, whence n muft be the common Divifor of ζ, η, θ; (289): and have the other general Limitations of N°. 290, 293, 294.

Now by the firft, fecond, and third Equations, as in Art. CXLII. by equating the Values of R, n is found to divide 2ζ, by $k \times \overline{\frac{1}{2}n^2k^3 - \alpha k - \frac{1}{2}p^2k + 2pl - 4m} - 2l^2$; wherefore k cannot divide $\frac{2\zeta}{n}$, except $-2l^2$ be made to vanifh. Alfo by firft and fourth Equations, $R = \frac{s + 2nlm}{\alpha + nk^2} =$ (by fecond Equation) $-\frac{1}{4}pnk^2 + nkl + \frac{1}{2}\beta$; whence by Multiplying by α, and tranfpofing, we have $\eta = (t - \frac{1}{2}\alpha\beta) - \frac{1}{4}\alpha pn^2k^4 + \alpha n^2lk^3 - 2nlm$; wherefore $n = \frac{\eta}{k \times \overline{\alpha pnk^3 + \alpha nk^3 - 2lm}}$; whence k cannot divide $\frac{\eta}{n}$, except $2lm$ be made to vanifh; by the fifth Equation, $R^2 = v + nm^2 =$ (by fecond Equation) $\frac{1}{16}p^2n^2k^4 - \frac{1}{2}pln^2k^3 - \overline{\frac{1}{4}p\beta n + l^2n^2} \times k^2 + l\beta nk - nm^2$; whence by dividing and multiplying by 2, $n =$

2 θ

$-aa = \alpha$. $4abb - a^3 = \beta$. $2a^3b + 2aabb - 4ab^3 - a^4 = \gamma$. $-b^4 + 2a^3b + 3aabb - 4ab^3 - \frac{5}{4}a^4 = \zeta$. $-\frac{1}{2}a^5 - 3a^3bb - 4ab^4 = n$, and $-aab^4 + a^4bb - \frac{1}{4}a^5 = \theta$. And the common Divisor of the Terms 2ζ, n, and 2θ, is $aa - 2bb$, or $2bb - aa$, according as aa or $2bb$ is the greater. But let aa be greater than $2bb$,

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$$\frac{2\theta}{k \times \frac{1}{8}p^2n^2k^3 - pln^2k^2 - \frac{1}{2}p\beta nk + 2n^2l^2k + l\beta n - 2m^2;}$$

whence k cannot divide $\frac{2\theta}{n}$, unless $2m^2$ is made to vanish. Whence that n may divide $\frac{2\zeta}{n}$, $\frac{n}{n}$, and $\frac{2\theta}{n}$; $-2l^2 + 2lm - 2m^2$ must vanish: But this will be done, if ζ be multiplied into θ, and $\frac{1}{4}$ of nn subducted from the Product; for $\zeta\theta - \frac{1}{4}nn = \frac{1}{64}p^2n^4 - \frac{1}{64}\alpha^2p^2n^4 \times k^8 + \frac{1}{8}\alpha^2pln^4 - \frac{1}{8}pln^4 \times k^7 + \frac{1}{2}l^2n^4 \frac{1}{4}\alpha^2l^2n^4 + \frac{1}{32}\alpha p^2n^3 - \frac{1}{64}p^4n^3 - p\beta n^3 \times k^6 + \frac{1}{4}l\beta n^3 - \frac{1}{4}\alpha lpn^3 + \frac{3}{16}lp^3n^3 - \frac{1}{8}mp^2n^3 \times k^5 - \frac{1}{4}nm^2 - \frac{1}{8}\alpha\beta pn^2 + \frac{1}{2}\alpha l^2n^3 + \frac{1}{16}\beta p^3n^2 - \frac{13}{16}p^2n^3l^2 + mpln^3 - \frac{1}{4}\alpha plnm \times k^4 + \frac{1}{2}\alpha l\beta n^2 - \frac{1}{2}l\beta p^2n^2 + \frac{3}{2}pl^3n^3 + \frac{1}{2}p\beta mn^2 - 2ml^2n^3 + \alpha ml^2n^3 \times k^3 - \frac{1}{2}\alpha m^2n^2 + \frac{1}{4}p^2m^2n^2 + \frac{5}{4}p\beta l^2n^2 - 2m\beta ln^2 - l^4n \times k^2 - plm^2n^2 + 2m^3n^3 - \beta l3n^2 \times k$, the Term $l^2m^2n^2$ having vanished, the Residue is Divisible both by n^2 and k; both which are in every Term. Wherefore writing $\zeta\theta - \frac{1}{4}nn = \lambda$, we have $k = \frac{\lambda}{2nn}$. Having found n, and k, $Q = \frac{nk^2 + \alpha}{2}$ (283), and $R = \frac{1}{2}r - \frac{1}{2}p\alpha - \frac{1}{4}pnk^2 + nkl$ (284) $=$ (by the fourth Equation) $\frac{t + 2nlm}{\alpha + nk2}$; whence by Multiplication and Transposition, $\frac{1}{2}nk^2 + \frac{1}{2}\alpha \times r - \frac{1}{4}n^2k^4 - \frac{1}{2}\alpha nk^2 - \frac{1}{4}\alpha\alpha$

 $\times p$

$2bb$, and $aa - 2bb$ will be n. For n muſt always be affirmative. Moreover, $\frac{\zeta}{n}$ is $-\frac{5}{4}aa + 2ab + \frac{1}{2}bb$, $\frac{\eta}{n}$ is $-\frac{1}{2}a^3 + 2abb$ and $\frac{\theta}{n}$ is $-\frac{1}{4}a^4 + \frac{1}{2}aabb$, and conſequently $\frac{\zeta}{2n} \times \frac{\theta}{n} - \frac{\eta\eta}{8nn}$ or $\frac{\lambda}{2nn}$, is $\frac{1}{8}a^6 - \frac{1}{4}a^5 b - \frac{1}{8}a^4 bb + \frac{1}{2}a^3 b^3 - \frac{3}{8}aab^4$, the Diviſors whereof are 1, a, aa; but becauſe $\sqrt{n} \times k$ cannot be of more than one Dimenſion, and the $\sqrt{n}$ is of one, therefore k will be of none; and conſequently can only be a Number. Wherefore, rejecting a and aa, there remains only 1 for k. Beſides, $\frac{1}{4}a + \frac{1}{2}nkk$ gives 0 for Q, and $\frac{Qr - QQp - t}{n}$ is alſo nothing; and conſequently l, which ought to be its Diviſor, will be nothing. Laſtly, $\frac{1}{2}r - \frac{1}{2}pQ + nkl$ gives abb for R. And $RR - v$ is $-2aab^4 + a^4bb$, which may be divided by n or $aa - 2bb$, and the Root of the Quotient $aabb$ be extracted,

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$\times p - t =$ (that is $rQ - pQ - t$) $l \times \overline{2nm - k^3n^2 - ank}$; wherefore $l = \frac{rQ - pQ - t}{n \times \overline{2m - k^3n - ak}}$: Whence k is a Diviſor of $\frac{rQ - pQ - t}{n}$, by $2m - k^3 - ak$: l being found, $R = \frac{1}{2}r - \frac{1}{2}pQ + nkl$ (284). Laſtly, $m =$ (by the fifth Equation) $\sqrt{\frac{R^2 - v}{n}}$; $=$ (by the 4th Equation) $\frac{QR - \frac{1}{2}t}{nl}$; $=$ (by the third Equation) $\frac{Q^2 + pR - nl^2 - s}{2nk}$. Wherefore if all coincide, for $x^6 + px^5 + qx^4 + rx^3 + sx^2 + tx + v = 0$, write $x^3 + \frac{1}{2}px^2 + Qx + R = \pm\sqrt{n} \times \overline{kx^2 + lx + m}$.

and

and that Root taken negatively, *viz.* $-ab$, is not unequal to the indefinite Quantity $\frac{QR-\frac{1}{2}pt}{nl}$ or $\frac{0}{0}$, but equal to the definite Quantity $\frac{QQ+pR-nll-s}{2nk}$. Wherefore that Root $-ab$ will be m, and in the Room of the Equation proposed, there may be writ $x^3-\frac{1}{2}pxx+Qx+R=\sqrt{n\times kxx}+lx+m$, that is, $x^3-axx+abb=\sqrt{aa-2bb}\times xx-ab$. *The Truth of which Conclusion you may prove by squaring the Parts of the Equation found, and taking away the Terms on the Right Hand from both Sides.* For from that Operation will be produced the Equation $x^6-2ax^5+2bbx^4+2abbx^3-2aabbxx+2a^3bxx-4ab^3xx+3aab^4-a^4bb=0$, which was proposed to be reduced.

CXLVII. *If the Equation is of eight Dimensions, let it be* $x^8+px^7+qx^6+rx^5+sx^4+tx^3+vxx+wx+z=0$, *and make* $q-\frac{1}{4}pp=\alpha$. $r-\frac{1}{2}p\alpha=\beta$. $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$. $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$. $v-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$. $w-\frac{1}{2}\beta\gamma=\zeta$, *and* $z-\frac{1}{4}\gamma\gamma=\eta$. *And seek of the Terms* 2δ, 2ε, 2ζ, 8η, *a common Divisor that shall be an Integer, and neither a square Number, nor divisible by a square Number, and which also divided by* 4 *shall leave Unity, provided any of the alternate Terms,* p, r, t, w *be odd. If there be no such common Divisor, it is certain, that the Equation cannot be reduced by the Extraction of a quadratick surd Root, and if it cannot be so reduced, there will scarce be found a common Divisor of all those four Quantities. The Operation therefore hitherto is a Sort of an Examination, whether the Equation be reducible or not; and consequently, since that Sort of Reductions are seldom possible, it will most commonly end the Work.* (*b*).

CXLVIII. *And, by a like Reason, if the Equation be of ten, twelve, or more Dimensions, the Impossibility of its Reduction may be known.*

CXLVII. (*b*) See N°. 295.

As if it be $x^{10}+px^9+qx^8+rx^7+sx^6+tx^5+vx^4+ax^3+bx^2+cx+d=0$, *you must make* $q-\frac{1}{4}pp=\alpha$, $r-\frac{1}{2}p\alpha=\beta$, $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$, $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$, $v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$, $a-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma=\zeta$, $b-\frac{1}{2}\beta\delta-\frac{1}{4}\gamma\gamma=\eta$, $c-\frac{1}{2}\gamma\delta=\theta$, $d-\frac{1}{4}\delta\delta=\kappa$, *and seek such a common Divisor to the five Terms*, 2ε, 2ζ, 8η, 4θ, 8κ, *as is an Integer, and not a Square, and which shall also leave* 1 *when divided by* 4, *if any one of the Terms* p, r, t, a, c, *be odd* (*i*).

CXLIX. *So if there be an Equation of twelve Dimensions, as* $x^{12}+px^{11}+qx^{10}+rx^9+sx^8+tx^7+vx^6+ax^5+bx^4+cx^3+dx^2+ex+f=0$ *make* $q-\frac{1}{4}pp=\alpha$, $r-\frac{1}{2}p\alpha=\beta$, $s-\frac{1}{2}p\beta-\frac{1}{4}\alpha\alpha=\gamma$, $t-\frac{1}{2}p\gamma-\frac{1}{2}\alpha\beta=\delta$, $v-\frac{1}{2}p\delta-\frac{1}{2}\alpha\gamma-\frac{1}{4}\beta\beta=\varepsilon$, $a-\frac{1}{2}p\varepsilon-\frac{1}{2}\alpha\delta-\frac{1}{2}\beta\gamma=\zeta$, $b-\frac{1}{2}\alpha\varepsilon-\frac{1}{2}\beta\delta-\frac{1}{2}\gamma\gamma=\eta$, $c-\frac{1}{2}\beta\varepsilon-\frac{1}{2}\gamma\delta=\theta$, $d-\frac{1}{2}\gamma\varepsilon-\frac{1}{4}\delta\delta=\kappa$, $e-\frac{1}{2}\delta\varepsilon=\lambda$, $f-\frac{1}{4}\varepsilon\varepsilon=\mu$, *and you must seek a common integer Divisor of the six Terms* 2ζ, 8η, 4θ, 8κ, 4λ, 8μ, *that is not a Square, but being divided by* 4 *shall leave Unity, provided any one of the Terms* p, r, t, a, c, e, *be odd*.

CL. *And thus you may go on* ad infinitum, *and the proposed Equation when it has no common Divisor, will be always irreduceable by the Extraction of the surd quadratick Root. But if at any Time such a Divisor* n *being found, there are Hopes of a future Reduction, it may be tried by following the Steps of the Operation we shewed in an Equation of eight Dimensions* (*k*).

CLI. *Seek a square Number, to which after it is multiplied by* n, *the last Term* z *of the Equation being added under its proper Sign, shall make a square Number. But that may be expeditiously performed if you add to* z, *when* n *is an even Number, or to* 4 z *when it is odd, these Quantities successively* n, 3n, 5n, 7n, 9n, 11n, *and so on till the Sum becomes equal to some Number in the Table of square Numbers, which I suppose to be ready at Hand* (*l*). *And if no such square*

CXLVIII. (*i*) See N°. 279.
CL. (*k*). See N°. 295.
CLI. (*l*) For the Sums of the odd Laterals are square Numbers. (43).

Number occurs before the square Root of that Sum, augmented by the square Root of the Excess of that Sum above the last Term of the Equation, is four Times greater than the greatest of the Terms of the proposed Equation p, q, r, s, t, v, &c. *there will be no Occasion to try any farther; for then the Equation cannot be reduced* (*m*). *But if such a square Number does accordingly occur, let its Root be* S, *if* n *is even, or* 2 S *if* n *be odd; and call* $\sqrt{\frac{SS - z}{n}} = h$. *But* S *and* h *ought to be Integers if* n *is even, but if* n *is odd, they may be Fractions that have* 2 *for their Denominator. And if one is a Fraction, the other ought to be so too. Which also is to be observed of the Numbers* R *and* m, Q *and* l, p *and* k *hereafter to be found. And all the Numbers* S *and* h, *that can be found within the prescribed Limit, must be collected in a Catalogue.*

Afterwards, for k *all the Numbers are to be successively tried, which do not make* $nk + \frac{1}{2}p$ *four Times greater than the greatest Term of the Equation, and you must in all Cases put* $\frac{nkk + \alpha}{2} = Q$. *Then you are to try successively for* l *all the Numbers that do not make* $nl \pm Q$ *four Times greater than the greatest Term of the Equation, and in every Trial put* $\frac{-npkk + 2\beta}{4} + nkl = R$. *Lastly, for* m *you must try successively all the Numbers which do not make* $nm \pm R$ *four Times greater than the greatest of the Terms of the Equation, and you must see whether in any Case if you make* $s - QQ - pR + nll = 2H$, *and* $H + nkm = S$, *let* S *be some of the Numbers which were before brought into the Catalogue for* S; *and besides, if the other Number answering to that* S, *which being set down for* h *in the same Catalogue, will be*

(*m*) For $4S^2 = 4z \pm 4nb^2$, if there is a fractional Term in the Root; now putting *y* the greatest Term of the Equation, $4S^2 - 4nb^2$, *i.e.* $4z \pm 4nb^2 - 4nb^2 = 4z$; if then $4z \pm 4nb^2 - 4nb^2$ be greater than $4y$, it is greater than $4z$, and the Equation irreducible; whence *a fortiori*, if $\sqrt{4z \pm 4nb^2} + \sqrt{4z \pm 4nb^2 - 4nb^2}$ be greater than $4y$, the Equation is irreducible.

 equal

equal to these three, $\frac{2RS - w}{2nm}$, $\frac{2QS + RR - v - nmm}{2nl}$, *and* $\frac{pS + 2QR - t - 2nlm}{2nk}$. *If all these Things shall happen in any Case, instead of the Equation proposed, you must write this* $x^4 + px^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + n}$.

For Example, Let there be proposed the Equation $x^8 + 4x^7 - x^6 - 10x^5 + 5x^4 - 5x^3 - 10xx - 10x - 5 = 0$, and you will have $q - \frac{1}{4}pp = -1 - 4 = -5 = \alpha$, $r - \frac{1}{2}p\alpha = -10 + 10 = 0 = \beta$. $s - \frac{1}{2}p\beta - \frac{1}{4}\alpha\alpha = 5 - \frac{25}{4} = -\frac{5}{4} = \gamma$. $t - \frac{1}{2}p\gamma - \frac{1}{2}\alpha\beta = -5 + \frac{5}{2} = -\frac{5}{2} = \delta$. $v - \frac{1}{2}\alpha\gamma - \frac{1}{4}\beta\beta = -10 - \frac{25}{8} = -\frac{105}{8}$. $w - \frac{1}{4}\beta\gamma = -10 = \zeta$. $z - \frac{1}{4}\gamma\gamma = -5 - \frac{25}{64} = -\frac{345}{64} = \eta$. Therefore 2δ, 2ε, 2ζ, 8η respectively are -5, $-\frac{105}{4}$, -20, and $-\frac{345}{8}$, and their common Divisor 5, which divided by 4, leaves 1, as it ought, because the Term s is odd. Since therefore the common Divisor n, or 5, is found, which gives hope to a future Reduction, and because it is odd, to $4z$, or -20, I successively add n, $3n$, $5n$, $7n$, $9n$, &c. or 5, 15, 25, 35, 45, &c. and there arises -15, 0, 25, 60, 105, 160, 225, 300, 385, 480, 585, 700, 825, 960, 1105, 1260, 1425, 1600. Of which only 0, 25, 225, and 1600 are Squares.

Wherefore the Halves of these Roots 0, $\frac{5}{2}$, $\frac{15}{2}$, 20, are to be collected in a Table for the Values of S, and the Values of $\sqrt{\frac{SS - z}{n}}$, that is, 1, $\frac{3}{2}$, $\frac{7}{2}$, 9, respectively for h. But because $S + nh$, if 20 be taken for S and 9 for h, becomes 65, a Number greater than four Times the greatest Term of the Equation; therefore I reject 20 and 9, and write only the rest in the Table as follows:

h	1	$\frac{3}{2}$	$\frac{7}{2}$
S	0	$\frac{5}{2}$	$\frac{15}{2}$

Then I try for k all the Numbers which do not make $\frac{1}{2}p \pm nk$, or $2 \pm 5k$, greater than 40, (four Times the greatest

greatest Term of the Equation) that is, the Numbers − 8, − 7, − 6, − 5, − 4, − 3, − 2, − 1, 0, 1, 2, 3, 4, 5, 6, 7, putting $\frac{nkk+a}{2}$, or $\frac{5kk-5}{2}$, that is, the Numbers $\frac{315}{2}$, 120, $\frac{175}{2}$, 60, $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, 60, $\frac{175}{2}$, 120, respectively for Q. But even since $Q \pm nl$, and much more Q, ought not to be greater than 40, I perceive I am to reject $\frac{315}{2}$, 120, $\frac{175}{2}$, and 60, and their Correspondents − 8, − 7, − 6, − 5, 5, 6, 7, and consequently that only − 4, − 3, − 2, − 1, 0, 1, 2, 3, 4, must respectively be tried for k, and $\frac{75}{2}$, 20, $\frac{15}{2}$, 0, $-\frac{5}{2}$, 0, $\frac{15}{2}$, 20, $\frac{75}{2}$, respectively for Q. Let us therefore try − 1 for k, and 0 for Q, and in this Case for l there will be successively to be tried all the Numbers which do not make $Q \pm nl$ greater than 40, that is, all the Numbers between 10 and − 10; and for R you are respectively to try the Numbers $\frac{2\beta - npkk}{4} + nkl$, or $-5-5l$, that is, − 55, − 50, − 45, − 40, − 35, − 30, − 25, − 20, − 15, − 10, − 5, 0, 5, 10, 15, 20, 25, 35, 40, 45, the three former of which and the last, because they are greater than 40, may be neglected. Let us try therefore − 2 for l, and 5 for R, and in this Case for m there will be besides to be tried all the Numbers which do not make $R \pm nm$, or $5 \pm 5m$, greater than 40, that is, all the Numbers between 7 and − 9, and see whether or not by putting $s - QQ - pR + nll$, that is $5 - 20 + 20$ or $5 = 2H$, it may be $H + nkm$ or $\frac{5}{2} - 5m = S$; that is, if any of these Numbers $\frac{-65}{2}$, $\frac{-55}{2}$, $\frac{-45}{2}$, $\frac{-35}{2}$, $\frac{-25}{2}$, $\frac{-15}{2}$, $\frac{-5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, $\frac{25}{2}$, $\frac{35}{2}$, $\frac{45}{2}$, $\frac{55}{2}$, $\frac{65}{2}$, $\frac{75}{2}$, $\frac{85}{2}$, is equal to any of the Numbers 0, $\pm\frac{5}{2}$, $\pm\frac{15}{2}$, which were brought into the Catalogue for S. And we meet with four of these $-\frac{15}{2}$, $-\frac{5}{2}$, $\frac{5}{2}$, $\frac{15}{2}$, to which answer $\pm\frac{7}{2}$, $\pm\frac{3}{2}$, $\pm\frac{3}{2}$, $\pm\frac{7}{2}$, written for h in the same Table, as also 2, 1, 0, − 1 substituted for m. But let us try

$-\frac{5}{2}$ for S, 1 for m, and $\pm\frac{3}{2}$ for b, and you will have $\frac{2RS - w}{2nm} = \frac{-25 + 10}{10} = -\frac{3}{2}$, and $\frac{2QS + RR - v - nmm}{2nl} = \frac{21 + 10 - 5}{20} = -\frac{1}{2}$, and $\frac{pS + 2QR - t - 2nlm}{2nk} = \frac{-10 + 5 + 20}{-10} = -\frac{3}{2}$

Wherefore, since there comes out in all Cases $-\frac{3}{2}$, or b, I conclude all the Numbers to be rightly found, and consequently that in room of the Equation proposed, you must write $x^4 + \frac{1}{2}p.x^3 + Qxx + Rx + S = \sqrt{n} \times \overline{kx^3 + lxx + mx + b}$, that is, $x^4 + 2x^3 + 5x - 2\frac{1}{2} = \sqrt{5} \times \overline{-x^3 - 2xx + x - 1\frac{1}{2}}$. For by squaring the Parts of this, there will be produced that Equation of eight Dimensions, which was at first proposed.

But if by trying all the Cases of the Numbers, all the aforesaid Values of b had not in any Case consented, it would be an Argument that the Equation could not be reduced by the Extraction of the surd quadratick Root.

But something might be here remarked for the Abbreviating of the Work, which however I pass over for the Sake of Brevity, seeing the Use of so great Reductions is very little, and I was willing to shew rather the Possibility of the Thing, than a Practice that was commodious. These therefore are the Reductions of Equations by the Extraction of the *surd quadratick Root.*

I might now join the Reductions of Equations by the *Extraction of the surd cubick Root*, but these as being seldom of Use, for Brevity I pass by.

CLII. Yet there are some Reductions of *cubick Equations* commonly known, which, if I should wholly pass over, the Reader might perhaps think us deficient. *Let there be proposed the cubick Equation* $x^3 * + qx + r = 0$, *the second Term whereof is wanting. For that every cubick Equation may be reduced to this Form, is evident from what we have said above. Let* x *be supposed* $= a + b$. *Then will be* $a^3 + 3aab + 3abb + b^3$ *(that is* x^3*)* $+ qx + r = 0$.

Let

Let 3aab + 3abb (*that is*, 3abx) + qx*be* = o, *and then will* $a^3 + b^3 + r$ *be* = o. *By the former Equation* b *is* $= -\frac{q}{3a}$, *and cubically* $b^3 = -\frac{q^3}{27a^3}$. *Therefore, by the latter*, $a^3 - \frac{q^3}{27a^3} + r$ *is* = o, *or* $a^6 + ra^3 = \frac{q^3}{27}$, *and by the Extraction of the adfected quadratick Root*, $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}rr + \frac{q^3}{27}}$. *Extract the cubic Root and you will have* a. *And above, you had* $-\frac{q}{3a} = b$, and $a + b = x$. *Therefore* $a - \frac{q}{3a}$ *is the Root of the Equation proposed* (*n*).

For

CLII. (*n*). Or thus, suppose $a^3 + b^3 + r = 0$, and $3ab + q = 0$; then $a + b = x$. For $\overline{3ab + q}\ (= 0) \times \overline{a + b} = 3a^2b + 3ab^2 + qa + bq\ (= 0)$, and this added to $a^3 + b^3 + r = 0$, the Sum is $a^3 + 3a^2b + 3ab^2 + b^3 + qa + qb + r = 0$; which is the transformed which would have resulted from the Substitution of $a + b$ for x in the Equation $x^3 + qx + r = 0$. Hence it follows, that if two Quantities can be found, which being substituted for a and b, will fulfil the Conditions, that $a^3 + b^3 + r = 0$; and that $3ab + q = 0$; then their Sum substituted for x will make also $x^3 + qx + r = 0$.

Because $a^3 + b^3 + r = 0$, therefore $b^3 = -a^3 - r$; and because $3ab + q = 0$, therefore $b = -\frac{q}{3a}$, and $b^3 = -\frac{q^3}{27a^3}$; wherefore $a^3 + r = \frac{q^3}{27a^3}$; and multiplying by a^3, $a^6 + a^3r = \frac{q^3}{27}$; and by extracting the

For Example, let there be proposed the Equation $y^3 - 6yy + 6y + 12 = 0$. To take away the second Term of this Equation, make $x + 2 = y$, and there will arise $x^3 * - 6x + 8 = 0$. Where q is $= -6$, $r = 8$, $\frac{1}{4}rr = 16$, $\frac{q^3}{27} = -8$, $a^3 = -4 \pm \sqrt{8}$, $a - \frac{q}{3a} = x$, and $x + 2 = y$, that is, $2 + \sqrt[3]{-4 \pm \sqrt{8}} + \frac{2}{\sqrt[3]{-4 \pm \sqrt{8}}} = y$.

CLIII. *And after this Way the Roots of all cubical Equations may be extracted wherein* q *is Affirmative; or also wherein* q *is Negative, and* $\frac{q^3}{27}$ *not greater than* $\frac{1}{4}rr$, *that is, when two of the Roots of the Equation are impossible. But where* q *is Negative, and* $\frac{q^3}{27}$ *at the same Time greater than* $\frac{1}{4}rr$, $\sqrt{\frac{1}{4}rr - \frac{q^3}{27}}$ *becomes an impossible Quantity; and so the Root of the Equation* x *or* y *will, in this Case, be impossible, viz. in this Case there are three possible Roots, which all of them are alike with respect to the Terms of the Equation* q *and* r, *and are indifferently denoted by the Letters* x *or* y, *and consequently all of them ought to be extracted by the same Method, and expressed the same Way as any one is extracted or*

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the affected quadratic Root, $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$; and by extracting the cubic Root, $a = \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}$; and $b = \frac{q}{3a}$. Therefore $b = \frac{q}{3\sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}}$; and therefore $x = a + b = a - \frac{q}{3a} = \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}} - \frac{q}{3\sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}}}$

This is the first Form of Cardan's Rule.

expressed;

expressed; but it is impossible to express all three by the Law aforesaid. The Quantity $a - \frac{q}{3a^2}$ *whereby* x *is denoted, cannot be manifold, and for that Reason the Supposition that* x, *in this Case wherein it is threefold, may be equal to the Binomial* $a - \frac{q}{3a}$, *or* $a + b$, *the Cubes of whose Terms* $a^3 + b^3$ *may together be* $= r$, *and the triple Rectangle* $3ab$ *be* $= q$, *is plainly impossible; and it is no Wonder that from an impossible Hypothesis, an impossible Conclusion should follow* (*o*).

CLIII. (*o*). 299. *The cube Root of any Quantity is threefold.* For let the Equation be $y^3 - 1 = 0$; because $y - 1 =$ divides it, giving the Quote $y^2 + y + 1 = 0$, therefore 1 is a Root; and resolving the quadratic Quote $y^2 + y + 1 = 0$; $y = \frac{-1 \mp \sqrt{-3}}{2}$: Wherefore the cube Roots of 1, are 1, $\frac{-1 - \sqrt{-3}}{2}$, and $\frac{-1 + \sqrt{-3}}{2}$. Now because the cube Root of any Quantity Z^3 may be $1 \times Z$, or $\frac{-1 - \sqrt{-3}}{2} \times Z$, or $\frac{-1 + \sqrt{-3}}{2} \times Z$; therefore supposing the cube Root of the Binome $-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$ to be $m \sqrt{n}$, its three Roots will be $1 \times \overline{m + \sqrt{n}}$, and $\frac{-1 - \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$, and $\frac{1 + \sqrt{-3}}{2} \times \overline{m + \sqrt{n}}$.

If the Equation $x^3 + qx + r = 0$ can have two impossible Roots, $\sqrt{\frac{1}{2}r^2 + \frac{q^3}{27}}$ will be possible (274) consequently either $\frac{q^3}{27}$ will be affirmative, or $\frac{1}{4}r^2$ will be greater than $\frac{q^3}{27}$ under a negative Sign. Whence in the cube Root of the Binome $\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2 + \frac{q^3}{27}}$, when

there

CLIV. *There is, moreover, another Way of expressing these Roots, viz. from* a^3+b^3+r, *that is, from nothing take* a^3+r, *or* $\frac{1}{2}r\pm\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}$, *and there will remain* $b^3 = -\frac{1}{2}r\mp\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}$. *Therefore* a *is* =

$\sqrt[3]{-\frac{1}{2}r+\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}$, *and* b =

$\sqrt[3]{-\frac{1}{2}r-\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}$; *or* a =

$\sqrt[3]{-\frac{1}{2}r-\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}$, and b =

$\sqrt[3]{-\frac{1}{2}r+\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}$, *and consequently the Sum of these*

$\sqrt[3]{-\frac{1}{2}r+\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}+$

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there are two impossible Roots, there can be no Expression of Impossibility. On the contrary, if all the Roots of the Equation $x^3+qx+r=0$ are real, $\sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}$ will be impossible (272); consequently $\frac{q^3}{27}$ will be negative, and $\frac{1}{4}r^2$ will be less than $\frac{q^3}{27}$ under the negative Sign: whence in the cube Root of $\frac{1}{2}r\pm\sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}$, when there are no impossible Roots, there will yet be an Expression of Impossibility. Whence it appears, that *tho' both Parts of the Binome are real, when the Roots are real; yet it is impossible, by this Method, to express the irrational Part as a real Quantity; and in this Sense it is that I would understand the Words,* Adeoque omnes eâdem lege deberent

erui

$\sqrt[3]{-\frac{1}{2}r-\sqrt{\frac{1}{4}rr+\frac{q^3}{27}}}$ *will be* $= x$ (*p*).

CLV. *Moreover the Roots of biquadratick Equations may be extracted and expressed by means of cubick ones.*

But first you must take away the second Term of the Equation. Let the Equation that then results be $x^4+qxx+rx+s=0$. *Suppose this to be generated by the Multiplication of these two* $xx+ex+f=0$, *and* $xx-ex+g=0$, *that is to be the same with this* $x^4 * \begin{matrix}+f\\+g\\-ee\end{matrix} xx \begin{matrix}+eg\\-ef\end{matrix} x +$

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erui et exprimi, quâ una aliqua eruitur et exprimitur: sed omnes tres lege præfatâ exprimere impossibile est. *He says not*, eruere impossibile est; *but*, exprimere impossibile est.

CLIV. (*p*). Or thus, because $a^3 = -\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}$, therefore $a^3+r = +\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}$, and consequently, $b^3 = -a^3-r-\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}$; and thence $b = \sqrt[3]{-\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}}$; consequently $x = a+b = \sqrt[3]{-\frac{1}{2}r \pm \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}} + \sqrt[3]{-\frac{1}{2}r \mp \sqrt{\frac{1}{4}r^2+\frac{q^3}{27}}}$. And this is the second Form of Cardan's Rule; by which it appears, that tho' there is an impossible Expression in each Part, yet, because it is affected with a contrary Sign, it will vanish in the Addition. This Rule, as it expresses rational Roots in the Form of irrationals, and commensurate as incommensurate, and as those Roots can be obtained by a shorter and easier Method (275, &c.) is of no Advantage, except in the Case of two impossible or two equal Roots.

$fg = 0$,

$fg = 0$, *and comparing the Terms you will have* $f + g - ee = q$, $eg - ef = r$, *and* $fg = s$. *Wherefore* $q + ee = f + g$, $\frac{r}{e} = g - f$, $\frac{q + ee + \frac{r}{e}}{2} = g$, $\frac{q + ee - \frac{r}{e}}{2} = f$, $\frac{qq + 2eeq + e^4 - \frac{rr}{ee}}{4} (= fg) = s$, *and by Reduction* $e^6 + 2qe^4 \begin{matrix} +qq \\ -4s \end{matrix} ee - rr = 0$. *For* ee *write* y; *and you will have* $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, *a cubick Equation, whose second Term may be taken away, and then the Root extracted either by precedent Rule or otherwise* (*q*). *Then that Root being had, you must go back again, by putting* $\sqrt{y} = e$, $\frac{q + ee - \frac{r}{e}}{2} = f$, $\frac{q + ee + \frac{r}{e}}{2} = g$, *and the two Equations* $xx + ex + f = 0$, *and* $xx - ex + g = 0$, *their Roots being extracted, will give the four Roots of the biquadratick Equation* $x^4 + qxx + rx + s = 0$, *viz.* $x = -\frac{1}{2}e \pm \sqrt{\frac{1}{4}ee - f}$, *and* $x = \frac{1}{2}e \pm \sqrt{\frac{1}{4}ee - g}$.

CLV. (*q*). For the Roots of the biquadratic being four, let the Sum of any two with the Sign changed be called *e*, the Binaries (35) of four Quantities, whereof two are equal to two with their Signs changed, are six; whence *e* is sixfold: Therefore the Equation whereby *e* may be found will have six Roots and Dimensions, and because (for the second Term of the Biquadratic is wanting) three of those six Roots are respectively equal to three with their Signs changed, the Terms of odd Dimensions will be wanting; and therefore the Equation can be transformed into a Cubic, by which the Squares of the Roots may be found.

CLVI. *Where*

CLVI. *Where note, that if the four Roots of the biquadratick Equation are possible, the three Roots of the cubick Equation* $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$ *will be possible also, and consequently cannot be extracted by the precedent Rule* (*r*).

CLVII. *And thus, if the affected Roots of an Equation of five or more Dimensions are converted into Roots that are not affected, the middle Terms of the Equation being some way or other taken away, that Expression of the Roots will be always impossible, where more than one Root in an Equation of odd Dimensions are possible, or more than two in an Equation of even Dimensions, which cannot be reduced by the Extraction of the surd quadratic Root, by the Method laid down above.*

Monsieur *Des Cartes* taught how to reduce a biquadratic Equation by the Rules last delivered. *E.g.* Let there be proposed the Equation reduced above, $x^4 - x^3 - 5xx + 12x - 6 = 0$. Take away the second Term, by writing $v + \frac{1}{4}$ for x, and there will arise $v^4 - 5\frac{3}{8}vv + \frac{25}{8}v - \frac{851}{256} = 0$. To take away the Fractions, write $\frac{1}{4}z$ for v, and there will arise $z^4 - 86zz + 600z - 851 = 0$. Here is $-86 = q$, $600 = r$, and $-851 = s$, and consequently $y^3 + 2qyy \begin{matrix} +qq \\ -4s \end{matrix} y - rr = 0$, substituting what is equivalent, will become $y^3 - 172yy + 10800y - 360000 = 0$. Where trying all the Divisors of the last Term 1, −1, 2, −2, 3, −3, 4, −4, 5, −5, and so onwards to 100, you will find at length $y = 100$. Which yet may be

CLVI. (*r*). If the Roots of the Biquadratic are all possible, or all impossible, the three Roots of the Cubic will be possible; and will be therefore more conveniently extracted by any other Method than the preceding Rule: But if the Biquadratic has two of its Roots impossible, two Roots of the Cubic will also be impossible, and will therefore be found conveniently by the preceding Rule.

found

found far more expeditiously by our Method above delivered. Then having got y, its Root 10 will be e, and $\frac{q+ee-\frac{r}{e}}{2}$, that is, $\frac{-86+100-60}{2}$ or -23 will be f, and $\frac{q+ee+\frac{r}{e}}{2}$ or 37 will be g, and consequently the Equations $xx+ex+f=0$, and $xx-ex+g=0$, writing z for x, and substituting equivalent Quantities, will become $zz+10z-23=0$, and $zz-10z+37=0$. Restore v in the room of $\frac{1}{4}z$, and there will arise $vv+2\frac{1}{2}v-\frac{23}{16}=0$, and $vv-2\frac{1}{2}v+\frac{37}{16}=0$. Restore, moreover, $x-\frac{1}{4}$ for v, and there will come out $xx+2x-2=0$, and $xx-3x+3=0$, two Equations, the four Roots whereof $x=-1\pm\surd 3$, and $x=1\frac{1}{2}\pm\surd-\frac{3}{4}$, are the same with the four Roots of the biquadratick Equation proposed at the Beginning, $x^4-x^3-5xx+12x-6=0$. *But these might have been more easily found by the Method of finding Divisors, explained before.* (s)

CLVII. (s). In the above Method of Des Cartes, the Biquadratic is supposed to be the Product of two Quadratics: In the Method of our Author, Art. CLIV. the Biquadratic is conceived to be the Difference of two complete Squares. Now though our Author refers in this Place to the Method of Divisors, yet it is plain, that if the Method of N°. 298 were used, it has the Advantage of the Method of Des Cartes: Because in that Method there is not the Trouble of exterminating the second Term; also the Equation, whereby Q is found, is more simple than that whereby $e^2=y$ is found, and lastly, which is the most considerable Advantage, Q is always commensurate and rational (and therefore the more easily to be found) not only when all the Roots of the Biquadratic are commensurate, but also when they are irrational and even impossible.

A P-

APPENDIX.

THE LINEAR CONSTRUCTION OF EQUATIONS.

HITHERTO I have ſhewn the Properties, Tranſmutations, Limits, and Reductions of all Sorts of Equations. I have not always joined the Demonſtrations, becauſe they ſeemed too eaſy to need it, and ſometimes cannot be laid down without too much Tedioufneſs. It remains now only to ſhew, how, after Equations are reduced to their moſt commodious Form, their Roots may be extracted in Numbers. And here the chief Difficulty lies in obtaining the two or three firſt Figures; which may be moſt commodiouſly done by either the geometrical or mechanical Conſtruction of an Equation. Wherefore I ſhall ſubjoin ſome of theſe Conſtructions.

The Antients, as we learn from *Pappus*, at firſt in vain endeavoured at the Triſection of an Angle, and the finding out of two mean Proportionals by a right Line and a Circle. Afterwards they began to conſider ſeveral other Lines, as the Conchoid, the Ciſſoid, and the Conick Sections, and by ſome of theſe to ſolve thoſe Problems. At length, having more throughly examined the Matter, and the Conick Sections being received into Geometry, they diſtinguiſhed Problems into three Kinds; viz. Into

Plane ones, which deriving their Original from Lines on a Plane, may be solved by a right Line and a Circle, into *Solid ones*, which were solved by Lines deriving their Original from the Consideration of a Solid, that is, of a Cone: And *Linear ones*, to the Solution of which were required Lines more compounded. And according to this Distinction, we are not to solve solid Problems by other Lines than the Conick Sections; especially if no other Lines but right ones, a Circle, and the Conick Sections, must be received into Geometry. But the Moderns advancing yet much farther, have received into Geometry all Lines that can be expressed by Equations, and have distinguished, according to the Dimensions of the Equations, those Lines into Kinds; and have made it a Law, that you are not to construct a Problem by a Line of a superior Kind, that may be constructed by one of an inferior one. In the Contemplation of Lines, and finding out their Properties, I approve of their Distinction of them into Kinds, according to the Dimensions of the Equations by which they are defined. But it is not the Equation, but the Description that makes the Curve to be a Geometrical one. The Circle is a Geometrical Line, not because it may be expressed by an Equation, but because its Description is a Postulate. It is not the Simplicity of the Equation, but the Easiness of the Description, which is to determine the Choice of our Lines for the Construction of Problems. For the Equation that expresses a Parabola, is more simple than that that expresses a Circle, and yet the Circle, by reason of its more simple Construction, is admitted before it. The Circle and the Conick Sections, if you regard the Dimension of the Equations, are of the same Order, and yet the Circle is not numbered with them in the Construction of Problems, but, by reason of its simple Description, is depressed to a lower Order, viz. that of a right Line; so that it is not improper to construct that by a Circle that may be constructed by a right Line. But it is a Fault to construct that by the Conick Sections which may be constructed by a Circle. Either therefore you must fix the Law to be observed in a Circle from the Dimensions of Equations, and so take away as vitious the Distinction between Plane and Solid Problems; or else

else you must grant, that that Law is not so strictly to be observed in Lines of superior Kinds, but that some by reason of their more simple Description, may be preferred to others of the same Order, and may be numbered with Lines of inferior Orders in the Construction of Problems. In Constructions that are equally Geometrical, the most simple are always to be preferred. This Law is beyond all Exception. But Algebraick Expressions add nothing to the Simplicity of the Construction. The bare Descriptions of the Lines only are here to be considered. These alone were considered by those Geometricians who joined a Circle with a right Line. And as these are easy or hard, the Construction becomes easy or hard. And therefore it is foreign to the Nature of the Thing, from any thing else to establish Laws about Constructions. Either therefore let us, with the Antients, exclude all Lines besides a right Line, the Circle, and perhaps the Conick Sections, out of Geometry, or admit all, according to the Simplicity of the Description. If the Trochoid were admitted into Geometry, we might, by its Means, divide an Angle in any given Ratio. Would you therefore blame those who should make use of this Line to divide an Angle in the Ratio of one Number to another, and contend that this Line was not defined by an Equation, but that you must make use of such Lines as are defined by Equations? If therefore, when an Angle was to be divided, for Instance, into 10001 Parts, we should be obliged to bring a Curve defined by an Equation of above an hundred Dimensions to do the Business; which no Mortal could describe, much less understand; and should prefer this to the Trochoid, which is a Line well known, and described easily by the Motion of a Wheel or a Circle, who would not see the Absurdity? Either therefore the Trochoid is not to be admitted at all into Geometry, or else, in the Construction of Problems, it is to be preferred to all Lines of a more difficult Description. And there is the same Reason for other Curves. For which Reason we approve of the Trisections of an Angle by a Conchoid, which *Archimedes* in his Lemma's, and *Pappus* in his Collections, have preferred to the Inventions of all others in this

Case; because we ought either to exclude all Lines, besides the Circle and right Line, out of Geometry, or admit them according to the Simplicity of their Descriptions, in which Case the Conchoid yields to none, except the Circle. Equations are Expressions of Arithmetical Computation, and properly have no place in Geometry, except as far as Quantities truly Geometrical (that is, Lines, Surfaces, Solids, and Proportions) may be said to be some equal to others. Multiplications, Divisions, and such sort of Computations, are newly received into Geometry, and that unwarily, and contrary to the first Design of this Science. For whosoever considers the Construction of Problems by a right Line and a Circle, found out by the first Geometricians, will easily perceive that Geometry was invented that we might expeditiously avoid, by drawing Lines, the Tediousness of Computation. Therefore these two Sciences ought not to be confounded. The Antients did so industriously distinguish them from one another, that they never introduced Arithmetical Terms into Geometry. And the Moderns, by confounding both, have lost the Simplicity in which all the Elegancy of Geometry consists. Wherefore that is *Arithmetically* more simple which is determined by the more simple Equations, but that is *Geometrically* more simple which is determined by the more simple drawing of Lines; and in Geometry, that ought to be reckoned best which is Geometrically most simple. Wherefore, I ought not to be blamed, if, with that Prince of Mathematicians, *Archimedes*, and other Antients, I make use of the Conchoid for the Construction of solid Problems. But if any one thinks otherwise, let him know, that I am here sollicitous not for a Geometrical Construction, but any one whatever, by which I may the nearest Way find the Roots of the Equations in Numbers. For the sake whereof I here premise this Lemmatical Problem.

To

To place the right Line BC *of a given Length, so between two other given Lines* AB, AC, *that being produced, it shall pass through the given Point* P. [See Fig. 90.]

If the Line BC turn about the Pole P, and at the same time moves on its End C upon the right Line AC, its other End B shall describe the Conchoid of the Antients. Let this cut the Line AB in the Point B. Join PB, and its Part BC will be the right Line which was to be drawn. And, by the same Law, the Line BC may be drawn, where, instead of AC, some Curve Line is made use of.

If any do not like this Construction by a Conchoid, another, done by a Conick Section, may be substituted in its room. From the Point P to the right Lines AD, AE, draw PD, PE, making the Parallelogram EADP, and from the Points C and D to the right Line AB let fall the Perpendiculars CF, DG, as also from the Point E to the right Line AC, produced towards A, let fall the Perpendicular EH, and making $AD = a$, $PD = b$, $BC = c$, $AG = d$, $AB = x$, and $AC = y$, you will have AD : AG : : AC : AF, and consequently $AF = \frac{dy}{a}$. Moreover, you will have AB : AC : : PD : CD, or $x : y : : b : a - y$. Therefore $by = ax - yx$, which is an Equation expressive of an Hyperbola. And again, by the 13th of the 2d *Elem*, BCq will be $= ACq + ABq - 2FAB$, that is, $cc = yy + xx - \frac{2dxy}{a}$. Both Sides of the former Equation being multiplied by $\frac{2d}{a}$, take them from both Sides of this, and there will remain $cc - \frac{2bdy}{a} = yy + xx - 2dx$, an Equation expressing a Circle, where x and y are at right Angles. Wherefore, if you make these two Lines an Hyperbola and a Circle, by the Help of these Equations, by their Intersection you will have x and y, or AB and

A C, which determine the Position of the right Line B C. But those right Lines will be compounded after this Way. [See Fig. 91.]

Draw any two right Lines, K L equal to A D, and K M equal to P D, containing the right Angle M K L. Compleat the Parallelogram K L M N, and with the Asymptotes L N, M N, describe through the Point K the Hyperbola I K X.

On K M produced towards K, take K P equal to A G, and K Q equal to B C. And on K L produced towards K, take K R equal to A H, and R S equal to R Q. Compleat the Parallelogram P K R T, and from the Center T, at the Interval T S, describe a Circle. Let that cut the Hyperbola in the Point X. Let fall to K P the Perpendicular X Y, and X Y will be equal to A C, and K Y equal to A B. Which two Lines, A C and A B, or one of them, with the Point P, determine the Position sought of the right Line B C. To demonstrate which Construction, and its Cases, according to the different Cases of the Problem, I shall not here insist.

I say, by this Construction, if you think fit, you may solve the Problem. But this Solution is too compounded to serve for any particular Uses. It is a bare Speculation, and Geometrical Speculations have just as much Elegancy as Simplicity, and deserve just so much Praise as they can promise Use. For which Reason, I prefer a Construction by the Conchoid, as much the simpler, and not less Geometrical; and which is of especial Use in the Resolution of Equations as by us proposed. Premising therefore the preceding Lemma, we Geometrically construct Cubick and Biquadratick Problems [*as which may be reduced to Cubick ones*] as follows. [See Fig. 92 and 93.]

Let there be proposed the Cubic Equation $x^3 * + qx + r = 0$, *whose second Term is wanting*, but the third is *denoted under its Sign* $+ q$, *and the fourth by* $+ r$.

Draw

Draw any right Line, K A, which call n. On K A, produced on both Sides, take K B $= \frac{q}{n}$ to the same Side as K A, if it be $+ q$, otherwise to the contrary Part. Bisect B A in C, and on K, as a Center with the Radius K C, describe the Circle C X, and in it accommodate the right Line C X equal to $\frac{r}{nn}$, producing it each Way. Join A X, which produce also both Ways. Lastly, between these Lines C X and A X inscribe E Y of the same Length as C A, and which being produced, may pass through the Point K; then shall X Y be the Root of the Equation. [See Fig. 94.] And of these Roots, those will be Affirmative which fall from X towards C, and those Negative which fall on the contrary Side, if it be $+ r$, but contrarily if it be $- r$.

Demonstration.

To demonstrate which, I premise these Lemma's.

LEMMA I.

Y X *is to* A K *as* C X *to* K E. For draw K F parallel to C X; then because of the similar Triangles A C X, A K F, and E Y X, E K F, it will be A C to A K as C X to K F, and Y X to Y E or A C as K F to K E, and therefore by perturbated Equality Y X to A K as C X to K E. Q. E. D.

LEMMA II.

Y X *is to* A K *as* C Y *to* A K + K E. For by Composition of Proportion Y X is to A K as Y X + C X (*i. e.* C Y) to A K + K E. Q. E. D.

Lemma III.

KE — BK *is to* YX *as* YX *to* AK.

For (by 12 *Elem.* 2.) $YKq - CKq$ is $= CYq - CY \times CX = CY \times YX$. That is, if the Theorem be resolved into Proportionals, CY to YK — CK as YK + CK to YX. But YK — CK is = YK — YE + CA — CK = KE — BK. And YK + CK = YK — YE + CA + CK = KE + AK. Wherefore CY is to KE — BK as KE + AK to YX. But by *Lemma* 2, it was CY to KE + AK as YX to AK. Wherefore by Equality it is YX to KE — BK as AK to YX. Or KE — BK to YX as YX to AK. Q. E. D.

These things being premised, the Theorem will be thus demonstrated.

In the *first Lemma* it was YX to AK as CX to KE, or $KE \times YX = AK \times CX$. In the *third Lemma* it was proved, that KE — BK was to YX as YX to AK. Wherefore, if the Terms of the first Ratio be multiplied by YX, it will be $KE \times YX - BK \times YX$ to XYq as YX : AK that is, $AK \times CX - BK \times YX$ to YXq as YX to AK, and by multiplying the Extremes and Means into themselves, it will be $AKq \times XC - AK \times BK \times YX = YX$ *cube*. Lastly, for YX, AK, BK, and CX, re-substituting x, n, $\frac{q}{n}$, and $\frac{r}{nn}$, this Equation will arise, viz. $r - qx = x^3$. Q. E. D. I need not stay to shew you the Variations of the Signs, for they will be determined according to the different Cases of the Problem.

Let now an Equation be proposed wanting the third Term, as $x^3 + pxx + r = 0$; and in order to construct it, n being assumed, take in any right Line two Lengths KA $= \frac{r}{nn}$, and KB $= p$, and let them be taken the same Way if r and p have like Signs; but otherwise, take them towards contrary Sides. Bisect BA in C, and on K, K, as a Center, with the Radius KC, describe a Circle, into which accommodate $CX = n$, producing it both Ways. Join also AX, and produce it both Ways. Lastly, between these Lines CX and AX inscribe EY $= CA$, so that if produced it may pass through the Point K, and KE will be the Root of the Equation. And the Roots will be Affirmative, when the Point Y falls on that Side of X which lies towards C; and Negative, when it falls on the contrary Side of X, provided it be $+r$; but if it be $-r$, it will be the Reverse of this.

To demonstrate this Proposition, look back to the Figures and Lemma's of the former; and then you will find it thus.

By *Lemma* 1. it was YX to AK as CX to KE, or $YX \times KE = AK \times CX$, and by *Lemma* 3, $KE - KB$ to YX as YX to AK, or (taking KB towards contrary Parts) $KE + KB$ to YX as YX to AK, and therefore $KE + KB$ multiplied by KE will be to $YX \times KE$ or $(AK \times CX)$ as YX to AK, or as CX to KE. Wherefore multiplying the Extreams and Means into themselves, KE *cube* $+ KB \times KEq$ is $= AK \times CXq$; and then for KE, KB, AK, and CK, restoring their Values assigned above, $x^3 + pxx = r$.

Let now an Equation having three Dimensions, and wanting no Term, be proposed in this Form, $x^3 + pxx + qx + r = 0$, *some of whose Roots shall be Affirmative, and some Negative.* [See Fig. 95.]

And first suppose q *Negative*, then in any right Line, as KB, let two Lengths be taken, as $KA = \frac{r}{q}$, and $KB = p$, and take them the same Way, if p and $\frac{r}{q}$ have contrary Signs; but if their Signs are alike, then take the Lengths contrary Ways from the Point K. Bisect AB in C, and there erect the Perpendicular CX equal to the Square Root of the Term q; then between the Lines A X and C X, produced infinitely both Ways, inſcribe the right Line E Y = A C, ſo that being produced, it may paſs through K; ſo ſhall K E be the Root of the Equation, which will be Affirmative when the Point X falls between A and E; but Negative when the Point E falls on that Side of the Point X which is towards A.

But if Q had been Affirmative, then in the Line K B you muſt have taken thoſe two Lengths thus, viz. K A = $\sqrt{\frac{-r}{p}}$, and K B = $\frac{q}{KA}$, and the ſame Way from K, if $\sqrt{\frac{-r}{p}}$ and $\frac{q}{KA}$ have different Signs; but contrary Ways, if the Signs are of the ſame Nature. B A alſo muſt be biſected in C; and there the Pependicular C X erected equal to the Term p; and between the Lines A X and C X, infinitely drawn out both Ways, the right Line E Y muſt alſo be inſcribed equal to A C, and made to paſs through the Point K, as before; then will X Y be the Root of the Equation; Negative when the Point X ſhould fall between A and E, and Affirmative when the Point Y falls on the Side of the Point X towards C.

The Demonſtration of the firſt Caſe.

By the firſt Lemma, K E was to C X as A K to Y X and (by Compoſition) ſo K E + A K, *i. e.* K Y + K C is to C X + Y X, *i. e.* C Y. But in the right-angled Triangle K C Y, $YCq = YKq - KCq = \overline{KY+KC} \times \overline{KY-KC}$; and by reſolving the equal Terms into Proportionals, K Y + K C is to C Y as C Y is to K Y — K C; or K E + A K is to C Y as C Y is to E K — K B. Wherefore ſince K E was to X C in this Proportion,

tion, by Duplication KEq will be to CXq as $KE+AK$ to $KE-KB$, and by multiplying the Extreams and Means by themselves $KE\,cube - KB\times KEq$ is $= CXq\times KE\times CXq\times AK$. And by restoring the former Values $x^3 - pxx = qx + r$.

The Demonstration of the second Case.

By the first Lemma, KE is to CX as AK is to YX, then by multiplying the Extreams and Means by themselves, $KE\times YX$ is $= CX\times AK$. Therefore in the preceding Case, put $KE\times YX$ for $CX\times AK$, and it will be $KE\,cube - KB\times KEq = CXq\times KE + CX\times KE\times YX$; and by dividing all by KE, there will be $KEq - KB\times KE = CXq + CX\times YX$; then multiplying all by AK, and you will have $AK\times KEq - AK\times KB\times KE = AK\times CXq + AK\times CX\times YX$. And again, put $KE\times YX$ instead of its equal $CX\times AK$, then $AK\times KEq - AK\times KB\times KE = EK\times CX\times YX + KE\times YXq$; whence all being divided by KE there will arise $AK\times KE - AK\times KB = YX\times CX + YXq$; and when all are multiplied by YX there will be $AK\times KE\times YX - AK\times KB\times YX = YXq\times CX + YX\,cube$. And instead of $KE\times YX$ in the first Term put $CX\times AK$, and then $CX\times AKq - AK\times BK\times YX = CX\times YXq + YX\,cube$, or, which is the same Thing, $YX\,cube + CX\times YXq + AK\times KB\times YX - CX\times AKq = o$. And by substituting for YX, CX, AK, and KB, their Values x, p, $\sqrt{\frac{-r}{p}}$, $q\sqrt{\frac{p}{-r}}$, there will come out, $x^3 + pxx + qx + r = o$, the Equation to be constructed.

These Equations are also solved, by drawing a right Line from a given Point, in such a Manner that the Part of it which is intercepted between another right Line and a Circle, both given in Position, may be of a given length. [See *Figure* 96.]

For, let there be proposed a Cubic Equation $x^3 * + qx + r = o$, *whose second Term is wanting.*

Draw

Draw the right Line KA at pleaſure, which call n. In KA produced both ways, take KB $= \frac{q}{n}$ on the ſame Side of the Point K as the Point A is if q be Negative, if not, on the contrary. Biſect BA in C, and from the Center A, with the Diſtance AC, deſcribe a circle CX. To this inſcribe the right Line CX $= \frac{r}{nn}$, and through the Points K, C, and X deſcribe the Circle KCXG. Join AX, and produce it till it again cuts the Circle KCXG laſt deſcribed in the Point G. Laſtly, between this Circle KCXG, and the right Line KC produced both ways, inſcribe the right Line EY = AC, ſo that EY produced paſs through the Point G. And EG will be one of the Roots of the Equation. But thoſe Roots are Affirmative which fall in the greater Segment of the Circle KGC, and Negative which fall in the leſſer KFC, if r is Negative, and the contrary will be when r is Affirmative.

In order to demonſtrate this Conſtruction, Let us premiſe the following *Lemmas*.

Lemma I.

All things being ſuppoſed as in the Conſtruction, CE *is to* KA *as* CE + CX *is to* AY, *and as* CX *to* KA.

For the right Line KG being drawn, AC is to AK as CX is to KG, becauſe the Triangles ACX and AKG are ſimilar. The Triangles YEC, YKG are alſo ſimilar; for the Angle at Y is common to both Triangles, and the Angles G and C are in the ſame Segment EGCK of the Circle KGC, and therefore equal. Whence CE will be to EY as KG to KY, that is, CE to AC as KG to KY, becauſe EY and AC were ſuppoſed equal. And by comparing this with the Proportionality above, it will follow by perturbated Equality that CE is to KA as CX to KY, and alternately CE is to CX as KA to KY. Whence, by Compoſition, CE + CX will be to CX as KA + KY to KY, that is, AY to

KY

KY, and alternately CE + CX is to AY as CX is to KY, that is, as CE to KA. Q. E. D.

Lemma II.

Let fall the Perpendicular CH *upon the right Line* GY, *and the Rectangle* 2 HEY *will be equal to the Rectangle* CE × CX.

For the Perpendicular GL being let fall upon the Line AY, the Triangles KGL, ECH have right Angles at L and H, and the Angles at K and E are in the same Segment CKEG of the Circle CGK, and are therefore equal; consequently the Triangles are similar. And therefore KG is to KL as EC to EH. Moreover, AM being let fall from the Point A perpendicular to the Line KG, because AK is equal to AG, KG will be bisected in M; and the Triangles KAM and KGL are Similar, because the Angle at K is common, and the Angles at M and L are right ones; and therefore AK is to KM as KG is to KL. But as AK is to KM so is 2 AK to 2 KM, or KG; (and because the Triangles AKG and ACX are similar) so is 2 AC to CX; also (because AC = EY) so is 2 EY to CX. Therefore 2 EY is to CX as KG to KL. But KG was to KL as EC to EH, therefore 2 EY is to CX as EC to EH, and so the Rectangle 2 HEY (by multiplying the Extreams and Means by themselves) is equal to EC × CX. Q. E. D.

Here we took the Lines AK and AG to be equal. For the Rectangles CAK and XAG are equal (by *Cor.* to 36 *Prop.* of the 3d Book of Eucl.) and therefore as CA is to XA so is AG to AK. But XA and CA are equal by Hypothesis; therefore AG = AK.

Lemma III.

All things being as above, the three Lines BY, CE, KA, *are continual Proportionals.*

For (by *Prop.* 12. *Book* 2. *Elem.*) CY*q* is = EY*q* + CE*q* + 2 EY × EH. And by taking EY*q* from both Sides

Sides, $CYq - EYq$ is $= CEq + 2\,EY \times EH$. But $2\,EY \times EH$ is $= CE \times CX$ (by *Lem.* 2.) and by adding CEq to both Sides, $CEq + 2\,EY \times EH$ becomes $= CEq + CE \times CX$. Therefore $CYq - EYq$ is $= CEq + CE \times CX$, that is, $\overline{CY + EY} \times \overline{CY - EY} = CEq + CE \times CX$. And by resolving the equal Rectangles into proportional Sides, it will be as $CE + CX$ is to $CY + EY$, so is $CY - EY$ to CE. But the three Lines EY, CA, CB, are equal, and thence $CY + EY = CY + CA = AY$, and $CY - EY = CY - CB = BY$. Write AY for $CY + EY$, and BY for $CY - EY$, and it will be as $CE + CX$ is to YA so is BY to CE. But (*by Lem.* 1.) CE is to KA as $CE + CX$ is to AY, therefore CE is to KA as BY is to CE, that is, the three Lines BY, CE, and KA are continual Proportionals. Q. E. D.

Now, by the Help of these three Lemmas, we may *demonstrate* the Construction of the preceding Problem, thus:

By Lem. 1. CE is to KA as CX is to KY, so $KA \times CX$ is $= CE \times KY$, and by dividing both Sides by CE, $\frac{KA \times CX}{CE}$ becomes $= KY$. To these equal Sides add BK, and $BK + \frac{KA \times CX}{CE}$ will be $= BY$. Whence (by *Lem.* 3.) $BK + \frac{KA \times CX}{CE}$ is to CE as CE is to KA, and thence, by multiplying the Extreams and Means by themselves, CEq is $= BK \times KA + \frac{KAq \times CX}{CE}$, and both Sides being multiplied by CE, CE *cube* becomes $= KB \times KA \times CE + KAq \times CX$. CE was called x, the Root of the Equation, KA was $= n$, $KB = \frac{q}{n}$, and $CX = \frac{r}{nn}$. These being substituted instead of CE, KA, KB, and CX, there will arise $x^3 = qx + r$, or $x^3 - qx - r = 0$, the Equation to be constructed;

ftructed; when q and r are Negative, KA and KB having been taken on the fame Side of the Point K, and the affirmative Root being in the greater Segment CGK. This is *one Cafe* of the Conftruction to be demonftrated. Draw KB on the contrary Side, that is, let its Sign be changed, or the Sign of $\frac{q}{n}$, or, which is the fame Thing, the Sign of the Term q, and there will be had the Conftruction of the Equation $x^3 + qx - r = 0$. *Which is the other Cafe.* In thefe Cafes CX, and the affirmative Root CE, fall towards the fame Parts of the Line AK. Let CX and the negative Root fall towards the fame Parts when the Sign of CX, or of $\frac{r}{nn}$, or (which is the fame Thing) the Sign of r is changed; and this will be the *third Cafe* $x^3 + qx + r = 0$, where all the Roots are Negative. And again, when the Sign of KB, or of $\frac{q}{n}$, or only of q, is changed, it will be the *fourth Cafe* $x^3 - qx + r = 0$. The Conftructions of all thefe Cafes may be run through, and particularly demonftrated after the fame Manner as the firft was. We having demonftrated one Cafe, thought it fufficient to touch flightly the reft. Thefe are demonftrated with the fame Words, by changing only the Situation of the Lines.

Now Let the Cubic Equation $x^3 + pxx * + r = 0$, whofe third Term is wanting, be to be conftructed.

In the fame Figure n being taken of any Length, take in any infinite right Line AY, $KA = \frac{r}{nn}$ and $KB = p$ and take them on the fame Side of the Point K, if the Signs of the Terms p and r are the fame, otherwife on contrary Sides. Bifect BA in C, and from the Center K with the Diftance KC defcribe the Circle CXG. And to it infcribe the right Line CX equal to n the affumed Length. Join AX and produce it to G, fo that AG may be equal to AK, and through the Points K, C, X, G, defcribe a Circle. And, laftly, between this Circle and the right Line KC, produced both Ways, inscribe the right Line $EY = AC$, so that being produced it may pass through the Point G; then the right Line KY being drawn, will be one of the Roots of the Equation. And those Roots are Affirmative which fall on that Side of the Point K, on which the Point A is on, if r is Affirmative; but if r is Negative, then the affirmative Roots fall on the contrary Side. And if the affirmative Roots fall on one Side, the negative fall on the other.

This Construction is demonstrated by the Help of the three last *Lemmas* after this Manner.

By the third Lemma, BY, CE, KA are continual Proportionals; and by *Lemma* 1, as CE is to KA so is CX to KY. Therefore BY is to CE as CX to KY. BY is $= KY - KB$. Therefore $KY - KB$ is to CE as CX is to KY. But as $KY - KB$ is to CE so is $\overline{KY - KB} \times KY$ to $CE \times KY$, by *Prop.* 1. *Book* 6. *Euc.* and because of the Proportionals CE to KA as CX to KY it is $CE \times KY = KA \times CX$. Therefore $\overline{KY - KB} \times KY$ is to $KA \times CX$ (as $KY - KB$ to CE, that is) as CX to KY. And by multiplying the Extreams and Means by themselves $\overline{KY - KB} \times KY\,q$ becomes $= KA \times CX\,q$; that is, $KY\,cube - KB \times KY\,q = KA \times CX\,q$. But in the Construction KY was x the Root of the Equation, KB was put $= p$, $KA = \frac{r}{nn}$, and $CX = n$. Write therefore x, p, $\frac{r}{nn}$, and n for KY, KB, KA, and CX respectively, and $x^3 - pxx$ will become $= r$, or $x^3 - pxx - r = 0$.

This Construction may be resolved into these four Cases of Equations, $x^3 - pxx - r = 0$, $x^3 - pxx + r = 0$, $x^3 + pxx - r = 0$, and $x^3 + pxx + r = 0$.

The first Case I have already demonstrated; the rest are demonstrated with the same Words, only changing the Situation of the Lines. To wit, as in taking KA and KB on the same Side of the Point K, and the affirmative Root KY on the contrary Side, has already produced $KY\,cube - KB \times KY\,q = KA \times CX\,q$, and thence

thence $x^3 - pxx - r = 0$; so by taking KB on the other Side the Point K there will be produced, by the like Reasoning, KY *cube* $+ KB \times KY q = KA \times CX q$, and thence $x^3 + pxx - r = 0$. And in these two Cases, if the Situation of the affirmative Root KY be changed, by taking it on the other Side of the Point K, by a like Series of Argumentation you will fall upon the other two Cases, KY *cube* $+ KB \times KY q = - KA \times CX q$, or $x^3 + pxx + r = 0$, and KY *cube* $- KB \times KY q = - KA \times CX q$, or $x^3 - pxx + r = 0$. Which were all the Cases to be demonstrated.

Now let this cubic Equation $x^3 + pxx + qx + r = 0$ *be proposed, wanting no Term (unless perhaps the third).* Which is constructed after this Manner: [See Fig. 97 and 98.]

Take the length n at Pleasure. Draw any right Line $GC = \frac{n}{2}$, and at the Point G erect a Perpendicular GD $= \sqrt{\frac{r}{p}}$, and if the Terms p and r have contrary Signs, from the Center C, with the Interval CD describe a Circle PBE. If they have the same Signs from the Center D, with the Space GC, describe an occult Circle, cutting the right Line GA in H; then from the Center C, with the distance GH, describe the Circle PBE. Then make $GA = - \frac{q}{n} - \frac{r}{nn}$ on the same Side the Point G that C is on, provided the Quantity $- \frac{q}{n} - \frac{r}{np}$ (the Signs of the Terms p, q, r, in the Equation to be constructed being well observed) should come out Affirmative; otherwise, draw GA on the other Sde of the Point G, and at the Point A erect the Perpendicular AY, between which and the Circle PBE already described, inscribe the right Line EY equal to the Term p, so that being produced, it may pass through the Point G; which being done, the Line EG will be one of the Roots of the Equation to be constructed. Those Roots are

Affirmative when the Point E falls between the Points G and Y, and Negative, when the Point E falls without, if p is Affirmative; and the contrary, if Negative.

In order to demonstrate this Construction, let us premise the following *Lemmas*.

Lemma I.

Let EF *be let fall perpendicular to* AG, *and the right Line* EC *be drawn*; $EGq + GCq$ is $= ECq + 2CGF$

For (by *Prop.* 12. *Book* 2. *Elem.*) EGq is $= ECq + GCq + 2GCF$. Let GCq be added on both Sides, and $EGq + GCq$ will become $= ECq + 2GCq + 2GCF$. But $2GCq + 2GCF$ is $= 2GC \times \overline{GC + CF} = 2CGF$. Therefore $EGq + GCq = ECq + 2CGF$. Q. E. D.

Lemma II.

In the first Case of the Construction, where the Circle PBE *passes through the Point* D, $EGq - GDq$ *is* $= 2CGF$.

For by the *first Lemma* $EGq + GCq$ is $= ECq + 2CGF$, and by taking CGq from both Sides, EGq is $= ECq - GCq + 2CGF$. But $ECq - GCq$ is $= CDq - GCq = GDq$. Therefore $EGq = GDq + 2CGF$, and by taking GDq from both Sides, $EGq - GDq$ is $= 2CGF$. Q. E. D.

Lemma III.

In the second Case of the Construction, where the Circle PBE *does not pass through the Point* D, $EGq + GDq$ is $= 2CGF$.

For in the *first Lemma*, $EGq + GCq$ was $= ECq + 2CGF$. Take ECq from both Sides, and it becomes $EGq + GCq - ECq = 2CGF$. But $GC = DH$, and $EC = CP = GH$. Therefore $GCq - ECq = DHq - GHq = GDq$, and so $EGq + GDq = 2CGF$. Q. E. D.

LEMMA IV.

$GY \times 2CGF$ *is* $= 2CG \times AGE$.

For by reason of the similar Triangles GEF and GYA, as GF is to GE so is AG to GY, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2CG \times AG$ is to $2CG \times GY$. Let the Extreams and Means be multiplied by themselves, and $2CG \times GY \times GF$ becomes $= 2CG \times AG \times GE$. Q. E. D.

Now, by the Help of these Lemmas, the Construction of the Problem may be thus demonstrated.

In the *first* Case, $EGq - GDq$ is $= 2CGF$ (by Lemma 2.) and by multiplying all by GY, $EGq \times GY - GDq \times GY$ becomes $= 2CGF \times GY = $ (by Lemma 4.) $2CG \times AGE$. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq - GDq \times EG - GDq \times EY$ becomes $= 2CGA \times EG$, or $EG\,cub. + EY \times EGq - \overline{GDq - 2CGA} \times EG - GDq \times GDq \times EY = 0$.

In the *second* Case, $EGq + GDq$ is $= 2CGF$ (by Lemma 3.) and by multiplying all by GY, $EGq \times GY + GDq \times GY$ becomes $= 2CGF \times GY = 2CG \times AGE$, by Lemma 4. Instead of GY write $EG + EY$, and $EG\,cub. + EY \times EGq + GDq \times EG + GDq \times EY$ will become $= 2CGA \times EG$, or $EG\,cub. + EY \times EGq + \overline{GDq - 2CGA} \times EG + GDq \times EY = 0$.

But the Root of the Equation EG was called x, $GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} - \frac{r}{np}$, that is, in the first Case, where the Signs of the Terms p and r are different; but in the second Case, where the Sign of one of the two, p or r, is changed, there is $-\frac{q}{n} + \frac{r}{np}, = GA$. Let therefore EG be put $= x$,

 GD =

$GD = \sqrt{\frac{r}{p}}$, $EY = p$, $2CG = n$, and $GA = -\frac{q}{n} \mp \frac{r}{np}$, and in the *first* Case it will be $x^3 + px^2 + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x - r = 0$; that is, $x^3 + px^2 + qx - r = 0$; but in the *second* Case, $x^3 + pxx + \overline{q + \frac{r}{p} - \frac{r}{p}} \times x + r = 0$, that is, $x^3 + px^2 + qx + r = 0$. Therefore in both Cases EG is the true Value of the Root x. Q. E. D.

But either Case may be distinguished into its several Particulars; as the former into these, $x^3 + px^2 + qx - r = 0$, $x^3 + px^2 - qx - r = 0$, $x^3 - px^2 + qx + r = 0$, $x^3 - px^2 - qx + r = 0$, $x^3 + px^2 - r = 0$, and $x^3 - px^2 + r = 0$; the latter into these, $x^3 + px^3 + qx + r = 0$, $x^3 + px^2 - qx + r = 0$, $x^3 - px^2 + qx - r = 0$, $x^3 - px^2 - qx - r = 0$, $x^3 + px^2 + r = 0$, and $x^3 - px^2 - r = 0$. The Demonstration of all which Cases may be carried on in the same Words with the two already demonstrated, by only changing the Situation of the Lines.

These are the chief Constructions of Problems, by inscribing a right Line given in Length so between a Circle and a right Line given in Position, that the inscribed right Line produced may pass through a given Point. And such a right Line may be inscribed by describing the *Conchoid* of the Antients, of which let that Point, through which the right Line given ought to pass, be the Pole, the other right Line given in Position be the Ruler or Asymptote, and the Interval be the Length of the inscribed Line. For this Conchoid will cut the Circle in the Point E, through which the right Line to be inscribed must be drawn. But it will be sufficient in Practice to draw the right Line between a Circle and a right Line given in Position by any mechanic Method.

But in these Constructions observe, that the Quantity n is undetermined, and left to be taken at Pleasure, that the Construction may be more conveniently fitted to particular Problems. We shall give Examples of this in finding two mean Proportionals, and in trisecting an Angle.

Let x *and* y *be two mean Proportionals to be found between* a *and* b. Because a, x, y, b are continual Proportionals, a^2 will be to x^2 as x to b, therefore $x^3 = aab$, or $x^3 - aab = 0$. Here the Terms p and q of the Equation are wanting, and $-aab$ is in the room of the Term r; therefore in the first Form of the Constructions, where the right Line EY tending to the given Point K, is drawn between other two right Lines EX and YC given in Position, and the right Line CX supposed $= \frac{r}{nn}$ that is $= \frac{-aab}{nn}$, let n be taken equal to a, and then CX will become $= -b$. From whence the following Construction comes out. [See Fig. 99.]

I draw any Line, KA $= a$, and bisect it in C, and from the Center K, with the Distance KC, describe the Circle CX, to which I inscribe the right Line CX $= b$, and between AK and CX, infinitely produced, I so inscribe EY $=$ CA, that EY being produced, may pass through the Point K. So KA, XY, KE, CX will be continual Proportionals, that is, XY and KE two mean Proportionals between a and b. This Construction is known. [See Fig. 100.]

But in the other Form of the Constructions, where the right Line EY converging to the given Point G is inscribed between the Circle GECX and the right Line AK, and CX is $= \frac{r}{nn}$, that is, (in this Problem) $= \frac{-aab}{nn}$, I put, as before, $n = a$, and then CX will be $= b$, and the rest are done as follows. [See Figure 101.]

I draw any right Line KA $= a$, and bisect it in C, and from the Center A, with the Distance AK, I describe the Circle KG, to which I inscribe the right Line KG $= 2b$, constituting the *Isosceles* Triangle AKG. Then, through the Points C, K, G, I describe the Circle, between the Circumference of which and the right Line AK produced, I inscribe the right Line EY $=$ CK tending to the Point G. Which being done, AK, EC, KY, $\frac{1}{2}$KG are continual Proportionals, that is, EC and KY are two mean Proportionals between the given Quantities a and b.

Let there be an Angle to be divided into three equal Parts; [See *Figure* 102.] *and let that Angle be* ACB, *and the Parts thereof to be found be* ACD, ECD, *and* ECB.

From the Center C, with the Distance CA, let the Circle ADEB be described, cutting the right Lines CA, CD, CE, CB in A, D, E, B. Let AD, DE, EB be joined, and AB cutting the right Lines CD, CE at F and H, and let DG, meeting AB in G, be drawn parrallel to CE. Because the Triangles CAD, ADF, and DFG are similar, CA, AD, DF, and FG are continual Proportionals. Therefore if AC be $= a$, and AD $= x$, DF will be equal to $\frac{xx}{a}$, and FG $= \frac{x^3}{aa}$. But AB is $=$ BH $+$ HG $+$ FA $-$ GF $=$ 3AD $-$ GF $= 3x - \frac{x^3}{aa}$. Let AB be $= b$, then b becomes $= 3x - \frac{x^3}{aa}$, or $x^3 - 3aax + aab = 0$. Here p, the second Term of the Equation, is wanting, and instead of q and r we have $-3aa$ and aab. Therefore in the first Form of the Constructions, where p was $= 0$, KA $= n$, KB $= \frac{q}{n}$, and CX $= \frac{r}{nn}$, that is, in this Problem, KB $= -\frac{3aa}{n}$, and CX $= \frac{aab}{nn}$, that these

Quantities

Quantities may come out as simple as possible, I put $n = a$, and so K B becomes $= - 3a$, and C H $= b$. Whence this *Construction* of the Problem comes out.

Draw any Line, K A $= a$, and on the contrary Side make K B $= 3a$. [*See* Figure 103.] Bisect B A in C, and from the Center K, with the Distance K C, describe a Circle, to which inscribe the right Line C X $= b$, and the Line A X being drawn, between that infinitely produced and the right Line C X, inscribe the right Line E Y = A C, and so that it being produced, will pass through the Point K. So X Y will be $= x$. But (see the last Figure) because the Circle A D E B = C X A, and the Subtense A B = Subtense C X, and the Parts of the Subtenses B H and X Y are equal; the Angles A C B, and C K X will be equal, as also the Angles B C H, X K Y; and so the Angle X K Y will be one third Part of the Angle C K X. Therefore the third Part X K Y of any given Angle C K X is found by inscribing the right Line E Y = A C the Diameter of the Circle, between the Chords C X and A X infinitely produced, and converging towards K the Center of the Circle.

Hence, if from K, the Center of the Circle, you let fall the Perpendicular K H upon the Chord C X, the Angle H K Y will be one third Part of the Angle H K X; so that if any Angle H K X were given, the third Part thereof H K Y may be found by letting fall from any Point X of any Side K X, the Line H X perpendicular to the other Side H K, and by drawing X E parallel to H K, and by inscribing the right Line Y E $= 2$ X K between X H and X E, so that it being produced may pass through the Point K. *Or thus.* [See *Figure* 104.]

Let any Angle A X K be given. To one of its Sides A X raise a Perpendicular X H, and from any Point K of the other Side X K let there be drawn the Line K E, the Part of which E Y (lying between the Side A X produced, and the Perpendicular X H) is double the Side

X K, and the Angle K E A will be one third of the given Angle A X K. Again, the Perpendicular E Z being raised, and K F being drawn, whose Part Z F, between E F and E Z, let be double to K E, and the Angle K F A will be one third of the Angle K E A; and so you may go on by a continual Trisection of an Angle *ad infinitum.* This Method is in the 32d *Prop.* of the 4th Book of *Pappus.*

But if you would trisect an Angle by the other Form of Constructions, where the right Line is to be inscribed between another right Line and a Circle: Here also will K B be $= \frac{q}{n}$, and C X $= \frac{r}{nn}$, that is, in the Problem we are now about, K B $= \frac{-3aa}{n}$, and C X $= \frac{aab}{nn}$; and so by putting $n = a$, K B will be $= -3a$, and C X $= b$. Whence this *Construction* comes out.

From any Point K let there be drawn two right Lines towards the same Way, K A $= a$, and K B $= 3a$. [*See* Figure 105.] Bisect A B in C, and from the Center A with the Distance A C describe a Circle. To which inscribe the right Line C X $= b$. Join A X, and produce it till it cuts the Circle again in G. Then between this Circle and the right Line K C, infinitely produced, inscribe the Line E Y $=$ A C, and passing through the Point G; and the right Line E C being drawn, will be equal to x the Quantity sought, by which the third Part of the given Angle will be subtended.

This Construction arises from the Form above; which, however, comes out better thus: Because the Circles A D E B and K X G are equal, and also the Subtenses C X and A B, the Angles C A X, or K A G, and A C B are equal, therefore C E is the Subtense of one third Part of the Angle K A G. Whence in any given Angle K A G, that its third Part C A E may be found, inscribe the right Line E Y equal to the Semi-Diameter A G of the Circle K C G, between the Circle and the Side K A,

of

of the Angle, infinitely produced, and tending to the Point G. Thus *Archimedes*, in *Lemma* 8, taught to trisect an Angle. The same Constructions may be more easily explained than I have done here; but in these I would shew how, from the general Constructions of Problems I have already explained, we may derive the most simple Constructions of particular Problems.

Besides the Constructions here set down, we might add many more, [*See* Figure 106.] *As if there were two mean Proportionals to be found between* a *and* b. Draw any right Line A K = *b*, and perpendicular to it A B = *a*. Bisect A K in I, and in A K put A H equal to the Subtense B I; and also in the Line A B produced, A C = Subtense B H. Then in the Line A K on the other Side of the Point A, take A D of any Length and D E equal to it, and from the Centers D and E, with the Distances D B and E C, describe two Circles, B F and C G, and between them draw the right Line F G equal to the right Line A I, and converging at the Point A, and A F will be the first of the two mean Proportionals that were to be found.

The Antients taught how to find two mean Proportionals by the *Cissoid*; but no Body that I know of hath given a good manual Description of this Curve. [*See* Figure 107.] Let A G be the Diameter, and F the Center of a Circle to which the *Cissoid* belongs. At the Point F let the Perpendicular F D be erected, and produced *in infinitum*. And let F G be produced to P, that F P may be equal to the Diameter of the Circle. Let the rectangular Ruler P E D be moved, so that the Leg E P may always pass through the Point P, and the other Leg E D must be equal to the Diameter A G, or F P, with its End D, always moving in the Line F D; and the middle Point C of this Leg will describe the *Cissoid* G C K which was desired, as has been already shewn. Wherefore, if between any two Quantities, *a* and *b*, there be two mean Proportionals to be found: Take A M = *a*, raise the Perpendicular M N = *b*. Join A N; and move the Rule P E D, as was just now shewn, until its

Point

Point C fall upon the right Line A N. Then let fall C B perpendicular to A P, take t to B H, and v to B G, as M N is to B C, and becauſe A B, B H, B G, B C, are continual Proportionals, a, t, v, b will alſo be continual Proportionals.

By the Application of ſuch a Ruler other ſolid Problems may be conſtructed.

Let there be propoſed the cubick Equation $x^3 \pm pxx + qx - r = 0$; where let q be always Affirmative, r Negative, and p of any Sign. Make $AG = \frac{r}{q}$, and biſect it in F, and take F R and $GL = \frac{p}{2}$, and that towards A if it be $+p$, if not towards P. Moreover, erect the Perpendicular F D, and in it take $FQ = \sqrt{q}$; to this erect alſo the Perpendicular Q C. And in the Leg E D of the Ruler, take E D and E C reſpectively equal to A G and A R, and let the Leg of the Ruler be applied to the Scheme, ſo that the Point D may touch the right Line F D, and the Point C the right Line Q C, then if the Parallelogram B Q be compleated, L B will be the ſought Root x of the Equation.

Thus far, I think, I have expounded the Conſtruction of ſolid Problems by Operations whoſe manual Practice is moſt ſimple and expeditious. So the Antients, after they had obtained a Method of ſolving theſe Problems by a Compoſition of ſolid Places, thinking the Conſtructions by the conic Sections uſeleſs, by reaſon of the Difficulty of deſcribing them, ſought eaſier Conſtructions by the Conchoid, Ciſſoid, the Extenſion of Threads, and by any Mechanic Application of Figures, preferring uſeful Things, though mechanical, to uſeleſs Speculations in Geometry, as we learn from Pappus. So the great Archimedes himſelf neglected the Triſection of an Angle by the conic Sections, which had been handled by other Geometricians before him, and taught how to triſect an Angle in his Lemmas after the Me-

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thod we have already explained. If the Antients had rather construct Problems by Figures not received in Geometry in that Time, how much more ought these Figures now to be preferred which are received by many into Geometry as well as the conic Sections?

However, I do not agree to this new sort of Geometricians, who receive all Figures into Geometry. Their Rule of admitting all Lines to the Construction of Problems in that Order in which the Equations, whereby the Lines are defined, ascend to the Number of Dimensions, is arbitrary, and has no Foundation in Geometry. Nay, it is false; for according to this Rule, the Circle should be joined with the conic Sections, but all Geometers join it with the right Line; and this being an inconstant Rule, takes away the Foundation of admitting into Geometry all analytic Lines in a certain Order. In my Judgment, no Lines ought to be admitted into plain Geometry besides the right Line and the Circle, unless some Distinction of Lines might be first invented, by which a circular Line might be joined with a right Line, and separated from all the rest. But truly plain Geometry is not then to be augmented by the Number of Lines: for all Figures are plain that are admitted into plain Geometry, that is, those which the Geometers postulate to be described *in plano*; and every plain Problem is that which may be constructed by plain Figures. So therefore admitting the conick Sections and other Figures more compounded into plain Geometry, all the solid and more than solid Problems that can be constructed by these Figures will become plane. But all plane Problems are of the same Order. A right Line is analytically more simple than a Circle; nevertheless, Problems which are constructed by right Lines alone, and those that are constructed by Circles, are of the same Order. These Things being postulated, a Circle is reduced to the same Order with a right Line. And much more the Ellipse, which differs much less from a Circle than a Circle from a right Line, by postulating in like manner the Description thereof *in plano*, will be reduced to the same Order with the Circle. If any, in considering

dering the Ellipse, should fall upon some solid Problem, and should construct it by the Help of the same Ellipse, and a Circle; this would be counted a plane Problem, because the Ellipse was supposed to be described *in plano*, and all the Construction besides will be solved by the Description of the Circle only. Wherefore for the same Reason, every plane Problem whatever may be constructed by a given Ellipse. For Example, [*See* Figure 108.] if the Center O of the given Ellipse A D F G be required, I would draw the two Parallels A B, C D meeting the Ellipse in A, B, C, D; and also two other Parallels E F, G H meeting the Ellipse in E, F, G, H, and I would bisect them in I, K, L, M, and produce I K, L M, till they meet in O. This is a real Construction of a plane Problem by an Ellipse. It imports nothing that an Ellipse is analytically defined by an Equation of two Dimensions: nor that it be generated geometrically by the Section of a solid Figure. The Hypothesis, only considering it as already described *in plano*, reduces all solid Problems constructed by it to the Order of plane ones, and concludes, that all plane ones may be rightly constructed by it: and this is the State of a *Postulate*. Whatever may be supposed done, it is permitted to assume it, as already done and given. Therefore let this be a Postulate to describe an Ellipse *in plano*, and then all those Problems that can be constructed by an Ellipse, may be reduced to the Order of plane ones, and all plane Problems may be constructed by the Ellipse.

It is necessary therefore that either plane and solid Problems be confounded among one another, or that all Lines be flung out of plane Geometry, besides the right Line and the Circle, unless it happens that sometime some other is given in the State of constructing some Problem. But certainly none will permit the Orders of Problems to be confused. Therefore the conick Sections and all other Figures must be cast out of plane Geometry, except the right Line and the Circle, and those which happen to be given in the State of the Problems. Therefore all these Descriptions of the Conicks *in plano*, which the

the Moderns are so fond of, are foreign to Geometry. Nevertheless, the conick Sections ought not to be flung out of Geometry. They indeed are not described geometrically *in plano*, but are generated in the plane Superficies of a geometrical Solid. A Cone is constituted geometrically, and cut by a geometrical Plane. Such a Segment of a Cone is a geometrical Figure, and has the same Place in solid Geometry, as the Segment of a Circle has in Plane, and for this Reason its Base, which they call a conick Section, is a geometrical Figure. Therefore a conick Section hath a Place in Geometry so far as it is the Superficies of a geometrical Solid; but is geometrical for no other Reason than that it is generated by the Section of a Solid, and therefore was not in former Times admitted but only into solid Geometry. But such a Generation of the conick Sections is difficult, and generally useless in Practice, to which Geometry ought to be most serviceable: therefore the Antients betook themselves to various mechanical Descriptions of Figures *in plano*; and we, after their Example, have framed the preceding Constructions. Let these Constructions be mechanical; and so the Constructions by conick Sections described *in plano* (as is wont now to be done) are mechanical. Let the Constructions by conick Sections given be geometrical; and so the Constructions by any other given Figures are geometrical, and of the same Order with the Constructions of plane Problems. There is no Reason that the conick Sections should be preferred in Geometry before any other Figures, unless so far as they are derived from the Section of a Cone; they being altogether unserviceable in Practice in the Solution of Problems. But least I should wholly neglect Constructions by the conick Sections, it will be proper to say something concerning them, in which also we will consider some commodious manual Description.

The Ellipse is the most simple of the conick Sections, most known, and nearest of Kin to a Circle, and easiest described by the Hand *in plano*. Many prefer the Parabola before it, for the Simplicity of the Equation by which it is expressed. But by this Reason the Parobola ought

ought to be preferred before the Circle itself, which it never is: therefore the reasoning from the Simplicity of the Equation will not hold. The modern Geometers are too fond of the Speculation of Equations. The Simplicity of these is of an analytic Consideration. We treat of Composition, and Laws are not given to Composition from Analysis. Analysis does lead to Composition: But it is not true Composition before its freed from Analysis. If there be never so little Analysis in Composition, that Composition is not yet real. Composition in itself is perfect, and far from a Mixture of analytick Speculations. The Simplicity of Figures depend upon the Simplicity of their Genesis and Ideas, and it is not an Equation but a Description (either geometrical or mechanical) by which a Figure is generated and rendered more easy to the Conception. Therefore we give the Ellipse the first Place, and shall now shew how to construct Equations by it.

Let there be any cubick Equation proposed, $x^3 = px^2 + qx + r$, *where* p, q, *and* r *signify given Co-efficients of the Terms of the Equation, with their Signs* + *and* —, *and either of the Terms* p *and* q, *or both of them, may be wanting.* For so we shall exhibit the Constructions of all cubick Equations in one Operation, which follows:

From the Point B in any given right Line, take any two right Lines, B C and B E, on the same Side the Point B, and also B D, so that it may be a mean Proportional between them. [*See* Figure 109.] And call B C, n, in the same right Line also take $BA = \frac{q}{n}$, and that towards the Point C, if $-q$, if not, the contrary Way. At the Point A erect a Perpendicular A I, and in it take $AF = p$, $FG = AF$, $FI = \frac{r}{nn}$, and F H to F I as B C is to B E. But F H and F I are to be taken on the same Side of the Point F towards G, if the Terms p and r have the same Signs; and if they have not the same Signs, towards the Point A. Let the Parallelograms I A C K and and H A E L be compleated, and from the Center K, with the Distance K G, let a Circle be described. Then in the

the Line HL let there be taken HR on either Side the Point H, which let be to H L as B D to BE; let G R be drawn, cutting E L in S, and let the Line G RS be moved with its Point R falling on the Line H L, and the Point S upon the Line E L, until its third Point G in describing the Ellipse, meet the Circle, as is to be seen in the Position of $\gamma\rho\sigma$. For half the Perpendicular γX let fall from γ the Point of meeting to A E will be the Root of the Equation. But G or γ the End of the Rule G R S, or $\gamma\rho\sigma$, can meet the Circle in as many Points as there are possible Roots. And those Roots are affirmative which fall towards the same Parts of the Line E A, as the Line F I drawn from the Point F does, and those are negative which fall towards the contrary Parts of the Line A E if r is affirmative; and contrarily if r is negative.

But this Construction is *demonstrated* by the Help of the following *Lemmas*.

LEMMA I.

All being supposed as in the Construction, $2CAX - AXq$ is $= \gamma Xq - 2AI \times \gamma X + 2AG \times FI$.

For from the Nature of the Circle, $K\gamma q - CXq$ is $= \overline{\gamma X - AI}|^2$. But $K\gamma q$ is $= GIq + ACq$, and $CXq = \overline{AX - AC}|^2$. that is, $= AXq - 2CAX + ACq$, and so their Difference $GIq + 2CAX - AXq$ is $= \overline{\gamma X - AI}|^2 = \gamma Xq - 2AI \times \gamma X + AIq$. Subtract GIq from both, and there will remain $2CAX - AXq = \gamma Xq - 2AI \times \gamma X + AIq - GIq$. But (by *Prop.* 4. *Book* 2. *Elem.*) AIq is $= AGq + 2AGI + GIq$, and so $AIq - GIq$ is $= AGq + 2AGI$, that is, $= 2AG \times \overline{\tfrac{1}{2}AG + GI}$, or $= 2AG \times FI$, and thence $2CAX - AXq$ is $= \gamma Xq - 2AI \times \gamma X + 2AG \times FI$. Q.E.D.

LEMMA

Lemma II.

All Things being constructed as above $2EAX - AXq$ is $= \frac{FI}{FH} X\gamma q - \frac{2FI}{FH} AH \times X\gamma + 2AG \times FI$.

For it is known, that the Point γ, by the Motion of the Ruler $\gamma\rho\sigma$ assigned above, describes an Ellipse, the Center whereof is L, and the two Axes coincide with the two right Lines LE and LH, of which that which is in LE is $= 2\gamma\rho$, or $= 2GR$, and the other which is in LH is $= 2\gamma\sigma$, or $= 2GS$. And the Ratio of these to one another is the same as that of the Line HR to the Line HL, or of the Line BD to the Line BE. Whence the *Latus Transversum* is to the principal *Latus Rectum*, as BE is to BC, or as FI is to FH. Wherefore since γT is ordinately applied to HL, it will be from the Nature of the Ellipse $GSq - LTq = \frac{FI}{FH} T\gamma$ squared. But LT is $= AE - AX$, and $T\gamma = X\gamma - AH$. Let the Squares of which be put instead of LTq and $T\gamma q$, and then $GSq - AEq + 2EAX - AXq$ will become $= \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$. But $GSq - AEq = \overline{GH + LS}|^2$, because GS is the Hypothenuse of a rectangled Triangle the Sides whereof are equal to AE and GH+LS. And (by reason of the similar Triangles RGH and RSL) LS is to GH as LR is to HR, and by Composition $GH + LS$ is to GH as HL is to HR, and by squaring the Proportions $\overline{GH + LS}|^2$ is to GHq as HLq is to HRq, that is, (by Construction) as BEq is to BDq, that is, as BE is to BC, or as FI is to FH, and so $\overline{GH + LS}|^2$ is $= \frac{FI}{FH} GHq$. Therefore $GSq - AEq$ is $= \frac{FI}{FH} GHq$, and so $\frac{FI}{FH} GHq + 2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq$.

Subtract

Subtract $\frac{FI}{FH}GHq$ from both Sides, and there will remain $2EAX - AXq = \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + AHq - GHq$. But AH is $= AG + GH$, and so $AHq = AGq + 2AGH + GHq$, and by subtracting GHq from both, there will remain $AHq - GHq = AGq + 2AGH$, that is, $= 2AG \times \overline{\tfrac{1}{2}AG + GH} = 2AG \times FH$, and therefore $2EAX - AXq$ is $= \frac{FI}{FH} \times X\gamma q - 2AH \times X\gamma + 2AG \times FH$, that is, $= \frac{FI}{FH}X\gamma q - \frac{2FI}{FH}AH \times X\gamma + 2AG \times FI$. Q. E. D.

Lemma III.

All Things standing as before, AX will be to $X\gamma - AG$ as $X\gamma$ is to 2BC.

For if from the Equals in the *second Lemma* there be subtracted the Equals in the *first Lemma*, there will remain $2CE \times AX = \frac{HI}{FH}X\gamma q - \frac{2FI}{FH}AH \times X\gamma + 2AI \times X\gamma$. Let both Sides be multiplied by FH, and $2FH \times CE \times AX$ will become $= HI \times X\gamma q - 2FI \times AH \times X\gamma + 2AI \times FH \times X\gamma$. But AI is $= HI + AH$, and so $2FI \times AH - 2FH \times AI = 2FI \times AH - 2FHA - 2FHI$. But $2FI \times HA - 2FHA = 2AHI$, and $2AHI - 2FHI = 2HI \times AF$. Therefore $2FI \times AH - 2FH \times AI = 2HI \times AF$, and so $2FH \times CE \times AX = HI \times X\gamma q - 2HI \times AF \times X\gamma$. And thence as HI is to FH, so is $2CE \times AX$ to $X\gamma q - 2AF \times X\gamma$. But by Construction HI is to FH as CE is to BC, and consequently as $2CE \times AX$ is to $2BC \times AX$, and therefore $2BC \times AX$ will be $= X\gamma q - AF \times X\gamma$, (by *Prop.* 9, *Book* 5, *Elem.*) But because the Rectangles are equal, the Sides are proportional, AX to $X\gamma - 2AF$, (that is, $X\gamma - AG$) as $X\gamma$ is to 2BC. Q. E. D.

 LEMMA

Lemma IV.

The same Things being still supposed, 2 F I *is to* A X — 2 A B *as* X γ *is to* 2 B C.

For if from the Equals in the *third Lemma*, to wit, $2BC \times AX = X\gamma q - 2AF \times X\gamma$, the Equals in the *first Lemma* be subtracted, there will remain $-2AB \times AX + AXq = 2FI \times X\gamma - 2AG \times FI$, that is, $AX \times \overline{AX - 2AB} = 2FI \times \overline{X\gamma - AG}$. But because the Rectangles are equal, the Sides are proportional, 2 F I is to A X — 2 A B as A X is to X γ — A G, that is, (by the *third Lemma*) as X γ is to 2 B C. Q. E. D.

At length, by the Help of these Lemmas, the Construction of the Problem is thus demonstrated.

By the *fourth Lemma*, X γ is to 2 B C as 2 F I is to A X — 2 A B, that is, (by *Prop.* 1. *Book* 6. *Elem.*) as $2BC \times 2FI$ is to $2BC \times \overline{AX - 2AB}$, or to $2BC \times AX - 2BC \times 2AB$. But by the *third Lemma*, A X is to X γ — 2 A F as X γ is to 2 B C, or $2BC \times AX = X\gamma q - 2AF \times X\gamma$, and consequently X γ is to 2 B C as $2BC \times 2FI$ is to $X\gamma q - 2AF \times X\gamma - 2BC \times 2AB$. And by multiplying the Means and Extreams into themselves, $X\gamma\ cub. - 2AF \times X\gamma q - 4BC \times AB \times X\gamma = 8BCq \times FI$. And by adding $2AF \times X\gamma q + 4BC \times AB \times X\gamma$ to both Sides $X\gamma\ cub.$ is $= 2AF \times X\gamma q + 4BC \times AB \times X + 8BCq \times FI$. But $\frac{1}{2}X\gamma$ in the Construction to be demonstrated was equal to the Root of the Equation $= x$, and $AF = p$, $BC = n$, $AB = \frac{q}{n}$, and $FI = \frac{r}{nn}$, and therefore $BC \times AB = q$. And $BCq \times FI = r$. Which being substituted, will make $x^3 = px^2 + qx + r$. Q. E. D.

Corol. Hence if A F and A B be supposed equal to nothing, by the third and fourth *Lemma*, 2 F I will be to A X as A X is to X γ, and X γ to 2 B C. From whence arises the

the Invention of two mean Proportionals between any two given Quantities, F I and B C.

Scholium. Hitherto I have only expounded the Construction of a cubick Equation by the Ellipse; but the Rule is of a more universal Nature, extending itself indifferently to all the conick Sections. For, if instead of the Ellipse you would use the Hyperbola, take the Lines B C and B E on the contrary Side of the Point B, then let the Points A, F, G, I, H, K, L, and R be determined as before, except only that F H ought to be taken on the Side of F not towards I, and that H R ought to be taken in the Line A I not in H L, on each Side the Point H, and instead of the right Line G R S, two other right Lines are to be drawn from the Point L to the two Points R and R for Asymptotes to the Hyperbola. With these Asymptotes L R, L R describe an Hyperbola through the Point G, and a Circle from the Center K with the Distance G K: And the halves of the Perpendiculars let fall from their Intersections to the right Line A E will be the Roots of the Equation proposed. All which, the Signs + and − being rightly changed, are demonstrated as above.

But if you would use the Parabola, the Point E will be removed to an infinite Distance, and so not to be taken any where, and the Point H will coincide with the Point F, and the Parabola will be to be described about the Axis H L with the principal *Latus Rectum* B C through the Points G and A, the Vertex being placed on the same Side of the Point F, on which the Point B is in respect of the Point C.

Thus the Constructions by the Parabola, if you regard analytick Simplicity, are the most simple of all. Those by the Hyperbola next, and those which are solved by the Ellipse, have the third Place. But if in describing of Figures the Simplicity of the manual Operation be respected, the Order must be changed.

But it is to be observed in these Constructions, that by the Proportion of the principal *Latus Rectum* to the *Latus Transversum*, the Species of the Ellipse and Hyperbola

 may

may be determined, and that Proportion is the same as that of the Lines BC and BE, and therefore may be assumed: But there is but one Species of the Parabola, which is obtained by putting BE infinitely long. So therefore we may construct any cubick Equation by a conick Section of any given Species. To change Figures given in Specie into Figures given in Magnitude, is done by encreasing or diminishing in a given Ratio, all the Lines by which the Figures were given in Specie, and so we may construct all cubick Equations by any given conick Section whatever.. Which is more fully explained thus.

Let there be proposed any cubick Equation $x^3 = pxx . qx . r$, *to construct it by the Help of any given conick Section.* [See *Figures* 110 and 111.]

From any Point B in any infinite right Line BCE, take any two Lengths BC, and BE towards the same Way, if the conick Section is an Ellipse, but towards contrary Ways if it be an Hyperbola. But let BC be to BE as the principal *Latus Rectum* of the given Section, is to the *Latus Transversum*, and call BC, n, take BA $= \frac{q}{n}$, and that towards C, if q be negative, and contrarily if affirmative. At the Point A erect a Perpendicular AI, and in it take AF $= p$, and FG $=$ AF; and FI $= \frac{r}{nn}$. But let FI be taken towards G if the Terms p and r have the same Signs, if not, towards A. Then make as FH is to FI so is BC to BE, and take this FH from the Point F towards I, if the Section is an Ellipse, but towards the contrary Way if it is an Hyperbola. But let the Parallelograms IACK and HAEL be compleated, and all these Lines already described transferred to the given conick Section; or, which is the same Thing, let the Curve be described about them, so that its Axis or principal transverse Diameter might agree with the right Line LH, and the Center with the Point L. These Things being done, let the Line KL be drawn as also GL cutting the conick Section in g. In LK take Lk, which

which let be to L K as L g to L G, and from the Center k, with the Distance kg, describe a Circle. From the Points where it cuts the given Curve, let fall Perpendiculars to the Line L H, whereof let T γ be one. Lastly, towards γ take T Y, which let be to T γ as L G to L g, and this T Y produced will cut A B in X, and $\frac{1}{2}$ X Y will be one of the Roots of the Equation. But those Roots are affirmative which lie towards such Parts of A B as F I lies from F, and those are negative which lie on the contrary Side, if r is $+$, and the contrary if r is $-$.

After this Manner are cubick Equations constructed by given Ellipses and Hyperbolas: But if a Parabola should be given, the Line B C is to be taken equal to the *Latus Rectum* itself. Then the Points A, F, G, I, and K, being found as above, a Circle must be described from the Center K with the Distance K G, and the Parabola must be so applied to the Scheme already described, (or the Scheme to the Parabola) that it may pass through the Points A and G, and its Axis through the Point F parallel to A C, the Vertex falling on the same Side of the Point F as the Point B falls of the Point C; these being done, if Perpendiculars were let fall from the Points where the Parabola intersects the Circle to the Line B C, their Halves will be equal to the Roots of the Equation to be constructed.

And take Notice, that where the second Term of the Equation is wanting, and so the *Latus Rectum* of the Parabola is the Number 2, the Construction comes out the same as that which *Des Cartes* produced in his Geometry, with this Difference only, that these Lines are the double of them.

This is a general Rule of Constructions. But where particular Problems are proposed, we ought to consult the most simple Forms of Constructions. For the Quantity p remains free, by the taking of which the Equation may, for the most part, be rendered more simple. One Example of which I will give.

Let there be given an Ellipse, and let there be two mean Proportionals to be found between the given Lines a and b. Let the first of them be x, and $a \,.\, x \,.\, \frac{xx}{a} \,.\, b$ will be continual Proportionals, and so $ab = \frac{x^3}{a}$, or $x^3 = aab$, is the Equation which you must construct. Here the Terms p and q are wanting, and the Term $r = aab$, and therefore B A and A F are $= 0$, and F I is $= \frac{aab}{nn}$. That the last Term may be more simple, let n be assumed $= a$, and let F I be $= b$. And then the Construction will be thus:

From any Point A in any infinite right Line A E [*See* Figure 112.] take A C $= a$, and on the same Side of the Point A take A C to A E as the principal *Latus Rectum* of the Ellipse is to the *Latus Transversum*. Then in the Perpendicular A I take A I $= b$, and A H to A I as A C to A E. Let the Parallelograms I A C K, H A E L be compleated. Join L A and L K. Upon this Scheme lay the given Ellipse, and it will cut the right Line A L in the Point g. Make L k to L K as L g to L A. From the Center k, with the Distance kg, describe a Circle cutting the Ellipse in γ. Upon A E let fall the Perpendicular γ X, cutting H L in T, and let that be produced to Y, that T Y may be to T γ as L A to L g. And so $\frac{1}{2}$ X Y will be equal to x the first of the two mean Proportionals. Q. E. I.

Of

Of the Methods by which you may approximate to the Roots of numeral Equations by their Limits; and to the Roots of literal Equations by the Method of Series.

By COLIN MACLAURIN.

300. WHEN *any Equation is proposed to be resolved, first find the Limits of the Roots* (*by* N°. 264) as for Example, if the Roots of the Equation $x^2 - 16x + 55 = 0$ are required, you find the Limits are 0, 8, and 17, by N°. 267; that is, the least Root is between 0 and 8, and the greatest between 8 and 17.

In order to find the first of the Roots, I consider that if I substitute 0 for x in $x^2 - 16x + 55$, the Result is positive, viz. $+55$, and consequently any Number betwixt 0 and 8 that gives a positive Result, must be less than the least Root, and any Number that gives a negative Result, must be greater. Since 0 and 8 are the Limits, I try 4, that is, the Mean betwixt them, and supposing $x = 4$, $x^2 - 16x + 55 = 16 - 64 + 55 = 7$, from which I conclude that the Root is greater than 4. So that now we have the Root limited between 4 and 8. Therefore I next try 6, and substituting it for x we find $x^2 - 16x + 55 = 36 - 96 + 55 = -5$; which Result being negative, I conclude that 6 is greater than the Root required, which therefore is limited now between 4 and 6. And substituting 5, the Mean between them in place of x, I find $x^2 - 16x + 55 = 25 - 80 + 55 = 0$; and consequently 5 is the least Root of the Equation. After the same Manner you will discover 11 to be the greatest Root of that Equation.

Thus *by diminishing the greater, or increasing the lesser Limit, you may discover the true Root when it is a commensurable Quantity. But by proceeding after this Manner, when you have two Limits, the one greater than the Root, the other* *lesser, that differ from one another but by unit, then you may conclude the Root is incommensurable.*

We may however, by continuing the Operation in Fractions, approximate to it. As if the Equation proposed is $x^2 - 6x + 7 = 0$, if we suppose $x = 2$, the Result is $4 - 12 + 7 = -1$, which being negative, and the Supposition of $x = 0$ giving a positive Result, it follows that the Root is betwixt 0 and 2. Next we suppose $x = 1$; whence $x^2 - 6x + 7 = 1 - 6 + 7 = +2$, which being positive, we infer the Root is betwixt 1 and 2, and consequently incommensurable. In order to approximate to it, we suppose $x = 1\frac{1}{2}$, and find $x^2 6x + 7 = 2\frac{1}{4} - 9 + 7 = \frac{1}{4}$; and this Result being positive, we infer the Root must be betwixt 2 and $1\frac{1}{2}$. And therefore we try $1\frac{3}{4}$, and find $x^2 - 6x + 7 = \frac{49}{16} - \frac{42}{4} + 7 = 3\frac{1}{16} - 10\frac{8}{16} + 7 = -\frac{7}{16}$, which is negative; so that we conclude the Root to be betwixt $1\frac{3}{4}$ and $1\frac{1}{2}$. And therefore we try next $1\frac{5}{8}$, which giving also a negative Result, we conclude the Root is betwixt $1\frac{1}{2}$ (or $1\frac{4}{8}$) and $1\frac{5}{8}$. We try therefore $1\frac{9}{16}$, and the Result being positive, we conclude that the Root must be betwixt $1\frac{9}{16}$ and $1\frac{10}{16}$, and therefore is nearly $1\frac{19}{32}$.

301. *Or you may approximate more easily by transforming the Equation proposed into another whose Roots shall be equal to* 10, 100, *or* 1000 *Times the Roots of the former* (*by* 246) *and taking the Limits greater in the same Proportion. This Transformation is easy; for you are only to multiply the second Term by* 10, 100, *or* 1000, *the third Term by their Squares, the fourth by their Cubes*, &c. The Equation of the last Example is thus transformed into $x^2 - 600x + 70000 = 0$, whose Roots are 100 Times the Roots of the proposed Equation, and whose Limits are 100 and 200. Proceeding as before, we try 150, and find $x^2\ 600x + 70000 = 22500 - 90000 + 70000 = 2500$, so that 150 is less than the Root. You next try 175, which giving a negative Result must be greater than the Root: and thus proceeding you find the Root to be betwixt 158 and 159: from which you

you infer that the least Root of the proposed Equation $x^3-6x+7=0$ is betwixt 1.58 and 1.59, being the hundredth Part of the Root of $x^3-600x+70000=0$.

If the cubic Equation $x^3-15x^2+63x-50=0$ is proposed to be resolved, the Equation of the Limits will be (by 267) $3x^2-30x+63=0$, or $x^2-10x+21=0$, whose Roots are 3, 7; and by substituting 0 for x the Value of $x^3-15x^2+63x-50$ is negative, and by substituting 3 for x, that Quantity becomes positive, $x=1$ gives it negative, and $x=2$ gives it positive, so that the Root is between 1 and 2, and therefore incommensurable. You may proceed as in the foregoing Examples to approximate to the Root. But *there are other Methods by which you may do that more easily and readily*; which we proceed to explain.

302. *When you have discovered the Value of the Root to less than an Unit (as in this Example, you know it is a little above 1) suppose the Difference betwixt its real Value and the Number that you have found nearly equal to it, to be represented by* f; *as in this Example. Let* $x=1+f$. *Substitute this Value for* x *in this Equation, thus,*

$$\begin{aligned} x^3 &= 1+3f+3f^2+f^3 \\ -15x^2 &= -15-30f-15f^2 \\ +63x &= 63+63f \\ -53 &= -50 \end{aligned}$$

$$x^3-15x^2+63x-50=-1+36f-12f^2+f^3=0.$$

Now because f *is supposed less than Unit, its Powers* f^2, f^3, *may be neglected in this Approximation; so that assuming only the two first Terms, we have* $-1+36f=0$, *or* $f=\frac{1}{36}=.027$; *so that* x *will be nearly* 1.027.

You may have a nearer Value of x *by considering, that seeing* $-1+36f-12f^2+f^3=0$, *it follows that*

$f=\frac{1}{36-12f+f^2}$ (*by substituting* $\frac{1}{36}$ *for* f) *nearly* $=$

$$\frac{1}{36-12\times\frac{1}{36}+\frac{1}{36}\times\frac{1}{36}}=\frac{1}{16}=\frac{1296}{46225}=.02803.$$

But

But the Value of f *may be corrected and determined more accurately by suppoſing* g *to be the Difference betwixt its real Value, and that which we laſt found nearly equal to it.* So that $f = .02803 + g$. Then by ſubſtituting this Value for f in the Equation $f^3 - 12f^2 + 36f - 1 = 0$, it will ſtand as follows,

$$\left.\begin{array}{rl} f^3 = & 0.0000220226 + 0.002357g + 0.08409g^2 + g^3 \\ -12f^2 = & -.00942816 \quad - 0.67272g \quad - 12g^2 \\ +36f = & 1\ 00908 \qquad + 36g \\ -1 = & -1. \end{array}\right\} = 0$$

$$= -0.0003261374 + 35.329637g - 11.9195g^2 + g^3 = 0.$$

Of which the firſt two Terms, neglecting the reſt, give $35.329637 \times g = 0.0003261374$, and $g = \frac{.0003261374}{35.329637}$ $= 0.00000923127$. So that $f = 0.02803923127$; and $x = 1 + f = 1.02803923127$; which is very near the true Root of Equation that was propoſed.

If ſtill a greater Degree of Exactneſs is required, ſuppoſe h *equal to the Difference betwixt the true Value of* g, *and that we have already found, and proceeding as above you may correct the Value of* g.

For another Example; let the Equation to be reſolved be $x^3 - 2x - 5 = 0$, and by ſome of the preceding Methods you diſcover one of the Roots to be between 2 and 3. Therefore you ſuppoſe $x = 2 + f$, and ſubſtituting this Value for it, you find

$$\left.\begin{array}{rl} x^3 = & 8 + 12f + 6f^2 + f^3 \\ -2x = & -4 - 2f \\ -5 = & -5 \end{array}\right\} = 0$$

$$= -1 + 10f + 6f^2 + f^3;$$

from which we find that $10f = 1$ nearly, or $f = 0.1$. Then to correct this Value, we ſuppoſe $f = 0.1 + g$, and find

$f^3 =$

$$\left.\begin{array}{rl} f^3 = & 0.001 + 0.03g + 0.3g^2 + g^3 \\ 6f^2 = & 0.06 + 1.2g + 6.g^2 \\ 10f = & 1. + 10.g \\ -1 = & -1. \end{array}\right\} = 0$$
$$= 0.061 + 11.23g + 6.3g^2 + g^3,$$

so that $g = \frac{-0.061}{11.23} = -0.0054.$

Then by supposing $g = -.0054 + h$, you may correct its Value, and you will find that the Root required is nearly 2.09455147.

It is not only one Root of an Equation that can be obtained by this Method, but, by making use of the other Limits, you may discover the other Roots in the same Manner. The Equation of 301, $x^3 - 15x^2 + 63x - 50 = 0$, has for its Limits 0, 3, 7, 50. We have already found the least Root to be nearly 1,028039. If it is required to find the middle Root, you proceed in the same Manner to determine its nearest Limits to be 6 and 7; for 6 substituted for x gives a positive, and 7 a negative Result. Therefore you may suppose $x = 6 + f$, and by substituting this Value for x in that Equation, you find $f^3 + 3f^2 - 9f + 4 = 0$, so that $f = \frac{4}{9}$ nearly. Or since $f = \frac{4}{9 - 3f - f^2}$, it is (by substituting $\frac{4}{9}$ for f) $f = \frac{4}{9 - \frac{4}{3} - \frac{16}{81}} = \frac{324}{605}$, whence $x = 6 + \frac{324}{605}$ nearly. Which Value may still be corrected as in the preceding Articles. After the same Manner you may approximate to the Value of the highest Root of the Equation.

303. *In all these Operations, you will approximate sooner to the Value of the Root, if you take the three last Terms of the Equation, and extract the Root of the quadratic Equation consisting of these three Terms.*

Thus, in N°. 302, instead of the two last Terms of the Equation $f^4 - 12f^3 + 36f^2 - 1 = 0$, if you take the

I three

three laſt and extract the Root of the quadratic $12f^2 - 36f + 1 = 0$, you will find $f = .028031$, which is much nearer the true Value than what you diſcover by ſuppoſing $36f - 1 = 0$.

It is obvious that this Method extends to all Equations.

304. *By aſſuming Equations affected with general Coefficients, you may, by this Method, deduce general Rules or Theorems for approximating to the Roots of propoſed Equations of whatever Degree.*

Let $f^3 - pf^2 + qf - r = 0$ repreſent the Equation by which the Fraction f is to be determined, which is to be added to the Limit, or ſubtracted from it, in order to have the near Value of x. Then $qf - r = 0$ will give $f = \frac{r}{q}$. But ſince $f = \frac{r}{f^2 - pf + q}$, by ſubſtituting $\frac{r}{q}$ for f, we have this Theorem for finding f nearly, viz.

$$f = \frac{r}{\frac{q^2}{q^2} - \frac{pr}{q} + q} = \frac{q^2 \times r}{q^3 - pqr + r^2}.$$

After the ſame Manner, if it is a biquadratic, by which f is to be determined, as $f^4 - pf^3 + qf^2 - rf + s = 0$, then f being very little, we ſhall have $f = \frac{s}{r}$; which Value is corrected by conſidering that $f = \frac{s}{r - qf + pf^2 - f^3}$ (by ſuſtituting $\frac{s}{r}$ for f) $= \frac{s}{r - \frac{qs}{r} + \frac{ps^2}{r^2} - \frac{s^3}{r^3}}$, whence we have this Theorem for all biquadratic Equations,

$$f = \frac{r^3 \times s}{-s^3 + ps^2 r - qsr^2 + r^4}.$$

Other

Other Theorems may be deduced by assuming the three Terms of the Equation, and extracting the Root of the Quadratic which they form.

Thus, to find the Value of f in the Equation $f^3 - pf^2 + qf - r = 0$ where f is supposed to be very little, we neglect the first Term f^3, and extract the Root of the quadratic $pf^2 - qf + r = 0$, or of $f^2 - \frac{q}{p}f + \frac{r}{p} = 0$; and we find $f = \frac{q}{2p} \pm \sqrt{-\frac{r}{p} + \frac{q^2}{4p^2}} = \frac{q \pm \sqrt{q^2 - 4pr}}{2p}$ nearly.

But this Value of f may be corrected by supposing it equal to m, and substituting m^3 for f^3 in the Equation $f^3 - pf^2 + qf - r = 0$, which will give $m^3 - pf^2 + qf - r = 0$, and $pf^2 - qf + r - m^3 = 0$; the Resolution of which quadratic Equation gives $f = \frac{q \pm \sqrt{q^2 - 4pr + 4pm^3}}{2p}$, very near the true Value of f.

After the same Manner you may find like Theorems for the Roots of biquadratic Equations, or of Equations of any Dimension whatever.

305. *In general, let* $x^n + px^{n-1} + qx^{n-2} + rx^{n-3} +$, *&c.* $+ A = 0$ *represent an Equation of any Dimensions* n, *where* A *is supposed to represent the absolute known Term of the Equation. Let* k *represent the Limit next less than any of the Roots, and supposing* $x = k + f$, *substitute the Powers of* $k + f$ *instead of the Powers of* x, *and there will arise* $\overline{k+f}^n + p \times \overline{k+f}^{n-1} + q \times \overline{k+f}^{n-2} + r \times \overline{k+f}^{n-3}$, *&c.* $+ A = 0$, *or by Involution, disposing the Terms according to the Dimensions of* f . . . .

$$\left.\begin{array}{l} k^n \quad + nk^{n-1}\times f + n\times\frac{n-1}{2}k^{n-2}f^2 +, \&c. \\ pk^{n-1} + p\times\overline{n-1}\,k^{n-2}\times f + p\times\overline{n-1}\times\frac{n-2}{2}k^{n-3}f^2, +, \&c. \\ qk^{n-2} + q\times\overline{n-2}\,k^{n-3}\times f + q\times\overline{n-2}\times\frac{n-3}{2}k^{n-4}f^2, +, \&c. \\ rk^{n-3} + r\times\overline{n-3}\,k^{n-4}\times f + r\times\overline{n-3}\times\frac{n-4}{2}k^{n-5}f^2, +, \&c. \\ \&c. \end{array}\right\} + A = 0,$$

where neglecting all the Powers of f *after the first two Terms, you find*

$f =$

$$f = \frac{-Ak^n - pk^{n-1} - qk^{n-2} - rk^{n-3}, \&c.}{nk^{n-1} + p \times \overline{n-1}\,k^{n-2} + q \times \overline{n-2}\,k^{n-3} + r \times \overline{n-3}\,k^{n-4}, \&c.}, \text{ and}$$

$$x\,(= k+f) = \frac{-A + \overline{n-1}\,k^n + p \times \overline{n-2}\,k^{n-1} + q \times \overline{n-3}\,k^{n-2} + r \times \overline{n-4}\,k^{n-3}, \&c.}{nk^{n-1} + p \times \overline{n-1}\,k^{n-2} + q \times \overline{n-2}\,k^{n-3} + r \times \overline{n-3}\,k^{n-4}, \&c.}$$

whence particular Theorems for extracting the Roots of Equations may be deduced.

306. "*By this Method you may discover Theorems for approximating to the Roots of pure Powers*;" as to find the n Root of any Number A; suppose k to be the nearest less Root

Root in Integers, and that $k+f$ is the true Root, then shall $k^n + nk^{n-1}f + n\times\frac{n-1}{2}k^{n-2}f^2$, &c. $=A$; and, assuming only the two first Terms, $f=\frac{A-k^n}{nk^{n-1}}$: or, more nearly, taking the three first Terms,

$$f=\frac{A-k^n}{nk^{n-1}+n\times\frac{n-1}{2}k^{n-2}f},$$ and (taking $\frac{A-k^n}{nk^{n-1}}=f$)

$$f=\frac{A-k^n}{nk^{n-2}+\frac{n^2-n}{2}k^{n-2}\times\frac{A-k^n}{nk^{n-1}}}=\frac{A-k^n}{nk^{n-1}+\frac{\overline{n-n}}{2k}\times\overline{A-k^n}}$$

(putting $m=A-k^n$) $=\frac{km}{nk^n+\frac{n-1}{2}\times m}$; *which is a rational Theorem for approximating to* f.

You may find an irrational Theorem for it, by assuming the three first Terms of the Power of $k+f$, viz. $k^n+nk^{n-1}f+n\times\frac{n-1}{2}k^{n-2}f^2=A$. For $nk^{n-1}f+n\times\frac{n-1}{2}k^{n-2}f^2=A-k^n=m$; and resolving this quadratic Equation, you find

$$f=-\frac{k}{n-1}\pm\sqrt{\frac{2m}{n\times\overline{n-1}\times k^{n-2}}+\frac{k^2}{\overline{n-1}|^2}}=$$

$$-\frac{k}{n-1}\pm\sqrt{\frac{2mn-2m+nk^n}{n\times\overline{n-1}|^2\,k^{n-2}}}.$$

In the Application of these Theorems, when a near Value of f *is obtained, then adding it to* k, *substitute the Aggregate in Place of* k *in the Formula, and you will, by a new Operation, obtain a more correct Value of the Root required; and, by thus proceeding, you may arrive at any Degree of Exactness.*

Thus

Thus, to obtain the cube Root of 2, suppose $k=1$, and $f\left(=\frac{km}{nk^n+\frac{n-1}{2}m}\right)=\frac{1}{4}=0.25$. In the second Place, suppose $k=1.25$, and f will be found by a new Operation, equal to 0.009921, and consequently, $\sqrt[3]{2}=1.259921$ nearly. By the irrational Theorem, the same Value is discovered for $\sqrt[3]{2}$.

Of the Method of *Series* by which you may approximate to the Roots of literal Equations.

307. *If there be only two Letters,* x *and* a, *in the proposed Equation, suppose* a *equal to Unit, and find the Root of the numeral Equation that arises from the Substitution, by the Rules already given. Multiply these Roots by* a, *and the Products will give the Roots of the proposed Equation.*

Thus the Roots of the Equation $x^2 - 16x + 55 = 0$ are found, in N°. 300, to be 5 and 11. And therefore the Roots of the Equation $x^2 - 16ax + 55a^2 = 0$, will be $5a$ and $11a$. The Roots of the Equation $x^3 + a^2x - 2a^3 = 0$ are found by enquiring what are the Roots of the numeral Equation $x^3 + x - 2 = 0$, and since one of these is 1, it follows that one of the Roots of the proposed Equation is a; the other two are *imaginary*.

308. *If the Equation to be resolved involves more than two Letters, as* $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$, *then the Value of* x *may be exhibited in a Series having its Terms composed of the Powers of* a *and* y *with their respective Coefficients; which will "converge the sooner the less* y *is in respect of* a, *if the Terms are continually multiplied by the Powers of* y, *and divided by those of* a." *Or, " will converge the sooner the greater* y *is in respect of* a, *if the Terms be continually multiplied by the Powers of* a, *and divided by those of* y." Since when y is very little in respect of a, the Terms y, $\frac{y^2}{a}$, $\frac{y^3}{a^2}$, $\frac{y^4}{a^3}$, $\frac{y^5}{a^4}$, &c. decrease very quickly. If y vanish in respect of a, the second Term will vanish in respect of the first, since $\frac{y^2}{a} : y :: y : a$. And after the same Manner $\frac{y^3}{a^2}$ vanishes in respect of the Term immmediately preceding it

But

But when y is vastly great in respect of a, then a is vastly great in respect $\frac{a^2}{y}$, and $\frac{a^2}{y}$ in respect of $\frac{a^3}{y^2}$; so that the Terms a, $\frac{a^2}{y}$, $\frac{a^3}{y^2}$, $\frac{a^4}{y^3}$, $\frac{a^5}{y^4}$, &c. in this Case *decrease* very swiftly. In either Case, the Series converge swiftly that consist of such Terms; and a few of the first Terms will give a near Value of the Root required.

309. *If a Series for* x *is required from the proposed Equation that shall converge the sooner, the less* y *is in respect of* a; *to find the first Term of this Series, we shall suppose* y *to vanish; and extracting the Root of the Equation* $x^3 + a^2x - 2a^3 = 0$, *consisting of the remaining Parts of the Equation that do not vanish with* y, *we find, by 307, that* $x = a$; *which is the true Value of* x *when* y *vanishes, but is only near its Value when* y *does not vanish, but only is very little. To get a Value still nearer the true Value of* x, *suppose the Difference of* a *from the true Value to be* p, *or that* $x = a + p$. *And substituting* $a + p$ *in the given Equation for* x, *you will find,*

$$\left.\begin{array}{r}
x^3 = a^3 + 3a^2p + 3ap^2 + p^3 \\
+a^2x = a^3 + a^2p \\
-2a^3 = -2a^3 \\
+ayx = a^2y + apy \\
-y^3 = -y^3
\end{array}\right\} = 0$$

$$\left.\begin{array}{r}
= 4a^2p + 3ap^2 + p^3 \\
a^2y + apy - y^3
\end{array}\right\} = 0.$$

But since, by Supposition, y and p are very little in respect of a, it follows that the Terms $4a^2p$, a^2y, where y and p are separately of the *least* Dimensions, are vastly great in respect of the rest; so that, in determining a near Value of p, the rest may be neglected: and from $4a^2p + a^2y = 0$, we find $p = -\frac{1}{4}y$. So that $x = a + p = a - \frac{1}{4}y$, nearly.

Then to find a nearer Value of p, *and consequently of* x, *suppose* $p = -\frac{1}{4}y + q$, *and substituting this Value for it in the last Equation, you will find,*

$$\left.\begin{aligned} p^3 &= -\tfrac{1}{64}y^3 + \tfrac{3}{16}y^2q - \tfrac{3}{4}yq^2 + q^3 \\ 3ap^2 &= \tfrac{3}{16}ay^2 - \tfrac{3}{2}ayq + 3aq^2 \\ 4a^2p &= -a^2y + 4a^2q \\ ayp &= -\tfrac{1}{4}ay^2 + ayq \\ a^2y &= a^2y \\ -y^3 &= -y^3 \end{aligned}\right\} =$$

$$\left.\begin{aligned} = -\tfrac{65}{64}y^3 + \tfrac{3}{16}y^2q - \tfrac{3}{4}yq^2 + q^3 \\ -\tfrac{1}{16}ay^2 - \tfrac{1}{2}ayq + 3aq^2 \\ +4a^2q \end{aligned}\right\} = 0.$$

And since, by the Supposition, q is very little in respect of p, which is nearly $= -\frac{1}{4}y$, therefore q will be very little in respect of y; and consequently all the Terms of the last Equation will be very little in respect of these two viz. $-\frac{1}{16}ay^2$, $+4a^2q$, where y and q are of least Dimensions separately: particularly the Term $-\frac{1}{2}ayq$ is little in respect of $4a^2q$, because y is very little in respect of a; and it is little in respect of $-\frac{1}{16}ay^2$, because q is little in respect of y.

Neglect therefore the other Terms, and supposing $-\frac{1}{16}ay^2 + 4a^2q = 0$, you will have $q = \frac{1}{64} \times \frac{y^2}{a}$; so that $x = a - \frac{1}{4}y + \frac{1}{64} \times \frac{y^2}{a}$. *And by proceeding in the same Manner you will find* $x = a - \frac{y}{4} + \frac{y^2}{64a} - \frac{131y^3}{512a^2} + \frac{509y^4}{16384a^3} -$, &c.

310. *When it is required to find a Series for* x *that shall converge sooner, the greater* y *is in respect of any Quantity* a, *you need only suppose* a *to be very little in respect of* y, *and proceed by the same Reasoning as in the last Example on the Supposition of* y *being very little.*

Thus, to find a Value for x in the Equation $x^3 - a^2x + ayx - y^3 = 0$ that shall converge the sooner the greater y is in respect of a, suppose a to vanish, and the remaining Terms will give $x^3 - y^3 = 0$, or $x = y$. So that when y is vastly great, it appears that $x = y$ nearly.

But

But to have the Value of x more accurately, put $x = y + p$, then

$$\left.\begin{aligned} x^3 &= y^3 + 3y^2p + 3yp^2 + p^3 \\ -a^2x &= -a^2y - a^2p \\ +ayx &= ay^2 + ayp \\ -y^3 &= -y^3 \end{aligned}\right\} = 0$$

$$= +3y^2p + 3yp^2 + p^3 - a^2y - a^2p + ay^2 + ayp:$$

where the Terms $3y^2p + ay^2$ become vastly greater than the rest, y being vastly greater than a or p; and consequently $p = -\frac{1}{3}a$ nearly.

Again, by supposing $p = -\frac{1}{3}a + q$, you will transform the last Equation into

$$\left.\begin{aligned} &-\tfrac{8}{27}a^3 + 3y^2q + 3yq^2 + q^3 \\ &-\ a^2y - ayq - aq^2 \\ &\quad -\tfrac{2}{3}a^2q \end{aligned}\right\} = 0;$$

where the two Terms $3qy^2 - a^2y$ must be vastly greater than any of the rest, a being vastly less than y, and q vastly less than a by the Supposition; so that $3qy^2 - a^2y = 0$, and $q = \frac{a^2}{3y}$ nearly. By proceeding in this Manner, you may correct the Value of y, and find that

$$x = y - \tfrac{1}{3}a + \frac{a^2}{3y} + \frac{a^3}{81y^2} - \frac{8a^4}{243y^3}, \text{ \&c.}$$

which Series converges the sooner the greater y is supposed to be taken in respect of a.

311. In the Solution of the first *Example* those Terms were always compared in order to determine p, q, r, &c. in which y and those Quantities p, q, r, &c. were *separately* of fewest Dimensions. But in the second *Example*, those Terms were compared in which a and the Quantities p, q, r, &c. were of least Dimensions separately. And these always are the proper Terms to be compared together, because they become vastly greater than the rest, in the respective Hypotheses.

In general; to determine the first, or any Term in the Series, such Terms of the Equation are to be assumed together only, as will be found to become vastly greater than the other Terms; that is, which give a Value of x, *which substituted for it in all the Terms of Equation shall raise the Dimensions of the other Terms all above, or all below, the Dimensions of the assumed Terms, according as* y *is supposed to be vastly little, or vastly great in respect of* a.

Thus to determine the first Term of a converging Series expressing the Value of x in the last Equation $x^3 - a^2x + ayx - y^3 = 0$, the Terms ayx and $-y^3$ are not to be compared together, for they would give $x = \frac{y^2}{a}$, which substituted for x, the Equation becomes

$$\frac{y^6}{a^3} - ay^2 + y^3 - y^3 = 0,$$

where the first Term is more Dimensions than the assumed Terms ayx, $-y^3$; and the second of fewer: so that the two first Terms cannot be neglected in respect of the two last, neither when y is very great nor very little, compared with a. Nor are the Terms x^3, ayx, fit to be compared together in order to obtain the first Term of a Series for x, for the like Reason.

But x^3 may be compared with $-a^2x$, as also $-a^2x$ with $-y^3$ for that End. These two give the first Term of a Series that converges the sooner the less y is; as $x^3 = y^3$ gives the first Term of a Series that converges the sooner the greater y is. The last Series was given in the preceding Article. The comparing x^3 with $-a^2x$ gives these two Series,

$$x = a - \tfrac{1}{2}y - \frac{y^2}{8a} + \frac{7y^3}{16a^2} - \frac{59y^4}{128a^3}, \text{ \&c.}$$

$$x = -a + \tfrac{1}{2}y + \frac{y^2}{8a} + \frac{9y^3}{16a^2} + \frac{69y^4}{128a^3}, \text{ \&c.}$$

The comparing $-a^2x$ with $-y^3$ gives

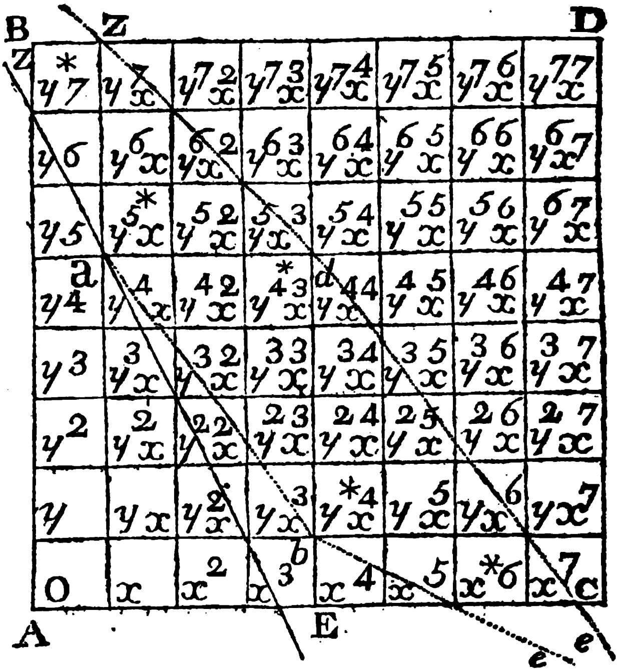
$$x = -\frac{y^3}{a^2} - \frac{y^4}{a^3} - \frac{y^5}{a^4} - \frac{y^6}{a^5} -, \text{ \&c.}$$

And

And these Series give three Values of x when y is very little; the last of which is itself also very little in that Case, as it appears indeed from the Equation, that when y vanishes, the three Values of x become $+a$, $-a$, and 0, because when y vanishes, the Equation becomes $x^3 - a^2x = 0$; whose Roots are a, $-a$, 0.

312. It appears sufficiently from what we have said, that when an Equation is proposed involving x and y, and the Value of x is required in a converging Series, the Difficulty of finding the first Term of the Series is reduced to this; "*To find what Terms assumed in order to determine a Value of* x *expressed in some Dimensions of* y *and* a *will give such a Value of it, as substituted for it in the other Terms will make them all of more Dimensions of* y, *or all of less Dimensions of* y, *than those assumed Terms.*"

To determine this, draw B A and A C at right Angles to each other, compleat the Parallelogram A B C D and divide it into equal Squares, as in the Figure. In these Squares place the Powers of x from A towards C, and the Powers of y from A towards B, and in any other Square place that Power of x that is directly below it in the line A C, and that Power of y that is in a parallel with it in the Line A B; so that in the Index of x in any Square may express its Distance from the Line A B, and the Index of y in any Square may express its Distance from the Line A C. Of this Square we are to observe,



1. *That the Terms are not only in geometrical Progression in the vertical Column* A B, *or the horizontal* A C, *and their parallels; but also in the Terms taken in any oblique strait Line whatever; for in any such Terms it is manifest that the Indices of* y *and* x *will be in arithmetical Progression. The Indices of* y, *because those Terms will remove equally from the Line* A C, *or approach equally to it, and the Indices of* y *in any such Terms are as their Distances from that Line* A C. *The Indices of* x *will also be in arithmetical Progression, because these Terms equally remove from, or approach to the Line* A B. Thus for Example, in the Terms y^7, y^5x, y^3x^2, yx^3, the Indices of y decreasing by the common Difference 2, while the Indices of x increase in the Progression of the natural Numbers, the common Ratio of the Terms is $\frac{x}{y^2}$. It follows,

2. From

2. *From the last Observation, that "if any two Terms be supposed equal, then all the Terms in the same straight Line with these Terms, will be equal;"* because by supposing these two Terms equal, the common Ratio is supposed to be a Ratio of Equality; *and from this it follows, that "if you substitute every where for* x *the Value that arises for it by supposing any two Terms equal, expressed in the Powers of* y, *the Dimensions of* y *in all the Terms that are found in the same straight Line will be equal;" but "the Dimensions of* y *in the Terms above that Line will be greater than in those in that Line;" and "the Dimensions of* y *in the Terms below the said Line will be less than its Dimensions in that Line."* Thus, by supposing $y^7 = yx^3$, we find $x^3 = y^6$, or $x = y^2$; and substituting this Value for x in all the Squares, the Dimensions of y in the Terms y^7, y^5x, y^3x^2, yx^3, which are all found in the same straight Line, will be 7, but the Dimensions in all the Terms above that Line will be more than 7, and in all the Terms below that Line will be less than 7.

313. *From these two Observations we may easily find a Method for discovering what Terms ought to be assumed from an Equation in order to give a Value for* x *which shall make the other Terms all of higher, or all of lower Dimensions of* y *than the assumed Terms: viz. "after all the Terms of the Equation are ranged in their proper Squares (by the last Article) such Terms are to be assumed as lie in a straight Line, so that the other Terms either lie all above the straight Line, or fall all below it."*

For Example, suppose the Equation proposed is $y^7 - ay^5x + y^4x^2 + a^2yx^4 - ax^6 = 0$, then marking with an Asterisk the Squares in the last Article which contain the same Dimensions of x and y as the Terms in the Equation, imagine a Ruler ZE to revolve about the first Square marked at y^7, and as it moves from A towards C, it will first meet the Term ay^5x, and while the Ruler joins these two Terms, all the other Terms lie above it: from which you infer, that by supposing these Terms equal, you shall obtain a Value of x, which substituted for it, will give all the other Terms of higher Dimensions of y, than those

Terms;

Terms: and hence we conclude that the Value of x deduced from supposing these Terms equal, *viz.* $\frac{y^2}{a}$, is the first Term of a Series that will converge the sooner the less y is in respect of a.

If the Ruler be made to revolve about the same Square the contrary Way from D towards C, it will first meet the Term y^4x^3, and by supposing $y^7 + y^4x^3 = 0$, we find $y = x$, which gives the first Term of a Series for x, that converges the sooner the greater that y is. *And this is the celebrated Rule invented by* Sir Isaac Newton *for this Purpose.*

314. *This Rule may be extended to Equations having Terms that involve Powers of* x *and* y *with fractional or surd Indices; "by taking Distances from* A *in the Lines* AC *and* AB *proportional to these Fractions and Surds," and thence determining the Situation of the Terms of the proposed Equation in the Parallelogram* ABCD.

It is to be observed also, that when the Line joining any two Terms has all the other Terms on one Side of it, by them you may find the first Term of a converging Series for x, *and thus "various such Series can be deduced from the same Equation."* As, in the last Example, the Line joining y^5x and yx^4 has all the Terms above it; and therefore supposing $-ay^5x + a^2yx^4 = 0$, we find $x^3 = \frac{y^4}{a}$, and $x = \frac{y^{\frac{4}{3}}}{a^{\frac{1}{3}}}$, which is the first Term of another converging Series for x. Again, the straight Line joining yx^4 and x^6 has all the other Terms above it, and therefore, supposing $a^2yx^4 - ax^6 = 0$, we find $ay = x^2$, and $x = a^{\frac{1}{2}}y^{\frac{1}{2}}$, the first Term of another Series for x, converging also the sooner the less y is. There are two Series converging the sooner the greater y is, to be deduced from supposing $y^7 = -y^4x^3$, or $y^4x^3 = ax^6$. And, *to find all these Series, "describe a polygon* Zabed, *having a Term of the Equation in each of its*

Angles, and including all the other Terms within it, then a Series may be found for x, *by supposing any two Terms equal that are placed in any two adjacent Angles of the Polygon.*"

315. *If the Ruler* Z E *be made to move parallel to itself, all the Terms which it will touch at once will be of the same Dimensions of* y: for they will bear the same Proportion to one another as the Terms in the Line Z E themselves. The Terms which the Ruler will touch first will have fewer Dimensions of *y*, than those it touches afterwards in the Progress of its Motion, if it moves towards D; but more Dimensions than they, if it moves towards A. *The Terms in the straight Line* Z E, *serve to determine the first Term of the converging Series required. These with the Terms it touches afterwards serve to determine the succeeding Terms of the converging Series; all the rest vanishing compared with these, when* y *is very little and the Ruler moves from* A *towards* D, *or when* y *is vastly great and the Ruler moves from* D *towards* A.

316. *The same Author gives another Method for discovering the first Term of a Series that shall converge the sooner the less* y *is.* "*Suppose the Term where* y *is separately of the fewest Dimensions to be* Dy^l; *compare it successively with the other Terms, as with* $Ey^m x^s$, *and observe where* $\frac{l-m}{s}$ *is found greatest; and putting* $\frac{l-m}{s} = n$, Ay^n *will be the first Term of a Series that shall converge the sooner the less* y *is:*" For in that Case Dy^l and $Ey^m x^s$ will be infinitely greater than any other Terms of the proposed Equation. Suppose $Fy^e x^k$ is any other Term of the Equation, and, by the Supposition, $\frac{l-m}{s}$ $(= n)$ is greater than $\frac{l-e}{k}$, and consequently, multiplying by k, you find nk greater than $l-e$, and $nk+e$ greater than l; now if for x you substitute Ay^n, then $Fy^e x^k = FA^k y^{nk+e}$, which therefore will vanish compared with Dy^l (since $nk+e$ is greater than l) when y is infinitely little. Thus therefore all the Terms will vanish compared with Dy^l and $Ey^m x^s$ which are supposed equal; and

and consequently they will give the first Term of a Series that will converge the sooner the less y is.

317. *If you observe "when* $\frac{l-m}{s}$ *is found least of all, and suppose it equal to* n, *then will* Ay^n *be the first Term of a Series that will converge the sooner the greater* y *is."* For in that Case Dy^l and Ey^mx^s will be infinitely greater than Fy^ex^k, because $\frac{l-m}{s}$ $(=n)$ being less than $\frac{l-e}{k}$, it follows that nk is less than $l-e$, and $nk+e$ less than l, and consequently Fy^ex^k $(=FA^ky^{nk+e})$ vastly less than $-Dy^l$, when y is very great.

After the same Manner, if you compare any Term Dy^lx^h, *where both* x *and* y *are found, with all the other Terms, and observe where* $\frac{l-m}{s-h}$ *is found greatest or least, and suppose* $\frac{l-m}{s-h}=n$, *then may* Ay^n *be the first Term of a converging Series.* For supposing that Fy^ex^k is any other Term of the Equation, if $\frac{l-m}{s-h}$ $(=n)$ is greater than $\frac{l-e}{k-h}$, then shall $nk-nh$ be greater than $l-e$, and $nk+e$ greater than $l+nh$. But $nk+e$ are the Dimensions of y in Fy^ex^k when $x=Ay^n$, and $l+nh$ are the Dimensions of y in Ey^mx^s; therefore Fy^ex^k is of more Dimensions of y than Ey^mx^s, and therefore vanishes compared to it when y is supposed infinitely little. In the same Manner, if $\frac{l-m}{s-h}$ is less than $\frac{l-e}{k-h}$, then will Ey^mx^s be infinitely greater than Fy^ex^k, when y is infinite.

318. *When the first Term* (Ay^n) *of the Series is found by the preceding Method, then by supposing* $x=Ay^n+p$, *and substituting this Binomial and its Powers for* x *and its Powers, there will arise an Equation for determining* p *the second Term of the Series. This new Equation may be treated in the same Manner as the Equation of* x, *and by the Rule of* 313, *the Terms that are to be compared in order to obtain a near Value*

of p, *may be discovered; by Means of which Terms* p *may be found: which suppose equal to* By^{n+r}, *then by supposing* p = $By^{n+r} + q$, *the Equation may be transformed into one for determining* q *the third Term of the Series, and by proceeding in the same Manner you may determine as many Terms of the Series as you please; finding* $x = Ay^{n} + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. *where the Dimensions of* y *ascend or descend according as* r *is positive or negative; and always "in arithmetical Progression, that this Value of* x *being substituted for it in the proposed Equation, the Terms involving* y *and its Powers may fall in with one another, so that more than one may always involve the same Dimension of* y, *which may mutually destroy each other and make the whole Equation vanish, as it ought to do."*

It is obvious that as the Dimensions of y in $Ay_{n} + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. are in an arithmetical Progression whose Difference is r, the Square, Cube, or any Power s of $Ay^{n} + By^{n+r} + Cy^{n+2r} + Dy_{n}+3r$ + &c. will consist of Terms wherein the Dimensions of y will constitute an arithmetical Progression having the same common Difference r; for these Dimensions will be sn, $sn+r$, $sn+2r$, $sn+3r$, &c. Therefore, *if in any Term* $Ey^{m}x^{s}$ *you substitute for* x *the Series* $Ay^{n} + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$, &c. *the Terms of the Series expressing* $Ey^{m}x^{s}$ *will consist of these Dimensions of* y, *viz.* $m+sn$, $m+sn+r$, $m+sn+2r$, $m+sn+3r$, &c. *and by a like Substitution in any other Term as* $Fy^{e}x^{k}$, *the Dimensions of* y *will be* $e+nk$, $e+nk+r$, $e+nk+2r$, $e+nk+3r$, &c. *The former Series of Indices must coincide with the latter Series, that the Terms in which they are found may be compared together, and be found equal with opposite Signs so as to destroy one another, and make the whole Equation vanish.*

The first Series consists of Terms arising by adding some Multiple of r *to* $m+sn$, *the latter by adding some Multiple of* r *to* $e+nk$; *and that these may coincide, some Multiple of* r *added to* $m+sn$ *must be equal to some other Multiple of* r *added to* $e+nk$. *From which it appears that the Difference of* *of* m + s n *and* e + n k *is always a Multiple of* r; *and consequently that* r *is a Divisor of the Difference of the Dimensions of* y *in the Terms* $Ey^m x^s$ *and* $Fy^e x^k$, *supposing* $x = Ay^n$. *It follows therefore* "*that* r *is a common Divisor of the Difference of Dimensions of* y *in the Terms of the Equation, when you have substituted* Ay^n *for* x *in all the Terms.*" *And if* r *be assumed equal to the greatest common Divisor* (*excepting some Cases afterward to be mentioned*) *you will have the true Form of a Series for* x. *And now the Dimensions* y^n, y^{n+r}, y^{n+2r}, y^{n+3r}, &c. *being known, there remains only, by Calculation, to determine the general Coefficients* A, B, C, D, &c. *in order to find the Series* $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r}$ &c. $= x$.

319. *This leads us to* Sir Isaac Newton's second general Method *of Series*; *which consists in assuming a Series with undetermined Coefficients expressing* x, *as* $Ay^n + By^{n+r} + Cy^{n+2r} +$, &c. *where* A, B, C, &c. *are supposed as yet unknown, but* n *and* r *are discovered by what we have already demonstrated*; *and substituting this every where for* x, *you must suppose, in the new Equation that arises, the Sum of all the Terms that involve the same Dimension of* y *to vanish, by which Means you will obtain particular Equations, the first of which will give* A, *the second* B, *the third* C, &c. *and these Values being substituted in the assumed Series for* A, B, C, &c. *the Series for* x *will be obtained as far as you please.*

Let us apply, for Example, this Method to the Equation (of 308) $x^3 + a^2x - 2a^3 + ayx - y^3 = 0$. Suppose it is required to find a Series converging the sooner the less y is: its first Term (by 310, or 312) is found to be a, so that $n = 0$. Substitute a for x in the Equation, and the Terms become $a^3 + a^3 - 2a^3 + a^2y - y^3$, and the Differences of the Indices are 0, 1, 2, 3; whose greatest common Measure is 1, so that $r = 1$. Assume therefore $x = A + By + Cy^2 + Dy^3$, &c. and substitute this Series for x in the Equation. Then

$$\begin{array}{llllll}
x^3 = A^3 & + 3A^2By & + 3AB^2y^2 & + B^3y^3 & + \&c. \\
 & & + 3A^2Cy^2 & + 3A^2Dy^3 & + \&c. \\
 & & & + 6ABCy^3 & + \&c. \\
+ a^2x = a^2A & + a^2By & + a^2Cy^2 & + a^2Dy^3 & + \&c. \\
+ ayx = & aAy & + aBy^2 & + aDy^3 & + \&c. \\
- 2a^3 = - 2a^3 & & & & \\
- y^3 = \ldots\ldots & & & - 1 \times y^3. &
\end{array}$$

Now since $x^3 + a^2x + ayx - 2a^3 - y^3 = 0$, it follows that the Sum of these Series involving y must vanish. But that cannot be if the Coefficient of every particular Term does not vanish. For every Term where y is infinitely little, is infinitely greater than the following Terms, so that if every Term does not vanish of itself, the Addition or Substraction of the following Terms which are infinitely less than it, or of the preceding Terms which are infinitely greater, cannot destroy it; and therefore the whole cannot vanish. It appears therefore that $A^3 + a^2A - 2a^3 = 0$, is an Equation for determining A, and gives $A = a$.

In order to determine B, you must suppose the Sum of the Coefficients affecting y to vanish, *viz.* $\overline{3A^2B + a^2B + aA} \times y = 0$, or, since $A = a$, $4a^2By + a^2y = 0$, and $B = -\frac{1}{4}$.

To determine C, in the same Manner suppose $3AB^2y^2 + 3A^2Cy^2 + a^2Cy^2 + aBy^2 = 0$, or, substituting for A and B their Values already found, $\frac{3ay^2}{16} + 4a^2Cy^2 - \frac{ay^2}{4} = 0$, and consequently $C = \frac{1}{64a}$. And, by proceeding in the same Manner, $D = \frac{131}{512a^2}$, so that $x = a - \frac{1}{4}y + \frac{1}{64a}y^2 + \frac{131}{512a^2}y^3$, &c. as we found before in N°. 309.

320. *By this Method you may transfer Series from one undetermined*

determined Quantity to another, and obtain Theorems for the Reversion *of Series.*

Suppose that $x = ay + by^2 + cy^3 + dy^4 +$ *&c. and it is required to express* y *by a Series consisting of the Powers of* x. *It is obvious that when* x *is very little,* y *is also very little, and that in order to determine the first Term of the Series, you need only assume* $x = ay$. *And therefore* $y = \frac{x}{a}$; *so that* $n = 1$.

By substituting $\frac{x}{a}$ *for* y, *you find the Dimensions of* x *in the Terms will be* 1, 2, 3, 4, &c. *so that* $r = 1$ *also. You may therefore assume* $y = Ax + Bx^2 + Cx^3 + Dx^4 +$ &c. *And by the Substitution of this Value of* y *you will find*

$$\begin{array}{llll} ay = aAx & + aBx^2 & + aCx^3 & + \&c. \\ by^2 = & bA^2x^2 & + 2bABx^3 & + \&c. \\ cy^3 = & & cA^3x^3 & + \&c. \\ \&c. & & & \&c. \end{array}$$

But the first Term being already found to be $\frac{x}{a}$, *you have* $A = \frac{1}{a}$; *and since* $aB + bA^2 = 0$, *it follows that* $B = -\frac{b}{a^3}$. *After the same Manner you will find* $C = \frac{2b^2 - ac}{a^5}$.

Whence $y = \frac{x}{a} - \frac{b}{a^3}x^2 + \frac{2b^2 - ac}{a^5}x^3 +$ &c.

321. *Suppose again you have* $ax + bx^2 + cx^3 + dx^4 +$ &c. $= gy + hy^2 + iy^3 + ky^4$, &c. *to find* x *in Terms of* y. *You will easily see, by* 313, *that the first Term of the Series for* x is $\frac{gy}{a}$, *that* $n = 1$, $r = 1$. *Therefore assume* $x = Ay + By^2 + Cy^3$, &c. *and by substituting this Value for* x *and bringing all the Terms to one Side, you will have*

$$\begin{array}{lllll} ax & = aAy + aBy^2 + & aCy^3 & + \&c. \\ bx^2 = & & bA^2y^2 + 2bABy^3 & + \&c. \\ cx^3 = & & & cA^3y^3 + \&c. \\ \&c & & & \&c. \\ -gy & = -gy & & \\ -hy^2 = & \ldots\ldots -hy^2 & & \\ -iy^3 = & \ldots\ldots\ldots\ldots -iy^3 & & \\ \&c. & & & \&c. \end{array}$$

From whence we see, first, that $aA = g$, *and* $A = \frac{g}{a}$. 2°. *That* $aB + bA^2 - h = 0$, *and* $B = \frac{h}{a} - \frac{bg^2}{a^3}$. 3°. *That* $aC + 2bAB + cA^3 - i = 0$, *and therefore* $C = \frac{i - 2bAB - cA^3}{a}$. *And thus the three first Terms of the Series* $Ay + By^2 + Cy^3$, &c. *are known.* [See Mr. De Moivre in Phil. Transf. 240.]

322. Before we conclude it remains to clear a Difficulty in this Method, that has embarrassed some late ingenious Writers, concerning "*the Value of* r *to be assumed when two or more of the Values of the first Term of a Series for expressing* x *are found equal;*" *a Correction of the preceding Rule being necessary in that Case.* And the Author of that Correction having only collected it from Experience, and given it us without Proof, it is the more necessary to demonstrate it here.

It is to be observed then, that in order that the Series $Ay^n + By^{n+r} + Cy^{n+2r} + Dy^{n+3r} +$ &c. *may express* x, *it is not only necessary that when it is substituted for* x *in the proposed Equation* $Dy^l + Ey^mx^s + Fy^ex^k = 0$, *the Indices* $m + ns$, $m + ns + r$, $m + ns + 2r$, &c. *should fall in with the Indices* $e + nk$, $e + nk + r$, $e + nk + 2r$, &c. *in order that the Terms may be compared together to determine the Coefficients* A, B, C, *&c. but it is also necessary, that in the particular Equations for determining any of those Coefficients, as* B *for Example, those Terms that involve* B *should not destroy each other.* Thus the Equation $3A^2B - 3A^2B - aA = 0$ can never determine B, because $3A^2B - 3A^2B = 0$, and thus B exterminates itself out of the Equation; besides the Contradiction arising from $-aA = 0$, when A perhaps has been determined already to be equal to some real Quantity.

In order to know how to avoid this Absurdity, let us suppose that the first Order of Terms in the proposed Equation are, as before, DY^{l}, $Ey^{m}x^{s}$, &c. and if Ay^{n} is found to be the first Term of a Series for x, then the Dimensions of y in the first order of Terms, arising by substituting in them Ay^{n} for x, will be $m+ns$, and the Dimensions of y arising by substituting $Ay^{n}+By^{n+r}+Cy^{n+2r}$, &c. for x will be $m+ns$, $m+ns+r$, $m+ns+2r$, &c. Suppose that $Fy^{e}x^{k}$ is the next Order of Terms; and, by the same Substitution, the Dimensions of y arising from it will be

(because $Fy^{e}x^{k}=Fy^{e}\times\overline{Ay^{n}+By^{n+r}+Cy^{n+2r}+\&c.}|^{k}=FA^{k}y^{e+nk}+kFBA^{k-1}y^{e+nk+r}$, &c.) $e+nk$, $e+nk+r$, $e+nk+2r$, &c. Now it is plain that $e+nk$ must coincide with some one of the Dimensions $m+ns$, $m+ns+r$, $m+ns+2r$, &c. that the Terms involving them may be compared together. And therefore, as we observed in 318, r *must be the Difference of* e+nk *and* m+ns, *or some Divisor of that Difference.* In general, r *must be assumed such a Divisor of that Difference as may allow not only* e+nk *to coincide with some one of the Series* m+ns, m+ns+r, m+ns+2r, &c. *but as may make all the Indices of the other Orders besides* e+nk *likewise to coincide with one of that Series: that is, if* $Gy^{f}x^{h}$ *is another Term in the Equation,* r *must be so assumed that the Series* f+nh, f+nh+r, f+nh+2r &c. *arising by substituting in it* $Ay^{n}+By^{n+r}+Cy^{n+2r}$, &c. *for* x, *may coincide somewhere with the first Series* m+ns, m+ns+r, m+ns+2r, *&c.* And therefore we said in 318, *that* r *must be assumed so as to be equal to some common Divisor of the Differences of the Indices* m+ns, e+nk, f+nh, &c. *which arise in the proposed Equation by substituting in it for* x *the first Term already known* Ay^{n}. *For by assuming* r *equal to a common Divisor of these Differences, the three Series*

m+ns, m+ns+r, m+ns+2r, m+ns+3r, &c.
e+nk, e+nk+r, e+nk+2r, e+nk+3r, &c.
f+nh, f+nh+r, f+nh+2r, f+nh+3r, &c.

will coincide with one another, since some Multiples of r *added to* m+ns *will give* e+nk *and all that follow it in the second Series, and some Multiples of* r *added to* m+ns *will also give* f+nh *and all that follow it in the third Series. It is also obvious, that, if no particular Reason hinder it,* r *ought*

ought to be assumed equal to the greatest common Measure of these Differences. For Example, if the Indices $m+ns$, $e+nk$, $f+nh$, happen to be in arithmetical Progression, then r ought to be assumed equal to the common Difference of the Terms, and the first of the second Series will coincide with the second of the first, and the first of the third Series will coincide with the second of the second Series, and with the third of the first, and so on.

323. These things being well understood, we are next to observe that after you have substituted $Ay^n+By^{n+r}+Cy^{n+2r}$, &c. for x in the first Order of Terms in the Equation, the Terms that involve $m+ns$ Dimensions of y will destroy one another; for $x-Ay^n$ must be a Divisor of the Aggregate of these Terms, since they give Ay^n as one Value of x: let $\overline{x-Ay^n}\times P$ represent that Aggregate, and, substituting for x its Value $Ay^n+By^{n+r}+Cy^{n+2r}$, &c. that Aggregate becomes $\overline{Ay^n+By^{n+r}+Cy^{n+2r}, \&c.-Ay^n}\times P=\overline{By^{n+r}+Cy^{n+2r}, \&c.}\times P$. Now the lowest Dimension in $\overline{x-Ay^n}\times P$ was supposed to be $m+ns$, whence the Dimension of P, in the same Terms, will be $m+ns-n$, and the lowest Dimension in $\overline{By^{n+r}+Cy^{n+2r}+\&c.}\times P$ will be $n+r+m+ns-n$ $=m+ns+r$. Suppose again that two Values of x, determined from the first Order of Terms, are equal, and then $\overline{x-Ay^n}|^2$ will be a Divisor of that Aggregate of the first Order of Terms. Suppose that Aggregate now $\overline{x-Ay^n}|^2\times P$, which by Substitution of $Ay^n+By^{n+r}+Cy^{n+2r}$, &c. for x will become $\overline{By^{n+r}+Cy^{n+2r}+\&c.}|^2\times P$, in which the lowest Term will now be of $m+ns$ Dimensions, since in $\overline{x-Ay^n}|^2\times P$ the lowest Term is supposed of $m+ns$ Dimensions; and consequently in these Terms, the Dimension of P itself, is $m+ns-2n$.

In general, if the Number of Values of x *supposed equal to* A y^n *be* p, *then must* $\overline{x-Ay^n}|^p$ *be a Divisor of the Aggregate of the Terms of the first Order. And that Aggregate being expressed by* $\overline{x-Ay^n}|^p\times P$, *in the lowest Terms, the Dimensions of* y *in* P *will be* $m+ns-pn$, *that in* $\overline{x-Ay^n}|^p$ *they may be* $m+ns$, *as we always suppose. Substitute in* $\overline{x-Ay^n}|^p\times P$ *for* $x-Ay^n$ *its Value* By^{n+r}

 $+Cy^n$

$+ Cy^{n+2r} +$ &c. *and in the Result* $\overline{By^{n+r} + Cy^{n+2r} + \&c.}\,|^{p}$ $\times P$ *the lowest Dimensions of* y *will be* $pn + pr + m + ns - pn = m + ns + pr$.

324. From what has been said we conclude, that when you have substituted for x in the first Order of Terms of the Equation proposed the Series $Ay^{n} + By^{n+r} + Cy^{n+2r} +$ &c. the first Term of which Ay^{n} is known, and the Values of x whose Number is p are found equal, then the Terms arising that involve $m+ns$, $m+ns+r$, $m+ns+2r$, &c. till you come to $m+ns+pr$, will destroy each other and vanish; so that the first Term, with which the Terms of the second Order $e+nk$ can be compared, must be that which involves $m+ns+pr$; and therefore supposing $e+nk = m+ns+pr$, or $r = \frac{e+nk-m-ns}{p}$, *the highest Value you can give* r *must be the Difference of* $e+nk$ *and* $m+ns$ *divided by* p *the Number of equal Values of the first Term of the Series. If this Value of* r *is a common Measure of all the Differences of the Indices, then is it a just Value of* r; *but if it is not, such a Value of* r *must be assumed, as may measure this and all the Differences: that is, such a Value as may be the greatest common Measure of the least Difference divided by* p (viz. $\frac{e+nk-m-ns}{p}$) *and of the common Measure of all the Differences.* For thus the Indices $m+ns$, $m+ns+r$, $m+ns+2r$, &c. will coincide with $e+nk$, $e+nk+r$, $e+nk+2r$, &c. and with $f+nh$, $f+nh+r$, $f+nh+2r$, &c. and you shall always have Terms to be compared together sufficient to determine B, C, D, &c. the general Coefficients of the Series assumed for x.

325. To all this, it may be added, that if $x - Ay^{n}$ be a Divisor of the Aggregate of the Terms of the second Order $Fy^{e}x^{k}$, &c. then, by substituting for x the Series $Ay^{n} + By^{n+r} + Cy^{n+2r} +$ &c. there vanish not only as many Terms of the Series involving $m+ns$, $m+ns+r$, $m+ns+2r$, &c. as there are equal Values of the first Term Ay^{n}; but the Terms involving $e+nk$ Dimensions of y vanish also; and therefore it is then only necessary that $e+nk+r$ coincide with $m+ns+pr$; so that, in that Case, you need only take $r = \frac{e+nk-m-ns}{p-1}$. And if

$x -$

$\overline{x - Ay^n}|^{p-1}$ be a Divisor of the Aggregate of the second Order of Terms, then the Terms (after substituting for x the Series $Ay^n + By^{n+r} + Cy^{n+2r}$ &c.) which involve $e+nk$, $e+nk+r$, $e+nk+2r$, &c. will vanish to the Term $e+nk+\overline{p-1}\times r$; so that, supposing $e+nk+\overline{p-1}\times r = m+ns+pr$, you have $r = e+nk-m-ns$, that is, to the least Difference of the Indices $m+ns$, $e+nk$, $f+nh$, &c. provided that Difference be a Measure of the other Differences; although there may be as many Values of the first Term of the Series equal, as there are Units in p. Or, if that does not happen, r must be taken, as formerly, equal to the greatest common Measure of the Differences.

326. Suppose that the Orders of Terms of the Equation can be expressed the first by $\overline{x-Ay^n}|^p \times P$, the second by $\overline{x-Ay^n}|^q \times Q$, the third by $\overline{x-Ay^n}|^l \times L$, &c. and suppose that $Ey^m x^s$ is one of the first, $Fy^e x^k$ one of the second, $Gy^f x^h$ one of the third, and so on; then it is plain that, substituting for x the Series $Ay^n + By^{n+r} + Cy^{n+2r} +$, &c. the lowest Term that will remain in the first will be $m+ns+pr$ Dimensions of y, the lowest Term that will remain in the second will be of $e+nk+qr$, and the lowest Term remaining in the third of $f+nh+lr$ Dimensions of y. For by the same Reasoning as we used in 323, to demonstrate that, in the first Order of Terms $\overline{x-Ay^n}|^p \times P$, the lowest Dimensions of y are $m+ns+pr$, we shall find that, in the subsequent Orders, the lowest Dimensions of y in the Terms $\overline{x-Ay^n}|^q \times Q = \overline{By^{n+r}+Cy^{n+2r}\&c.}|^q \times Q$ must be $e+nk-qn+qn+qr = e+nk+qr$, and so of the other Terms $\overline{x-Ay^n}|^l \times L$ the lowest Dimensions must be $f+nh+lr$. The Indices therefore of the Terms that do not vanish being

$$* \; * \; m+ns+pr,$$
$$* \; * \; * \; * \; e+nk+qr,$$
$$* \; * \; * \; * \; * \; * \; f+nh+lr,$$

if r be taken equal to $\frac{e+nk-m-ns}{p-q}$, then will $m+ns+pr$ and $e+nk+qr$ coincide: and if at the same Time r be a Divisor of $f+nh-m-ns$, and be found in it a Number of Times greater than $p-l$, or if r be less than

$f+nh$

$\frac{f+nh-m-ns}{p-l}$, then r will be rightly assumed. In general, *take all the Quotients* $\frac{e+nk-m-ns}{p-q}$, $\frac{f+nh-m-ns}{p-l}$; *and either the least of these, or a Number whose Denominator exceeding* $p-q$ *by an Integer, measures it and all the Differences* $f+nh-m-ns$, *gives* r; *supposing* p, q, *and* l *Integers. But if* p, q, *and* l *are Fractions, you are to take* r *so that it be equal to* $\frac{e+nk-m-ns}{p-q+K}=\frac{f+nh-m-ns}{p-l+M}$, *and so that* K *and* M *may be Integers.* Suppose, for Example, $m+ns=\frac{7}{3}$, $p=\frac{5}{2}$; $e+nk=\frac{10}{3}$, $q=\frac{3}{2}$; $f+nh=\frac{9}{2}$, and $l=\frac{1}{2}$: then putting ----- $(r=)\ \frac{e+nk-m-ns}{p-q+K}=\frac{1}{1+K}=\frac{f+nh-m-ns}{p-l+M}=\frac{\frac{13}{6}}{2+M}$, $M=\frac{1}{6}+\frac{13}{6}K$; whence it is easily seen, that 5 and 11 are the least Integers that can be assumed for K and M. And that $r=\frac{1}{1+K}=\frac{1}{6}$; and therefore $m+ns+pr=\frac{33}{12}$, $e+nk+qr=\frac{43}{12}$, and $f+nh+lr=\frac{55}{12}$. That is, the Terms of the first Series whose Dimensions are $m+ns+\overline{p+K}\times r$, $m+ns+\overline{p+M}\times r$ fall in with the first Terms of the second and third Series respectively.

OF

OF THE MEASURES OF RATIOS.

OF THE MEASURES OF RATIOS.

Translated from the Latin of

The late JAMES MAGUIRE, *A. M.* of the University of DUBLIN.

A POSTHUMOUS WORK.

MAGNITUDES are said to be equal, which being placed one upon the other, are, or seem to be, congruous: as *Lines*, *Angles*, *Surfaces*, which being compared by mental Apposition, are seen upon Account of some given Circumstances to coincide; and *Solids*, which penetrate each other, and coalesce into one. But *Magnitudes, so named in a looser Sense* because they can be increased or diminished, *are said to be equal, which considered as Causes, produce the same Effects*; as the *Times*, in which a Body moved uniformly is carried over equal Spaces; the *Velocities*, with which Bodies moved forward are carried over equal Spaces in a given Time; and *Forces*, which when they are opposed destroy each other.

And since it is manifest, that the Form of any given Magnitude whatsoever may be varied to Infinity, the Quantity being still preserved, by changing the Situation of the Parts: Therefore, *when the Congruity of Magnitudes cannot be immediately perceived* upon Account of the Variety of Forms and Positions, *it is* nevertheless to be *investigated* by the Mind many Ways; *by adding to Congruents, or by* *ſubducting from them, ſome other Congruents*, that an Equality may appear between the reſulting Magnitudes; *or by dividing them into Parts, whoſe Equality can be more eaſily made known*; viz. Lines into ſmaller Lines, Surfaces, into Triangles, and Solids, into Pyramids; *or by a ſucceſſive Appoſition*, *or by a mental Transformation*. Thus is made known, the Equality between Parallelograms, which have the ſame Baſe and an equal Altitude; between a Circle, and a Triangle under the Radius and Periphery; between the Periphery of a Circle, and a Line, which it always touches while it perfects a Revolution by going forward in the Manner of a Wheel; between the curve Surface of a right Cylinder, and a Rectangle under its Side and the Periphery of its Baſe.

But, becauſe *Magnitudes of the ſame Kind*, whatever may be their Form, *can be equal*: It follows, that *of any two Magnitudes whatſoever* A, B; *either the one* A *meaſures the other* B; *or* that *the one* A *conſiſts of the other* B *preciſely, along with ſome Part of it*; *or*, laſtly, that *the one* A *conſiſts of a Multiple of* B, *with ſome Part of it*; and, *conſequently*, that *thoſe Magnitudes will have ſome Ratio to each other*. Now *a Ratio is a certain Habitude of two Magnitudes with Regard to Quantity*. And *becauſe this Habitude regards the Relation of Quantity alone* excluſive of all Circumſtances of Forms and Species, it comes to paſs, that *it can be expreſſed by no Method, but by Numbers*; to wit, by the moſt general Ideas of the Magnitudes themſelves.

The Ratio therefore *of the incommenſurables* A, B, *is ineffable*: For, if they were to each other, as Number to Number; 'tis plain, that they wou'd be Multiples of ſome ſame Meaſure, which is contrary to Hypotheſis. But this *Ineffability ariſes from the Incommenſurability*: For, ſince all Magnitude is eſtimated ſo great, as the Aggregate ariſing from the Repetition of ſome other Magnitude of the ſame Kind (or at leaſt, from the Repetition of ſome aliquot Part of it) may be, (which is itſelf look'd upon as the primary Meaſure, becauſe that the Mind acquieſces in the

Contemplation

Contemplation of it as being intuitively known; and which is therefore named by an abſolute and moſt general Name, Unity): It is manifeſt, that *every Magnitude, which is incommenſurable with this Meaſure Unity, is ineffable.* And *the Incommenſurability ariſes from the Diviſibility of the Magnitudes:* to wit, If of the Magnitude A, there is taken any aliquot Part *p*, ever ſo ſmall, and it be ſubducted ſo often as it can be, from the Magnitude B, there can always remain ſome Part leſs than the diviſible Magnitude *p*.

And hence alſo *one or other* A, *of the Incommenſurables is commenſurable with ſome other Magnitude* C, *which approaches nearer to* B, *than by any given Difference.*

And therefore *the ineffable Ratios of Incommenſurables are the Limits towards which the effable Ratios of Commenſurables approach to Infinity, and to which they can attain nearer than by any definite Difference, but yet never accurately,* by Reaſon of the Diviſibility of Magnitudes.

'Tis the Province of Arithmeticians, to expreſs any given Magnitudes related in any Manner to any given Magnitudes. But becauſe the Nature of Numbers and of Magnitudes can not admit this to be done perfectly, they are under a Neceſſity of flying to the Aſſiſtance of Approximations; that, although they are unable to attain to the accurate Values of Magnitudes, they yet may exhibit other Magnitudes, which may approach nearer to thoſe, which were to be expreſſed, than by any definite Difference. *When* therefore, in what follows, *Ratios are conſidered as compounded of ſome ſame Ratio* $\frac{a}{b}$, *we would have* underſtood, *not the Ratio of the determinate Magnitudes* a, b; *but indefinitely, a Ratio, of which thoſe Ratios may be compounded, which approach nearer to any given Ratios, than by any definite Difference.*

Now *a Ratio is compounded of Ratios, by expreſſing thoſe Ratios;* viz. by changing the Terms into Numerals, and then by multiplying Antecedents into Antecedents, and

Consequents into Consequents; and *a Ratio is* said to be *compounded of any Ratio, when it is the duplicate, or triplicate, or quadruplicate,* &c. *of it. And that, which is compounded of two equal Ratios, is called the Duplicate of either of them; viz. the Duplicate of that Ratio, which is compounded of one of them and of the Ratio of Equality,* which in Composition makes no Change. *That which is compounded of three equal Ratios is called the triplicate of that which is compounded of one; that which of four, quadruplicate;* &c. and *that, which is compounded of three, is called the sesquiplicate of that, which is compounded of two:* And universally, *the Relations of Ratios are derived from the Ratios between the Numbers of the equal component Ratios,* but relative Names are wanting. And *the Ratio between any Extremes whatsoever in a geometrical Series* a, b, c, d, e, &c. *is said to be compounded of the intermediate Ratio,* for the Ratio $\frac{a}{e}$ is compounded of all the intermediate Ratios $\frac{a}{b}$, $\frac{b}{c}$, $\frac{c}{d}$, $\frac{d}{e}$, which are equal the one to the other *.

PROP. I.

All Ratios of the same Inequality are analogous one to the other: that is, The Ratio $\frac{m}{n}$ is duplicate, or triplicate, or sesquiplicate, or subduplicate, &c. of the Ratio $\frac{A}{B}$, or of the Reciprocal $\frac{B}{A}$. viz. The Ratio $\frac{m}{n}$ is compounded of some Ratio, of which also is compounded the Ratio $\frac{A}{B}$, or the Reciprocal $\frac{B}{A}$.

* Eucl. V. Definitions.

I call

I call *Ratios of the same Inequality* those whose Antecedents are all greater, or whose Antecedents are all less, than their Consequents.

$q, r, s, m, a, b, c, d, e, f, g, h,$

without changing the Ratio $\frac{m}{n}$, let the Magnitudes m, n, become homogeneous with the Magnitudes A, B, and between m, and n, let there be a Number of mean Proportionals a, b, c, &c. indefinitely great, viz. that the Ratio $\frac{a-m}{m}$ may be taken less than any given Ratio, or (which is the same Thing) that the intermediate Ratio $\frac{a}{m}$ may approach nearer to the Ratio of Equality, than any predefinite Ratio of Inequality, and let the Series be infinitely continued each Way: And (because m, a, b, &c. ascending increase to a Magnitude greater than any given one, and m, s, r, q, &c. descending decrease to a Magnitude less than any given one) if either one or the other A, of the Terms A, B, is not in the Series, let it be between g and h, viz. let it be greater than g, and less than h, and the Ratio $\frac{A}{g}$ will be less than the Ratio $\frac{h}{g}$; that is, than the Ratio $\frac{a}{m}$; and therefore the Ratio $\frac{A-g}{g}$ will be less than the Ratio, $\frac{a-m}{m}$, which could have been assumed less than any given Ratio. Therefore a Series can be exhibited in which the Terms m, n, are placed, and of which some Term will have to the Term A a Ratio which is less short of the Ratio of Equality, than any given Ratio of Inequality; that is, the Terms m, n, A, B, are placed in a geometric Series. And thence it appears, that the Ratios $\frac{m}{n}$, $\frac{A}{B}$, are compounded of the same Ratio $\frac{m}{a}$, or of the reciprocals, $\frac{m}{a}$, $\frac{a}{m}$.

 PROP.

PROP. II.

The Analogy which is between Ratios of the same Inequality is immutable; viz. *the Numbers of the component Ratios are in a given Ratio.* For let $\frac{p}{z}$ be the ultimate Ratio of which the Ratios $\frac{m}{n}$, $\frac{A}{B}$, of the same Inequality can be compounded, (for we suppose that they cannot be compounded of any Ratio, which is much short of the Ratio of Equality, and thence that they cannot be compounded of any other Ratio, beside the subduplicate, or subtriplicate, &c. of the Ratio $\frac{p}{z}$, viz. some Ratio, of which $\frac{p}{z}$ is also compounded) now if the Numbers of equal Ratios of which the Ratios $\frac{m}{n}$, $\frac{A}{B}$, are compounded be always called indefinitely n and N, and that they be supposed to be afterwards compounded of the subduplicate, or subtriplicate &c. of the Ratio $\frac{p}{z}$, 'tis evident, that n and N are both increased in a double, or triple, &c. Ratio; that is, the Numbers of the component Ratios are always in a given Ratio.

All Magnitudes which are in that Ratio are called the Measures of the compound Ratios.

Cor. 1. *If any Magnitude* M *be put for the Measure of any Ratio whatsoever, the Measures of all other Ratios of the same Inequality will be thence determined*; because of the invariable Ratios which they have to the Measure M: For the Measure of the duplicate Ratio, will be double; of the triplicate, triple; of the sesquiplicate, sesquialterate; of the subduplicate, subduple; of the subtriplicate, subtriple; &c. &c. therefore the Measures of those Ratios which verge to the Ratio of Equality are infinitely

infinitely diminished. And because *the Measure of a Ratio is* defined *a Magnitude proportional to the Number of equal Ratios, which are componnded with the Ratio of Equality**: the Measure of the Ratio of Equality itself will vanish. But *the Systems of Measures are varied by increasing or diminishing in any given Ratio the assumed Magnitude* M, and thence also the Measures which are determined by it of all the other Ratios; *Or* also, *by assuming for Measures other Magnitudes heterogeneous to the Magnitude* M.

Cor. 2. *In a given System the Measure of the compound Ratio* $\frac{AC}{BD}$ *is the Sum of the Measures of the component Ratios* $\frac{A}{B}$, $\frac{C}{D}$. For let three Ratios be compounded of some same Ratio $\frac{g}{h}$, of which let z be the Measure, and let n and N be the Numbers of the Ratios $\frac{g}{h}$ of which the Ratios of the same Inequality $\frac{A}{B}$, $\frac{C}{D}$, are compounded, and (because a Ratio compounded of Ratios is compounded also of all the Ratios of which they themselves are compounded) $n + N$ will be the Number of the Ratios $\frac{g}{h}$, of which the Ratio $\frac{AC}{BD}$ is compounded; therefore (Cor. 1.) $n \times z$, $N \times z$, and $\overline{n + N} \times z$, are the Measures of the Ratios $\frac{A}{B}$, $\frac{C}{D}$, and $\frac{AC}{BD}$. But because from the Composition of the Reciprocals, $\frac{A}{B}$, $\frac{B}{A}$, the Ratio of Equality is made, of which there is no Measure, and because these are compounded of the reciprocals $\frac{c}{d}$, $\frac{d}{c}$, equal in Numbers, *therefore that their*

* Preceding Definition.

their Measures may be distinguished, and that the Measure of the compound Ratio may still also remain equal to the Sum of the Measures of the component Ratios; whatever may be the Measures of the Ratios of either Inequality, the Measures of all their Reciprocals ought to be the same, with their Signs changed.

Cor. 3. $\begin{matrix} n-2z, & n-z, & n, & n+z, & n+2z, & n+3z, & n+4z, \\ a, & b, & c, & d, & e, & f, & g, \end{matrix}$

If the Magnitudes $n-2z$, $n-z$, n, &c. in an arithmetical Series, be adjoined to the Magnitudes a, b, c, &c. in a geometrical Series, one to one respectively, the Measure of the Ratio between any Terms whatsover in the geometrical Series, will be the Difference of the Terms adjoined. For if the Number of Terms in the Series, whose Beginning and End are a and d, be called N; and if the common Difference z be put for the Measure of the intermediate Ratio $\frac{a}{b}$; the Number of the Ratios $\frac{a}{b}$, of which the Ratio $\frac{a}{d}$ is compounded, will be $N-1$, whose Measure therefore (Cor. 1.) is $\overline{N-1} \times z$; which, from the Nature of the arithmetical Series, is the Difference of the adjoined Terms.

Scholium.

The numeral Measures of Ratios are called Logarithms.

327 These are usually so disposed in a Table, that in the Column opposite to each integer Number the Logarithm is placed of the Ratio which that Number has to Unity. For *all Numbers are in a geometrical Series, to which (as they continually increase) Numbers continually increasing in an arithmetical Series are understood to be adjoined*; whence the Measure of the Ratio between any Terms whatsoever in the geometrical Series is the Difference of the Terms adjoined (a). viz. the Logarithm of the Ratio between the Numerator and Denominator of any

(a) Cor. 3.

any Fraction whatsoever (*b*), will be the Term affixed to the Numerator diminished by the Term affixed to the Denominator. And because in the vulgar Tables the Term affixed to Unity is a Cypher, the Logarithm of the Ratio of any Number whatsoever to Unity, will be the Term itself, which is adjoined. And hence is had a most useful Compendium for arithmetical Operations. For

328. *If a Number is to be multiplied by a Number,* because the Ratio of the Product to Unity is compounded of the Ratios, which the Factors have to Unity; *if the Logarithms of the Ratios which the Factors have to Unity be added together, the Sum will be the Logarithm of the Ratio of the Product to Unity* (*c*), *and the corresponding Number will be the Product itself.*

329. *And if a Number is to be divided by a Number, the Difference of the Logarithms of the Ratios of the Dividend and the Divisor to Unity is the Logarithm of the Ratio of the Quote to Unity* (*d*), *and the corresponding Number is the Quote itself.*

Let r, s, be integer Numbers. It is plain, that the Number of Ratios, of which is compounded the Ratio which the Power $n^{\frac{1}{s}}$ has to Unity, is to the Number of Ratios of which is compounded the Ratio which n has to Unity, as 1 to s; or as $\frac{1}{s}$ to 1 (*e*); and therefore that the Number of Ratios of which is compounded the Ratio which the Power $n^{\frac{r}{s}}$ has to Unity, is to the Number of Ratios of which is compounded the Ratio which n has to Unity, as $r \times \frac{1}{s}$ to 1; viz. as $\frac{r}{s}$ to 1. And hence, *if any Power of any Number whatsoever is sought, the Logarithm of the Ratio of that Number to Unity, multiplied*

(*b*) Number 143. 76. (*c*) Number 79. Cor. 2.
(*d*) Number 81. Cor. 2. (*e*) Number 62.

multiplied into the Index of the Power, whether Integer or Fraction (f), is the Logarithm of the Ratio of the Power to Unity; and the Number corresponding is the Power itself.

331. Hence, *if the Logarithms of the Ratios, which the Numbers* A, B, *have to Unity, be called* L, *and* R; *then* $A^R = B^L$: For the Logarithm of the Ratio which either Power has to Unity is the Product of the Logarithms R L (*g*).

332. And hence *Equations, in which Indices are sought, can be reduced to those, in which Roots are sought.* As if $a^n + b^n = c$; and a, b, c, being given, the Index n be sought: Let q and p be the Logarithms of the Ratios of a to 1, and of b to 1; and let $\frac{g}{1}$ be the Ratio whose Logarithm is n: Therefore $a^n = g^q$, and $b^n = g^p$ (*h*); whence $g^q + g^p = c$, where the rest being given, g is sought, whose corresponding Logarithm is n, the Index sought.

PROP. III.

Q

T V

Z R I

n p q y

m i k l o

r s t u w x

a b c d e f g

A B C D E F G H

In the Series of Magnitudes A, B, C, *&c. the Differences, resulting from the continual Subduction of the preceding Terms from those which immediately follow, constitute the Series* a, b, c, *&c. whose Terms, subducted from those immediately following, give the Series* r, s, t, *&c. &c. The Excess* B — A,

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(*f*) Number 83. 85. 145. Cor. 3. (*g*) Eucl. V. 9. Number 83. (*h*) Number 331.

B — A, *multiplied into the Number of Terms less one, in the Series whose Beginning and End are* A, H, *is deficient of the Excess* H — A, *by the Sum* $x + 2w + 3u + 4t$, *&c. or (if the Number of Terms of the Series* A, B, C, *&c. be* N ; *and the first of the third Differences, be* m ; *the first of the fourth,* n ; *and so of the rest) by the Sum* $\frac{N-1}{1} \times \frac{N-2}{2} r + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} m + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times \frac{N-4}{4} n$ + &c. &c.

Part 1st. Because (by Hypothesis) $B - A = a$, $C - B = b$, and so on; it is manifest that the Difference of the Extremes H — A is equal to $a + b + c$ &c. the Sum of the Differences: But (by Hypothesis) $b = a + r$, $c = b + s$ and so on; therefore $c = a + r + s$, and $d = a + r + s + t$, and so on continually; whence the Difference H — A is equal to the Series following

a
$a+r$
$a+r+s$
$a+r+s+t$
$a+r+s+t+u$
$a+r+s+t+u+w$
$a+r+s+t+u+w+x$; and it is plain, that the Sum of this Series is equal to the Sum $\overline{N-1} \times a + \overline{N-2} \times r + \overline{N-3} \times s + \overline{N-4} \times t$, &c. whence $\overline{N-1} \times a$ is deficient of the Excess H — A by the Sum $\overline{N-2} \times r + \overline{N-3} \times s + \overline{N-4} \times t$, &c. or (computing backward) by the Sum $x + 2w + 3u + 4t$ &c.

Part 2d. Because $s = r + m$, and $t = r + m + i$, and so on; and after the same Manner, $i = m + n$, and $k = m + n + p$, and so on continually; the former Series will be changed into the following one.

a
$a+r$
$a+2r+m$
$a+3r+3m+n$
$a+4r+6m+4n+Z$
$a+5r+10m+10n+5Z+T$
$a+6r+15m+20n+15Z+6T+Q$; where 'tis ob-

vious that the Coefficients of the Term a viz. Units, are Figurates of the firſt Order, whoſe Number is N — 1, in *Paſchal's* arithmetical Triangle; and the Coefficients of the Term r, viz. 1, 2, 3, &c. are Figurates of the ſecond Order, whoſe Number is N — 2; the Coefficient of the Term m, are Figurates of the third Order, whoſe Number is N — 3; and ſo on continually (*a*): Whence by the common Methods of ſumming the Series of theſe Figurates (*b*), the Sum of the Series is equal to the Sum

$$\frac{N-1}{1}a + \frac{N-1}{1} \times \frac{N-2}{2}r + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3}m + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times \frac{N-4}{4}n + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times \frac{N-4}{4} \times \frac{N-5}{5}Z + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times \frac{N-4}{4} \times \frac{N-5}{5} \times \frac{N-6}{6}T + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times \frac{N-4}{4} \times \frac{N-5}{5} \times \frac{N-6}{6} \times \frac{N-7}{7}Q = H - A.$$

Cor. 1.

$r, \quad s, \quad t, \quad u, \quad w, \quad x,$

$a, \quad b, \quad c, \quad d, \quad e, \quad f, \quad g,$

A, B, C, D, E, F, G, H,

If A, B, C, *&c. be continued Proportionals, and* N *remain unchanged, the nearer the intermediate Ratio* $\frac{A}{B}$ *approaches to the Ratio of Equality,* i. e. *the leſs the Ratio* $\frac{a}{A}$ *of the Difference of the Terms is to the leſs Term, the nearer always will the Ratio* $\frac{\overline{N-1} \times a}{H-A}$ *approach to the Ratio of Equality.*

For Magnitudes continually proportional are proportional to their Differences (*c*): Therefore, becauſe B : A :: b : a, by

(*a*) Number 23. (*b*) Number 27. (*c*) Eucl. V. 19.

by Division B — B : A :: b — a : a, that is, a : A :: r : a; therefore the Ratio $\frac{a}{A}$ being diminiſhed, the Ratio $\frac{r}{a}$ is alſo diminiſhed; and thence alſo the Ratio, which any Multiple of the Magnitude r has to a; the Ratio alſo $\frac{B}{A} = \frac{s}{r}$ (for B : A :: s : r) is diminiſhed, therefore the Ratios $\frac{s}{r}$, and $\frac{r}{a}$, are diminiſhed, whence the Ratio $\frac{s}{a}$ will be diminiſhed; and thence the Ratio, which any Multiple of the Magnitude s has to a, will be diminiſhed; and the Ratio $\frac{s}{r}$ being diminiſhed, it's Duplicate $\frac{t}{r}$ (for r, s, t, are continually proportional) is diminiſhed; therefore the Ratios $\frac{t}{r}$ and $\frac{r}{a}$ are diminiſhed, and thence the Ratio, which any Multiple of the Magnitude t has to a, is diminiſhed. After the ſame Manner the Ratio, which any Multiple of the Magnitude u, has to a is diminiſhed; and ſo on continually. Now all theſe Ratios will be conſiderably diminiſhed, becauſe the Ratios $\frac{B}{A}$ and $\frac{r}{a}$ are conſiderably diminiſhed; and therefore the Ratio which the Sum $x + 2w + 3u$ &c. has to a will be conſiderably diminiſhed, and thence alſo the Ratio which this Sum $x + 2w + 3u$, &c. has to $\overline{N - 1} \times a$ will be conſiderably diminiſhed; and the Ratio $\frac{\overline{N-1} \times a}{H - A}$ will verge to the Ratio of Equality.

Cor. 2.

A, B, C, D, E, F, G, H, - - - - I, K, L, M, N, O, P, Q,

The Extremes verging to an Equality, and N *remaining unchanged,*

unchanged, the Ratio $\overline{N-1}\times\frac{a}{H-A}$ *verges to the Ratio of Equality*; *and thence the Excesses, by which the Terms* B, C, D, *&c. exceed the Term* A, *divided by that Term, viz.* $\frac{B-A}{A}$, $\frac{C-A}{A}$, $\frac{D-A}{A}$, *&c. verge to the Measures of the Ratios* $\frac{B}{A}$, $\frac{C}{A}$, $\frac{D}{A}$, *&c.* For if the Number of Terms in a Series whose Beginning and End are any Extremes whatsoever A and E be called N, and $\frac{B-A}{A}$ be put for the Measure of the intermediate Ratio $\frac{B}{A}$; then $N-1$ will be the Number of the Ratios $\frac{B}{A}$ of which the Ratio $\frac{E}{A}$ is compounded, whose Measure therefore (Cor. 1. P. 2.) will be $\overline{N-1}\times\frac{B-A}{A}$ which verges to the Quantity $\frac{E-A}{A}$. But Quantities continually proportional are proportional to their Differences, viz. $\frac{B-A}{A}$, $\frac{C-B}{B}$, $\frac{D-C}{C}$, &c. are constantly equal to each other, and the Measures of equal Ratios are equal; and therefore if the Terms I, K, L, &c. be placed in a continued Order any where in the Series continued, $\frac{K-G}{G}$ will be the Measure of the Ratio $\frac{K}{G}$; and $\frac{K-I}{I}$, $\frac{L-I}{I}$, will verge to the Measures of the Ratios $\frac{K}{I}$, $\frac{L}{I}$, &c. in the same System.

Cor.

Cor. 3. $r, s, t, u, w, x,$
$a, b, c, d, e, f, g,$
A, B, C, D, E, F, G, H,

The same Things being supposed, the Measure of the Ratio between any determinate Extremes whatsoever, H, *and* A, *will be always less than* $\frac{H - A}{A}$, for the Sum $x + 2w + 3u$, &c. will not vanish.

Cor. 4. *The Extremes verging to an Equality, and the intermediate Ratio remaining unchanged, the Ratio* $\frac{\overline{N - 1} \times a}{H - A}$ *verges to the Ratio of Equality.* For the Difference of the Extremes H and A is equal to the Sum of the Differences $g + f + e$, &c. and in like Manner the Difference of the Extremes F and A is equal to the Sum of the Differences $e + d + c$, &c; but (because the Differences a, b, c, &c. increase perpetually) the Ratio of the greater Sum of the Differences $g + f + e$, &c. to the less Sum of the Differences $e + d + c$, &c. is greater, than the Ratio which the greater Number of them has to the less Number of them; that is, (if the Number of Terms in the Series whose Beginning and End are A and F be called n) the Ratio of H — A to F — A is greater than the Ratio of N — 1 to n — 1, or than the Ratio of $\overline{N - 1} \times a$ to $\overline{n - 1} \times a$ (*d*); and by Alternation, the Ratio of H — A to $\overline{N - 1} \times a$ is greater than the Ratio of F — A to $\overline{n - 1} \times a$; viz. the Ratio $\frac{F - A}{\overline{n - 1} \times a}$ is less short of the Ratio of Equality.

Cor. 5. A, B, C, H, *first Series*
A, D, E, F, H, second Series
$a, b, c, d, e, f, g, h, i, k, l, m,$
A, g, p, *D*, B, s, *E*, m, C, *F*, y, z, H, *third Series*
The Extremes remaining unchanged, and N *being increased,*
B — A

(*d*) Eucl. viii. 18.

$B - A \times N - 1$ *is diminished.* For let there be two Series, having their Beginning and End the same, A, and H; and let the Number of Terms be N, and n: And if a third Series be made whose Beginning and End are also the same A and H, and in which the Number of Terms less one is a Number P, which $N - 1$ and $n - 1$ will measure; the Terms of either of the former will be the same with those Terms of the third Series, which shall be taken at Intervals, by omitting a Number of the Terms intervening, equal to the Quote of the Number P divided by $N - 1$, or $n - 1$, respectively. Let the Differences of the Terms in this third Series be a, b, c, &c; the Number of Differences a, b, c, &c. of which either Difference $B - A$, or $D - A$, consists, is P divided by the respective $N - 1$, or $n - 1$, and therefore the Numbers of the Differences a, b, c, &c. of which the Differences $B - A$ and $D - A$, consist, are as $n - 1$ to $N - 1$ (e), that is, inversely as the Numbers of the Terms in the Series, A, B, C, &c. and A, *D*, *E*, &c. diminished by Unity. But (upon Account of the perpetual Increase of the Differences) the Ratio which $d + c + b + a$ the greater Sum of Differences has to the less Sum of Differences $c + b + a$ is greater, than the Ratio which their greater Number has to their less Number; viz. $N - 1$ being increased in any Ratio, $B - A$ is diminished in a greater Ratio; and consequently $B - A \times N - 1$ is diminished.

Cor. 6. $\quad a, \; b, \; c, \; d, \; e, \; f, \; g,$
A, B, C, D, E, F, G, H,

The Ratio of $H - A$ *to* $N - 1 \times a$ *is less than the Ratio of the Extremes* $\frac{H}{A}$. For (upon Account of the Differences a, b, c, &c. always increasing) the Ratio of the Sum of all the Differences $g + f + e$, &c. to the least Difference multiplied into their Number, viz. to $N - 1$ $\times a$

(e) 60.

$\times a$, is leſs than the Ratio of the greateſt Difference to the leaſt, viz. than the Ratio of $\frac{g}{a}$, or $\frac{G}{A}$; and therefore leſs than the Ratio $\frac{H}{A}$.

Cor. 7. *The Extremes remaining unchanged, and* N *being ſufficiently great,* $N - 1 \times a$ *will approach nearer to an invariable Quantity, than by any definite Difference:* to wit, N *may be aſſumed of a Magnitude ſo great, that though it ſhould be again and ever ſo much increaſed, yet the Ratio between* $N - 1 \times a$ *firſt aſſumed, and* $N - 1 \times a$ *again aſſumed, ſhall differ from the Ratio of Equality leſs than by any aſſigned Difference.* For N may be indefinitely increaſed, while (Cor. 5.) $N - 1 \times a$ is in the mean Time perpetually diminiſhed; but the Ratio of the given Quantity $H - A$ to $N - 1 \times a$ is always leſs than the given Ratio $\frac{H}{A}$ (Cor. 6.); therefore if N ſhould be perpetually increaſed, $N - \times 1\ a$ will verge to an invariable Quantity.

Cor. 8. *The finite Number of Terms* N *remaining unchanged, let the Ratio* $\frac{a}{A}$ *be indefinitely diminiſhed; the Extremes* A *and* H *will come nearer to an Equality than by any predefinite Difference; and* $N - 1 \times a$ (Cor. 1.) *will be equal to the Difference of the Extremes* $H - A$; *(to wit, their Ratio will approach nearer to the Ratio of Equality than any predefinite Ratio of Inequality will) and thence it appears, that the Terms* A, B, C, *&c. will be equidifferent; and that, if* $\frac{B - A}{A}$ *be put for the Meaſure of the Ratio* $\frac{B}{A}$, then $\frac{H - A}{A}$ *will be the Meaſure of the Ratio* $\frac{H}{A}$: *And the contrary.* Now *if the Extremes remaining unchanged,* N *be any how changed, the Ratio between* $H - A$

 and

and N — 1 × a *will* (Cor. 6.) *remain nearer always to the Ratio of Equality, than the Ratio* $\frac{H}{A}$; *and therefore if* $\frac{B-A}{A}$ *be always put for the Measure of the intermediate Ratio* $\frac{B}{A}$, *the Measure of the Ratio* $\frac{H}{A}$ *will remain unvaried*; to wit, it will differ less from a given Quantity, than by any predefinite Difference; *and therefore, if* N *be understood to be indefinitely increased, so that every Magnitude of the same Kind, which is neither less than* A *nor greater than* H, *may be placed in this Series, tis evident, that the Measure of the Ratio between any such Magnitude whatsoever, and the Term* A, *will be the Difference of the Terms divided by the less Term.*

Cor. 9.

A, B, C, D, E, F, G, H, - - - I, K, L, M, N, O, P, Q,

Let the Series now be continued on, untill the Excesses by which the Terms I, K, L, *&c. exceed the Term* A, *may have some considerable Ratios to the Term* A; (but you are to conceive a Number of Terms indefinitely great to be placed between H and I,) and (because that in approaching to the Term H, $\frac{Q-A}{A}$, $\frac{P-A}{A}$, &c. perpetually decrease and verge (Cor. 4.) to the Measures of the Ratios $\frac{Q}{A}$, $\frac{P}{A}$, &c. and because every Magnitude of the same Kind is placed in this Series continued each way) tis evident, that *the less the Quote is of the Difference of any two Magnitudes whatsoever divided by the smaller, the nearer always will this Quote approach to the Measure of the Ratio of the Magnitudes*; and therefore the nearer also will the double, or triple, &c. of this Quote, approach to the Measure of the duplicate, or triplicate, &c. Ratio; wherefore *the Extremes remaining unaltered, and the Number of mean Terms being in any Manner varied, let the Difference between the two first Terms divided by the smaller*

be

be always indefinitely called x, *and let the Number of Terms be always* N; *by how much less* x *is, by so much always will* $\overline{N-1} \times x$ *approach nearer to an invariable Measure of the Ratio* $\frac{Q}{A}$: *And it will approach nearer than by any assigned Difference, if the second Term ascends not beyond the Term* H. (*f*)

Cor. 10. x, y, z, A, B, C, D, E, F, G, H,

And if the Terms, A, z, y, *&c. be placed in the Series continued the contrary Way, and if the intermediate Ratio verges to the Ratio of Equality;* $\frac{A-B}{A}$, $\frac{A-C}{A}$, &c. will verge (Cor. 2. P. 2.) to the Measures of the Ratios $\frac{z}{A}$, $\frac{y}{A}$, &c. which are the Reciprocals of the Ratios $\frac{B}{A}$, $\frac{C}{A}$, &c. But they also verge to the Quantities $\frac{z-A}{A}$, $\frac{y-A}{A}$, &c. because the Series y, z, A, B, continued each Way, and finite, verges to an arithmetical Series; therefore $\frac{z-A}{A}$, $\frac{y-A}{A}$, &c. will verge to the Measures of the Ratios $\frac{z}{A}$, $\frac{y}{A}$, &c; and *all Things which were before demonstrated in the Series* A, B, C, *&c. have Place also (with the proper Changes* (*g*)) *in the Series,* A, z, y, *&c.*

Cor. 11.

A, B, C, D, E, F, G, H, - - - I, K, L, M, N, O, P, Q,

Let all the Terms, A, B, C, *&c. be now changed into Numeral, and the Series be conceived to be continued each Way to Infinity,*

(*f*) Cor. 8. (*g*) Cor. 2. P. 2.

Infinity, all Numbers will be placed in this Series; and if $\frac{B-A}{A}$ *be put for the Logarithm of the Ratio* $\frac{B}{A}$, then $\frac{H-A}{A}$ *will be the Logarithm of the Ratio* $\frac{H}{A}$; and $\frac{B-A}{A} \times N-1$ *will be the Logarithm of the Ratio between any Extremes whatsoever,* Q *and* A; *and will remain invariable, however the Number of mean Terms be varied, so that the second Term ascends not beyond* H (*h*).

Logarithms of this Kind the most noble Lord Napier *Baron of* Merchiston *in* Scotland *invented, and they are called Hyperbolic* (*i*).

Cor. 12. *Hence, if* $\frac{B-A}{A}$ *be given, the Logarithm of any Ratio whatsoever* $\frac{B}{A}$, *will be as* N — 1; *and if* N *be given, the Logarithm will be as* $\frac{B-A}{A}$; *that is, if* A *also be given, as* B — A; *that is, if* A *be Unity, as the Excess above Unity of the Root of the Term* Q *whose Index is* N — 1. For if Unity begins the Series, every subsequent Term is a Power of the second Term, whose Index is the Number of Terms reckon'd to that Term exclusive (*k*), *viz. if from two Numbers, which are both greater or both less than Unity, be extracted the Roots of the same Power whose Index is indefinitely great, the Excesses of these Roots above Unity, will be as the Logarithms of the Ratios which those Numbers have to Unity; that is, the greater the Index of the Power is, the nearer will the Ratio of the Excesses approach to the Ratio of the Logarithms.* These Things are all evident in the hyperbolic System, and by (Prop. 2.) the same have Place in every other System.

Cor. 13. And *if for the Logarithm of any Ratio whatsoever be put it's hyperbolic Logarithm multiplied into any numeral*

(*h*) Cor. 8. (*i*) Cor. 3. (*k*) 76.

numeral Quantity M, *the Logarithms of all other Ratios (Prop. 2.) will be their hyperbolic Logarithms multiplied into the ſame Quantity; but the hyperbolic Logarithm of any given Ratio is invariable* (*l*), *that is, there is but one Syſtem of hyperbolic Logarithms; therefore the Logarithm of any given Ratio is as that aſſumed Quantity* M, which is called *the Module of the Syſtem.*

Cor. 14. A, B, C, D, E, F, G, H,
Becauſe the Logarithm of the Ratio $\frac{H}{A}$ is $\frac{H-A}{A} \times M$: If that Logarithm be called L; then $\frac{H-A}{A} \times M = L$, and $H - A \times M = L \times A$; whence $H - A : L :: A : M$ (*m*), that is, *the Exceſs by which any Number whatſoever exceeds the Subtrahend* A, *is to the Logarithm of the Ratio, which it has to the Subtrahend, as the Subtrahend is to the Module of the Syſtem*; provided that this Ratio approaches very nearly to the Ratio of Equality. And hence *the Exceſſes by which any two Numbers whatſoever exceed a given Subtrahend* A *are as the Logarithms of the Ratios which they have to the Subtrahend* (*n*), *provided that they are placed in the Series on the ſame Side of the Number* A, *and that the three Numbers are very nearly equal*; viz. the nearer they approach to an Equality, the nearer the Ratio of the Exceſſes approaches to the Ratio of the Logarithms.

The Ratio whoſe Logarithm is the Module of the Syſtem is called *the modular Ratio.*

Cor. 15. *The modular Ratio is the ſame in all Syſtems.* For the Modules of two Syſtems are as (Cor. 13.) the Logarithms of any given Ratio in thoſe Syſtems; and thence 'tis evident, that the Modules themſelves are the Logarithms of ſome ſame Ratio, to wit, If the Modules

(*l*) Cor. 7. 9. (*m*) Eucl. vii. 19. (*n*) Eucl. v. ii.

be called L, and M, and $\frac{p}{g}$ be the modular Ratio in the System whose Module is L, L will be to M, as L to the Logarithm of the Ratio $\frac{p}{g}$ in the System, whose Module is M, therefore that Logarithm is M: viz. $\frac{p}{g}$ is also the modular Ratio in the System whose Module is M *.

Cor. 16. *Every Number is a Power of the Number, whose Ratio to Unity, is the Modular; the Index of which Power is the hyperbolic Logarithm of the Ratio of that Number to Unity.* For the Logarithm of the Ratio of the Number to Unity multiplied into the Index of the Power gives the Logarithm of the Ratio of the Power to Unity (*o*), and *the hyperbolic Logarithm of the modular Ratio, is Unity*, which in Multiplication makes no Change.

By Number I every where understand, every numeral Quantity, whether Integer, or Fraction.

SCHOLIUM.

333. *The Logarithms of all Ratios cannot be accurately exhibited.* For if they could be perfectly expressed, and integer Numbers shou'd be taken in the Ratio of the Logarithms, it wou'd follow, that all Ratios cou'd be compounded of some same Ratio, and that the Numbers of the component Ratios were those aforesaid Integers respectively; that is, all Numbers cou'd be accurately placed in a geometrical Series, whose intermediate Ratio shou'd be that of determinate Magnitudes: And thence also there cou'd no Ratio exist, which shou'd approach nearer to

* This modular Ratio is that of 2,7182812, &c. to Unity, that is, of some Number between 2 and 3 (but nearer to 3 than 2) because 2,718, &c. is greater than $2,5 = 2\frac{1}{2}$ to Unity. See Cote's Harmonia Mensurarum, Prop. 1. Schol. 2. and the next Prop. of this.

(*o*) Number 330.

to the Ratio of Equality, than that intermediate Ratio; viz. the Ratio of the Difference of two Terms any where adjacent in a Series to either Term cou'd not be any further diminished, and that Difference wou'd be invisible. Which is absurd (*a*).

334. *But* because a Series can be exhibited, in which are placed the Terms m, n, of any given Ratio $\frac{m}{n}$, and also some Terms, whose Ratios to the Terms of any other given Ratio $\frac{A}{B}$, shall approach nearer to the Ratios of Equality, than by any definite Difference; it follows, that *if any given Number whatsoever be put for the Logarithm of the Ratio* $\frac{m}{n}$, *the Logarithm of a Ratio, which shall come nearer to the Ratio* $\frac{A}{B}$ *than by any given Difference can be exhibited* (*b*).

335. *But the Logarithms of all Ratios, whose Analogy to the Ratio* $\frac{m}{n}$ *can be accurately expressed, may be accurately found*; viz. *whose Terms are placed in the same Series with the Terms* m, *and* n, *the intermediate Ratio of which is that of determinate Magnitudes.* Thus if 2 be put for the Logarithm of the Ratio of 5 to 3, that is, of the Ratio of the Fraction $\frac{5}{3}$ to 1; the Logarithms of all Ratios, whose Terms are placed in the same Series with the Terms $\frac{5}{3}$ and 1, will be effable, viz. the Logarithms of all the Ratios, which the Powers of the Number $\frac{5}{3}$, whose Indices are rational, have to Unity:

(*a*) Definitions. (*b*) P. 1.

Because, every one of those Powers is placed in the same Series, with the Terms $\frac{5}{3}$ and 1, whose intermediate Ratio is that of determinate Magnitudes. As for Example 1, $\frac{5}{3}$, and $\frac{5}{3}^{\frac{9}{11}}$, are placed in a Series, whose intermediate Ratio, is that of the determinate Magnitudes $\frac{5}{3}^{\frac{1}{11}}$ and 1, viz. the Ratio, which the Root of the Number, $\frac{5}{3}$, of the eleventh Power, has to Unity: Now because the Number of the Ratios $\frac{5}{3}^{\frac{1}{11}}$ to 1, of which the Ratio $\frac{5}{3}$ to 1, is compounded, is 11; and the Number of the same Ratios, of which the Ratio $\frac{5}{3}^{\frac{9}{11}}$ to 1, is compounded, is 9; and because $11 : 9 :: 2 : \frac{18}{11}$; and because (by Hypothesis) 2 is the Logarithm of the Ratio of $\frac{5}{3}$ to 1; therefore $\frac{18}{11}$ will be the Logarithm of the Ratio of $\frac{5}{3}^{\frac{9}{11}}$ to 1. And after the same Manner the Logarithms of the Ratios which the other Powers have to Unity will come out, by multiplying their respective Indices by the Logarithm of the Ratio which $\frac{5}{3}$ has to 1.

336. *Thus also, if Unity be put for the Logarithm of the Ratios of* 10 *to* 1, *the Logarithms of all the Ratios, which the Powers of the Number* 10 *whose Indices are rational*

onal have to 1, *will be effable.* As for Example, the Logarithms, will be effable of the Ratios $\frac{10^2}{1}$, $\frac{10^3}{1}$, $\frac{10^{\frac{1}{2}}}{1}$, $\frac{10^{\frac{1}{3}}}{1}$, $\frac{10^{\frac{5}{11}}}{1}$, viz. of the Ratios $\frac{100}{1}$, $\frac{1000}{1}$, $\frac{\sqrt{10}}{1}$, $\frac{\sqrt[3]{10}}{1}$, $\frac{\sqrt[11]{100000}}{1}$, and these Logarithms will be 2, 3, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{5}{11}$, respectively; because 1, 10, and all those Powers are placed in a Series, whose intermediate Ratio is that of the determinate Magnitudes $\sqrt[66]{10}$ and 1; viz. the Ratio, which the Root of the Number 10 of the sixty-sixth Power, has to Unity: *And the natural Numbers,* 1, 2, 3, *&c. will be the Logarithms of all Ratios, which are compounded in the Ratio* $\frac{10}{1}$; viz. of the Ratios $\frac{10}{1}$, $\frac{100}{1}$, $\frac{1000}{1}$, &c.

337. *But the Logarithms of all the other effable Ratios will be ineffable.* Example, the Logarithm of the Ratio of 6 to 1 will be ineffable, because no Series can be exhibited, in which 1, 10, and 6, are placed; but 6 is a Power of 10, whose Index is the ineffable Logarithm of the Ratio $\frac{6}{1}$: that is, Because 1 and 10 are placed in the same Series with some Number n, whose Ratio to 6 can come nearer to the Ratio of Equality than by any assigned Difference, and thence, *because the Logarithm of the Ratio* $\frac{n}{1}$ *can always be accurately exhibited, if there be taken a Power of* 10, *whose Index is that Logarithm, the Number* n *will come out* (*c*): *And if again there be taken the Logarithm of*

(*c*) Number 331.

of a Ratio which comes still nearer to the Ratio of $\frac{6}{1}$, *viz. if* (as it is usually expressed, the Logarithm of the Ratio of $\frac{6}{1}$, be to be found more accurately) *there be taken a Power of the Number* 10, *whose Index is the last found Logarithm, there will come out a Number* n *nearer to the Number* 6 *than before, and so on to Infinity.*

338. But *in the hyperbolic System the Logarithm of no given Ratio is assumed:* But they are all to be determined from their Ratios to the infinitely small Logarithm of that Ratio, of which all Ratios are supposed to be compounded (*d*).

339. *In this System therefore the Logarithm of no given Ratio can be accurately found.* Let the Ratio $\frac{Q}{1}$ be given, and between 1 and Q let there be taken ever so many mean Proportionals: And if the first of those mean Proportionals be s and the Number of the Terms of the Series be N, viz. if s be the Root of the Power of the Number Q whose Index is $N - 1$, and if $s - 1$ be put for the Logarithm of the Ratio $\frac{s}{1}$, then $\overline{s - 1} \times \overline{N - 1}$ will be the Logarithm of the Ratio $\frac{Q}{1}$ (*e*).

340. After the same Manner *the Logarithms of any given Ratios whatsoever may be accurately determined, but in different Systems: And the Indices of the Powers being indefinitely increased, all those Systems will verge to Identity, and at last coalesce into one, viz. the Hyperbolic* (*f*).

341.

(*d*) Cor. 11. (*e*) Cor. 11. (*f*) Cor. 7.

341. Therefore *if there be taken the Root of any Number whatsoever, (of a Power whose Index is sufficiently great; and the Excess by which the Root exceeds Unity* (*g*), *be multiplied by the Index of the Power* (*h*); *the hyperbolic Logarithm of the Ratio of that Number to Unity will come out very nearly* (*i*). *And the Logarithm found after this Manner of the decuple Ratio, is* 2.302585, *&c.*

This Hyperbolic System was the first, which Lord Napier the most sagacious Inventor of Logarithms hit upon. He afterwards found out others more commodious for practical Uses: but, whilst he was intent upon bringing them to Perfection, he changed this Life for a better. Mr. *Henry Briggs* the first of the Savilian Professors of Geometry at Oxford *perfected and published his Logarithms less than the hyperbolic by more than Half.*

For *he put Unity for the Logarithm of the decuple Ratio, to the End that the Logarithms corresponding to the Numbers continued in the decuple Ratio,* viz. 1, 10, 100, 1000, &c. *might be* 0, 1, 2, 3, *&c. which are called Characteristics,* because they shew how many Places the Numbers corresponding go beyond the Place of Unity.

Therefore, the Logarithm corresponding to every Number, greater than Unity and less than Ten, will be less than Unity; viz. a Fraction; *and the Logarithm corresponding to every Number, placed between* 10 *and* 100, *will be between* 1 *and* 2; viz. Unity with a Fraction adjoined, *and so on continually.*

342. And because the Division of any Number whatsoever (whether integer, decimal, or mixed) by any Power of Ten, may be done by moving the separating Line so many Places toward the left Hand, as there are Units in the Index of the Power (*k*); and because the Logarithm of the

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(*g*) Cor. 12. (*h*) 337. (*i*) 340. (*k*) 133.

the Ratio of a Quote to Unity, is the Difference of the Logarithms of the Ratios of the Dividend and Divisor to Unity (*l*); therefore *the Logarithms, which correspond to Decimals, or mixed Numbers, are found from the given Logarithms of Integers by subducting so many Units from the Characteristic, as there are decimal Places cut off.* As may be seen in the following Series.

Numbers	*Logarithms*
83749	4. 922979
8374,9	3. 922979
837,49	2. 922979
83,749	1. 922979
8,3749	0. 922979
,83749	—1. 922979
,083749	—2. 922979

343. *The Logarithms of Decimals thus emerging will have their integral Parts, viz. their Characteristics negative; and the other Parts affirmative.* And if the last Figure to the left be a Decimal, or a Centesimal, &c. the Characteristic will be —1, or —2, &c. (*m*)

344. Whence *if the Number is sought which agrees to a given Logarithm, having rejected the Characteristic, let the given Logarithm, or the Logarithm next to the given one, be sought in the Table,* (and it will be most certainly found at the last Characteristic in the Table, for if it be at the less Characteristic, it will without Doubt be at the greater; but not conversely, because that in every subsequent Class are placed all the Numbers which are produced by multiplying the Numbers of the next preceding Class by 10, and moreover, the other intermediate Numbers,) *and the Number which is set over against it will be either accurately, or very nearly, that which is sought, whose last Figure to the left will proceed so many Places beyond that of Unity, as the given Characteristic indicates.* Thus if the Number is sought whose

(*l*) 329. (*m*) XXVIII.

whose Logarithm is 2.922979. To the Part of it 922979 there stands opposite in the Table the Number 83749, whose last Figure is in the Place of Hundreds, viz. two Places beyond the Place of Units, because of the Characteristic 2; and so the Number is 837,49. But had the Characteristic been — 2, the last Figure must have been in the Place of hundredth Parts; viz. two Places short of the Place of Unity, because of the negative Characteristic — 2; and the Number sought wou'd have been ,083749. Whence the following Rule is given, to wit. *If the Characteristic of the given Logarithm is — 1, the Number corresponding will be wholly decimal; if the Characteristic be — 2, it will be wholly decimal, and also one Cypher is to be prefixed; if it be — 3, two Cyphers are to be prefixed, and so on continually.*

345. *But if the given Logarithm which corresponds to the Fraction which is sought be wholely negative, let the given Logarithm changed into affirmative be sought in the Table, and a Fraction, whose Denominator is the Number corresponding to it, and whose Numerator is Unity, will be that which is sought.* For Example let the Logarithm — 2 + 922979 be resumed, which is equal to the totally negative Logarithm — 1.077021 (*n*). I seek at the last Characteristic in the Table the Part ,077021 of 1,077021, and I find the next to it to be 077040, to which corresponds the Number 11941, whose last Figure to the left will be in the Place of Tens, because of 1 the Characteristic of the Logarithm 1.077021; and therefore the Number corresponding to the Logarithm 1.077021 will be 11,941 very nearly, whose Reciprocal $\frac{1}{11,941}$ (= ,083749) will be nearly that which is sought; to wit, which corresponds (Cor. 2. P. 2.) to the negative Logarithm —1.077021 (*o*).

346. But

(*n*) For. 1 — 922979 = — 1,077021 (XXVI) & — ✓ subducted from — 2 leaves — 1. (*o*) Cor. 10.

346. But *if a Decimal equal to the sought Fraction is to be found, let the given Logarithm changed into affirmative be subducted from the next greater Characteristic, or (which is the same) let the fractional Part of it be subducted from Unity, and the Number, which in the Table corresponds to the Residue will be the Decimal which was to be found, having so many Cyphers prefixed to it, as there are Units in the Characteristic of the given Logarithm.* For Example, Let the negative Logarithm — 1.077021 be resumed, if this be added to the Characteristic 2 (*p*), viz. the Logarithm, which corresponds to the Power of the Number 10, or (which is the same) if with the Sign changed it be subducted from 2; there will result 0,922979 the Logarithm corresponding (Cor. 2. p. 2.) to the Product of the Decimal to be found into the Power of the Number 10: Therefore the significant Figures of the Decimal to be found will correspond in the Table to the Part 922979, but the Number corresponding is 83749, to which one Cypher is to be prefixed upon Account of the Characteristic — 1, of the Logarithm —1.077021. For the Logarithms of the Ratios $\frac{10}{1}$, $\frac{100}{1}$, $\frac{1000}{1}$, &c. being supposed to be 1, 2, 3, &c. the Logarithms of their Reciprocals $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c. will be — 1, — 2, — 3 &c. And as the Logarithm corresponding to every Number, which is greater than 10 and less than 100, is Unity with a Fraction adjoined; so the negative Logarithm corresponding to every Fraction, which is less than $\frac{1}{10}$ and greater than $\frac{1}{100}$, will be Unity with a Fraction adjoined; and the negative Logarithm corresponding to every Fraction, which is less than $\frac{1}{100}$ and greater

(*p*) XXIV.

greater than $\frac{1}{1000}$, will be 2 with a Fraction adjoined; and ſo on continually. But every Fraction, which is placed between $\frac{1}{10}$ and $\frac{1}{100}$ when changed into a Decimal, will have one Cypher prefixed to it's ſignificant Figures; and which is placed between $\frac{1}{100}$ and $\frac{1}{1000}$ will have two Cyphers prefixed; and ſo on continually: Therefore, ſo many Cyphers are to be prefixed to the Number found, as there are Units in the Characteriſtic of the given Logarithm. And ſhould the given Logarithm conſiſt entirely of a Characteriſtic, tis manifeſt, that the Decimal ſought will be Unity with ſo many Cyphers prefixed, as is indicated by the Characteriſtic diminiſhed by Unity.

From what has been ſaid it appears, that *the Method firſt mentioned is far the beſt, to wit, that which ſuppoſes the Logarithm correſponding to a Fraction to be partly affirmative, and partly negative:* becauſe it at once exhibits a decimal Fraction equivalent to the ſought Fraction. Wherefore that this Form may always be followed.

347. *If a greater Logarithm is to be ſubducted from a leſs, let the Operation be performed in the ſame Manner, as if the Subtrahend was greater than the Minuend* (q): *And if any Thing is to be transferr'd to the Characteriſtic of the greater Logarithm, the Difference between it's Characteriſtic thus increaſed and the Characteriſtic of the leſs Logarithm will be negative; and the reſt of the Reſidue affirmative.* Example, let the Logarithm 2.89265 be ſubducted from the Logarithm 1.32221

1. 32221
2. 89265

the Reſidue will be $-2.+42956$: for the Part 89265, ſubducted

(q) XXV.

subducted from the Part 32221 augmented by Unity, leaves the affirmative Residue 42956, and the Characteristic 2 augmented by Unity and subducted from the Characteristic 1, gives the negative Residue — 2.

348. *In Addition, Subduction, or Multiplication, if any thing is to be transferred to a negative Characteristic, that which is transferred (being affirmative) diminishes the negative Value of the Characteristic* (r). Thus, if the Logarithm — 2. 42956 is subducted from the Logarithm 1. 32221

— 2. 42956

the Remainder is 2. 89265

because Unity transferred to — 2 makes — 1, which subducted from 1, leaves the Characteristic 2. or, if the Logarithm — 1. 66847118 be multiplied by 5, the Product will be — 2. 3423559; for 3 transferred to the negative Product 5 × — 1 from the next preceding Product 5 × 6, makes the negative Characteristic — 2.

349. *In Division, if the Divisor cannot measure a negative Characteristic, let the Characteristic be increased until the Divisor can measure it, and the Number by which it measures it will be the negative Characteristic of the Quote; and the Augment together with the next Figure of the Dividend, adjoined to it on the right Hand, will constitute the next Dividual.* For thus the Logarithm to be divided will be diminished, and again augmented, which will make no Change. Example, let the sursolid Root of the Fraction $\frac{13}{591}$, be required,

the Logarithm of the Numerator is 1. 11394335
the Log. adjoin'd to the Denominator is 2. 77158748

the Difference of these Logarithms is — 2. 34235587 the Logarithm corresponding to the Fraction $\frac{13}{591}$: Let the Logarithm — 2. 34235587 be divided by 5, and the Quote will be — 1. 66847117: for the Characteristic — 2 increased

creased by the Negative — 3 and divided by 5 gives — 1 the Quote; and the other Part 3423 &c. augmented by the Affirmative 3 prefixed and divided by 5, gives the Quote 66847117; therefore the Logarithm corresponding to the Root sought is — 1. 66847117: I look for the Part 66847117 at the last Characteristic of the Table, and find the next less 66846978, to which the Number 46609 corresponds, which will be wholly decimal, because of the Characteristic — 1 of the given Logarithm; and therefore the Root required is 0,46609.

350. *If having rejected the Characteristic the given Logarithm cannot be accurately found in the Table, let the next greater, and the next less be taken, and their Difference found: This Difference will be to the Difference of their corresponding Numbers, viz. Unity, as the Excess of the given Logarithm above the next less Logarithm, to a fourth Number; which is usually called the* proportional Part: *And which being added to the less Number will very nearly give the Number sought, if the Table be sufficiently extensive,* viz. that the Difference of contiguous Logarithms may be small enough, or (which comes to the same) that the corresponding Numbers be so great that they come very nearly to an Equality: For great Numbers, whose Difference is not greater than Unity, approach to an Equality, and (Cor. 14. P. 3.) the Excesses by which two exceed a third, are as the Logarithms of their Ratios to the third; that is, as the Excesses, by which their corresponding Logarithms exceed the Logarithm corresponding to the third: Thus, if the sursolid Root of the Fraction $\frac{13}{59}$ be to be found more accurately, the Logarithm corresponding to the Root is — 1. 66847117, the one next less, rejecting their Characteristics, is 66846978, and next greater 66847910, their Difference 00000932 is to Unity, as 00000139 (to wit the Excess of 66847117 above 66846978) is to the decimal Fraction

149 (*s*), therefore 149 is the proportional Part, to be annexed to 46609; and therefore the significant Figures, which correspond to the Part 66847117 of the given Logarithm, are 46609149; whence, because of the Characteristic — 1, the Root required is 0,46609149.

351. *If the Logarithm corresponding to a given Number is sought, and the Number is not contained within the Limits of a pretty extensive Table; let the given Number be changed into a mixed, by cutting off the redundant Figures for Decimals; and let there be taken the Logarithm corresponding to the Figures remaining to the left, and also the Logarithm next greater: Then Unity will be to the Difference of these, as the Decimal to a fourth Number, which is called also the* proportional Part; *and which added to the less Logarithm, makes the required Logarithm very nearly.* Example, Let the Logarithm be required corresponding to the Number 46609149, the Logarithm, rejecting the Characteristic, corresponding to the Number contained in the Table 46609, is 6684697852, and that which is the next greater, is 6684791029, and their Difference is 0000093177; and Unity is to 0000093177, as 149 (*t*) to 0000013883, which

```
(s) for (139) 932) 1390 (, 149
                   932
                   ---
                   458
                   372
                   ---
                    86
                    81
                   ---
```

```
(t) (55) 0000093177
                941
         ----------
               9317
               3727
                839
         ----------
         0000013883
```

which added to 6684697852 makes 6684711735; which corresponds to the Figures of the given Number. Whence, because that given Number is an Integer consisting of eight Figures, the Logarithm required will be 7.6684711735 (*u*).

352. *The Logarithm thus found ought to be something less than the true one. But it will come out accurate enough for Practice; especially if the last Figure to the left of the corresponding Number be* 8 *or* 9 (*w*).

353. And it will be found more accurately by the following Method. *Let double the aforesaid Fraction be divided by the Sum of the Fraction and of the Integer squared, and the Quote multiplied into the Module of the vulgar System, to wit, into the decimal Fraction* 0,4342944819 *&c. and the fractional Product added to the less Logarithm; and the Logarithm required will come forth.*

Because all Logarithms, and consequently also the corresponding Numbers cannot be accurately found in the Table, and because an intermediate Number corresponding to a given Logarithm may be to be found, 'tis necessary to make a Correction by a proportional Part: Doctor *Wallis* observed, that as a Remedy to this Inconvenience; there was *wanting an antilogarithmic Canon, in which the Logarithms being wrote in Order, from* 1 *to* 100000, *the Numbers corresponding to them, should be wrote opposite; so that by the Assistance of this Canon the Number corresponding to a given Logarithm might be found with the same Ease, as we are used to find by the Assistance of the vulgar Tables the Logarithm corresponding to a given Number.* But in the vulgar System the Numbers corresponding to those Logarithms are 10, 100, 1000, &c. which it wou'd be both useless and impossible to annex: therefore *this most learned Person mean'd without Doubt, that to the Logarithms from* .00001 *to Unity, there*

(*u*) Number 341. (*w*) Number 344.

there should be annexed the corresponding Numbers, to wit, that there should be exhibited 99999 *mean Proportionals between Unity and* 10.

354. *He has told us moreover, that such a Canon was many Years since made*, and was either begun by *Harriot*, or was both begun and finished by *Walter Warner*, who had *Harriot*'s Papers, and published his Algebra. That he had these Accounts from Doctor *John Pell*, who was *Warner*'s Friend and Assistant in that Calculus; and that he had only a View of this Work among *Harriot*'s Papers; but that afterwards this Canon was in the Hands of *Richard Busbey* the Schoolmaster, who gave Hopes of publishing it under the Care of Doctor *Pell*, provided that *Wallis* wou'd, if *Pell* shou'd die, succeed him in the Care of it, *but that he feared least the Work shou'd be entirely lost, especially as no Person wou'd take upon himself the Expence of an Edition.* And *in Truth tho' such a Canon had been made publick, yet it wou'd have been neglected*; for I cannot see to what Use it wou'd serve. For the Logarithms which usually occur in arithmetical Operations are irrational, (to wit, resulting from the Addition, Subduction, Multiplication, and Division, of the Logarithms corresponding to the Numbers which are in the Table,) agreeing with those rational ones to five Places of Figures: Therefore *the Numbers in such a Canon, altho' exhibited to an infinite Number of Places of Figures, (for they wou'd be irrational) wou'd be always less than the required Numbers; and wou'd agree with them, often not to five Places of Figures, very seldom to six, but never, if the first redundant Figures of the given Logarithm shou'd make a Number, not less than the Number* 43429, *consisting of the first five Figures in the Module of the vulgar System.* Example, 'tis certain that the Numbers corresponding to the Logarithms 01246 and 012464343 will not agree to six Places of Figures, because 43430 is not less than 43429.

355. *But the Number corresponding to any given Logarithm may be found to five Places of Figures in the vulgar Tables. And also the Logarithm itself may be found most commonly*

commonly to five Places of Figures; and to four always. For the Difference of any two Logarithms whatsoever is the Logarithm of the Ratio of their corresponding Numbers, and the Difference of two contiguous Numbers in the second Class (Cor. 3. 5. P. 3.) viz. of those which proceed from 10 to 100, divided by the less Number, is greater than the hyperbolic Logarithm of the Ratio of those Numbers, and therefore much greater than the vulgar Logarithm of the same Ratio. Example, $\frac{1}{13}$ is greater than the hyperbolic Logarithm, and therefore greater also, than the vulgar Logarithm, of the Ratio $\frac{14}{13}$. But every Fraction, whose Numerator is Unity, and whose Denominator is placed between 10 and 100, is less than $\frac{1}{10}$; therefore the Difference of any contiguous Logarithms whatsoever, which are annexed to the *Characteristic* 1, is less than $\frac{1}{10}$. But in ascending these Logarithms come to the Logarithm 2; (for 1, and 2, are the Logarithms corresponding to 10 and 100) and therefore 'tis plain, that *in the Initials of these Logarithms*, (neglecting their Characteristics) *there will be single Figures of every Kind*, 0, 1, 2, &c. Also the Difference of two contiguous Numbers in the third Class divided by the less, is greater than the hyperbolic Logarithm of the Ratio between the Numbers, and therefore greater also than the vulgar Logarithm of the same Ratio. Example, $\frac{1}{324}$ is greater than the hyperbolic Logarithm, and therefore greater than the vulgar Logarithm, of the Ratio $\frac{325}{324}$. But every Fraction, whose Numerator is Unity, and whose Denominator is placed between 100 and 1000, is less than $\frac{1}{100}$: therefore the Difference of any contiguous Logarithms whatsoever, which are annexed to the *Characteristic*

Characteristic 2, is less than $\frac{1}{100}$. But in increasing those Logarithms attain to the Logarithm 3; and therefore it appears, *that in the Initials of these Logarithms, there will be Binaries of Figures of all Kinds*, 00, 01, 02, &c. 10, 11, 12, &c. After the same Manner; *in the Initials of those which are annexed to the Characteristic* 3, *there will be Ternaries of Figures of all Kinds, and so on continually. And therefore every Logarithm can be found to one Place of Figures, at the Characteristic* 1; *to two Places, at the Characteristic* 2; *to three Places, at the Characteristic* 3: *and so on to Infinity.*

356. *Moreover, in the Table also may be found to five Places of Figures every Logarithm, which is not less than the Logarithm which corresponds to the Number* 43429; *to wit, the first five Figures in the Module of the vulgar System.* For 100000 is to 43429, as $\frac{1}{43429}$ to $\frac{1}{100000}$; and the Fraction $\frac{1}{43429}$ exceeds by very little the hyperbolic Logarithm of the Ratio $\frac{43430}{43429}$; (for it must necessarily differ from it, by a Quantity, which is more than twice less than a Fraction, whose Numerator is Unity, to wit, the Square of the Difference of the Terms 43430 and 43429, and whose Denominator is the Square of the Term 43429, yet, if it be supposed equal to it, the Error thence arising will be more than compensated by a contrary Error, which will arise by supposing, for the Ratio of the hyperbolic Logarithm to the vulgar, a Ratio, which is something greater; that of 100000 to 43429: So that the vulgar Logarithm of the Ratio $\frac{43430}{43429}$, viz. the Difference of the Logarithms corresponding to the Numbers 43430, and 43429, will be something less than the Fraction $\frac{1}{100000}$: And the Differences of the subsequent contiguous Logarithms are perpetually diminished.

But

But those Logarithms in ascending attain to *the Logarithm* 5; and therefore *in their Initials will be Quinaries of Figures of every Kind, which are not less than the first Quinary of Figures, in the Logarithm which corresponds to* 43429. *But the Limit also will be less than that Logarithm;* for it will sooner come to pass, that the Difference between any contiguous Quinaries whatsoever shou'd not be greater than Unity, than that the Differences of the Logarithms in decreasing shou'd become not greater than $\frac{1}{100000}$.

Mr. *Henry Briggs* published Logarithms computed to fourteen Places of Figures for all Numbers from 1 to 20000, and from 90000 to 100000. These diminished of their four Figures to the right Hand, together with the Logarithms from 20000 to 90000 computed to ten Places, *Adrian Vlacque* had printed, and published the Table compleated. He gave his Reason for omitting the four Figures, that beside the remaining being sufficient for common Use, the Table might not require more Space, and thereby become more expensive, which wou'd be an Injury to him, because the Number of Buyers wou'd be diminished. But he wou'd have done better, if he had published the Logarithms for all Numbers from 10000 to 100000 computed to fourteen Places, and omitted all the rest; because they are superfluous, and increase the Bulk of the Table.

But *Briggs endeavour'd to explain the two Methods of computing Logarithms, which were found out by Napier. The former is after this Manner.*

357. *Let the Number to which the corresponding Logarithm is required, be supposed to be involved to the Power, whose Index is* 100 000 000 000 000; *and the Number of Figures in this Power diminished by Unity is the Logarithm required.* But because it wou'd be impossible to arrive at so great a Power, by a continued Multiplication by the Root; to avoid that insuperable Labour, many interme-

diate Numbers being omitted, *such Powers are to be multiplied into each other, whose Indices added together will make the Index of the Power required. But it is neither* possible, *nor necessary, to exhibit these Powers perfectly, but so many Figures only to the left Hand in each, as are sufficient to shew the Numbers of Places in the Powers following; to the End that the Number of Places in the last Power may be computed.* For there will be as many Places in any Product, as there are in both Factors taken together, unless it shou'd happen, that the Product of the last Figures to the left, increased by the Increment of the Figures to the right, can be expressed by a single Figure; in which Case, the Number of Places in the Product is equal to the Number of Places in the Factors less by Unity (*a*). That all those Things might more easily appear, He *distinguished the Powers with their Indices and Numbers of Places into Tetrads,* in the Manner following

Powers	Indices	Numbers of Places
4	2	1
16	4	2
256	8	3
1024	10	4
1048576	20	7
1099511627776	40	13
120892581961463	80	25
126765060022823	100	31
160693804425899	200	61
258224987808685	400	121
666801443287940	800	241
107150860718618	1000	302

2 multiplied into itself makes 4, whose Index is 2; 4 multiplied into itself makes 16, whose Index is 4; 16 into itself makes 256, whose Index is 8; and 256 multiplied into 4 makes 1024, whose Index 10 is equal to the Indices

(*a*) Number 54.

Indices of the Factors. The Number of Places in this last Product is 4. Now these four Numbers 4, 16, 256, 1024, constitute the first Tetrad. Another Tetrad is then to be made, whose first Number is produced, by multiplying the last Number of the preceding Tetrad into itself; the second is the Square of the first; the third the Square of the second; but the fourth is the Product of the third multiplied into the first. And the Indices of these four are 20, 40, 80, 100; and the Number of Places 7, 13, 25, 31. After the same Manner the remaining Tetrads are finished. So that the Index of the last Member of every Tetrad may be 1000, or 10000, &c. untill *the fourth Member of the last Tetrad may have it's Index* 100 000 000 000 000; *but the Number of Places in this fourth Number will be* 30102999566399; *and therefore the Logarithm agreeing to the Number* 2, *will be* 30102999566398.

In Order to explain this Method, he premised the three following Lemmas.

First, *In a Series of Numbers continually proportional from Unity, any two Terms involved according to their alternate Indices give equal Powers.* Which is true, because either Power is equal to the second Term of the Series raised to a Power whose Index is the Product of the Indices (*b*).

Second, *If any Term be continually divided by it's Side, viz. the Second Term of the Series, as often as it can be, viz. untill the Quote is Unity, the Number of Divisions will be the Index of the Term.* Which is also true, being the Definition of the *Index of any Term whatsoever in that Series.*

Third, *If any Term whatsoever* A *be raised to a Power, whose Index is the same with the Index of any other Term whatsoever* B, *and the Power be divided as long as it can be by*

(*b*) Number 83.

*by*B, *the Number of Divisions will be the Index of the Term* A *in the same Series* (*c*). And this is also true, for it is contained in the two preceding: By the Affiftance of these Lemmas he explained the Method, as follows.

If in a Series whose Beginning is Unity, the Index appointed to the Number 10 be 100 000 000 000 000, viz. if between 1 and 10 there are 99 999 999 999 999 mean Proportionals; all Numbers will very nearly be placed in this Series. And *any two Terms involved according to their alternate Indices give equal Powers*; *whence if any Number whatsoever, for Example,* 2, *be raised to a Power whose Index is* 100 000 000 000 000, *there will come out that Power of the Number* 10 *whose Index is the same with the Index of the Number* 2 *in that Series*; *whence if that Power be continually divided as often as it can be by* 10, *the Number of Divisions will be the Index of the Number* 2; *but the Number of Divisions by* 10 *is the Number of Places in the Dividend less by Unity* (*d*); *therefore the Number of Places diminished by Unity is the Index of the Number* 2 *in that Series*; *or, is the Logarithm corresponding to* 2; *if the Indices be put for the Logarithms.* 'Tis to be observed however, *that the Number* 2 *is not placed in that Series, but some Number* n, *which, tho' it is but a little exceeded by the Number* 2, *nevertheless* n *and* 2, *involved according to so great an Index, will give Powers not equal, but very different: But* n *and* 10 *involved according to their alternate Indices, give equal Powers*; *therefore those Powers of the Numbers* 2 *and* 10 *will be unequal. And altho' the Index appointed to the Number* 10 *shou'd be increased indefinitely, in order that* n *may always approach nearer to* 2, *and that there shou'd be always taken Powers of* n *and* 2 *involved according to the increased Indices, these Powers will not upon Account of the Indices thus increased verge to an Equality.*

But if n *and* 2 *verge to an Equality, their Powers also raised according to any invariable and finite Index whatsoever will*

(*c*) Number 85. (*d*) Number 133.

will verge to an Equality. Wherefore if in the Place of the Index of the Number 10 *there is always substituted some invariable Number* r, *and in the Place of the Index of the Number* n *there is always put a Number, which has the same Ratio to* r, *which the Index of the Number* n *has to the Index of the Number* 10; n *and* 10 *involved alternately according to those Numbers proportional to their Indices will give equal Powers; and the similar Powers of* 2 *and* 10 *will verge to an Equality; that is,* 2 *and* 10 *involved according to their alternate Logarithms give equal Products.* Example, If the Index of the Number 10 be 1000, viz. if between 1 and 10 there be 999 mean Proportionals, the Index of the Number *n* will be 301; now for the Index of the Number 10, let 10 be substituted, and in the Place of the Index of the Number *n* there will be 3,01; for 1000 is to 301, as 10 to 3,01. Let the tenth Power of the Number 2 be taken, viz. 1024, and if that Power of the Number 10 be taken, whose Index is 3,01, viz. if the Cube of 10 be multiplied into its Root of the hundredth Power; a Number will come out something less than 1024. And if the Index of the Number 10 be 10000, the Index of the Number *n* will be 30102; let 10 be put for the Index of the Number 10, and in the Place of the Index of the Number *n*, there will come out 3,0102: And if there be taken a Power of the Number 10 whose Index is 3,0102, there will come out a Number nearer to the Number 1024 than before; and so on to Infinity. Therefore if 2 be raised to any Power whatsoever whose Index may be the Logarithm corresponding to the Number 10, there will come out a Power of 10, whose Index is the Logarithm corresponding to the Number 2. *And if any Number* A (*greater than Unity*) *shou'd be conceived as the Power of any other Number* (*greater than Unity*) B, *and* A *shou'd be divided continually by* B *untill the Quote becomes less than the Divisor, the Number of Divisions will be the integral Part of the Index. Whence if any Integer* p *be put for the Logarithm of the Ratio* $\frac{10}{1}$, *and the Power of the Number* 2 *whose Index is* p, *be continually divided*

vided by 10 *until the Quote is less than the Divisor, the Number of Divisions will be the integral Part of the Logarithm of the Ratio* $\frac{2}{1}$; *but the Number of Divisions of any Integer whatsoever by* 10 *is the Number of Places in the Dividend less one; therefore the Number of Places in the aforesaid Power less one will be the integral Part of the Logarithm corresponding to the Number* 2. Example, if 1000 be put for the Logarithm of the Ratio $\frac{10}{1}$, and 2 be raised to the thousandth Power, the Number of Places in that Power less by Unity, viz. 301 will be the integral Part of the Logarithm corresponding to 2: And if for the Logarithm of the Ratio $\frac{10}{1}$ be put 100 000 000 000 000, and 2 be raised to a Power whose Index is that Logarithm, the Number of Places less one viz. 30102999566398 will be the integral Part of the Logarithm corresponding to the Number 2.

358. *But the same might have been more briefly proved by the Assistance of the following Lemma, to wit. If there be a Series beginning from Unity of Numbers increasing in continued Proportion, and any Number whatsoever* A *greater than Unity, which is not in the Series, shou'd be raised to a Power whose Index is the same with the Index of any Term of that Series whatsoever* B, *and that Power of* A *shou'd be divided by* B *untill the Quote shou'd be less than the Divisor; the Number of Divisions will be the Index of that Term of the Series which is next less than* A. Example, Let the Series be $\overset{0}{1}, \overset{1}{3}, \overset{2}{9}, \overset{3}{27}, \overset{4}{81}, \overset{5}{243}$; and let 25, which is not in the Series, be raised to a Power whose Index is the same with the Index of the Term 243, viz. to the 5th Power, this Power will be 9765625; let 9765625 be divided by 243 until the Quote be less than the Divisor, and the Number of Divisions will be 2, to wit, the Index of the Term 9 in the Series, which is next less than the Number 25. *The same also appears from this, that if any integer Number whatsover be raised to any integer Power whatsoever,*

whatsoever, the Number of Figures in the Power diminished by Unity will be the integral Part, to wit, the Characteristic of the vulgar Logarithm corresponding to the Power; but the same Logarithm will also correspond to the Root, if the Index of the Power be substituted for the Logarithm of the decuple Ratio; that is, if the System be changed by multiplying all the vulgar Logarithms by that Index; which, if it be 10, 100, *or* 1000, *&c. will not change the Figures of the Logarithms by Multiplication. If it be* 10, *the integral Part of the Product will consist of the Characteristic of the vulgar Logarithm corresponding to the Root, and of one Figure of Decimals: If it be* 100, *the integral Part of the Product will consist of the Characteristic, and two Figures of Decimals: If it be* 1000, *the integral Part of the Product will consist of the Characteristic, and three Places of Decimals, and so on continually. And therefore if any integer Number whatsoever be raised to the tenth Power, the Number of Places in the Power diminished by Unity will exhibit the Figures of the vulgar Logarithm corresponding to the Number, to one Place of Decimals: And if it be raised to the hundredth Power, the Number of Places less one will exhibit the Figures of the Logarithm to two Places of Decimals: If to the thousandth, the Number of Places less by Unity will give the Figures of the Logarithm to three Places of Decimals; and so on to Infinity.*

359. How troublesome it must be to exhibit the Logarithm corresponding to a given Number to fourteen Places of Figures, by the Method above treated of, every Person must see: And *as this Method contains the Involution, so again the other, which is scarce easier, requires the Evolution of a Power sufficiently high; This Mr. Briggs performed by the continued Extraction of the Square Root.*

For he extracted the square Root of the Number 10, and the square Root of this Root, viz. the biquadratic Root of the Number 10, and the Root of this Root, viz. the Root of the Number 10 of the eighth Power; and so on continually; until *after having finished fifty-four Extractions, he attained the Root of the Number* 10 *of the Power, whose Index is* 18014398509481984, *express-*

Methods of Fluxions, or by certain geometrical Lines. Wherefore we have not so nicely examined their Principles, and the Validity of the Conclusions. But *Halley*, who is the only Person, as far as I know, that has attempted to deduce arithmetically these Constructions from the Nature of Logarithms, published (in Philosoph. Transf. N. 216) a Method of Computation eminent above all others, and mostly celebrated. Wherefore it seemed proper faithfully to transcribe what he has said concerning the Nature of Logarithms.

The old one (viz. Definition of Logarithms) Numerorum proportionalium æquidifferentes Comites, *seems scanty to define them fully. They may more properly be said to be* Numeri Rationum Exponentes: *Wherein we consider* Ratio *as a* Quantitas sui Generis *beginning from the* Ratio *of Equality, or* 1 *to* 1 = 0; *being affirmative when the Ratio is increasing, as of Unity to a greater Number, but negative, when decreasing; and these* Rationes *we suppose to be measured by the Number of* Ratiunculæ *contained in each. Now these* Ratiunculæ *are so to be understood as in a continued Scale of Proportionals infinite in Number between the two Terms of the* Ratio, *which infinite Number of mean Proportionals is to that infinite Number of the like and equal* Ratiunculæ *between any other two Terms, as the Logarithm of the one* Ratio *is to the Logarithm of the other. Thus, if there be supposed between* 1 *and* 10 *an infinite Scale of mean Proportionals, whose Number is* 100000, *&c.* in infinitum; *between* 1 *and* 2 *there shall be* 30102, *&c. of such Proportionals, and between* 1 *and* 3 *there will be* 47712, *&c. of them; which Numbers therefore are the Logarithms of the* Rationes *of* 1 *to* 10, 1 *to* 2, *and* 1 *to* 3; *and not so properly to be called the Logarithms of* 10, 2, *and* 3.

But if instead of supposing the Logarithms composed of a Number of equal Ratiunculæ *proportional to each* Ratio, *we shall take the* Ratio *of Unity to any Number to consist always of the same infinite Number of* Ratiunculæ, *their Magnitude, in this Case, will be as their Number in the former; wherefore if between Unity and any Number proposed, there be taken*

taken any Infinity of mean Proportionals, the infinitely little Augment or Decrement of the first of these Means from Unity, will be a Ratiuncula, *that is the* Momentum *or Fluxion of the Ratio of Unity to the same Number: And seeing that in these continual Proportionals all the* Ratiunculæ *are equal, their Sum or the whole Ratio will be as the said Momentum is directly; that is, the Logarithm of each Ratio will be as the Fluxion thereof. Wherefore if the Root of any infinite Power be extracted out of any Number, the* Differentiola *of the said Root from Unity, shall be as the Logarithm of that Number. So that the Logarithms thus produced may be of as many Forms as you please to assume infinite* Indices *of the Power whose Root you seek: as if the Index be supposed* 100000, *&c. infinitely, the Roots shall be the Logarithms invented by the Lord* Napier; *but if the said Index were* 2302585, *&c. Mr.* Briggs's *Logarithms would immediately be produced, &c.*

Thus wrote *Halley*, in which I am not only unable to perceive the Force of the Arguments, but I cannot even from the Language extract any consistent Meaning. And that I may begin with the Definition: The Expression *Rationum* seems here to signify the same as *Logarithmorum*, and I think I may collect this from his saying that *Ratio is affirmative and negative*, and that *Rationes are measured by the Number of* Ratiunculæ *contained in each*; where by *Ratiunculæ* he seems to mean the Logarithm of the intermediate Ratio in a geometrical Series; although the Words *which infinite Number of mean Proportionals is to that infinite Number of the like and equal* Ratiunculæ *between any other two Terms, as the Logarithm of the one* Ratio *is to the Logarithm of the other* seem to mean the same, as if he had called the *Ratiunculæ* the mean Proportionals themselves, (at least if the Word *equal* wou'd permit it) and had said, the *Ratiunculæ* were in the Scale of Proportionals; but this he does not say, only that *they are to be understood as in a continued Scale*; for he calls *the infinitely little Augment or Decrement a Ratiuncula*, and *the Fluxion of the Ratio*; and *the Sum of the Ratiunculæ, the whole*

Ratio, and *the Logarithm*. But if *Ratio* and *Logarithm* signify the same, to what End is it to define Logarithms to be *Numeri rationum Exponentes*? This is to say, that *Logarithms are the numeral Exponents of Logarithms*, or to admonish us (whatever be the Force of the Words) that *therein we consider Ratio as a Quantitas sui Generis*? For as the Sense of those who write by Cyphers is used to be extorted, by seeking, in various Trials upon the most convenient Parts of the Writing, what Words are necessarily implied by the Characters; and by transferring those, when they are found, to the same Characters in other Parts, that the Words signified by the neighbouring Characters may also be more easily detected: so we, who now explore Things which are hidden, are obliged to distinguish, in the Places which seem to give some Light, what Ideas may be implyed by the Words, that those being found out, and transferred to the same Words in Places more obscure, we may trace, if it may be done, the Sense in them also, and so at length the Sense of the continued Discourse. This however we can by no Means accomplish, for perhaps we are ourselves rather slow, and there is some Obscurity in the Expressions. Nor is this extraordinary: For nothing is more usual, than most successfully to deduce Calculations from Principles, as if they were understood, and proved, (than which often times nothing is more easily done) when nevertheless the Principles themselves are opened to us in such a Manner, that the Mind is blunted in considering and examining them.

But the following Words *beginning from the Ratio of Equality, or* 1 *to* 1 = 0; *being affirmative when the Ratio is increasing, as of Unity to a greater Number, but negative when decreasing* seem to signify, that there is something in the Nature of Logarithms, which shou'd cause that Logarithms of Ratios of lesser Inequality shou'd be affirmative, and of greater Inequality, negative; which is nevertheless intirely arbitrary.

But

But the Words *which Numbers therefore are the Logarithms of the* Rationes *of* 1 *to* 10, 1 *to* 2, *and* 1 *to* 3; *and not so properly to be called the Logarithms of* 10, 2, *and* 3; What can they signify, if not, that those Numbers are not properly called the Logarithms of the Numbers 10, 2, and 3, but *the Ratios of the Ratios*, or, *the Logarithms of the Logarithms of* 1 *to* 10, *of* 1 *to* 2, *of* 1 *to* 3?

Or is the Import of the Word *Ratio* vague in this Treatise? now tho' we shou'd agree to this, and let that Word denote one Thing now, and another then, as the Sense shall require, yet certainly in the Definition, it will denote the same as *Logarithm*. For *Logarithms are* Numeri Rationum Exponentes: and those same Ratios *are supposed to be measured by the Number of* Ratiunculæ *contained in each*: therefore they contain *Ratiunculæ*, and are said to be *their Sums* or *whole Ratios* which are *as their Fluxions*, that is, *the Logarithm of any Ratio whatsoever is as its Fluxion*. Thus we are reconducted by one unbroken Thread to this, that we must conclude the Word *Ratio* in the Definition to signify the same as *Logarithms*.

But if any one shou'd be of Opinion, that by *Ratios*, *the Measures of Ratios* are there denoted, so as that the Definition shou'd import the same, as if he had said that *Logarithms are the numeral Exponents of the Measures of Ratios*, that wou'd be, not to define the Thing, but after a Manner to declare the Signification of a Name. And those Words, *wherein we consider* Ratio *as a* Quantitas sui Generis, *beginning from the Ratio of Equality*, &c. wou'd be equivalent to the following, *In which the Measure of a Ratio* (that is, by exhibiting the Measures) *the Logarithm is considered as a Quantity of its own Kind, beginning from the Measure of the Ratio or the Logarithm, of Equality, or,* 1 *to* 1 = 0, *being affirmative when the Ratio is increasing, but negative when decreasing*, &c. whereas it is manifest, that whatever that is, which in these Words is called *Ratio*, is the same with that *Ratio*, whatever it may be, which is called *Ratio* in the Definition.

But what that Diversity of infinite Numbers can be, in what it may consist, or whence it can arise, when he says, that *if there be supposed between* 1 *and* 10 *an infinite Scale of mean Proportionals whose Number is* 100000 *&c. in infinitum*; *between* 1 *and* 2 *there shall be* 30102 *&c. of such Proportionals, and between* 1 *and* 3 *there will be* 47712 *&c.* I am intirely at a Loss to know. I know indeed that 1 and 10 can be placed in the same Series with any Number, which shall differ from the Number 2 by any Difference ever so small; if that Number be always called indefinitely *n*, and there be between 1 and 10 taken 999 mean Proportionals, there will be 300 mean Proportionals between 1 and *n*; and if between 1 and 10 there are taken 9999 mean Proportionals, there will be 3009 mean Proportionals between 1 and *n*; that if between 1 and 10 there are taken 99999 mean Proportionals, there will be 30101 mean Proportionals between 1 and *n*, and so on to Infinity.

But these Words, *if instead of supposing the Logarithms composed of a Number of equal* Ratiunculæ, *proportional to each* Ratio, *we shall take the* Ratio *of Unity to any Number to consist always of the same infinite Number of* Ratiunculæ, *their Magnitude, in this Case, will be as their Number in the former*; *wherefore if between Unity and any Number proposed, there be taken an Infinity of mean Proportionals, the infinitely little Augment or Decrement of the first of those Means from Unity, will be a* Ratiuncula, *that is, the* Momentum *or Fluxion of the Ratio of Unity to the same Number*: *and seeing that in these continued Proportionals all the* Ratiunculæ *are equal, their Sum, or the whole Ratio will be as the said Momentum is directly*; *that is, the Logarithm of each Ratio will be as the Fluxion thereof*; *wherefore if the Root of any infinite Power be extracted out of any Number the* Differentiola *of the said Root from Unity, shall be as the Logarithm of that Number.*

Let him who pleases attempt to explain, and deduce from the Premises.

But in the following, *so that Logarithms thus produced may be of as many Forms as you please to assume infinite Indices of the Power whose Root you seek*, the Expression, *thus produced*, is something obscure, for the Particle *thus* seems to signify a Relation to some Method heretofore mentioned; but it has not yet been told, by what Method Logarithms might be produced.

But if Logarithms may be understood to be produced, by assuming for the Logarithms of the Ratios which the Roots have to Unity, *the Differences, by which they exceed Unity*: It appears (Cor. 8. P. 3.) that there is no Diversity between those Indices to diversify the Species of Logarithms. If the Indices be determinate, there will arise Logarithms in different Systems; but if the least of the Indices be sufficiently great, the Ratio between the Logarithms of any given Ratio whatsoever, in any two Systems whatsoever, will come nearer to the Ratio of Equality than any predefinite Ratio of Inequality shall come. Whence as the Indices, to wit, *Numbers*, differ from each other by *Excesses* only, a Thought most difficult to me occurs; Of what Kind may that *Diversity* be which is *between infinite Indices?* if infinite may be defined that than which there is no greater.

And if Logarithms may be understood to be produced, by assuming for the Logarithms corresponding to Powers *the Differences by which the Roots exceed Unity multiplied into any given numeral Quantity*, that is, by putting for the Logarithms which correspond to the Roots, not the Excesses themselves, but those Excesses variously multiplied according to the various Systems: there is no Doubt, but that the Logarithms thus produced, may be of so many Forms as you please to assume Indices indefinitely great of the Power whose Root is sought, to wit, If the Indices be assumed sufficiently great, the Systems will be various according to the various Indices: But all the Logarithms which will come out by using any one Index in one System, shall remain without a Deviation, which is not less than any predefinite one. Thus, if the

Index be 2 302 585092 994045, and the Excesses by which the Roots exceed Unity be always multiplied into 1, 000 000 000 000 000, there will come out the Logarithms, very nearly, which correspond to the Powers in the vulgar System; for to multiply those Excesses into 1, 000 000 000 000 000, is to multiply by 2 302 585092 994045 the Products of those Excesses into the Quote of 1000 000 000 000 000 divided by 2 302 585092 994045; and this Quote is nearly the Module of the vulgar System: For 1 is the vulgar Logarithm of the decuple Ratio, and 2 302 585092 994045 is, very nearly, the hyperbolic Logarithm of the same Ratio. But these Excesses by which the Roots exceed Unity multiplied into the Module of the vulgar System give the vulgar Logarithms corresponding to the Roots; and therefore these Logarithms multiplied into the Index of the Power give the vulgar Logarithms corresponding to the Powers. But it is evident in the same Manner, that the Index remaining unchanged, the Logarithms will come out in any other preassigned System whatsoever; if instead of 1, 000 000 000 000 000, any other proper Multiplier be made use of.

But the following Words *if the Index be supposed 100000, &c. infinitely, the Roots shall be the Logarithms invented by the Lord* Napier; *but if the said Index were 2302585, &c. Mr.* Briggs's *Logarithms wou'd immediately be produced*, are altogether erroneous.

Halley proceeds, and when (by *Newton*'s Theorem for finding the Coefficients of Powers) he had extracted the Root of an infinite Power, he deduces from the resulting Series logarithmic ones, by *rejecting* Numbers divided by *a Number infinitely infinite*; that is, by *a Number infinitely greater, than a Number infinitely great. Retaining* however those, divided by an *infinite Number*.

But we shall endeavour to derive the same Series from another Fountain.

Prop.

Prop. 4.

$$
\begin{array}{ccccccccccccccc}
 & & & & & n, & & p, & & q, & & & & & \\
 & & & & Z, & & I, & & D, & & Q, & & & & \\
 & & & m, & & i, & & k, & & l, & & s, & & & \\
 & & r, & & s, & & t, & & u, & & w, & & x, & & \\
 & a, & & b, & & c, & & d, & & e, & & f, & & g, & \\
A, & & B, & & C, & & D, & & E, & & F, & & G, & & H,
\end{array}
$$

To find the Logarithm of any given Ratio whatsoever in a given System.

Let A, B, C, &c. be continual Proportionals; a, b, c, &c. the first Differences; r, s, t, &c. the second; m, i, k, &c. the third Differences; and so on continually. If a be affirmative, then A, a, r, &c. will be continual Proportionals; if a be negative, viz. if B be less than A, then a, m, n, &c. viz. intermitting one will be negative, and being changed into affirmative, A, a, r, &c. will be continual Proportionals:

Therefore in both Cases if $(A = 1)$ A be Unity, we shall have $r = a^2$, $m = a^3$, $Z = a^4$; and so on continually; and consequently if the Number of Terms in a Series, whose Beginning and End are A and H, be called N, then (Prop. 3.) $1 + \frac{N-1}{1} \times a + \frac{N-1}{1} \times \frac{N-2}{2} \times a^2 + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times a^3$, &c. $=$ H. Let N be increased to Infinity; and the Terms of the Series $N-1$, $N-2$, $N-3$, &c. being continued to any finite Number whatsoever, will verge to an Equality among one another; and the Series $1 + \frac{N-1}{1} \times a + \frac{N-1}{1} \times \frac{N-2}{2} \times a^2 + \frac{N-1}{1} \times \frac{N-2}{2} \times \frac{N-3}{3} \times a^3$, &c. will be infinite: And $\overline{N-1} \times a$ will be (Cor. 11. P. 3.) the hyperbolic Logarithm of the Ratio of H to 1; and

and therefore if that Logarithm be called L, then $1 + \frac{L}{1} + \frac{L^2}{1 \times 2} + \frac{L^3}{1 \times 2 \times 3} + \frac{L^4}{1 \times 2 \times 3 \times 4}$, &c. $= H$: Which Series is a general Expression of the Number H, whose Ratio to 1 is that, of which L is the hyperbolic Logarithm, whether that Number be greater, or less, than 1. If it is less, viz. if L be negative, for L let it's negative Value be particularly exhibited, and it will be $1 - \frac{L}{1} + \frac{L^2}{1 \times 2} - \frac{L^3}{1 \times 2 \times 3} + \frac{L^4}{1 \times 2 \times 3 \times 4}$, &c. $= H$.

And if L be put for the Logarithm of the Ratio $\frac{H}{1}$ in any System whatsoever, whose Modnle is M, it will be (Def. of Module) $\overline{M - 1} \times a = \frac{L}{M}$; and thence $1 + \frac{L}{1 \times M} + \frac{L^2}{1 \times 2 \times M^2} + \frac{L^3}{1 \times 2 \times 3 \times M^3} + \frac{L^4}{1 \times 2 \times 3 \times 4 \times M^4}$, &c. $= H$. And if $L = M$, viz. if the Logarithm of the Ratio $\frac{H}{1}$ be the Module of the System, that is, if that Ratio be the Modular, it will be $1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120}$, &c. $= H = 2{,}718281$, &c. therefore the modular Ratio is that of 2,7182812, &c. to Unity.

But to return to what was proposed, Let $H - 1 = b$, and if the Series $L + \frac{L^2}{2} + \frac{L^3}{6} + \frac{L^4}{24}$, &c. $= b$ be reversed, according to the Method of the most celebrated De Moivre (*a*), there will be found a Series exhibiting the

(*a*) Number 321.

the Quantity L, viz. the hyperbolic Logarithm of the Ratio 1 + b to 1. Therefore let there be put

$$L = Rb + Sb^2 + Tb^3 + Vb^4 + Wb^5 \&c.$$

$$\text{therefore } \frac{L^2}{2} = + \frac{R^2}{2}b^2 + RSb^3 + \begin{matrix}\frac{1}{2}S^2\\+RT\end{matrix} b^4 + \begin{matrix}RW\\+ST\end{matrix} b^5 \&c.$$

$$\text{And } \frac{L^3}{6} = \frac{R^3}{6}b^3 + \frac{R^2S}{2}b^4 + \begin{matrix}\frac{R^2T}{2}\\\frac{RS^2}{2}\end{matrix} b^5 \&c.$$

$$\text{And } \frac{L^4}{24} = + \frac{R^4}{24}b^4 + \frac{R^3S}{6}b^5 \&c.$$

therefore $Rb + \overline{S + \frac{R^2}{2}} \times b^2 + \overline{T + RS + \frac{R^3}{6}} \times b^3 + \overline{V + \frac{S^2}{2} + RT + \frac{R^2S}{2} + \frac{R^4}{24}} \times b^4$ &c. $= b = b + 0 \times b^2 + 0 \times b^3 + 0 \times b^4$ &c. And thence, by equating the corresponding Terms, $R = 1$; $S + \frac{R^2}{2} = S + \frac{1}{2} = 0$, therefore $S = -\frac{1}{2}$; also $T + RS + \frac{R^3}{6} = T - \frac{1}{2} + \frac{1}{6} = 0$, whence $T = \frac{1}{3}$ also $V + \frac{S^2}{2} + RT + \frac{R^2}{2}S + \frac{R^4}{24} = V + \frac{1}{8} + \frac{1}{3} - \frac{1}{4} + \frac{1}{24} = 0$, whence $U = -\frac{1}{4}$; therefore the Law of the Continuation is known; and consequently $L = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5} - \frac{b^6}{6}$ &c. which is the Series which *Nicholas Mercator* formerly found out for the Quadrature of the Hyperbola; and which indiscriminately expresses the Logarithm of the Ratio 1 + b to 1, whe-

ther

ther h be affirmative or negative, *viz.* whether the Quantity added to Unity in the Binome $1 + h$ be affirmative or negative, that is, whether H be greater or less than 1. If it be less, let for h it's negative Value be specially substituted, and $L = -h - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4}$ &c; and therefore in a System whose Module is M, the Logarithm of the Ratio, which $1 + h$ greater than Unity has to 1, will be M into $-h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4} + \frac{h^5}{5} - \frac{h^6}{6}$ &c. and the Logarithm of the Ratio, which $1 - h$ less than Unity has to 1, will be M into $-h - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4}$ &c.

Cor. 1. *Let* x *be the Difference of two Quantities, of which* a *is the smaller and* b *the greater: Because* $b - x = a$, *the Ratio* $\frac{a}{b}$ *viz. the Ratio of* $b - x$ *to* b, *will be the same with the Ratio of* $1 - \frac{x}{b}$ *to* 1; *and the Ratio of* b (*to wit,* $a + x$) *to* a, *will be the same with the Ratio of* $1 + \frac{x}{a}$ *to* 1; *Therefore for* h *in the Series* M *into* $-h - \frac{h^2}{2} - \frac{h^3}{3} - \frac{h^4}{4}$ &c. *let* $\frac{x}{b}$ *be substituted, and* M *into* $-\frac{x}{b} - \frac{x^2}{2b^2} - \frac{x^3}{3b^3} - \frac{x^4}{4b^4}$ *&c. will be the Logarithm of the Ratio* $\frac{a}{b}$. *Also if in the Series* M *into* $h - \frac{h^2}{2} + \frac{h^3}{3} - \frac{h^4}{4}$ *&c.* $\frac{x}{a}$ *be substituted for* h, *then* M *into* $\frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ *&c. will be the Logarithm of the Ratio* $\frac{b}{a}$:

But

But the same Logarithm (to wit that of the Ratio which is the Reciprocal of the Ratio $\frac{a}{b}$*) will be* (Cor. 2. Prop. 2.) M *into* $\frac{x}{b} + \frac{x^2}{2b^2} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4}$ &c.

Cor. 2. *Because the Ratio* $\frac{b}{a}$ *is compounded of the Ratio, which* b *has to the arithmetical Mean between* b *and* a, *and of the Ratio of that Mean to the less Term* a, *that is,* (*if* a + b *be called* z) *compounded of the Ratios* $\frac{b}{\frac{1}{2}z}$ *and* $\frac{\frac{1}{2}z}{a}$ (*b*); *therefore the Sum of their corresponding Logarithms will be the Logarithm of the Ratio* $\frac{b}{a}$ (*c*); *and because* $\frac{1}{2}$x *is the Difference between the Terms of either Ratio* (*d*), *if in the Series of the foregoing Corollary be put* $\frac{\frac{1}{2}x}{\frac{1}{2}z} = \frac{x}{z}$ *for* $\frac{x}{a}$ *and* $\frac{x}{b}$; *the Logarithm of the Ratio* $\frac{b}{\frac{1}{2}z}$ *will be* M *into* $\frac{x}{z} - \frac{x^2}{2z^2} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6}$ &c. that of the Ratio $\frac{\frac{1}{2}z}{a}$ will be M into $\frac{x}{z} + \frac{x^2}{2z^2} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6x^6}$ &c. the Sum of which, to wit, 2 M into $\frac{x}{z} + \frac{x^3}{3z^3} + \frac{x^5}{5z^5} + \frac{x^7}{7z^7}$ &c. *will be the Logarithm of the*

(*b*) Number 40. (*c*) Number 238.

(*d*) for $\frac{b}{\frac{1}{2}z} = \frac{a+x}{a+\frac{1}{2}x}$ by Cor. 1. also $\frac{\frac{1}{2}z}{a} = \frac{a+\frac{1}{2}x}{a}$ by the same.

of the Ratio $\frac{b}{a}$, and this Series converges most swiftly when the Ratio $\frac{b}{a}$ is not much distant from the Ratio of Equality.

Scholium.

The Logarithms of composite Numbers may be found from the Addition of the Logarithms corresponding to the prime Numbers which compound them.

361. *And the Logarithm corresponding to any prime Number whatsoever may with sufficient Expedition be found, if the Logarithms be given which correspond to the Numbers next adjacent on each Side. For let the Product of the adjacent Numbers be taken, and the Square of the Number itself, which will exceed the Product only by Unity* (*e*), *and to the Logarithm of the Ratio of the Square to the Product, found by the Series of the 2d Corollary, let there be added the Sum of the Logarithms corresponding to the adjacent Numbers, to wit, the Logarithm of the Ratio of the Product to Unity, and there will come out the Logarithm of the Ratio of the Square to Unity, the half of which will be the Logarithm required* (*f*). Thus, if the Logarithms are given, which correspond to the Numbers 2, 3, 5, and the Logarithm corresponding to the Number 19, be required. The Square of 19 is 361, and the Product of 18 and 20, the Numbers next adjacent, is 360; but the Logarithms of the Ratios $\frac{18}{1}$ and $\frac{20}{1}$ are given; (for $2 \times 3 \times 3 = 18$. and $2 \times 2 \times 5 = 20$) to the Sum of which let the Logarithm of the Ratio $\frac{361}{360}$ be added, and the half Sum will be the Logarithm required.

The

(*e*) Number 29. (*f*) Number 330.

The Series in those Corollaries verge more swiftly to the Logarithms of Ratios, which are more near to the Ratio of Equality: Wherefore the Logarithms which correspond to the less prime Numbers being compleated, there can be found very expeditiously, as well the Logarithms which correspond to the greater Numbers, as those also, after the Table is compleated, which correspond either to intermediate Fractions, or to Numbers, which run out beyond the Limits of the Table.

362. And there is an Artifice given by *Cotes* *, which makes the Computation of the Logarithms corresponding to the Primes 2, 3, 5, 7, something lighter: to wit, *If the hyperbolic Logarithms of the Ratios* $\frac{126}{125}$, $\frac{225}{224}$, $\frac{2401}{2400}$, $\frac{4375}{4374}$, *be found, (which may be done, because the Series of Cor. 2. will converge most swiftly,) and called* p, q, r, s, *the hyperbolic Logarithm of the decuple Ratio will be* $239p + 90q - 63r + 103s$, *which being found, the Module of the vulgar System will be given; which let be called* M, *then* M *into* $72p + 27q - 19r + 31s$; M *into* $114p + 43q - 30r + 49s$; M *into* $167p + 63q - 44r + 72s$; M *into* $202p + 76q - 53r + 87s$; *will be the vulgar Logarithms of the Ratios* $\frac{2}{1}$, $\frac{3}{1}$, $\frac{5}{1}$, $\frac{7}{1}$. For $\frac{126}{125} = \frac{2 \times 3^2 \times 7}{5^3}$; $\frac{225}{224} = \frac{3^2 \times 5^2}{2^5 \times 7}$; $\frac{2401}{2400} = \frac{7^4}{2^5 \times 3 \times 5^2}$, and $\frac{4375}{4374} = \frac{5^4 \times 7}{2 \times 3^7}$; and thence, if the Logarithms of the Ratios $\frac{2}{1}$, $\frac{3}{1}$, $\frac{5}{1}$, $\frac{7}{1}$, be called, *a*, *b*, *c*, *d*, the following Equations will come out.

* Harmonia Mensurarum, Prop. 1. Schol. 3.

$$p = a + 2b + d - 3c$$
$$q = -5a + 2b - d + 2c$$
$$r = -5a - b + 4d - 2c$$
$$s = -a - 7b + d + 4c$$

and thence, by Analysis

$$a = 72p + 27q - 19r + 31s$$
$$b = 114p + 43q - 30r + 49s$$
$$c = 167p + 63q - 44r + 72s$$
$$d = 202p + 76q - 53r + 87s$$

and therefore $a + c$ viz. the Logarithm of the decuple Ratio will be $239p + 90q - 63r + 103s$.

363. *The Number corresponding to a given vulgar Logarithm may be found*, if the Table of Logarithms is at hand, by the proportional Part; but more *accurately in the following Manner. Let the Number required be* n, *and (neglecting the Characteristic) the Number corresponding to the next less Logarithm (to be distinguished by the Characteristic of the given Logarithm) be* a; *and the hyperbolic Logarithm (which we suppose given,) of the Ratio* $\frac{n}{a}$ (*viz. the Difference by which the given Logarithm exceeds the Logarithm of the Ratio* $\frac{a}{1}$) *divided by the Module of the System be called* L; *then will* $\frac{1 + \frac{1}{2}L}{1 - \frac{1}{2}L} \times a = n$ *very nearly*. For since the Ratio $\frac{n}{a}$ differs little from the Ratio of Equality, if $n - a$ and $n + a$ be called x and z, the Fraction $\frac{x}{z}$ will be small, and thence (Cor. 2.) $\frac{2x}{z} = \frac{2n - 2a}{n + a} = L$, very nearly; therefore $n = \frac{1 + \frac{1}{2}L}{1 - \frac{1}{2}L} \times a$. But if a more perfect Computation is requisite, it will be $n = \frac{1 + \frac{1}{2}L - \frac{1}{24}L^3}{1 - \frac{1}{2}L + \frac{1}{24}L^3} \times a$ abundantly accurate. And if you would have the Operation still

ftill more accurate, take $n = \frac{1+\frac{1}{2}L-\frac{1}{24}L^3+\frac{1}{240}L^5}{1-\frac{1}{2}L+\frac{1}{24}L^3-\frac{1}{240}L^5}+a$.

364. *The Number correſponding to any given Logarithm whatſoever may be found by the Series* $1+\frac{L}{M}+\frac{L^2}{2M^2}+\frac{L^3}{6M^3}$, *&c. which will converge the ſlower, if* $\frac{L}{M}$ (*viz. the hyperbolic Logarithm of the required Ratio*) *be not ſmall. It muſt be obſerved however, that if the given Logarithm be a vulgar one* (or if the Module of the vulgar Syſtem be at hand, ſo that the given Logarithm may be readily reduced to a vulgar one) *the Operation will generally become ſufficiently eaſy*; to wit, always eaſier, than if the given Logarithm was the hyperbolic one of the decuple Ratio. For if the Number correſponding to the Logarithm diminiſhed of its Characteriſtic be found; the Figures of the Number which correſponds to the given Logarithm will come out; whence (becauſe the Characteriſtic is known) the Number required will itſelf be known.

FINIS.

E R R A T A.

PAGE 5, line 10, *for* Numerators *read* Denominators. P. 8, l. 11, *f.* Numerators *r.* Denominators. P. 10, l. 36, *f.* Binominal *r.* Binomial. P. 12, l. 20. *f.* often to be taken *r.* often taken. P. 13, l. 7, *f.* $y-2b\times y+b$ *r.* $\overline{y-2b}\times\overline{y+b}$. P. 23, l. ult. *f.* effected *r.* affected. P. 28, l. 3, *f.* nearer any Part *r.* nearer any lesser Part. l. 12, *f.* any adjacent *r.* the adjacent. P. 40, l. 30, *f.* 3,0925 *r.* 3,9025. l. 33, *f.* 25060 *r.* 25090. P. 43, l. penult. *f.* (48) *r.* (47) P. 52, l. 14. *f.* $\sqrt{x-14}$, *r.* $\sqrt[3]{x-14}$. P. 57, l. 27, *f.* $\frac{n-1}{n}$ *r.* $\frac{n-1}{2}$. P. 59, l. 5, *f.* $\frac{2ax}{c}-\sqrt{\frac{aab}{c}}$ *r.* $\frac{2ax}{c}\sqrt{\frac{aab}{c}}$. P. 60, l. 16, *f.* Unicæ *r.* Unciæ. P. 71, l. 8, *f.* $x^3\mp\frac{19}{3}x^2$ *r.* $x^3\pm\frac{19}{3}x^2$. P. 77, l. 11, *f.* Nnmber *r.* Number. P. 80, l. 25, *f.* particnlar *r.* particular. P. 119, l. 12, *f.* Members *r.* Numbers. P. 140, l. 4 and 6, *f.* $xxy-z^c$ *r.* $\overline{xxy-z}^c$. P. 143, l. 22, *f.* c is even *r.* and c even. P. 146, l. 5, *f.* 8 *r.* 12. l. 18, *f.* $\frac{5}{2}$ *r.* $\frac{5}{2}$. P. 183, l. 18, *f.* $10\frac{1}{4}$ *r.* $10\frac{3}{4}$. P. 190, l. antepenult, *f.* aec *r.* aecc. P. 191, l. last, *f.* : *r.* . P. 197, l. 7, *f.* $\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$ *r.* $s\sqrt{\frac{1}{4}b-\frac{1}{4}ss}$. P. 208, l. 18, *f.* reaion *r.* reafon. P. 226, l. 20, *f.* $\frac{2\sqrt{bb-xx}}{b}$ *r.* $\frac{x\sqrt{bb-xx}}{b}$. l. 21, *f.* (Eucl. V. 40) *r.* (Eucl. V. 4. Cor.) P. 236, l. 25, *f.* BC^2-DC^2 *r.* BD^2-DC^2. P. 247, l. 22, *f.* $\sqrt{\ }5\frac{a^4}{4}$ *r.* $\sqrt{5\frac{a^4}{4}}$. P. 268, l. 12, *dele* it. P. 279, l. 17, *f.* G *r.* E. P. 322, l. 20, *f.* $\perp 4br-8aa$ *r.* $\perp 4br=8aa$. P. 325, l. 20, *f.* $-143b$ *r.* $-143b^4$. P. 361, l. 24,

f. by

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f. by the Square *r.* by a positive Quantity exclusive of the Square. P. 364, l. 16, *f.* descending or ascending to *r.* descending to or ascending from. P. 365, l. 16, *f.* Fractions *r.* Fraction. P. 375, l. penult, *f.* $\overline{y+e}$ *r.* $y\pm e$. P. 384, l. 10, *f.* $\sqrt{\frac{1}{2}5p^2-\frac{1}{5}q}$ *r.* $\sqrt{\frac{1}{25}p^2-\frac{1}{5}q}$. P. 385, l. 28, *f.* $\frac{qa^2}{b^2}$ *r.* $\frac{qa^2}{b^2}y$. P. 387, l. 21, *f.* $p\sqrt[3]{q^2}\sqrt[4]{r^2}\sqrt[5]{s^2}p\sqrt[2]{pq}$ *r.* $p\sqrt[3]{q^2}\sqrt[4]{r^2}\sqrt[5]{s^2},\ p\sqrt[2]{pq}$. l. 25, *f.* $p^4qr^2s^2A$ *r.* $p^4q^3r^2s^2A$ P. 396, l. 3, *f.* z into z y n *r.* z into x y n. P. 399, l. 15, *f.* $x^2\times y^3$ *r.* x^3+y^3. l. 17, *f.* $x^2\times y^2$ into $x^4\times y^4$ *r.* x^2+y^2 into x^4+y^4. P. 405, l. 15, *f.* $\frac{R+c}{2}$ *r.* $\frac{R\pm c}{2}$. P. 409, l. 16, *f.* Square *r.* Squares. P. 415, l. 20, *f.* Sign *r.* Signs. P. 423, l. 3, *f.* $\frac{1}{17}q^3$ *r.* $\frac{1}{27}q^3$. P. 425 l. 8, *f.* $f^2-\frac{g^2}{3}$ *r.* $f^2+\frac{g^2}{3}$. l. 26, *f.* $f^3+g^2=\frac{r}{2}$ *r.* $f^3+3g^2f=\frac{r}{2}$. P. 428, l. 4, *f.* $\varepsilon=\frac{1}{2}p\,\delta-\frac{1}{2}a\gamma$, &c. *r.* $\varepsilon=v-\frac{1}{2}p\,\delta-\frac{1}{2}a\gamma$, &c. l. 20, *f.* $nk^2\times x^{r-2}$ *r.* $nk^2\times x^{2r-2}$. P. 429, l. last and P. 430, l. 2, *f.* $\beta=2R-\frac{1}{4}pnk^2+2nkl$ *r.* $\beta=2R+\frac{1}{4}pnk^2-2nkl$. P. 430, l. 5, *f.* $+^2\beta=\frac{1}{2}r$ *r.* $+\frac{1}{2}\beta=\frac{1}{2}r$. l. 9 and 11, *f.* $\frac{1}{4}p^2nk^2+2nm-pnl-\frac{1}{2}an\times k$ *r.* $\overline{\frac{1}{4}p\,n-\frac{1}{2}an\times k^2+2nm-pnl}\times k$. l. 16, *f.* $-\frac{1}{4}ank$ *r.* $-\frac{1}{4}ank^2$. l. 17, *f.* $\frac{-2ank}{8}$ *r.* $\frac{-2ank^2}{8}$. P. 432, l. 6, *f.* $\frac{(k^2)}{64}$ *r.* $\frac{k^2}{64}$. l. last, *f.* $(\overline{2-r+1}$ *r.* $(\overline{2r+1}$. P. 442, l. 19, *f.* $-05\frac{5}{8}$ *r.* $-50\frac{5}{8}$. P. 444. l. 25, *f.* $\frac{1}{2}$, pk *r.* $\frac{1}{2}$pk. P. 446, l. 12, *f.* $\frac{8}{2}$ *r.* $\frac{9}{2}$. P. 506, l. 12, *f.* $x^3 6x$ *r.* x^3-6x. P. 507, l. last, *dele* $=\frac{1}{16}$. P. 509, l. last, *f.* $36f^2$ *r.* 36f. P. 510, l. 1, *f.* $12f^3$ *r.* $12f^2$ l. 17, *f.* $\frac{q^2}{q^2}$ *r.* $\frac{r^2}{q^2}$. P. 511, l. 8, *f.* $f=\frac{q}{2p}\sqrt{-\frac{r}{p}+\frac{q^2}{4p^2}}$ *r.* $f=\frac{q}{2p}\pm\sqrt{-\frac{r}{p}+\frac{q^2}{4p^2}}$. P. 513, *f.* $f=\frac{-Ak^n,\ \&c.}{nk^{n-1},\ \&c.}$ *r.* $f=\frac{A-k^n,\ \&c.}{nk^{n-1},\ \&c.}$ P. 514, l. 6, *f.*

f =

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$f=\frac{A-k^n,\ \&c.}{nk^{n-2},\ \&c.}$ *r.* $f=\frac{A-k^n,\ \&c.}{nk^{n-1},\ \&c.}$ P. 524, l. 7, *f.* y^4x^5 *r.* y^4x^6 ibid. *f.* $y+y^4x^3=0$ *r.* $y^7+y^4x^3=0$. P. 526, l. 9, *f.* $-Dy^1$ *r.* Dy^1. P. 528, l. 29, *f.* (by 310, *r.* (by 309.

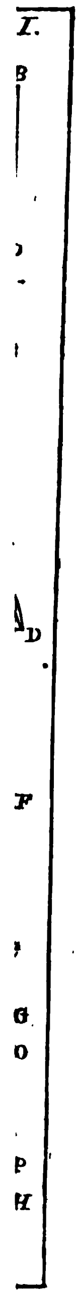
MEASURES *of* RATIOS.

P. 13, l. 1, *f.* B—B *r.* B—A. P. 20, l. 26, *f.* by (Prop. 2.) *r.* (by Prop. 2.) P. 22, l. antepenult, *f.* nearer to 3 than 2) because 2,718, &c. is greater than $2,5=2\frac{1}{2}$ to Unity *r.* nearer to 3 than to 2, because 2,718, &c. is greater than $2,5=2\frac{1}{2}$) to Unity. P. 23, l. 9, *f.* Ratios *r.* Ratio. P. 29, l. penult, *f.* –1,077021 (XXVI) &—V *r.* +0,077021 (XXVI) &—1. P. 32 at bottom, *r.* (r) XXIV. P. 33, l. 29, *f.* $\frac{13}{59}$ *r.* $\frac{13}{591}$. P. 48, l. 13, *f.* seems scanty *r.* seems too scanty. *f.* p. 59, *r.* p. 56, l. 2, *f.* $\frac{L^2}{1\times2\times3}$ *r.* $\frac{L^3}{1\times2\times3}$. l. 11, *f.* $M-1\times a=\frac{L}{M}$ *r.* $N-1\times a=\frac{L}{M}$. P. 57, l. 4, *f.* $\frac{RW}{ST}h^5$ *r.* $\frac{RV}{ST}h^5$. P. 58, l. 8, *f.* M into–h *r.* M into h. P. 62, l. penult, *f.* $\frac{-\frac{1}{24}L^3}{+\frac{1}{2}4L^3}$ *r.* $\frac{-\frac{1}{24}L^3}{+\frac{1}{24}L^3}$.

P. 64, MULTIPLICATION, l. 2, *f.*

$x^{\overline{-n}} \mp \frac{n}{1} x^{\overline{-n-1}} a + \frac{n}{1}\times\frac{n+2}{2} x^{\overline{-n-2}} a^2 \mp {}_1 \times\frac{n+1}{2}\times\frac{n+2}{2} x^{\overline{-n-3}}$

r. $x^{\overline{-n}} \mp \frac{n}{1} x^{\overline{-n-1}} a + \frac{n}{1}\times\frac{n+1}{2} x^{\overline{-n-2}} a^2 \mp \frac{n}{1}\times\frac{n+1}{2}\times\frac{n+2}{3} x^{\overline{-n-3}}$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