Weber's Electrodynamics

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Weber's Electrodynamics

by

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To the memory of my grandfather,
Eng. David Koch,
for all that I owe to him.

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Preface

"Great progress has been made in electrical science, chiefly in Germany, by cultivators of the theory of action at a distance. The valuable electrical measurements of W. Weber are interpreted by him according to this theory, and the electromagnetic speculation which was originated by Gauss, and carried on by Weber, Riemann, F. and C. Neumann, Lorenz, etc., is founded on the theory of action at a distance, but depending either directly on the relative velocity of the particles, or on the gradual propagation of something, whether potential or force, from the one particle to the other. The great success which these eminent men have attained in the application of mathematics to electrical phenomena, gives, as is natural, additional weight to their theoretical speculations, so that those who, as students of electricity, turn to them as the greatest authorities in mathematical electricity, would probably imbibe, along with their mathematical methods, their physical hypothesis.

These physical hypotheses, however, are entirely alien from the way of looking at things which I adopt, and one object which I have in view is that some of those who wish to study electricity may, by reading this treatise, come to see that there is another way of treating the subject, which is no less fitted to explain the phenomena, and which, though in some parts it may appear less definite, corresponds, as I think, more faithfully with our actual knowledge, both in what it affirms and in what it leaves undecided.

In a philosophical point of view, moreover, it is exceedingly important that two methods should be compared, both of which have succeeded in explaining the principal electromagnetic phenomena, and both of which have attempted to explain the propagation of light as an electromagnetic phenomenon and have actually calculated its velocity, while at the same time the fundamental conceptions of what actually takes place, as well as most of the secondary conceptions of the quantities concerned, are radically different."

These are the words of James Clerk Maxwell, in the Preface of his major book, *A Treatise on Electricity and Magnetism.* As we can see from these words, Maxwell perceived a conceptual difference between his conceptions, derived in great measure from those of Faraday; and the conceptions of Gauss, Weber, etc. Maxwell knew that both formulations succeeded in explaining the main phenomena of electromagnetism, and he emphasized the great importance in the comparison of the two methods.

And the goal of this book is exactly to follow this general idea. Our basic intention is to present in a fairly complete way Weber's Electrodynamics. As Maxwell said and showed more than once, Weber's theory is compatible with what we call Maxwell's equations (namely, laws of Gauss, Ampère and Faraday), although it is completely different from Maxwell's conceptions in philosophical matters. In this book we show how Maxwell's equations can be derived from Weber's force and the limitations of this compatibility.

In Maxwell's time the electrodynamic researches in the Continent were centered on the action at a distance laws of Coulomb, Ampère, Weber, Neumann, etc. In these theories only the charges, current carrying circuits and magnets, as well as their distances, velocities and accelerations are important. The ether or the field concept are not necessary. Maxwell had different conceptions, based essentially on the ether, and was trying to show that this new model could also explain the known facts of electromagnetism, as we can see from the middle paragraph quoted above. Nowadays we have the opposite situation. We only talk of fields, local action, finite velocity of propagation of the interactions, etc. The aim of this book is summarized in Maxwell's middle paragraph, but now reversing the methods or physical hypotheses.

Maxwell's admiration of Weber's work can also be seen by observing that he dedicated the last chapter of his most important book (the *Treatise*) to present Weber's Electrodynamics and to show its compatibility with the main known facts of electromagnetism.

This book is intended for students and scientists in the areas of physics, engineering, mathematics, history and philosophy of science. This work is intended to be complete in the sense that no previous knowledge of Weber's law is required to follow the text. A first Chapter on Vector Analysis including the main mathematical tools utilized in the text is included for completeness.

The subject of this work is within classical physics. For this reason we did not deal here with quantum mechanics nor with Einstein's theories of relativity. These topics are beyond the scope of this book.

At the end of the book a large bibliography has been included to allow interested readers further studies. It is not intended to be complete but only to indicate some of the subjects being researched nowadays along these lines and to mention authors working in this field. These recent references can be utilized as topics of research by graduate students. In the text each reference is indicated by the author's name and year of publication. For instance: (Edwards, Kenyon and Lemon, 1976).

This book can be utilized in a one or two semesters course. We have taught courses on Weber's Electrodynamics at undergraduate and graduate levels, and this book grew out of these experiences. We wrote a *Course of Weber's Electrodynamics*, with exercises, which has been utilized in these courses (Assis, 1992 a). The reception of the students to this material has been very encouraging and they always mention it has been helpful in their formation in science.

Whenever possible we present historical information relevant to the topic which is being treated. The reason is to give the historical context of some discoveries and to make a critical analysis of some topics. The sources of the major part of this information are the original papers, and the excellent books of Whittaker (*A History of the Theories of Aether and Electricity*), O'Rahilly (*Electromagnetic Theory* - A Critical Examination of Fundamentals), and Mach (*The Principles of Physical Optics*).

In this book we utilize the International System of Units. When we define any physical concept we utilize "$\equiv$" as a symbol of definition.

Acknowledgments - To the undergraduate and graduate students who followed our courses on Weber's Electrodynamics for the many constructive remarks they presented. To our students who are developing researches in this area. To all those who read a first version of this work and gave their feedback. To Drs. Peter and Neal Graneau, Thomas E.

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Finally I wish to thank my parents, my wife and children for the stimulus they always gave me.

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June, 1994

Chapter 1 / Vector Analysis

In this Chapter we review briefly the main ideas and theorems related to vector analysis which will be utilized in this book. This Chapter is intended as a short summary of a subject familiar to most students and scientists. No formal development or general proofs will be presented here as they can be easily found elsewhere.

1.1. Definitions and Notation

The majority of the physical concepts dealt with in this work can be mathematically described in terms of scalar and vector quantities. A scalar is a quantity which is completely characterized by its magnitude. Examples are electrical charge q, mass m, time t, temperature T, etc. A vector is a quantity which is completely characterized by its magnitude and direction. While scalars are represented in this book by ordinary type, vectors are represented by arrows. Examples of vectors are position from a fixed origin $\vec{r}$, velocity $\vec{v}$, acceleration $\vec{a}$, force $\vec{F}$, electric and magnetic fields $\vec{E}$ and $\vec{B}$, etc. Unit vectors (magnitude equal to one) are represented by hats, like $\hat{r}$. The magnitude of an arbitrary vector $\vec{G}$ is represented by $|\vec{G}|$ of simply by G.

Extensions of these ideas are scalar fields and vector fields. A scalar field is a function of position which is completely specified by its magnitude at all points in space. For instance, the temperature of a rigid body may change from point to point inside the body. An arbitrary scalar field ϕ is represented by $\phi(\vec{r})$. A vector field is a function of position which is completely specified by its magnitude and direction at all points in space. For instance, the velocity of the water relative to the earth may change from point to point inside a river. An arbitrary vector field $\vec{G}$ is represented by $\vec{G}(\vec{r})$.

1.2. Vector Algebra

Usually vectors are represented in a three-dimensional Cartesian coordinate system. This coordinate system is represented in Figure 1.1 (a).

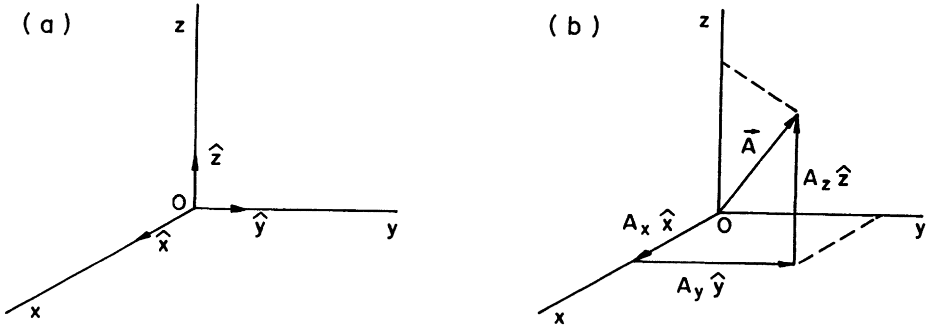

Figure 1.1

It has an origin O and three orthogonal axes x, y and z. The unit vectors along these axes are represented by $\hat{x}$, $\hat{y}$ and $\hat{z}$, respectively. An arbitrary vector $\vec{A}$ is specified by its components along these axes, namely, $\vec{A} = (A_x,\ A_y,\ A_z)$. Its magnitude is given by $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. The components of $\vec{A}$ are its projections along the three coordinate axes: $A_x = A\cos\alpha_x$, $A_y = A\cos\alpha_y, A_z = A\cos\alpha_z$, where the α's are the angles between $\vec{A}$ and the appropriate coordinate axes. The vector $\vec{A}$ can be expanded as $\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$, Figure 1.1 (b).

The sum and subtraction of two vectors $\vec{A}$ and $\vec{B}$ are represented in Figure 1.2 (a) and (b). It is easily seen that $\vec{A} + \vec{B} = \vec{B} + \vec{A} = (A_x + B_x)\hat{x} + (A_y + B_y)\hat{y} + (A_z + B_z)\hat{z}$. This is equivalent to the familiar parallelogram rule for vector addition. We also have $\vec{A} - \vec{B} = (A_x - B_x)\hat{x} + (A_y - B_y)\hat{y} + (A_z - B_z)\hat{z}$. The multiplication of $\vec{A}$ by a scalar c is given by $c\vec{A} = (cA_x)\hat{x} + (cA_y)\hat{y} + (cA_z)\hat{z}$.

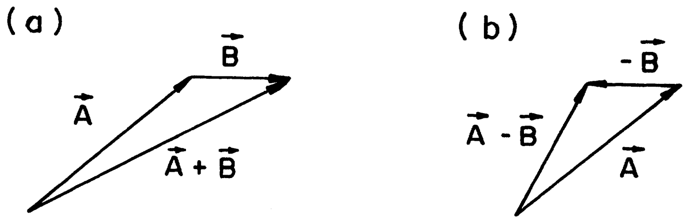

Figure 1.2

The scalar product of two vectors $\vec{A}$ and $\vec{B}$, $\vec{A}\cdot\vec{B}$, is given by

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}=AB\cos\theta=A_xB_x+A_yB_y+A_zB_z\ . \tag{1.1}$$

In this expression θ is the angle between $\vec{A}$ and $\vec{B}$, Figure 1.3 (a).

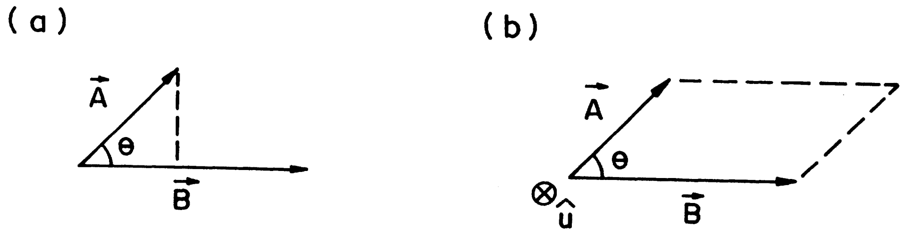

Figure 1.3

The scalar product is also called dot product or inner product.

The vector product (also called cross or outer product) of two vectors $\vec{A}$ and $\vec{B}$, $\vec{A}\times\vec{B}$, is given by

$$\vec{A} \times \vec{B} = AB \sin\theta \hat{u} \ , \tag{1.2}$$

where $\hat{u}$ is a unit vector pointing perpendicular to the plane of $\vec{A}$ and $\vec{B}$ according to the right-hand rule, Figure 1.3 (b). The magnitude of $\vec{A} \times \vec{B}$ is the area of the parallelogram generated by $\vec{A}$ and $\vec{B}$. In terms of Cartesian components we have

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z} \ , \tag{1.3}$$

From (1.2) or (1.3) we can see that the vector product is not commutative, namely

$$\vec{B} \times \vec{A} = - \vec{A} \times \vec{B} \ . \tag{1.4}$$

Utilizing these rules we can find the results of the scalar triple product and show that

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) \ . \tag{1.5}$$

Moreover,

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = (\vec{A} \times \vec{B}) \cdot \vec{C} \ . \tag{1.6}$$

The vector triple product is found to be

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) \ , \tag{1.7}$$

$$(\vec{A} \times \vec{B}) \times \vec{C} = -\vec{A}(\vec{B} \cdot \vec{C}) + \vec{B}(\vec{A} \cdot \vec{C}) \ . \tag{1.8}$$

1.3. Gradient

An arbitrary scalar field $\phi = \phi(\vec{r})$ can also be written as $\phi(x,\ y,\ z)$. An infinitesimal variation of ϕ can then be expressed as

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz \ . \tag{1.9}$$

The position vector $\vec{r}$ and an infinitesimal displacement $d\vec{r}$ are represented in Cartesian coordinates by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \ , \tag{1.10}$$

$$d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz \ . \tag{1.11}$$

We can then write (1.9) as

$$d\phi = (\nabla\phi)\cdot d\vec{r} = |\nabla\phi||d\vec{r}|\cos\theta \ , \tag{1.12}$$

where $\nabla\phi$ (or *grad*ϕ, as it is usually represented) is called the gradient of ϕ and is defined by

$$\nabla\phi \equiv \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} \ . \tag{1.13}$$

In (1.12) θ is the angle between $\nabla\phi$ and $d\vec{r}$. As $|\cos\theta|$ has a minimum value of 0 and a maximum value of 1, the magnitude of $d\phi$ will be maximized when $\theta = 0$ and $\cos\theta = 1$. It will be minimized when $\theta = 90^o$ and $\cos\theta = 0$. This means that $\nabla\phi$ is a vector which points in the direction of maximum increase of ϕ and whose magnitude is the derivative of ϕ along this maximal direction at the point being considered.

We can easily see that for two scalar fields $f(\vec{r})$ and $g(\vec{r})$:

$$\nabla(fg) = f\nabla g + g\nabla f \ . \tag{1.14}$$

If $r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$ is the distance from the origin to the point $\vec{r} = (x,\ y,\ z)$ and $\hat{r} \equiv \vec{r}/r$ is a unit vector in the direction of $\vec{r}$, it can be shown that

$$\nabla(r^n) = nr^{n-1}\hat{r} \ . \tag{1.15}$$

We can define a vector operator ∇ by

$$\nabla \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \ . \tag{1.16}$$

It is called nabla or del. We can think of $\nabla\phi$ as ∇ acting upon the function or scalar field ϕ. Although it is not a vector, as it has no meaning by itself but only when operating upon a function, it behaves algebraically in many situations like a vector. This is a useful notation which simplifies many expressions.

1.4. Divergence, Curl and Laplacian

The divergence of an arbitrary vector field $\vec{G}(\vec{r})$, $\nabla \cdot \vec{G}$ or $div\vec{G}$, is defined by

$$\nabla \cdot \vec{G} \equiv \frac{\partial G_x}{\partial x} + \frac{\partial G_y}{\partial y} + \frac{\partial G_z}{\partial z} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right) \cdot (\hat{x}G_x + \hat{y}G_y + \hat{z}G_z) \ . \quad (1.17)$$

The result of this operation is clearly a scalar field.

We can show that if $\phi(\vec{r})$ is an arbitrary scalar field then

$$\nabla \cdot (f\vec{G}) = f(\nabla \cdot \vec{G}) + (\nabla f) \cdot \vec{G} \ . \quad (1.18)$$

It can also be shown that

$$\nabla \cdot (r^n \hat{r}) = (2+n)r^{n-1} \ , \ \text{for } n \neq -2 \ . \quad (1.19)$$

The case for $n = -2$ is dealt with in Section 1.6.

The curl of an arbitrary vector field $\vec{G}(\vec{r})$, $\nabla \times \vec{G}$ or $curl\vec{G}$, is another vector field defined by

$$\nabla \times \vec{G} \equiv \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_x & G_y & G_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z} \ . \quad (1.20)$$

We can also show that if $\vec{H}(\vec{r})$ is another arbitrary vector field then

$$\nabla \times (\phi\vec{G}) = \phi(\nabla \times \vec{G}) - \vec{G} \times (\nabla\phi) \ , \quad (1.21)$$

$$\nabla \cdot (\vec{G} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{G}) - \vec{G} \cdot (\nabla \times \vec{H}) \ , \quad (1.22)$$

$$\nabla(\vec{G} \cdot \vec{H}) = \vec{G} \times (\nabla \times \vec{H}) + \vec{H} \times (\nabla \times \vec{G}) + (\vec{G} \cdot \nabla)\vec{H} + (\vec{H} \cdot \nabla)\vec{G} \ , \quad (1.23)$$

$$\nabla \times (\vec{G} \times \vec{H}) = (\vec{H} \cdot \nabla)\vec{G} - (\vec{G} \cdot \nabla)\vec{H} + \vec{G}(\nabla \cdot \vec{H}) - \vec{H}(\nabla \cdot \vec{G}) \ . \tag{1.24}$$

It can also be shown that

$$\nabla \times (r^n \hat{r}) = 0 \ . \tag{1.25}$$

We can have second derivatives of these fields applying the operator ∇ twice. We can then show that if $\phi(\vec{r})$ and $\vec{G}(\vec{r})$ are arbitrary but reasonably well behaved fields (so that we can change the order of the derivatives, etc.) then

$$\nabla \times (\nabla \phi) = 0 \ , \tag{1.26}$$

$$\nabla \cdot (\nabla \times \vec{G}) = 0 \ . \tag{1.27}$$

The divergence of the gradient of a function, $\nabla \cdot (\nabla \phi)$, appears frequently in physics and received the name Laplacian of ϕ. It is also represented by $\nabla^2 \phi$ and is given in Cartesian coordinates by

$$\nabla \cdot (\nabla \phi) \equiv \nabla^2 \phi = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \cdot \left(\frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} \right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \ . \tag{1.28}$$

We also have

$$\nabla \times (\nabla \times \vec{G}) = \nabla(\nabla \cdot \vec{G}) - \nabla^2 \vec{G} \ . \tag{1.29}$$

This expression may also be utilized to define the Laplacian of a vector, $\nabla^2 \vec{G}$.

1.5. Integral Calculus

There are three main theorems related to the gradient, divergence and curl of scalar and vector fields.

The theorem for the gradient states that the line integral of $\nabla\phi$ from a point in space $a = (a_x,\ a_y,\ a_z)$ to a point $b = (b_x,\ b_y,\ b_z)$ through an arbitrary path of integration C depends only on the values of ϕ at the end points, see Figure 1.4.

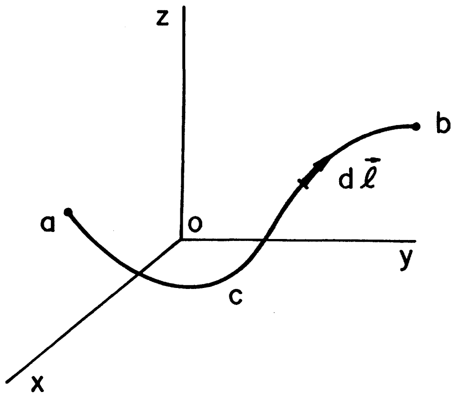

Figure 1.4

Mathematically this theorem is written as

$$\int_a^b (\nabla\phi) \cdot d\vec{l} = \phi(b) - \phi(a)\ , \tag{1.30}$$

where $d\vec{l}$ is an infinitesimal displacement along the line of integration C. This theorem states that this line integral is independent of the path taken from a to b.

A corollary of this theorem is that the line integral of $\nabla\phi$ over a closed circuit of arbitrary form is zero, namely (Figure 1.5):

$$\oint_C (\nabla\phi) \cdot d\vec{l} = 0\ . \tag{1.31}$$

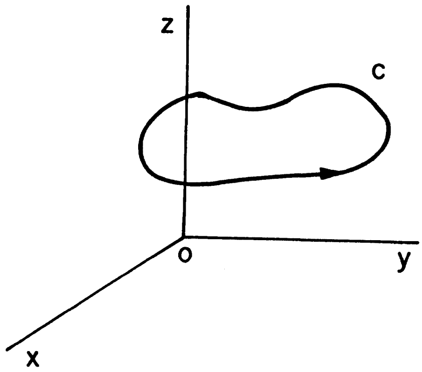

Figure 1.5

The second theorem is called Gauss's theorem, or the divergence theorem. According to this theorem the surface integral of the normal component of an arbitrary vector field $\vec{G}(\vec{r})$ over the closed surface S bounding a volume V is equal to the volume integral of the divergence of $\vec{G}$ over the volume V (Figure 1.6):

$$\oiint_S \vec{G} \cdot d\vec{a} = \int\int\int_V (\nabla \cdot \vec{G}) dV \; . \tag{1.32}$$

In this equation dV is an element of volume and $d\vec{a}$ is an infinitesimal element of area, whose magnitude is the area of the element and whose direction is perpendicular to the surface S in each point, pointing outward by convention.

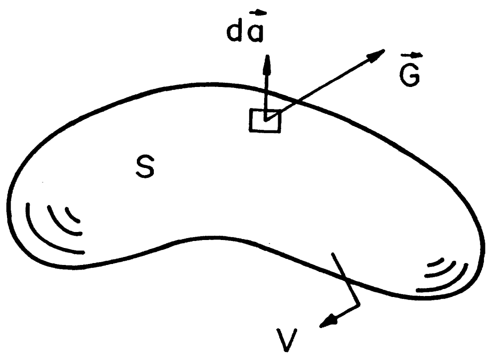

Figure 1.6

The left-hand side of (1.32) represents the flux of $\vec{G}$ over the surface S. This shows that the divergence of a vector quantity is connected with the amount of this quantity which passes through S.

The third theorem is related with the curl of $\vec{G}(\vec{r})$ and is known as Stoke's theorem. According to it the line integral of $\vec{G}$ around a closed curve C is equal to the integral of the normal component of $\nabla \times \vec{G}$ over *any* surface S bounded by the curve (Figure 1.7):

$$\oint_C \vec{G} \cdot d\vec{l} = \int\int_S (\nabla \times \vec{G}) \cdot d\vec{a} \, . \tag{1.33}$$

In this equation the infinitesimal element of area $d\vec{a}$ is normal to the *open* surface S at each point and is related to the infinitesimal displacement $d\vec{l}$ along the curve of integration C by the right-hand rule, by convention.

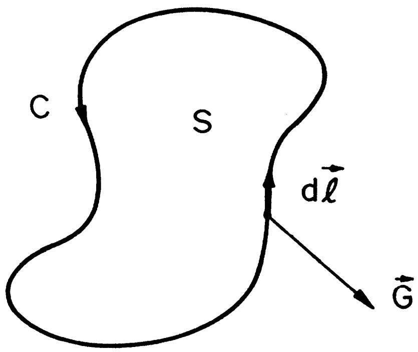

Figure 1.7

The left-hand side of (1.33) indicates the circulation of $\vec{G}$ along the closed curve C. This means that the curl of $\vec{G}$ is related to its circulation around a closed path.

A corollary of this theorem is that the surface integral of $\nabla \times \vec{G}$ over a closed surface of arbitrary form is zero:

$$\oint\!\!\oint_S (\nabla \times \vec{G}) \cdot d\vec{a} = 0 \, . \tag{1.34}$$

1.6. The Dirac Delta Function

The function $\delta(\vec{r}-\vec{r}_o)$ is known as the three-dimensional Dirac delta function. Its main properties:

$$\delta(\vec{r}-\vec{r}_o) = 0 \text{ for } \vec{r}_o \neq \vec{r} \ , \tag{1.35}$$

$$\int\int\int_V \phi(\vec{r})\delta(\vec{r}-\vec{r}_o)dV$$

$$= \begin{cases} 0, & \text{if V does not contain } \vec{r}_o \ , \\ \phi(\vec{r}_o), & \text{if V contains } \vec{r}_o \ . \end{cases} \tag{1.36}$$

An important example where it appears in electromagnetism is in $\nabla \cdot (\hat{r}/r^2)$:

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta(\vec{r}) \ . \tag{1.37}$$

This completes the relation (1.19) for all n's.

By (1.15) and (1.28) we obtain

$$\nabla^2 \frac{1}{r} = -4\pi\delta(\vec{r}) \tag{1.38}$$

We can see that $\nabla \cdot (\hat{r}/r^2)$ can not be zero everywhere applying the divergence theorem to the volume V and area S of a sphere of radius R centered on the origin. In this case $d\vec{a} = \hat{r}da$ so that

$$\int\int\int_V \left(\nabla \cdot \frac{\hat{r}}{r^2}\right) dV = \oint\!\!\oint_S \frac{\hat{r}}{r^2} \cdot \hat{r}da = \frac{1}{R^2} \oint\!\!\oint_S da = \frac{S}{R^2} = 4\pi \ , \tag{1.39}$$

where $S = 4\pi R^2$ is the area of the surface bounding the sphere of radius R. If $\nabla \cdot (\hat{r}/r^2)$ were zero (this would happen if (1.19) were valid for $n = -2$) even at the origin we would obtain $0 = 4\pi$, which can not be the case. With (1.36) and (1.37) we find that the triple integral in (1.39) is in fact equal to 4π, as it should be.

1.7. Cylindrical and Spherical Coordinates

In problems in which there is cylindrical or spherical symmetries it is usually helpful to utilize what are called cylindrical coordinates (Figure 1.8) or spherical coordinates.

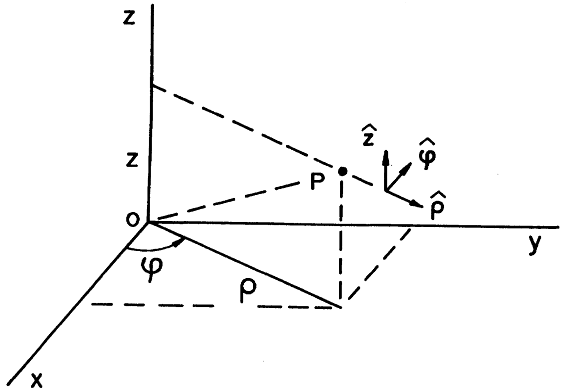

Figure 1.8

The coordinates of a point P in cylindrical coordinates are $(\rho,\ \varphi,\ z)$. The relations to Cartesian coordinates are:

$$x = \rho\cos\varphi,\ y = \rho\sin\varphi,\ z = z\ . \tag{1.40}$$

The opposite relations are:

$$\rho = \sqrt{x^2 + y^2},\ \varphi = \tan^{-1}\frac{y}{x},\ z = z\ . \tag{1.41}$$

An infinitesimal element of volume dV is represented in Cartesian and cylindrical coordinates by

$$dV = dxdydz\ , \tag{1.42}$$

$$dV = \rho d\rho d\varphi dz\ . \tag{1.43}$$

The unit vectors $\hat{\rho}$ and $\hat{\varphi}$ along the directions of ρ and φ are related to $\hat{x}$ and $\hat{y}$ by

$$\hat{\rho} = \hat{x}\cos\varphi + \hat{y}\sin\varphi \ , \tag{1.44}$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi \ . \tag{1.45}$$

From these relations we see that $\hat{\rho} = \hat{\rho}(\varphi)$ and $\hat{\varphi} = \hat{\varphi}(\varphi)$.

The opposite relations are

$$\hat{x} = \hat{\rho}\cos\varphi - \hat{\varphi}\sin\varphi \ , \tag{1.46}$$

$$\hat{y} = \hat{\rho}\sin\varphi + \hat{\varphi}\cos\varphi \ . \tag{1.47}$$

The position vector of a point P, $\vec{r}$, and an infinitesimal displacement, $d\vec{r}$, are represented in cylindrical coordinates by

$$\vec{r} = \rho\hat{\rho} + z\hat{z} \ , \tag{1.48}$$

$$d\vec{r} = \hat{\rho}d\rho + \hat{\varphi}\rho d\varphi + \hat{z}dz \ . \tag{1.49}$$

If $\vec{r} = \vec{r}(t)$ represents the position of a particle or material point moving in space then its velocity $\vec{v}$ and acceleration $\vec{a}$ relative to a frame of reference S are given in Cartesian coordinates by (see (1.10) and (1.11)):

$$\vec{v} = \frac{d\vec{r}}{dt} = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} + \hat{z}\frac{dz}{dt} \equiv \hat{x}v_x + \hat{y}v_y + \hat{z}v_z \equiv \hat{x}\dot{x} + \hat{y}\dot{y} + \hat{z}\dot{z} \ . \tag{1.50}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \hat{x}\frac{d^2x}{dt^2} + \hat{y}\frac{d^2y}{dt^2} + \hat{z}\frac{d^2z}{dt^2} \equiv \hat{x}a_x + \hat{y}a_y + \hat{z}a_z \equiv \hat{x}\ddot{x} + \hat{y}\ddot{y} + \hat{z}\ddot{z} \ . \tag{1.51}$$

In cylindrical coordinates $\vec{v}$ and $\vec{a}$ are represented by

$$\vec{v} = \hat{\rho}\frac{d\rho}{dt} + \hat{\varphi}\rho\frac{d\varphi}{dt} + \hat{z}\frac{dz}{dt} \equiv \hat{\rho}\dot{\rho} + \hat{\varphi}\rho\dot{\varphi} + \hat{z}\dot{z} \ . \tag{1.52}$$

$$\vec{a} = \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\varphi}{dt}\right)^2\right]\hat{\rho} + \left(\rho\frac{d^2\varphi}{dt^2} + 2\frac{d\rho}{dt}\frac{d\varphi}{dt}\right)\hat{\varphi} + \frac{d^2z}{dt^2}\hat{z}$$

$$\equiv (\ddot{\rho} - \rho\dot{\varphi}^2)\hat{\rho} + (\rho\ddot{\varphi} + 2\dot{\rho}\dot{\varphi})\hat{\varphi} + \ddot{z}\hat{z} \ . \tag{1.53}$$

The spherical coordinates are given in Figure 1.9:

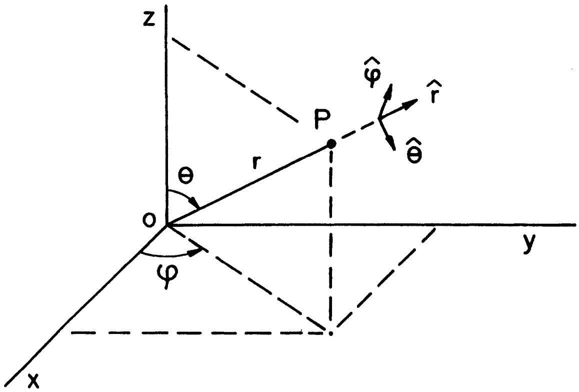

Figure 1.9

The coordinates of a point P in this system are $(r,\ \theta,\ \varphi)$. The relations analogous to (1.40) until (1.53) are, respectively:

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta \ . \tag{1.54}$$

$$r = \sqrt{x^2+y^2+z^2},\ \theta = \cos^{-1}\frac{z}{\sqrt{x^2+y^2+z^2}},\ \varphi = \tan^{-1}\frac{y}{x} \ . \tag{1.55}$$

$$dV = r^2\sin\theta dr d\theta d\varphi \ . \tag{1.56}$$

$$\hat{r} = \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta \ , \tag{1.57}$$

$$\hat{\theta} = \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta \ , \tag{1.58}$$

$$\hat{\varphi} = -\hat{x}\sin\varphi + \hat{y}\cos\varphi \ , \tag{1.59}$$

$$\hat{x} = \hat{r}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi \ , \tag{1.60}$$

$$\hat{y} = \hat{r}\sin\theta\sin\varphi + \hat{\theta}\cos\theta\sin\varphi - \hat{\varphi}\cos\varphi \ , \tag{1.61}$$

$$\hat{z} = \hat{r}\cos\theta - \hat{\theta}\sin\theta \ , \tag{1.62}$$

$$\vec{r} = r\hat{r} \ , \tag{1.63}$$

$$d\vec{r} = \hat{r}dr + \hat{\theta}rd\theta + \hat{\varphi}r\sin\theta d\varphi \ , \tag{1.64}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\varphi}\sin\theta\hat{\varphi} \ , \tag{1.65}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\varphi}^2\sin^2\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$$- r\dot{\varphi}^2\sin\theta\cos\theta)\hat{\theta} + (r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta)\hat{\varphi} \ . \tag{1.66}$$

The maximum ranges of integration in Cartesian, cylindrical and spherical coodinates are, respectively:

$$x: \ -\infty \text{ to } \infty, \ y: \ -\infty \text{ to } \infty, \ z: \ -\infty \text{ to } \infty \ , \tag{1.67}$$

$$\rho: \ 0 \text{ to } \infty, \ \varphi: \ 0 \text{ to } 2\pi, \ z: \ -\infty \text{ to } \infty \ , \tag{1.68}$$

$$r: \ 0 \text{ to } \infty, \ \theta: \ 0 \text{ to } \pi, \ \varphi: \ 0 \text{ to } 2\pi \ . \tag{1.69}$$

This completes an extremely brief overview of the main mathematical tools which will be employed in this book.

Chapter 2 / Review of Classical Electromagnetism

2.1. Introduction

The study of nature and the laws which it follows is one of the main goals of scientists. Physicists, in particular, dedicate themselves to research mechanical, gravitational, electric, magnetic, optical, and nuclear phenomena. In this Chapter we will concentrate on the study of classical electromagnetism. Most of the historical information presented here is from the original sources which we quote and from the following authoritative books: (O'Rahilly, 1965), (Whittaker, 1973) and (Mach, 1926).

This is the name given to the science which deals with the interaction between electric charges, magnets, electric currents and electromagnetic radiation (visible light, X rays, radio waves, etc.) from a unified point of view. Although since the Greeks some electric and magnetic phenomena had been known (Thales, circa 600 b.C., observed that amber, after being rubbed, attracted small objects; they knew that loadstone attracted pieces of iron), the knowledge and larger development of this science began effectively around 1600. In this year Willian Gilbert (1540 - 1603) published the important book *de Magnete*, which deals with magnetism and electricity. An English translation of this important book can be found in (Gilbert, 1952). In it Gilbert presents his great discovery that the earth itself is a permanent magnet, and so explains why the magnetic needles point in a single direction. His book has already been translated to English (Gilbert, 1952). We owe to him also the clear distinction between electric and magnetic attractions. Optics also experienced a great development from this time onwards. Although the Greeks knew the law of reflection (incidence and reflexion angles are equal) and the phenomenon of refraction, the law of refraction was only discovered by Snell (1591 - 1626) in 1621. The first publication of this law occurs in 1637 in the appendix *Dioptrics*, of the famous and pleasant book *Discours on the Methode*, by Rene Descartes (1596 - 1650), see (Descartes, 1965).

From then on these branches were developed more or less independently of one another. The existence of two kinds of electricity (vitreous and resinous, or positive and negative, as we say nowadays) is due to du Fay (1698 - 1739) in 1733-4. The principle of conservation of electric charges is due to Benjamin Franklin (1706 - 1790) in his experiments of 1747, see (Franklin, 1941). The inverse square law for electrostatics is due to Priestley (1733 - 1804), in 1767, and to Coulomb (1736 - 1806), in 1785. The same kind of law relative to magnetic poles is due to Michell (1724 - 1793), in 1750, and to Coulomb, in 1785. Isaac Newton (1642 - 1727) discovered the decomposition of white light in the colours of the spectrum (rainbow) in 1666. He presented this discovery in his first published paper, of 1672 (reprinted in (Newton, 1978, pp. 47 - 59)). He was also the first to measure the periodicity of light, namely, what we today call the wavelength of light, although to him light was not an undulatory perturbation in a medium (ether) but a flux of particles (ballistic theory). Also due to Newton is the first correct interpretation of the polarization of light, in 1717. He published his second great book (the first being the *Mathematical Principles of Natural Philosophy*), the *Optics*, in 1704, see (Newton, 1952 a and b). The discovery that light propagates in time (and not instantaneously), and the first value of the velocity of light are due to Roemer (1644-1710), in 1675.

The interconnexion between electric and magnetic phenomena, although foreseen by many, was only discovered by Oersted (1777-1851) in 1820 (an English translation of this epoch making paper can be found in (Oersted, 1820)). Following this discovery there appear the great works of Ampère (1775-1836), during 1820-27, and Faraday (1791-1867), beginning in 1831, see (Ampère, 1823) and (Faraday, 1952). The interconnexion of electric and magnetic phenomena with those of light, although also foreseen by many, occurs unambiguously for the first time in Faraday's discovery of 1845 that the plane of polarization of a beam of light was rotated when the beam travelled through a dielectric (glass) parallel to the lines of force of the magnetic field. Another relation between these two branches became evident in 1856 with the first measurement, by W. Weber (1804 - 1891) and R. Kohlrausch (1809 - 1858), of the ratio of electromagnetic and electrostatic units of charge. The value they obtained, $3.1 \times 10^8 m/s$, was essentially the same as the known value of the velocity of light in air. The idea that light is an electromagnetic perturbation propagating in the ether is due to Maxwell (1831-1879), during 1861-64. The experimental confirmation of the theoretical predictions of Maxwell happened with Hertz

(1857-1894), during 1885-89. His papers and the description of his experiments can be found in (Hertz, 1962).

These works constitute the foundations of classical electromagnetism. In this Chapter we review this subject. As there are dozens of books dealing with this area, at all levels, we make only a short survey of some topics, and especially of those which will be relevant in the discussion of Weber's theory. Due to the goal of this book we will not treat many important subjects of modern electromagnetic theory, but these subjects can be found fairly well described in many specialized books. What we want in writing this Chapter is to furnish a background for the introduction of Weber's theory. After that we will be equipped to make a detailed comparison between Weber's electrodynamics and classical electromagnetism.

2.2. Equations of Motion

After this short historical account we return to the main subject of this Chapter, namely, a description of classical electromagnetism. From a general point of view it can be said that it has four main parts, independent from one another, but all of them necessary for a complete formulation of the theory. They are:

(A) Equation of motion,

(B) Lorentz's force,

(C) Maxwell's equations,

(D) Constitutive relations of the medium.

The constitutive equations or relations of the medium are empirical descriptions of the properties of materials. As such they do not depend on which theory we are working with, and are likewise valid in all theoretical formulations. Examples: Ohm's law ($V = RI$ or $\vec{J} = \sigma\vec{E}$), $\vec{D} = \varepsilon\vec{E}, \vec{B} = \mu\vec{H}$, etc. In these relations σ, ε, and μ are characteristic properties of each medium, and are empirically measured. From now on we will concentrate only on aspects (A), (B) and (C).

A typical problem in physics is to describe the motion of material bodies when under the influence of forces. The usual way of dealing with this problem is to utilize the three famous axioms or laws of motion formulated by Newton. These laws are here presented as Newton first formulated them in 1687 (Newton, 1952 a) in the book *Mathematical Principles of Natural Philosophy.* This book, also known by its first Latin name *Principia*, originally written in Latin, is considered by many as the greatest scientific work of all time. His axioms or laws of motion:

Law I: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it. (2.1)

Law II: The change of motion is proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed. (2.2)

Law III: To every action there is always opposed an equal reaction: or, the mutual actions of two bodies upon each other are always equal, and directed to contrary parts. (2.3)

Corollary I: A body, acted on by two forces simultaneously, will describe the diagonal of a parallelogram in the same time as it would describe the sides by those forces separately.

In modern vectorial language these three laws could be rewritten as:

Law I: If $\vec{F}_R = 0$ then the body remains at rest or in uniform rectilinear motion. (2.4)

Law II: $\vec{F}_R = d(m\vec{v})/dt$. (2.5)

Law III: $\vec{F}_{AB} = -\vec{F}_{BA}$. (2.6)

In (2.4) and (2.5), $\vec{F}_R$ is the resultant force acting on the body of inertial mass m, and $\vec{v}$ is the velocity of this body relative to absolute space according to Newton. Nowadays we would say velocity relative to an inertial frame of reference. In all this work $\vec{F}_{ji}$ will mean the force exerted by body j on body i (that is, the force on i due to j). So in (2.6) $\vec{F}_{AB}$ is the force exerted by A on B, and $\vec{F}_{BA}$ is the force exerted by B on A.

If the mass is a constant, Newton's second law can be recast as

$$\vec{F}_R = m\vec{a} \ , \tag{2.7}$$

where $\vec{a}$ is the acceleration of the body with inertial mass m. In this book we will concentrate on this last situation and we will not deal with some of the usual problems of mass variation in classical mechanics (as the problem of the truck which is losing sand, or that of the rocket which is expelling fuel and so changing its mass).

Newton's first corollary is called nowadays the principle of superposition of forces. It states that forces add like vectors (rule of the parallelogram).

Before going on we would like to make a few comments regarding the acceleration which appears in (2.7). This is the acceleration of the body relative to absolute space, as formulated by Newton. It can also be said that this is the acceleration of the body relative to an inertial frame. Although the earth is not an inertial frame (we know this because it rotates relative to the "fixed stars," because it is flattened on the poles, and due to experiments like that of Foucault's pendulum), in the majority of cases it can be considered as such. In practice this means that in general we can utilize Newton's laws in the laboratory frame of reference (the effects due to the lack of inertiality are usually small compared with what is being observed). This is valid in most situations in which the motions are restricted to a small area and have a short duration as compared with 24 hours. In this sense an observer with constant velocity (in magnitude and direction) relative to the earth can also be considered as inertial. When we want to study the rotation of the earth around its axis or the planetary motion, a very good inertial frame is the one defined by the "fixed stars." This is the reference frame relative to which the collection of stars belonging to the Milky Way has no translational acceleration and does not rotate as a whole.

There are two forms of Newton's third law: action and reaction in the strong and weak forms. In the first case the force is directed along the line joining the two bodies, and in the second case it has at least one of the components which is not along the line joining the bodies (Figure 2.1).

In this figure we show two examples where Newton's third law is valid in the strong form, two in the weak form, and two ficticious examples where it is not satisfied.

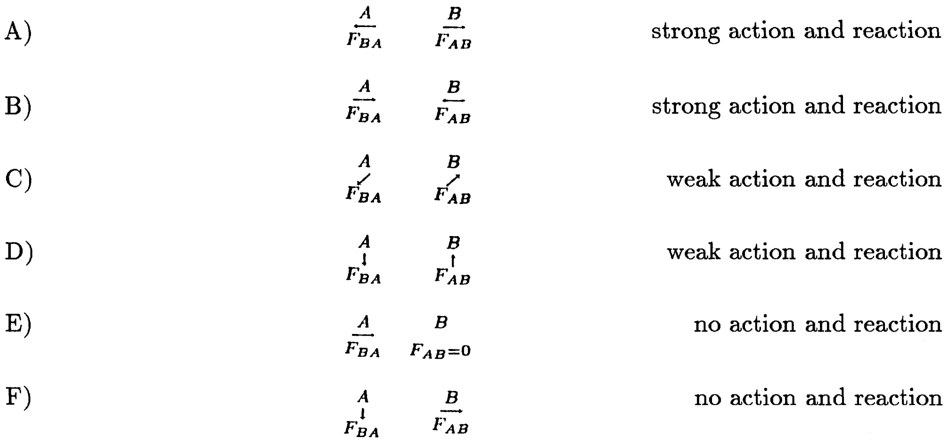

Figure 2.1

To solve any problem in mechanics we usually use (2.7). In order to do that we need precise relations for the force, and these will depend on the kind of interaction to which the body is subjected. Here are some examples:

I) Gravitational force (also given by Newton in 1687):

$$\vec{F}_{ji} = -Gm_i m_j \frac{\hat{r}_{ij}}{r_{ij}^2} \ , \tag{2.8}$$

where

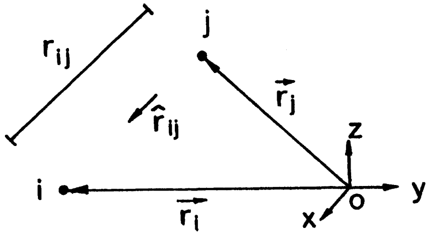

Figure 2.2

In equation (2.8) we have:

$$\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j = (x_i - x_j)\hat{x} + (y_i - y_j)\hat{y} + (z_i - z_j)\hat{z} \ ,$$
$$r_{ij} \equiv |\vec{r}_{ij}| = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2]^{1/2} \ , \tag{2.9}$$
$$\hat{r}_{ij} \equiv \vec{r}_{ij}/r_{ij} \ .$$

Moreover, G is the universal constant of gravitation ($G = 6.67 \times 10^{-11} Nm^2/kg^2$) and m_i (m_j) is called the gravitational mass of particle i (j). In (2.8) and (2.9) $\hat{r}_{ij}$ is the unit vector pointing from body j to i, r_{ij} is the distance between them, $\vec{r}_i$ ($\vec{r}_j$) is the vector which points from the origin of the cordinate system to body i (j), and $\vec{r}_{ij}$ is the vector pointing from j to i.

A typical example of a gravitational force is that of a body of gravitational mass m interacting with the earth (weight $\equiv \vec{P}$). This force is represented by

$$\vec{P} = m\vec{g} \ , \tag{2.10}$$

where $\vec{g}$ is the gravitational field of the earth. If the body is near the earth's surface we have $g \equiv |\vec{g}| = GM/R^2 \simeq 9.8\ ms^{-2}$, where M is the earth's gravitational mass and R its radius.

II) Elastic force:

Here we have (Fig. 2.3):

$$\vec{F} = -k\vec{x} \ , \tag{2.11}$$

where k is the elastic constant of the spring ($k > 0$), and x is the distance of the body to the equilibrium position ($x = l - l_o$, where l_o is the relaxed length of the spring and l its stretched or compressed length).

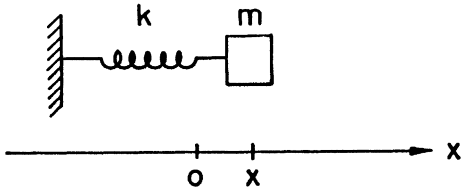

Figure 2.3

III) Force of dynamic friction:

$$\vec{F} = -b\vec{v} \,, \tag{2.12}$$

where b is the coefficient of friction ($b > 0$) between the body and the fluid (air or water, for instance), and $\vec{v}$ is the velocity of the body relative to this medium. Usually the dynamic or moving friction in a fluid is better represented by $\vec{F} = -b_1 v^2 \hat{v}$, where b_1 is a positive constant and $\hat{v} \equiv \vec{v}/|\vec{v}|$. The linear expression (2.12) is however much easier to handle and by a suitable choice of b works reasonably well.

These are the most common forces that we find in mechanics. In the next two sections we will see the forces which appear in electromagnetism. Putting together these expressions for the force and (2.7) allows us to describe the motion of bodies under the usual interactions.

The formulation presented here is the classical Newtonian one. There are other formulations to describe the motion of bodies in space, as the special and general theories of relativity of Einstein. In this book we will not discuss these alternative formulations as they are beyond the scope of this work.

2.3. Electric and Magnetic Forces

In this Section we discuss the forces which appear in electromagnetism.

I) Coulomb's force:

This is the electrostatic force, obtained by Coulomb in 1785, and which describes the force between two point charges q_i and q_j at rest relative to one another and to the laboratory:

$$\vec{F}_{ji} = \frac{q_i q_j}{4\pi\varepsilon_0} \frac{\hat{r}_{ij}}{r_{ij}^2} , \tag{2.13}$$

where $\hat{r}_{ij}$ and r_{ij} were defined in (2.9), see Figure 2.2, and ε_0 is a constant called the permittivity of free space ($\varepsilon_0 = 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$).

It is usually claimed that Eq. (2.13) was obtained by Coulomb from his measurements with the torsion balance. In an interesting paper published recently, Heering argued that Coulomb did not find the inverse square law by the doubtful measurements from his torsion balance experiments, but by theoretical considerations in analogy with Newton's law of gravitation (Heering, 1992).

If there are N charges at rest interacting with q_0 we obtain from (2.13) and from the principle of the superposition of forces that the resultant force acting on q_0 is given by

$$\vec{F} = q_0 \vec{E} , \tag{2.14}$$

where

$$\vec{E}(\vec{r}_0) \equiv \sum_{j=1}^{N} \frac{q_j}{4\pi\varepsilon_0} \frac{\hat{r}_{0j}}{r_{0j}^2} . \tag{2.15}$$

In (2.15) $\vec{E}$ is known as the electrostatic field obtained from Coulomb's law.

This force can also be derived from the potentials. Lagrange (1736 - 1813) had introduced the scalar potential function in gravitation in 1777. In 1782 Laplace (1749 - 1827) obtained the equation satisfied by this potential in free space, and the result was published in 1785. In 1811 Poisson (1781 - 1840) introduced the scalar potential in electromagnetism and obtained a more general result than that of Laplace, when he derived (1813) the equation satisfied by the potential in regions where there is matter and free charges. The electrostatic potential is given by

$$\phi(\vec{r}_0) \equiv \sum_{j=1}^{N} \frac{q_j}{4\pi\varepsilon_0} \frac{1}{r_{0j}} \ . \tag{2.16}$$

In (2.16) ϕ is known as the electrical scalar potential at the point $\vec{r}_0$ due to the charges q_j. Applying the gradient function in ϕ, operating at the point $\vec{r}_0, \nabla_0\phi$, yields the electric field of (2.15):

$$\nabla_0\phi \equiv \hat{x}\frac{\partial\phi}{\partial x_0} + \hat{y}\frac{\partial\phi}{\partial y_0} + \hat{z}\frac{\partial\phi}{\partial z_0} \ , \tag{2.17}$$

$$\vec{E} = -\nabla_0\phi \ . \tag{2.18}$$

In general the potential will change from point to point in space. The quantity $\nabla\phi$ is a vector which points, at each point in space, in the direction of the largest increase in ϕ around this point. Positive charges in a region of variable potential will move from the larger to the smaller potential if they are not under the influence of other forces (that is, they will move in the same direction given by $\vec{E}$). Negative charges will move in the opposite direction.

Applying (2.18) in (1.30) yields

$$V_{AB} \equiv \phi(\vec{r}_A) - \phi(\vec{r}_B) = -\int_A^B (\nabla\phi)\cdot d\vec{l} = \int_A^B \vec{E}\cdot d\vec{l} \ , \tag{2.19}$$

where V_{AB} is the potential difference between two points A and B. It is also called the voltage between A and B.

If we have a continuous distribution of charges in the volume V, (2.15) to (2.18) will become

$$\phi(\vec{r}_o) = \frac{1}{4\pi\varepsilon_o} \iiint_V \frac{\rho(\vec{r}_j)dV}{r_{oj}} \ , \tag{2.20}$$

$$\vec{E}(\vec{r}_o) = -\nabla_o \phi = \frac{1}{4\pi\varepsilon_o} \iiint_V \rho(\vec{r}_j) \frac{\hat{r}_{oj}}{r_{oj}^2} dV . \quad (2.21)$$

In these equations we replaced the summations by triple integrals and q_j by $dq = \rho(\vec{r}_j)dV$, where $\rho(\vec{r}_j)$ is the charge per unit volume at the point $\vec{r}_j$.

II) Magnetic force acting on a charge:

$$\vec{F} = q_0 \vec{v}_0 \times \vec{B} . \quad (2.22)$$

In Appendix A can be found a discussion of the historical origins and meanings of this expression.

In this expression $\vec{B}$ is the magnetic field at the position of q_0, which is generated by magnets or by electric currents. On the other hand $\vec{v}_0$ is the velocity of the charge q_0 *relative to an arbitrary observer or frame of reference.* When we apply it together with Newton's second law of motion in the form (2.7) then the observer or frame of reference must be an arbitrary inertial one. This is an important property (characteristic) of this expression and which has received little emphasis in the usual textbooks. When most books present this equation they usually say only the following: "A charge q moving with velocity $\vec{v}$ in a region with magnetic field $\vec{B}$ will experience a magnetic force given by $q\vec{v} \times \vec{B}$." That is, in general they do not specify what is this velocity which appears in (2.22). But obviously velocity is not an intrinsic property of any body, it is a relation between the charge and a certain body relative to which it is moving. For this reason one and the same charge can have several different velocities simultaneously. For instance, it can be simultaneously at rest relative to the earth, approaching itself to another charge, turning away with a larger velocity from a certain magnet, etc. Unfortunately most textbooks do not specify relative to what is to be understood the velocity $\vec{v}_0$ of the charge q_0 which appears in (2.22). We discussed this subject in (Assis and Peixoto, 1992).

We illustrate the many possibilities of interpretation in Figure 2.4:

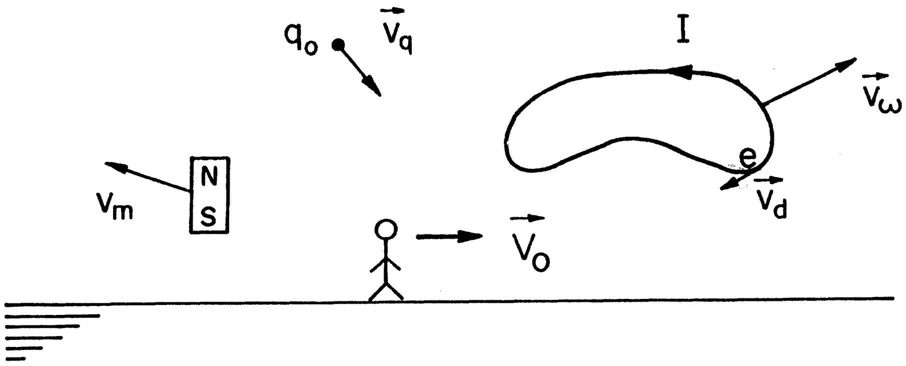

Figure 2.4

In this Figure the velocities are relative to the earth or laboratory. We have a charge q_o moving with velocity $\vec{v}_q$, an observer O moving with velocity $\vec{V}_O$, a magnet moving with velocity $\vec{v}_m$, a current carrying wire moving with velocity $\vec{v}_w$, and a typical electron in this wire moving with drifting velocity $\vec{v}_d$ relative to this wire. Which one of these velocities, or which combination (like $\vec{v}_q - \vec{v}_m$, $\vec{v}_q - \vec{V}_O$, etc.) should we insert in the place of $\vec{v}_o$ in (2.22)?

Facing this indefinition the student usually becomes confused between several possibilities: velocity of the charge relative to the magnet or wire which generate $\vec{B}$; relative to the earth or the laboratory; relative to any inertial observer or frame of reference; relative to the magnetic field; relative to the average velocity of the microscopic charges (electrons) which generate $\vec{B}$; relative to the $\vec{B}$-field detector; etc. Only in the chapters dealing with special relativity in these textbooks can we discover the meaning of the velocity which appears in (2.22), namely, velocity relative to an arbitrary inertial frame. Examples of this initial lack of definition of the velocity $\vec{v}_0$ can be found in several books: (Symon, 1971, p. 141); (Feynman, Leighton and Sands, 1964, Vol. 2, pp. 1.2 and 13.1); (Jackson, 1975, pp. 2 and 238); (Reitz and Milford, 1967, p. 148); (Sears, 1958, pp. 227-9); (Griffiths, 1989, p. 198); (Purcell, 1965, p. 150); (Panofsky and Phillips, 1964, p. 182); etc. The fact that the velocity which appears in (2.22) is relative to an inertial frame, and therefore it has different values for different inertial observers, is responsible for many of the typical characteristics of classical electromagnetism which we will discuss in the following.

It should be emphasized that the magnetic force is given by a vectorial product between the velocity of the charge and the magnetic field at its location, and this product is defined by the well known right hand rule of vectorial analysis. Moreover: $|\vec{F}| = |q_0 v_0 B \sin\theta|$, where θ is the angle between $\vec{v}_0$ and $\vec{B}$.

We gave in (2.15) and (2.16) the values of the electric Coulombian field and of the scalar electric potential. We give now the value of the magnetic field generated by a current element $I_j d\vec{l}_j$ belonging to metallic wires. It is given by the law of Biot (1774 - 1862) and Savart (1791 - 1841), of 1820-24, see (Whittaker, Vol. 1, pp. 82 - 83; Tricker, 1962, see especially pp. 453-455). In Chapter four we present a further discussion of this expression:

$$d\vec{B}(\vec{r}_0) \equiv \frac{\mu_0}{4\pi} I_j d\vec{l}_j \times \frac{\hat{r}_{0j}}{r_{0j}^2} , \tag{2.23}$$

In this expression μ_0 is called the vacuum permeability ($\mu_0 \equiv 4\pi \times 10^{-7} kgmC^{-2}$), and $I_j d\vec{l}_j$ is a current element of the circuit C_j. The magnetic field is then obtained by an integration over all the closed circuit C_j of arbitrary form:

$$\vec{B}(\vec{r}_0) = \frac{\mu_o}{4\pi} \oint_{C_j} I_j d\vec{l}_j \times \frac{\hat{r}_{oj}}{r_{oj}^2} . \tag{2.24}$$

The original works of Biot and Savart related with this law and their English translations can be found in (Biot and Savart, 1820 and 1824).

III) General electric force:

In 1729 Gray discovered electric conduction, that is, the passage of electric currents through metals, and then releasing the charge which had been generated by friction. This allowed the classification of materials as conductors and insulators. In 1780 Galvani discovered the electric current generated chemically (he found an electric current passing through frog nerves) and this allowed Volta (1745 - 1827) to begin the building of the first chemical batteries in 1792. This was the beginning of the study of electric currents. Previous to that there was only a systematic study of electrostatic and magnetostatic (study of natural magnets) phenomena.

To explain how the chemical battery works Volta introduced the concept of electromotive force. A clear discussion of this concept and its distinction from the electrostatic potential difference can be found in (Varney and Fisher, 1980).

In 1826 Ohm (1789 - 1854) discovered the law which bears his name: if a battery generates a voltage V then the electric current I which will pass through the metallic circuit connected to the terminals of this battery will depend on the resistance R of the wire according to the relation

$$I = \frac{V}{R} \; . \tag{2.25}$$

An English translation of this important work by Ohm can be found in (Ohm, 1966). See also (Jungnickel and McCormmach, 1986, Vol. 1, pp. 51 - 55).

This equation may be written in differential form as (using (2.19)):

$$\vec{J} = -\,\sigma\nabla\phi \; . \tag{2.26}$$

In this equation $\vec{J}$ is the current density in each point of the wire (its units are $Am^{-2} = Cs^{-1}m^{-2}$). It is related to the amount of current I which pass through the cross section S of the wire by

$$I = \int\int_S \vec{J}\cdot d\vec{a} \; . \tag{2.27}$$

Moreover, in (2.26) σ is the conductivity of the medium (material of which the wire is composed). The reciprocal of σ is called the resistivity of the medium and is sometimes represented by ρ (do not confuse it with the charge density): $\rho = 1/\sigma$. The unit of ρ is Ωm (Ohm-meter). Typical values of ρ for good conductors like metals (copper, silver, aluminium, gold) is of the order of $10^{-8}\Omega m$ at room temperature. For a semiconductor like saturated salt water we have $\rho = 0.044\Omega m$. On the other hand an insulator like pure water at room temperature has $\rho = 2.5 \times 10^5 \Omega m$. Good insulators like wood, glass and rubber have ρ ranging from $10^8 \Omega m$ to $10^{16} \Omega m$. All these values are for room temperature.

The relation between the resistance R of a wire of length L, arbitrary but uniform cross section of area A and made of a material with conductivity σ and resistivity ρ is:

$$R = \frac{L}{\sigma A} = \frac{\rho L}{A} \ . \tag{2.28}$$

For direct currents (dc), $\vec{E}$ and $\vec{J}$ are uniform within the wire, so that they have the same value for any point belonging to a cross section of the wire, although they may vary in magnitude and direction for different cross sections of the same wire. This means that dc currents flow uniformly through each cross section of the wire.

In 1831 Faraday (1791 - 1867) discovered that an electric current is generated not only by a battery but also when the magnetic flux across the closed circuit is varied (for instance, approaching or removing a magnet from this circuit, or varying the intensity of the current of a secondary circuit which generates $\vec{B}$ according to (2.24)). Faraday's law of induction can be written (when the circuit of resistance R is not connected to a battery) as (see Faraday, 1952):

$$I = \frac{\text{emf}}{\text{R}} \ , \tag{2.29}$$

where

$$\text{emf} \ \equiv -\frac{d}{dt}\Phi_M \ , \tag{2.30}$$

$$\Phi_M \equiv \int\int_S \vec{B} \cdot d\vec{a} \ . \tag{2.31}$$

In (2.29) and (2.30) "emf" is known as the induced electromotive force. Although it has the name of a force it is in reality a voltage of non-electrostatic origin, which has the Volt as its unit ($1V = 1kg \ m^2C^{-1}s^{-2}$). In (2.30) we introduced the minus sign to conform this law to Lenz's rule of 1834 which states that when we change the magnetic flux over a circuit the induced current flows in a direction such that the resultant force acting on the circuit tends to oppose the variation of the flux. In (2.30) and (2.31) Φ_M is the magnetic flux over the primary circuit, where the current is being induced, due to the magnetic field generated at the secondary circuit.

In 1845 Franz Neumann (1798 - 1895) introduced for the first time the magnetic vector potential $\vec{A}$. It is given by

$$\vec{A}(\vec{r}_0) \equiv \frac{\mu_0}{4\pi} \oint_{C_j} I_j \frac{d\vec{l}_j}{r_{0j}} \; . \tag{2.32}$$

The magnetic field at $\vec{r}_0$ can be found by applying the curl operator in $\vec{A}$:

$$\vec{B}(\vec{r}_0) = \nabla_0 \times \vec{A} \; . \tag{2.33}$$

When the curl acts in a vectorial field it generates a new vectorial field. In Cartesian coordinates we have:

$$\vec{B}(\vec{r}_0) = \hat{x}B_x + \hat{y}B_y + \hat{z}B_z = \nabla_0 \times \vec{A}$$

$$= \hat{x}\left(\frac{\partial A_z}{\partial y_0} - \frac{\partial A_y}{\partial z_0}\right) + \hat{y}\left(\frac{\partial A_x}{\partial z_0} - \frac{\partial A_z}{\partial x_0}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x_0} - \frac{\partial A_x}{\partial y_0}\right) \; . \tag{2.34}$$

The result of (2.34) with $\vec{A}$ given by (2.32) is (2.24).

The discovery of Neumann was to express Faraday's law only in terms of the vector potential he had created, namely (in Chapter five we describe in greater detail Faraday's law and $\vec{A}$):

$$\Phi_M = \oint_C \vec{A} \cdot d\vec{l} \, , \tag{2.35}$$

$$\text{emf} = -\frac{d}{dt}\Phi_M = \oint_C \left(-\frac{\partial \vec{A}}{\partial t}\right) \cdot d\vec{l} \; . \tag{2.36}$$

In this last expression we are supposing a static circuit, and this is the reason the total derivative could go inside the integral symbol and became a partial derivative.

Comparing (2.19) with (2.36) it follows that $-\partial\vec{A}/\partial t$ has the same role as an ordinary electrostatic field because both generate a voltage, and this voltage can give rise to a current.

After Neumann's work the theory of electric circuits was generalized to take into account the effects of self-induction. The main works along this line were done by Kirchhoff (1824 - 1887) during 1849-57; W. Thomson (1824 - 1907), also known as Lord Kelvin, during 1853-4 (together with Stokes); and Heaviside (1850 - 1925), in 1876. Already in 1857 Kirchhoff wrote the general electric force as we know it today, namely, including the influences of the scalar and vector potentials.

Kirchhoff considered the force acting on a charge q inside a current carrying wire as being due to two parts. The first one was the usual electrostatic force due to the free electricity along the surface of the wire (the distribution of free electricity was not neutralized everywhere along the surface of the wire due to the battery), and could be written as $-q\nabla\phi$. The second part was due to the alteration of the intensity of the current in all portions of the wire. It could be written as $-q\partial\vec{A}/\partial t$. He was here taking into account the self-induction of the wire. He then assumed a generalization of Ohm's law (2.26) stating that even when the current is not stationary the density of the current would be equal to the product of the conductivity and electromotive force per unit charge considering both contributions, namely, the electrostatic one and that due to the self-induction. Mathematically he assumed that

$$\vec{J} = -\,\sigma\left(\nabla\phi + \frac{\partial\vec{A}}{\partial t}\right) . \tag{2.37}$$

We can write this equation as

$$\vec{J} = \sigma\vec{E} , \tag{2.38}$$

where $\vec{E}$ is a generalized electric field defined by

$$\vec{E}(\vec{r}_0) \equiv -\nabla_0\phi - \frac{\partial\vec{A}}{\partial t} . \tag{2.39}$$

The general electric force is then given by

$$\vec{F} = q\vec{E} , \tag{2.40}$$

with $\vec{E}$ defined by (2.39).

Kirchhoff's important works related to these topics and their English translations can be found in (Kirchhoff, 1850, 1857 a and b).

2.4. Lorentz's Force Law

In classical electromagnetism the general expression for the electromagnetic force acting on a charge is known as Lorentz's force. It includes electric and magnetic fields and is given by

$$\vec{F} = q_0\vec{E} + q_0\vec{v}_0 \times \vec{B} \ , \tag{2.41}$$

where the electric field $\vec{E}$ is given by (2.39) and the magnetic $\vec{B}$ by (2.33). Putting together this expression with Newton's second law, (2.7), allows the classical description of the motion of a charge interacting with arbitrary electric and magnetic fields, namely:

$$q_o\vec{E} + q_o\vec{v}_o \times \vec{B} = m_o\vec{a}_o \ , \tag{2.42}$$

or

$$-q_o\left(\nabla_o\phi + \frac{\partial\vec{A}}{\partial t}\right) + q_o\vec{v}_o \times (\nabla_o \times \vec{A}) = m_o\vec{a}_o \ . \tag{2.43}$$

The expression (2.41) was first obtained by H. A. Lorentz (1853 - 1928), a Dutch theoretical physicist, in 1895: (Lorentz, 1895), (Pais, 1982, p. 125; and Pais, 1986, p. 76). This happened after Maxwell's death (1879). In this work Lorentz presented a microscopic corpuscular structure for Maxwell's formulation of electromagnetism, which was all based on the continuum. Lorentz began to describe the sources of the fields as discrete entities, namely, charges and current elements.

Let us now make a first analysis of Lorentz's force (a detailed discussion is presented in Chapter 6). In the first place we observe that the scalar potential, and Coulomb's electric field (2.15) and (2.16), depend only on the distances between interacting charges, but they do not depend on their velocities. On other hand the magnetic field $\vec{B}$ in (2.24) depends not only on the distances but also on the electric current. As current is charge in motion, $\vec{B}$ is a function of the velocity of the source charges (namely, the charges which generate the field) and of the distances between interacting charges. The vector potential is directly related to $\vec{B}$ through (2.33) and so it is also a function of the velocity of the source charges and of the distances between interacting charges.

There are then three components in Lorentz's force (2.41): (I) Coulomb's force, $-q_0\nabla_0\phi$, which depends on the relative positions of charges at rest. (II) The magnetic force, $q_0\vec{v}_0 \times \vec{B}$, which depends on the velocity $\vec{v}_0$ of the test charge (that is, the charge which experiences this force) and on the velocity of the source charges (through $\vec{B}$). (III) The induction force, $-q_0\partial\vec{A}/\partial t$, which has a component depending on the acceleration of the source charges ($\vec{A}$ connected to $\vec{B}$, which depends on the velocity, and in this component of the force there appears $\partial\vec{A}/\partial t$), but which does not depend on the velocity nor on the acceleration of the test charge. The other component of the induction force depends on the velocity of the charges which generate $\vec{B}$. This can be seen observing that induction happens not only when the intensity of the current varies (as above, acceleration different from zero), but also when the intensity of the current is constant but a magnet approaches the primary circuit. In this last situation, for induction to happen the existence of $\vec{B}$ is necessary, and this shows that this component of Lorentz's force will depend on the velocity of the source charges.

In Chapter 6 we make a comparison between the forces of Lorentz and of Weber.

2.5. Maxwell's Equations

To solve our initial problem, which is to describe the motion of charges in space relative to one another, the first step has been done. That is, we have the equation of motion (2.7) and the corresponding electromagnetic force (2.41). So if a particle (material point) of charge q and inertial mass m is moving with velocity $\vec{v}$ relative to an arbitrary inertial frame in a region where there are electric and magnetic fields $\vec{E}$ and $\vec{B}$, its motion in an inertial coordinate system is described by the equation

$$q\vec{E} + q\vec{v} \times \vec{B} = m\vec{a} \ . \tag{2.44}$$

Usually the fields $\vec{E}$ and $\vec{B}$ which appear in this equation are not those generated by the charge q itself, but are generated by other charges and current distributions, called the sources of $\vec{E}$ and $\vec{B}$. To solve the complete problem self-consistently (that is, to describe the motion of an ensemble of charges interacting with one another in the absence of external electromagnetic fields) it is necessary to know how the sources generate the fields. Given a charge and a current distribution, we need to obtain the fields $\vec{E}$ and $\vec{B}$ generated by this system. This is exactly the function of Maxwell's equations.

We represent the volumetric charge density by ρ (its units are Cm^{-3}) and the current density by $\vec{J}$ (its units are $Am^{-2} = Cs^{-1}m^{-2}$). The amount of charge inside a volume V and the amount of current which pass through a surface S are given by

$$Q = \iiint_V \rho dV \ , \tag{2.45}$$

$$I = \iint_S \vec{J} \cdot d\vec{a} \ . \tag{2.46}$$

In these expressions dV is an element of volume and $d\vec{a}$ is a vectorial element of area, always normal to the surface S at each point. By convention if S is a closed surface then $d\vec{a}$ will point outward at all points, and the double integral over a closed surface is represented by $\oiint$. In (2.46) I is obtained by a scalar or inner product between $\vec{J}$ and $d\vec{a}$, which is defined by the usual rules of vectorial analysis. By convention $\vec{J}$ points in the opposite direction from the motion of the negative charges (usually electrons). If in a region of space we have positive and negative charges in motion relative to an inertial frame, with velocities $\vec{v}_+$ and $\vec{v}_-$, respectively, this yields

$$\vec{J} = \rho_+\vec{v}_+ + \rho_-\vec{v}_- \ . \tag{2.47}$$

In this equation $\rho_+(\rho_-)$ is the positive (negative) charge density, which is moving with velocity $\vec{v}_+(\vec{v}_-)$. Usually we have only macroscopic neutral currents (as is the case of the current in metallic wires or in gaseous plasmas) so that $\rho_- = -\rho_+$. In these cases $\vec{J} = \rho_+(\vec{v}_+ - \vec{v}_-)$. Moreover in usual metallic currents only the electrons move so that $\vec{J} = \rho_-\vec{v}_- = -\rho_+\vec{v}_-$. In Maxwell's equations ρ and $\vec{J}$ are the basic sources which generate $\vec{E}$ and $\vec{B}$.

Maxwell's equations are usually presented in two ways: differential and integral forms. Initially we present them in a differential form, supposing the sources and the fields in vacuum. All quantities involved here are in general functions of position and of time: $\varphi = \varphi(x,\ y,\ z,\ t)$, $\vec{G} = \hat{x}G_x(x,\ y,\ z,\ t) + \hat{y}G_y(x,\ y,\ z,\ t) + \hat{z}G_z(x,\ y,\ z,\ t)$. Maxwell's equations are:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \ , \tag{2.48}$$

$$\nabla \times \vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} \ , \tag{2.49}$$

$$\nabla \cdot \vec{B} = 0 \ , \tag{2.50}$$

$$\nabla \times \vec{E} = -\frac{\partial\vec{B}}{\partial t} \ . \tag{2.51}$$

In (2.49) the constant c is given by

$$c = \frac{1}{\sqrt{\varepsilon_o\mu_o}} \ . \tag{2.52}$$

It has the same value as the velocity of light in vacuum. The first time this electromagnetic constant c appeared in electromagnetism was in W. Weber's force of 1846, to be described in the next Chapter. The first to measure this quantity were W. Weber and R. Kohlrausch, in 1856. They found $c = 3.1 \times 10^8 m/s$. Maxwell measured this quantity only in 1868 and found $c = 2.8 \times 10^8 m/s$, (Maxwell, 1954, Vol. 2, Arts. [786 - 787], pp. 435 - 436; Maxwell, 1965, pp. 125 - 143).

Some comments on these equations: (2.48) is known as Gauss's law and it is essentially equivalent to Coulomb's force, (2.13) to (2.21), and we show this in Section 2.6. The second equation is known as "Ampère's" circuital law. Although it has nowadays this name, it was never derived or written down by Ampère. The first to obtain this equation, **even without the term in the displacement current**, was Maxwell in his first paper dealing with electromagnetism of 1855, twenty years after Ampère's death (Maxwell, 1965, pp. 155 - 229; Whittaker, 1973, Vol. 1, pp. 242 - 245). Due to this fact we will call this equation the magnetic circuital law instead of "Ampère's" circuital law. To arrive at this equation Maxwell utilized the results Ampère had obtained with his force between current elements (to be described in Chapter four). Maxwell introduced the term in the displacement current (the term with Weber's constant c in (2.49)) in his second paper on electromagnetism of 1861/2 (Maxwell, 1965, pp. 451 - 513; Whittaker, 1973, Vol. 1, pp. 247 - 255). The term $\varepsilon_o \partial \vec{E}/\partial t$ is also called a current, as is $\vec{J}$, because although it does not indicate a net transport of electric charge, it has been observed that a varying electric field generates a magnetic field, and this is one of the fundamental properties of ordinary electric currents. The third equation, (2.50), represents the experimental observation that we cannot separate spatially the north and south poles of a magnet or current. It is the mathematical equation describing the non-existence of magnetic monopoles. The fourth and last equation is known as Faraday's law. It was Faraday who discovered this induction law in 1831.

From this short account we can see that what we call Maxwell's equations are in fact laws due to other researchers, and which were known in Maxwell's time. Maxwell discovered that this set of equations formed a coherent ensemble and introduced the displacement current. This last feat was really his great discovery, because the displacement current is essential for the obtaining of electromagnetic waves through Maxwell's equations. In this way he unified optics with electromagnetism, identifying light with an electromagnetic radiation. Maxwell introduced this term so that this set of equations could become compatible with the equation of continuity for electric charges, which is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \ . \tag{2.53}$$

To our knowledge the first to write down this equation describing the conservation of charges was Kirchhoff in 1857, in linear form (Kirchhoff, 1857 a) and also in the form (2.53), (Kirchhoff, 1857 b).

Applying $\partial/\partial t$ in (2.48) and inverting the order of derivatives yields

$$\nabla \cdot \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{1}{\varepsilon_o} \frac{\partial \rho}{\partial t} \ . \tag{2.54}$$

Utilizing (2.49), (2.52) and (1.27) in (2.54) yields

$$\nabla \cdot \left(\frac{1}{\varepsilon_o \mu_o} \nabla \times \vec{B} - \frac{1}{\varepsilon_o} \vec{J} \right) = - \frac{1}{\varepsilon_o} \nabla \cdot \vec{J} = \frac{1}{\varepsilon_o} \frac{\partial \rho}{\partial t} \ . \tag{2.55}$$

And this is the equation of continuity (2.53) in view of (2.47) and the fact that $\rho = \rho_+ + \rho_-$. This means that to obtain (2.53) from Maxwell's equations the displacement current in (2.49) is essential.

We present now Maxwell's equations in integral form, obtained from (2.48) to (2.51) and utilizing the theorems of Gauss and Stokes, (1.32) and (1.33):

$$\oiint_S \vec{E} \cdot d\vec{a} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int\int_V\int \rho dV \ , \tag{2.56}$$

$$\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I + \frac{1}{c^2} \frac{d}{dt} \Phi_E$$

$$= \mu_0 \int\int_S \vec{J} \cdot d\vec{a} + \frac{1}{c^2} \frac{d}{dt} \int\int_S \vec{E} \cdot d\vec{a} \ , \tag{2.57}$$

$$\oiint_S \vec{B} \cdot d\vec{a} = 0 \ , \tag{2.58}$$

$$\text{emf} \ \equiv \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int\int_S \vec{B} \cdot d\vec{a} \ . \tag{2.59}$$

To arrive at these equations in integral form the theorems of Gauss and Stokes, (1.32) and (1.33), were utilized. The complete history of these theorems seems to have never been written. Gauss's theorem was utilized in particular cases by Lagrange in 1760-61, was then formulated in more definite form by Gauss in 1813 and 1839-40 and by Ostrogadsky in 1828-31. By the 1840's it was widely known. Stokes's theorem was utilized in particular cases by Ampère in 1820-27, and was first formulated by Kelvin in 1850 and by Stokes in 1854. It was given by Stokes as an examination question for the Smith's Prize Examination of 1854. Among the candidates for the prize was Maxwell, who learned of the theorem this way, and utilized it in his papers dealing with electromagnetism which he wrote in the following years. By the 1870's this theorem was frequently employed. For these facts see (Crowe, 1985, pp. 146 - 147, note 29; and Jungnickel and McCormmach, 1986, Vol. 1, pp. 66 - 69).

To get the equations of electromagnetic waves it is only necessary to apply the curl operator to both sides of (2.49) and (2.51), and to utilize the vectorial identity (1.29). This yields

$$\nabla(\nabla \cdot \vec{B}) - \nabla^2\vec{B} = \mu_o\nabla \times \vec{J} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla \times \vec{E}) \ , \tag{2.60}$$

$$\nabla(\nabla \cdot \vec{E}) - \nabla^2\vec{E} = -\ \frac{\partial}{\partial t}(\nabla \times \vec{B}) \ . \tag{2.61}$$

Applying (2.48) to (2.51) in (2.60) and (2.61) yields, after rearranging the terms

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{B} = -\ \mu_o\nabla \times \vec{J} \ , \tag{2.62}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = \frac{1}{\varepsilon_o}\nabla\rho + \mu_o\frac{\partial \vec{J}}{\partial t} \ . \tag{2.63}$$

These equations describe the electromagnetic waves obtained through Maxwell's equations.

Before closing this Section let us discuss a little more the potentials ϕ and $\vec{A}$. As we saw, Maxwell's equations depend only on $\vec{E}$ and $\vec{B}$, and also Lorentz's force (2.41) depends only on $\vec{E}$ and $\vec{B}$. This means that these are the real fields of classical electromagnetism, that is, those which influence the force and the motion of charges. In (2.33) and (2.39) we expressed $\vec{E}$ and $\vec{B}$ in terms of ϕ and $\vec{A}$. As the gradient of a constant is zero, we can add or subtract a constant to ϕ without changing the value of the electric field or of the force. Likewise we can add a gradient of a scalar function φ to $\vec{A}$ without changing the value of $\vec{B}$, due to (1.26). This allows a certain freedom in the choice of $\vec{A}$ and ϕ, which receives the name of gauge. Below we present the gauges of Coulomb and of Lorentz, characterized by the definition of $\nabla \cdot A$:

$$\text{Coulomb's gauge} : \nabla \cdot \vec{A} \equiv 0 \ , \tag{2.64}$$

$$\text{Lorentz's gauge} : \nabla \cdot \vec{A} \equiv -\frac{1}{c^2}\frac{\partial \phi}{\partial t} \ . \tag{2.65}$$

It should be remarked that in both gauges Lorentz's force and Maxwell's equations are exactly the same.

Utilizing these gauges and Maxwell's equations we can obtain wave equations for ϕ and $\vec{A}$.

Applying (2.33) and (2.39) in Maxwell's equations yield, using also (1.26) to (1.29):

$$\nabla^2 \phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon_o} \ , \tag{2.66}$$

$$\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_o \vec{J} - \frac{1}{c^2}\nabla\left(\frac{\partial \phi}{\partial t}\right) - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} \ . \tag{2.67}$$

In Coulomb's gauge (2.64) these equations become

$$\nabla^2 \phi = -\frac{\rho}{\varepsilon_o} \ , \tag{2.68}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_o\vec{J} + \frac{1}{c^2}\nabla\left(\frac{\partial\phi}{\partial t}\right) . \tag{2.69}$$

In Lorentz's gauge (2.65) we have

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\phi = -\frac{\rho}{\varepsilon_o} , \tag{2.70}$$

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{A} = -\mu_o\vec{J} . \tag{2.71}$$

In a region without charges and currents we obtain from (2.62), (2.63), (2.70) and (2.71) that all rectangular components of $\vec{E}$, $\vec{B}$, $\vec{A}$, and ϕ itself, satisfy the same equation, namely:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\xi = 0 . \tag{2.72}$$

Although equation (2.68) is different from (2.70), and so they have different solutions (the same happens with (2.69) and (2.71)), this is not a problem in classical electromagnetism as the really important fields are $\vec{E}$ and $\vec{B}$, and not ϕ and $\vec{A}$ (we are not considering here the Aharonov-Bohm effect and other aspects of quantum mechanics). And in both gauges $\vec{E}$ and $\vec{B}$ satisfy the same equations, (2.62) and (2.63).

From a general point of view we can say that Newton's second law connected with Lorentz's force, together with Maxwell's equations and the constitutive relations of the medium constitute the kernel of classical electromagnetism. As we said previously, Maxwell's equations are independent of Lorentz's force. This means that Maxwell's equations could remain valid even if we had a force law different from that of Lorentz. Another relevant fact is that Maxwell's equations are independent from one another. This means that we cannot, for instance, derive Faraday's law of induction (2.51) from the magnetic circuital law (2.49) or vice-versa.

2.6. Derivation of Gauss's Law

In this section we derive Gauss's law from Coulomb's one, (2.13). When q_0 is interacting with N other charges this force can be written as (2.14), with the electric field given by (2.15). From (2.15) we observe that the electric field of each charge $q_j, \vec{E}_j \equiv q_j \hat{r}_{0j}/(4\pi\varepsilon_0 r_{0j}^2)$, is radial with center in this charge, and falls with the square of the distance. To arrive at Gauss's law we begin supposing a single charge q_j and a closed surface S_0 as indicated in Figure 2.5.

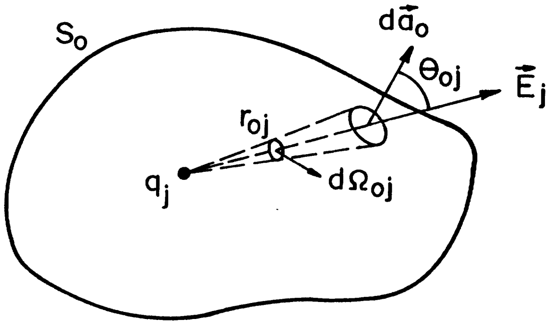

Figure 2.5

By convention $d\vec{a}_0$ is an element of area in this surface localized at $\vec{r}_0$ and pointing outwards. It is easily seen that

$$\vec{E}_j \cdot d\vec{a}_0 = \frac{q_j}{4\pi\varepsilon_0} \frac{\cos\theta_{0j}}{r_{0j}^2} da_0 \ , \tag{2.73}$$

where θ_{0j} is the angle between $\hat{r}_{0j}$ and $d\vec{a}_0$. As $\vec{E}_j$ is radial from the charge q_j it turns out that $\cos\,\theta_{0j} da_0 = r_{0j}^2 d\Omega_{0j}$, where $d\Omega_{0j}$ is the element of spherical angle subtended by da_0 at the position of q_j (see Figure 2.5). So

$$\vec{E}_j \cdot d\vec{a}_0 = \frac{q_j}{4\pi\varepsilon_0} d\Omega_{0j} \ . \tag{2.74}$$

Integrating $\vec{E}_j \cdot d\vec{a}_0$ over all of surface S_0 it is easily seen that $\oint_{S_o} d\Omega_{0j} = 4\pi$ if q_j is inside S_0, or it will be equal to zero if q_j is outside S_0. Using the principle of superposition to sum the contribution of the N charges results in

$$\oiint_{S_0} \vec{E} \cdot d\vec{a}_0 = \frac{1}{\varepsilon_0} \sum_{j=1}^{m} q_j = \frac{1}{\varepsilon_0} \iiint_{V_0} \rho dV_0 \ , \tag{2.75}$$

where the summation $\sum_{j=1}^{m}$ extends only to the internal charges, and the last equality is obtained supposing a continuous charge distribution, $q \to \rho dV$, where V_0 is the volume enclosed by S_0. This is Gauss's law in integral form, (2.56).

To arrive at a differential form, (2.48), we only need to utilize Gauss's theorem, (1.32). Then it comes

$$\iiint_{V_0} \left(\nabla \cdot \vec{E} - \frac{\rho}{\varepsilon_0} \right) dV_0 = 0 \ . \tag{2.76}$$

As this equation remains valid for any volume V_0, the integrand needs to be zero everywhere, and so yielding (2.48).

Another proof using more advanced properties of vectorial calculus can be obtained through the electrostatic potential, (2.16), together with (2.18) and (1.28):

$$\iiint_{V_0} (\nabla_0 \cdot \vec{E}) dV_0 = - \iiint_{V_0} [\nabla_0 \cdot (\nabla_0 \phi)] dV_0$$

$$= - \iiint_{V_0} \nabla_0^2 \phi = \sum_{j=1}^{m} \frac{-1}{4\pi\varepsilon_0} \iiint_{V_0} q_j \left(\nabla_0^2 \frac{1}{r_{0j}} \right) dV_0 \ . \tag{2.77}$$

Equation (1.38) can be written as

$$\nabla_0^2 \frac{1}{r_{0j}} = -4\pi\delta(\vec{r}_0 - \vec{r}_j) \ . \tag{2.78}$$

Applying (1.36) and (2.78) in (2.77) yields (2.75) or (2.76) if we also utilize Gauss's theorem (1.32). And from this point we can arrive at Gauss's law in differential form.

Chapter 3 / Weber's Electrodynamics

3.1. Wilhelm Weber and His Electromagnetic Researches

In this chapter we present a short description of Weber's life and some of his experimental researches in electromagnetism. For a biography of Weber and further references see, for instance, (Woodruff, 1976), (Jungnickel and McCormmach, 1986), (Atherton, 1989), (Thomson, 1885), (O'Rahilly, 1965, Vol. 2, Chapter 11), (Whittaker, 1973, Vol. 1, Chapter 7), (Wise, 1981), (Assis, 1991 a), (Harman, 1982, pp. 32, 96 and 103 - 107), (Archibald, 1989).

Wilhelm Eduard Weber was a German experimental physicist born in Wittenberg in 1804. His family moved to Halle in 1813. He entered the University of Halle in 1822 and there he wrote his doctoral dissertation in 1826 under J. S. C. Schweigger, working with acoustics: experimental and theoretical investigation of the acoustic coupling of tongue and air cavity in reed organ pipes. He became assistant professor at Halle University in 1828. In this year he met C. F. Gauss (1777 - 1855) at a scientific convention in Berlin. In 1831 he obtained the professorship of physics at Göttingen University, where Gauss had a post since 1807. It was Gauss who interested Weber in electromagnetic problems, a subject on which he had never worked until this time. Weber stayed there until 1837 and established a close friendship and collaboration with Gauss. In 1837 Weber lost his position at Göttingen University due to political problems with the new king of Hannover, Ernst August. He continued to work in Göttingen in collaboration with Gauss and also travelled to London and Paris. He became a professor of physics at the University of Leipzig in 1843 and stayed there unitl 1848. At Leipzig University he joined his brothers Ernst Heinrich and Eduard, both anatomists, and also G. T. Fechner, a friend of the family. Ernst Heinrich and Fechner established in 1834 and 1860 what is known as the Weber-Fechner law in psychology. Fechner was an atomist and influenced W. Weber in scientific matters. It was in Leipzig that Weber formulated his force between electrical charges in 1846 and the corresponding potential energy in 1848 (there is an English translation of this work, which is an outline of his 1846 memoir where he describes how he arrived at his force law, in (Weber, 1848 a)). He returned to his old position at the University of Göttingen in 1848 after the revolution which happened this year in Italy and Germany (Germany was unified as an independent state in 1871). He became director of the astronomical observatory and was closely associated with Rudolph Kohlrausch with whom he performed important electromagnetic measurements in 1855 - 1856. He ceased research in the 1870's, retired from teaching in 1873 and died at the age of eighty-six in Göttingen in 1891.

In 1832 Gauss presented a paper, written with Weber's assistance, where he introduced absolute units of measurement into magnetism. This means that the measurement of the strength of a magnetic property was reduced to measurements of length, time and mass. In this way it was reproducible anywhere in the world without the need of a particular precalibrated magnetic instrument. In Gauss's paper he obtained the absolute measures of bar magnetism and of terrestrial magnetism. In 1840 Weber defined the absolute electromagnetic unit of current in terms of the deflection of the magnetic needle of a tangent galvanometer. He determined the amount of water decomposed by the flow of a unit of current for one second, that is, by a unit of charge. In 1851 he defined an absolute measure for electrical resistance (there is an English translation of this work in (Weber, 1851)). In 1855 and 1856 he collaborated with R. Kohlrausch in the measurement of the ratio between electrodynamic and electrostatic units of charge. Maxwell elogiated Weber for these accomplishments with these words: "The introduction, by W. Weber, of a system of absolute units for the measurement of electrical quantities is one of the most important steps in the progress of the science. Having already, in conjunction with Gauss, placed the measurement of magnetic quantities in the first rank of methods of precision, Weber proceeded in his *Electrodynamic Measurements* not only to lay down sound principles for fixing the units to be employed, but to make determinations of particular electrical quantities in terms of these units, with a degree of accuracy previously unattempted. Both the electromagnetic and the electrostatic systems of units owe their development and practical applications to these researches" (Maxwell, 1954, Vol. 2, Art. [545], pp. 193 - 194).

Let us quote Weber when he describes these absolute units of measure into electromagnetism:

"If there are measures for time and space, a special *fundamental measure of velocity* is not necessary; and in like manner no special *fundamental measure for electric resistance* is needed if there are measures for electromotive force and for intensity of the current; for then *that resistance can be taken as unit of measure*, which a closed *conductor possesses in which the unit of measure of electromotive force produces the unit measure of intensity.* Upon this depends the reduction of the measurements of electric resistance to an absolute standard.

It might be thought that this reduction would be more simply effected by reverting to the special dimensions, length and section, and adhering to that metal (*copper*) which is best fitted and is most frequently used for such conductors. In that case the absolute unit of measure of resistance would be that resistance which a copper conductor possesses whose length is equal to the measure of length, and whose section is equal to the measure of surface, in which, therefore, besides measure of length and surface, *the specific resistance of copper must be given as unit* for the specific resistance of conducting surfaces. Thus a special *fundamental measure for specific resistances* would be necessary, the introduction of which would be open to question. *First*, because there would be no saving in the number of the fundamental measures if, in order to do without a fundamental measure for the absolute resistance, another fundamental measure must be introduced which is otherwise superfluous. And *secondly*, neither the copper nor any other metal is fitted for use in establishing a fundamental measure for resistances. Jacobi says that there are differences in the resistances of even the chemically purest metals, which cannot be explained by a difference in the dimensions; and that, accordingly, if one physicist referred his rheostat and multiplicator to copper wire a metre in length and 1 millimetre thick, other physicists could not be sure that his copper wire and theirs had the same *coefficient of resistance*, that is, whether the *specific* resistance of all these wires was the same. The reduction of measurements of galvanic resistance to an absolute measure can therefore only have an essential importance, and find practical application, if it takes place in the first mentioned way, in which no other measures are presupposed than those for *electromotive force* and for *intensity*.

The question then arises, as to what are the measurements of *electromotive forces* and *intensities*? In measuring these magnitudes, no specific *fundamental measures* are requisite, but they can be referred to *absolute measure* if the magnetic measures for *bar magnetism* and *terrestrial magnetism*, as well as *measure of space and time*, are given.

As an absolute unit of measure of electromotive force, may be understood *that electromotive force which the unit of measure of the earth's magnetism exerts upon a closed conductor, if the latter is so turned that the area of its projection on a plane normal to the direction of the earth's magnetism increases or decreases during the unit of time by the unit of surface.* As an absolute unit of intensity, can be understood *the intensity of that current which, when it circulates through a plane of the magnitude of the unit of measure, exercises, according to electro-magnetic laws, the same action at a distance as a bar-magnet which contains the unit of measure of bar magnetism.* The absolute measures of *bar magnetism* and of *terrestrial magnetism* are known from the treatise of Gauss, "Intensitas Vis Magneticae Terrestris ad mensuram absolutam revocata," Göttingae, 1833 (Poggendorff's *Annalen*, vol. xxviii. pp. 241 and 591).

From this statement it is clear that the measures of electric resistances can be referred to an absolute standard, provided measures of *space, time*, and *mass* are given as fundamental measures; for the absolute measures of *bar magnetism* and of *terrestrial magnetism* depend simply on these three fundamental measures. (...)" (Weber, 1851).

For a discussion of absolute measures in electromagnetism see also (Jungnickel and McCormmach, 1986, Vol. 1, pp. 63 - 75 and 122 - 145).

Diamagnetism was discovered by Faraday in 1845, who initially explained it by the hypothesis of diamagnetic polarity. Later on he abandoned this conception. It was, however, accepted by Weber in 1848, who also demonstrated experimentally in 1852 the existence of the effect. He also extended Ampère's theory of magnetism to cover the phenomenon of diamagnetism. Weber postulated a radical distinction between the natures of paramagnetism and diamagnetism, which was later confirmed by many experimental facts. English translations of these two papers of 1848 and 1852 can be found in (Weber, 1848 b) and (Weber, 1852). Weber's theory on diamagnetism was advocated by Maxwell (Whittaker, 1973, Vol. 1, pp. 194 - 195 and 208 - 211; Maxwell, 1954, Vol. 2, Chapter VI, "Weber's Theory of Induced Magnetism", Arts. [442 - 448], pp. 79 - 94) and is adopted in its main parts until today.

Weber introduced his electrodynamometer in 1846. Descriptions can be found in (Weber, 1848 a), (Maxwell, 1954, Vol. 2, Art. [725], pp. 367 - 371, "Weber's Electrodynamometer") and (Harman, 1982, pp. 32 - 33). According to Maxwell in these pages, "The experiments which he [Weber] made with it furnish the most complete experimental proof of the accuracy of Ampère's formula as applied to closed currents, and form an important part of the researches by which Weber has raised the numerical determination of electrical quantities to a very high rank as regards precision. Weber's form of electrodynamometer, in which one coil is suspended within another, and is acted on by a couple tending to turn it about a vertical axis, is probably the best fitted for absolute measurements." Due to these facts for some time the name "Weber" was used for the unit of current. At an international congress held in Paris in 1881 on the electrical units, H. von Helmholtz (1821 - 1894), the leader of the German delegation, proposed the name "Ampère" for the unit of current instead of "Weber," and this was accepted. The term "Weber" was officially introduced for the practical unit of magnetic flux in 1935. Weber and Helmholtz were leading German scientists in the last century and their careers were similar in many respects. Despite this fact their personal relations were always difficult. Maybe this was one of the reasons why Helmholtz never accepted Weber's electrodynamics. Later on we will discuss other aspects related to Helmholtz and Weber.

Coulomb's force of 1785, (2.13), can be written for charges e and e' in electrostatic units separated by a distance r as

$$F = \frac{ee'}{r^2} \ . \tag{3.1}$$

Repulsion or attraction occurs accordingly as this expression has a positive or negative value.

In 1823 Ampère obtained his force between the current elements ids and $i'ds'$ when they are separated by a distance r. In the next chapter we discuss this expression in detail.

In electrodynamic units it can be written as

$$d^2F = -\frac{ii'dsds'}{r^2}\left(\cos\varepsilon - \frac{3}{2}\cos\theta\cos\theta'\right) . \tag{3.2}$$

Repulsion or attraction occurs as this expression has a positive or negative value. In (3.2) ε is the angle between the positive direction of the currents in ds and ds', and θ and θ' are the angles between these positive directions and the connecting right line between them.

In order to unify electrostatics (Coulomb's force (3.1)) with electrodynamics (Ampère's force (3.2)), Weber proposed in 1846 that each current element in metallic conductors should be considered as the usual charges in motion. He considered then the current i as eua, where u is the velocity of the charge e and a is a constant factor of dimensions sm^{-1}. He also assumed Fechner hypothesis of 1845 according to which the current in metallic conductors is due to an equal amount of positive and negative charges moving in opposite directions relative to the wire with equal velocities (Fechner, 1845) This was Fechner's last paper on physics (Jungnickel and McCormmach, 1986, Vol. 1, pp. 137 - 138). With these ideas and working algebraically beginning from (3.2) he arrived at the following expression in 1846 for the force between two charges in relative motion (Weber, 1846 and 1848 a):

$$F = \frac{ee'}{r^2}\left[1 - \frac{1}{c_W^2}\left(\frac{dr}{dt}\right)^2 + \frac{2}{c_W^2}r\frac{d^2r}{dt^2}\right] . \tag{3.3}$$

The mathematical procedure followed by Weber in order to derive (3.3) from (3.2) and (3.1) can also be found in (Maxwell, 1954, Vol. 2, Chapter 23, Arts. [846 - 851], pp. 480 - 483); (Whittaker, 1973, Vol. 1, pp. 201 - 203).

Weber's paper of 1846 was the first of his eight major publications between 1846 and 1878 under the series title "Determination of Electrodynamic Measures."

The constant c_W which appears in (3.3) is the ratio between the electrodynamic and electrostatic units of charge. It has the dimension of a velocity (m/s). In his papers of 1846 and 1848 Weber represented it by $4/a$. In 1856 he was writing c instead of $4/a$, as was doing Kirchhoff in 1857 (Kirchhoff, 1857 a). But Weber's $c = 4/a$ is not our $c = (\varepsilon_o\mu_o)^{-1/2} = 3 \times 10^8 m/s$, but $\sqrt{2}$ times this last quantity. To avoid confusion we wrote c_W in (3.3), following the procedure adopted by Rosenfeld (Rosenfeld, 1957). So in (3.3) we have $c_W = \sqrt{2}c = \sqrt{2}/\sqrt{\varepsilon_o \mu_o}$.

Although Weber introduced this constant in 1846, the first measurement of the ratio between the electrodynamic and electrostatic units of charge, c_W, happened only in 1855 and 1856 (Jungnickel and McCormmach, 1986, Vol. 1, pp. 145 - 146). It was performed by W. Weber in collaboration with R. Kohlrausch (Weber and Kohlrausch, 1856). What they found was $c_W = 4.39 \times 10^8 m/s$. This means that $c = c_W/\sqrt{2} = (\varepsilon_o \mu_o)^{-1/2}$, which is the ratio of electromagnetic and electrostatic units of charge, was then found to be $c = 3.1 \times 10^8 m/s$. The value they obtained was essentially the same as the known value of the velocity of light in air. This result of 1856 and Faraday's discovery in 1845 of the rotation of the plane of polarization of a beam of light travelling in a material embedded in a magnetic field were the first quantitative proofs of a connection between optics and electromagnetism. It is worth while quoting Kirchner here in a paper where he describes the experimental procedure of this extremely important paper by Weber: "Considering that this ratio [between electrodynamic and electrostatic units of charge] was then not even known as to its order of magnitude, that we deal therefore with a real pioneering effort, and if one realizes furthermore the primitive equipment they had to work with, one has to admire the work by Weber and Kohlrausch as a masterpiece in the art of experimentation, very few of which exist in the history of our science" (Kirchner, 1957).

Soon afterwards Weber and Kirchhoff, working independently of one another, but both utilizing Weber's electrodynamics, predicted the existence in a conducting circuit of negligible resistance of periodic modes of oscillation of the electric current whose velocity of propagation had the same value $c_W/\sqrt{2} = c$ as the velocity of light. This result was independent of the cross section of the wire, of its conductivity, and of the density of electricity in the wire. Kirchhoff's works were published in 1857 (Kirchhoff, 1857 a and b), and they have been translated to English. Weber's simultaneous and more thorough work was delayed in publication and appeared only in 1864: (Jungnickel and McCormmach, 1986, Vol. 1, pp. 144-146 and 296-297), (Rosenfeld, 1957 and 1973). From Weber's electrodynamics, from the equation of the conservation of charges, (2.53), and from the generalized Ohm's law, (2.37), they were the first to derive the wave equation describing perturbations in the current propagating along a wire, namely

$$\frac{\partial^2 I}{\partial s^2} - \frac{1}{c^2}\frac{\partial^2 I}{\partial t^2} = K\frac{\partial I}{\partial t} \ . \tag{3.4}$$

In this equation I is the current, s is the distance along the wire from a fixed origin and K is a constant proportional to the resistivity of the wire. A similar equation was obtained for the density of free charge along the surface of the wire. What is amazing is that they obtained this result with Weber's action-at-a-distance force. They did not utilize the concepts of an ether, of the displacement current, nor of retarded time. And this was accomplished before Maxwell wrote down his equations in complete form, which happened only in 1861 - 1864. This is a remarkable historical fact which should be always kept in mind.

Maxwell introduced the term in the displacement current in the circuital law for the magnetic field, the term with c^2 in (2.49), in 1861/2. In this paper, his second one dealing with electromagnetism, he also obtained that an electromagnetic disturbance would be propagated in the electromagnetic medium with a velocity $c = (\varepsilon_o\mu_o)^{-1/2}$. At that time those who worked with ether models had one ether to transmit light, the luminiferous one; another to transmit electric and magnetic effects, the electromagnetic one; another one to transmit the gravitational force; etc. With his model Maxwell could unify at least two of them. Here are his words (his italics): "The velocity of transverse undulations in our hypothetical medium, calculated from the electro-magnetic experiments of MM. Kohlrausch and Weber, agrees so exactly with the velocity of light calculated from the optical experiments of M. Fizeau, that we can scarcely avoid the inference that *light consists in the transverse undulations of the same medium which is the cause of electric and magnetic phenomena* (Maxwell, 1965, pp. 451 - 513, see especially p. 500). Analogously in 1864 he wrote: "By the electromagnetic experiments of MM. Weber and Kohlrausch, $v = 310,740,000$ metres per second. (...) The velocity of light in air, by M. Fizeau experiments, is $V = 314,858,000$. (...) The agreement of these results seems to shew that light and magnetism are affections of the same substance, and that light is an electromagnetic disturbance propagated through the field according to electromagnetic laws" (Maxwell, 1965, pp. 526 - 597, see especially pp. 579 - 580). This indicates the great relation between Weber's measurement and Maxwell's theory. It should be remembered that Maxwell only measured c_W or c in 1868 (Maxwell, 1965, Vol. 2, pp. 125 - 143).

For further discussion of these points see (Kirchner, 1957), (Rosenfeld, 1957 and 1973), (Wise, 1981), (Woodruff, 1968 and 1976), (Harman, 1982).

Weber's Collected Works have already been published: (Weber, 1892 - 1894). English translations of some of his most important papers can be found in (Weber, 1848 a and b, 1851, 1852 and 1871). Some other English translations of his works can be found in the 7 volumes of the *Scientific Memoirs* (originally published between 1837 and 1853, and reprinted by Johnson Reprint Corporation, New York, 1966), edited by R. Taylor, J. Tyndall and W. Francis.

After this short historical introduction we begin the formal presentation of Weber's electrodynamics.

3.2. Weber's Force

In this Chapter we discuss Weber's force and some of its main characteristics.

According to Weber, the force exerted by an electric charge q_j on another q_i separated by a distance r_{ij} is given by (using vectorial notation and in the International System of Units):

$$\vec{F}_{ji} = \frac{q_i q_j}{4\pi\varepsilon_0} \frac{\hat{r}_{ij}}{r_{ij}^2} \left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2} \right) , \tag{3.5}$$

where

$$\dot{r}_{ij} \equiv \frac{d}{dt} r_{ij} , \tag{3.6}$$

$$\ddot{r}_{ij} \equiv \frac{d^2}{dt^2} r_{ij} = \frac{d}{dt} \dot{r}_{ij} , \tag{3.7}$$

This force first appeared in (Weber, 1846), and an English translation of an outline of this paper can be found in (Weber, 1848 a).

The constant c which appears in (3.5) is the ratio of electromagnetic and electrostatic units of charge, as we have seen. In the International System of Units it can be written as $c = (\varepsilon_o \mu_o)^{-1/2}$ and its measured value is $c = 3 \times 10^8 m/s$.

We now define explicitly the other quantities which appear in (3.5). Relative to an arbitrary frame of reference S, the charges q_i and q_j are located at $\vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$, $\vec{r}_j = x_j\hat{x} + y_j\hat{y} + z_j\hat{z}$. So the vector pointing from q_j to q_i, $\vec{r}_{ij}$, and its magnitude or the distance between the charges, r_{ij}, are given by

$$\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j = (x_i - x_j)\hat{x} + (y_i - y_j)\hat{y} + (z_i - z_j)\hat{z} , \tag{3.8}$$

$$r_{ij} \equiv |\vec{r}_{ij}| = \sqrt{\vec{r}_{ij} \cdot \vec{r}_{ij}} = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right]^{1/2} . \tag{3.9}$$

The unit vector pointing from j to i is then given by

$$\hat{r}_{ij} \equiv \frac{\vec{r}_{ij}}{r_{ij}} . \tag{3.10}$$

The velocities and accelerations of the charges are given by $\vec{v}_m \equiv d\vec{r}_m/dt$ and $\vec{a}_m \equiv d\vec{v}_m/dt = d^2\vec{r}_m/dt^2$; $m = i,\ j$. The relative velocity between them is defined by

$$\vec{v}_{ij} \equiv \vec{v}_i - \vec{v}_j = \frac{d\vec{r}_{ij}}{dt} , \tag{3.11}$$

$$\vec{a}_{ij} \equiv \vec{a}_i - \vec{a}_j = \frac{d\vec{v}_{ij}}{dt} = \frac{d^2\vec{r}_{ij}}{dt^2} . \tag{3.12}$$

From (3.6) to (3.12) we can obtain by simple time derivatives the relative **radial** velocity, $\dot{r}_{ij}$, and the relative **radial** acceleration, $\ddot{r}_{ij}$, which appear in Weber's force (3.5). We utilize that $x_{ij} \equiv x_i - x_j$, $y_{ij} \equiv y_i - y_j$, $z_{ij} \equiv z_i - z_j$, and that $\dot{x}_{ij} \equiv dx_{ij}/dt$, $\dot{y}_{ij} \equiv dy_{ij}/dt$, $\dot{z}_{ij} \equiv dz_{ij}/dt$, $\ddot{x}_{ij} \equiv d^2x_{ij}/dt^2$, $\ddot{y}_{ij} \equiv d^2y_{ij}/dt^2$, $\ddot{z}_{ij} \equiv d^2z_{ij}/dt^2$. We then have

$$\dot{r}_{ij} \equiv \frac{dr_{ij}}{dt} = \frac{x_{ij}\dot{x}_{ij} + y_{ij}\dot{y}_{ij} + z_{ij}\dot{z}_{ij}}{r_{ij}} = \hat{r}_{ij} \cdot \vec{v}_{ij} , \tag{3.13}$$

$$\ddot{r}_{ij} \equiv \frac{d\dot{r}_{ij}}{dt} = \frac{d^2r_{ij}}{dt^2} = \frac{1}{r_{ij}} \left[\vec{v}_{ij} \cdot \vec{v}_{ij} - (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 + \vec{r}_{ij} \cdot \vec{a}_{ij}\right] . \tag{3.14}$$

It should be observed that if we have rotation of one of the charges relative to the other, or rotation of the frame of reference, $\dot{r}_{ij}$ will be different from $\sqrt{\vec{v}_{ij} \cdot \vec{v}_{ij}} = (\dot{x}_{ij}^2 + \dot{y}_{ij}^2 + \dot{z}_{ij}^2)^{1/2}$, and $\ddot{r}_{ij}$ will be different from $\sqrt{\vec{a}_{ij} \cdot \vec{a}_{ij}} = (\ddot{x}_{ij}^2 + \ddot{y}_{ij}^2 + \ddot{z}_{ij}^2)^{1/2}$ and from $\hat{r}_{ij} \cdot \vec{a}_{ij} = (x_{ij}\ddot{x}_{ij} + y_{ij}\ddot{y}_{ij} + z_{ij}\ddot{z}_{ij})/r_{ij}$.

The main properties of Weber's force are:

(A) It obeys Newton's third law (action and reaction) in the strong form, for any state of motion of the charges. That is, the force is always along the straight line joining the two charges and $\vec{F}_{ji} = -\vec{F}_{ij}$. In Section 3.4 we will see that this implies conservation of linear and angular momentum.

(B) Coulomb's force is only a particular case of Weber's force, obtained when the charges are at rest relative to one another. That is, when $\dot{r}_{ij} = 0$ and $\ddot{r}_{ij} = 0$ the equation (3.5) reduces to (2.13). As the first of Maxwell's equations, Gauss's law, essentially is Coulomb's force written in differential form (see Section 2.6), it turns out that from Weber's force we can derive the first of Maxwell's equations, together with the first part of Lorentz's force, $-q\nabla\phi$ in (2.41) and (2.39).

(C) The equation of motion is obtained from (3.5) together with Newton's second law, (2.5) or (2.7).

(D) The velocity and acceleration which appear in (3.5) are only the radial velocities and accelerations *between the two charges*, as given by (3.6) and (3.7). From this aspect we can obtain the last and fundamental property of Weber's electrodynamics:

(E) In Weber's force there are only relational quantities to specify the position and motion of charges. That is, to know the force it is only necessary to evaluate $\vec{r}_i - \vec{r}_j$, $r_{ij} = |\vec{r}_i - \vec{r}_j|$, dr_{ij}/dt, and d^2r_{ij}/dt^2. This means that each term of Weber's force has the same value to *all* observers, even for non inertial ones. The vector $\vec{r}_i$ which joins an observer O to a particle i may be different from a vector $\vec{r}_i\,'$ which joins the same charge to another observer O'. But the vector $\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j$ which joins charge j to charge i is the same for both observers, namely $\vec{r}_{ij}\,' = \vec{r}_{ij}$. The same happens with $r_{ij}, \dot{r}_{ij}$ and $\ddot{r}_{ij}$. For this reason we say that these are relational concepts, which depend only on the relations between interacting bodies, but which do not depend on frames of reference nor on observers. We prefer the term "relational," instead of "relative," to avoid confusion with Einstein's theories of relativity. The relational quantities are: $\vec{r}_{ij}$, r_{ij}, $\hat{r}_{ij}$, $\dot{r}_{ij}$ and $\ddot{r}_{ij}$. On the other hand here are some quantities which are not relational, so that they can have simultaneously different values for different observers: $\vec{r}_i$, $\vec{r}_j$, $\vec{v}_i$, $\vec{v}_j$, $\vec{a}_i$, $\vec{a}_j$, $\vec{v}_{ij}$, $\vec{a}_{ij}$, $|\vec{v}_{ij}| = \sqrt{\vec{v}_{ij} \cdot \vec{v}_{ij}}$, $|\vec{a}_{ij}| = \sqrt{\vec{a}_{ij} \cdot \vec{a}_{ij}}$, $\hat{r}_{ij} \cdot \vec{a}_{ij}$, $\vec{r}_{ij} \cdot \vec{a}_{ij}$, etc.

Let us prove this. Suppose we have two frames of reference S and S' whose origins of coordinates are O and O'. Suppose that at the time t the origin O' is located at a distance $\vec{R}$ from O, moving relative to O with a velocity $\vec{V} = d\vec{R}/dt$ and with a translational acceleration $\vec{A}_t = d\vec{V}/dt = d^2\vec{R}/dt^2$. Suppose moreover that S' is rotating relative to S with an angular velocity $\vec{\omega}$. If $\vec{A}_t \neq 0$ or $\vec{\omega} \neq 0$ then at least one of these frames is obviously non inertial from the point of view of classical mechanics. The position, velocity and acceleration of a particle j ($j = 1,\ 2$) relative to S (S') are, respectively: $\vec{r}_j$, $\vec{v}_j$ and $\vec{a}_j$ ($\vec{r}_j\,'$, $\vec{v}_j\,'$ and $\vec{a}_j\,'$). In general $\vec{r}_j = x_j\hat{x} + y_j\hat{y} + z_j\hat{z}$ and $\vec{r}_j\,' = x_j\,'\hat{x}\,' + y_j\hat{y}\,' + z_j\hat{z}\,'$, etc. The relation between these quantities are (Symon, 1971, Chapter 7):

$$\vec{r}_j = \vec{r}_j^{\,'} + \vec{R} \ , \tag{3.15}$$

$$\vec{v}_j = \vec{v}_j^{\,'} + \vec{\omega} \times \vec{r}_j^{\,'} + \vec{V} \ , \tag{3.16}$$

$$\vec{a}_j = \vec{a}_j^{\,'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_j^{\,'}) + 2\vec{\omega} \times \vec{v}_j^{\,'} + \frac{d\vec{\omega}}{dt} \times \vec{r}_j^{\,'} + \vec{A}_t \ . \tag{3.17}$$

It is then easy to see that

$$\vec{r}_{ij} = \vec{r}_{ij}^{\,'}, \ r_{ij} = r_{ij}^{'}, \ \hat{r}_{ij} = \hat{r}_{ij}^{\,'} \ . \tag{3.18}$$

Moreover,

$$\vec{v}_{ij} = \vec{v}_{ij}^{\,'} + \vec{\omega} \times \vec{r}_{ij}^{\,'} \ , \tag{3.19}$$

so that even when $\vec{v}_{ij} \neq \vec{v}_{ij}^{\,'}$ and $\sqrt{\vec{v}_{ij} \cdot \vec{v}_{ij}} \neq \sqrt{\vec{v}_{ij}^{\,'} \cdot \vec{v}_{ij}^{\,'}}$ we will have

$$\dot{r}_{ij} = \frac{dr_{ij}}{dt} = \hat{r}_{ij} \cdot \vec{v}_{ij} = \hat{r}_{ij}^{\,'} \cdot \vec{v}_{ij}^{\,'} = \frac{dr_{ij}^{'}}{dt} = \dot{r}_{ij}^{'} \ . \tag{3.20}$$

We also have

$$\vec{a}_{ij} = \vec{a}_{ij}^{\,'} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{ij}^{\,'}) + 2\vec{\omega} \times \vec{v}_{ij}^{\,'} + \frac{d\vec{\omega}}{dt} \times \vec{r}_{ij}^{\,'} \ , \tag{3.21}$$

$$\vec{r}_{ij} \cdot \vec{a}_{ij} = \vec{r}_{ij}^{\,'} \cdot \vec{a}_{ij}^{\,'} + (\vec{\omega} \cdot \vec{r}_{ij}^{\,'})^2 - (\vec{r}_{ij}^{\,'} \cdot \vec{r}_{ij}^{\,'})(\vec{\omega} \cdot \vec{\omega}) + 2\vec{r}_{ij}^{\,'} \cdot (\vec{\omega} \times \vec{v}_{ij}^{\,'}) \ , \tag{3.22}$$

so that even when $\vec{a}_{ij} \neq \vec{a}_{ij}^{\,'}$, $\sqrt{\vec{a}_{ij} \cdot \vec{a}_{ij}} \neq \sqrt{\vec{a}_{ij}^{\,'} \cdot \vec{a}_{ij}^{\,'}}$, $\vec{r}_{ij} \cdot \vec{a}_{ij} \neq \vec{r}_{ij}^{\,'} \cdot \vec{a}_{ij}^{\,'}$ and $\sqrt{\hat{r}_{ij} \cdot \vec{a}_{ij}} \neq \sqrt{\hat{r}_{ij}^{\,'} \cdot \vec{a}_{ij}^{\,'}}$ we will have

$$\ddot{r}_{ij} = \frac{d^2 r_{ij}}{dt^2} = \frac{1}{r_{ij}} \left[\vec{v}_{ij} \cdot \vec{v}_{ij} - (\hat{r}_{ij} \cdot \vec{v}_{ij})^2 + \vec{r}_{ij} \cdot \vec{a}_{ij}\right]$$

$$= \frac{1}{r_{ij}^{'}} \left[\vec{v}_{ij}^{\,'} \cdot \vec{v}_{ij}^{\,'} - (\hat{r}_{ij}^{\,'} \cdot \vec{v}_{ij}^{\,'})^2 + \vec{r}_{ij}^{\,'} \cdot \vec{a}_{ij}^{\,'}\right] = \frac{d^2 r_{ij}^{'}}{dt^2} = \ddot{r}_{ij}^{'} \ . \tag{3.23}$$

We gave a larger emphasis to this last aspect because Weber's force is one of the few formulations ever proposed to embrace the electric and magnetic phenomena, if not the only one, which have this property. The other formulations as those of Gauss, Riemann, Clausius, Ritz, Lorentz, etc., either they depend on the velocity and acceleration of the charge relative to an ether, or they depend on the velocity and acceleration of the charge relative to an observer (frame of reference). We can see an example of this with Lorentz's force. If in a certain region of space there is only a stationary magnetic field (generated, for instance, by a magnet at rest relative to an inertial frame of reference), and an observer on this frame sees a charge q moving with velocity $\vec{v}$ in this region of space, then he will observe the charge experiencing a magnetic force given by (2.22) or by the last term of (2.41). To another observer O' which at the same time is moving with a constant velocity $\vec{v}$ relative to this frame there will be no magnetic force acting on the charge. The reason is that at this moment q is at rest relative to him, $\vec{v}\,' = 0$, so that $q\vec{v}\,' \times \vec{B}' = 0$ for him. From this short analysis we can observe that the magnetic force in classical electromagnetism can be different for two inertial observers.

Using (3.13) and (3.14) Weber's force can be put in the form

$$\vec{F}_{ji} = \frac{q_i q_j}{4\pi\varepsilon_0}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} - \frac{3}{2}(\hat{r}_{ij}\cdot\vec{v}_{ij})^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right)\right] . \qquad (3.24)$$

3.3. Weber's Potential Energy

Weber's force was the first historical example of a force between charges which depended not only on the distance between them but also on their velocities. At that time this was criticized by some scientists because they thought this force would not be compatible with the principle of conservation of energy. Two years later, in 1848, Weber was able to show that his force could be derived from a potential energy defined by

$$U \equiv \frac{q_i q_j}{4\pi\varepsilon_0} \frac{1}{r_{ij}} \left(1 - \frac{\dot{r}_{ij}^2}{2c^2}\right) . \tag{3.25}$$

The first term of this energy is the usual Coulombian potential energy. The second term is a mixture of kinetic and potential energies because it depends not only on the distance between the charges but also on their velocities. This was also the first example which appeared in the literature of a velocity dependent potential energy. As we mentioned above, the English translation of this very important paper by Weber where he introduced (3.25) can by found in (Weber, 1848 a). As with Coulomb's potential energy, U given by (3.25) can be thought as the energy spent to form the system. That is, U is the energy spent to bring from infinity q_i and q_j (where they are supposed to be initially at rest) up until the separation r_{ij} with relative radial velocity $\dot{r}_{ij}$. This energy is spent against Weber's force (3.5) acting between the charges.

Let us quote here how Weber presented (3.25) for the first time. His general fundamental principle for the whole theory of electricity, his force (3.5) or (3.3), was written by him in this work as (remembering that $a^2/16 = 1/c_W^2 = 1/2c^2$):

$$\frac{EE'}{RR}\left(1 - \frac{aa}{16}\frac{dR^2}{dt^2} + \frac{aa}{8}R\frac{ddR}{dt^2}\right) ,$$

where E and E' are the charges, R their separation, $dR^2/dt^2 = \dot{r}^2$ and $ddR/dt^2 = \ddot{r}$. Then he says:

"For a definite magnitude assumed for the purpose of measuring the time, in which $a = 4$, this expression becomes

$$\frac{EE'}{RR}\left(1 - \frac{dR^2}{dt^2} + 2R\frac{ddR}{dt^2}\right) .$$

Moreover, supposing that both R and dR/dt are functions of t, consequently that dR/dt is to be regarded as a function of R, which we shall denote by [R], we may also say that the *potential* of the mass E, in regard to the situation of the mass E', is

$$= \frac{E}{R}\left(1 - [R]^2\right) \ ;$$

for the partial differential coefficients of this expression, with respect to the three coordinates x, y, z, yield the components of the decomposed accelerating force in the directions of the three coordinate axes" (Weber, 1848 a).

The simplest way of deriving the force from this potential energy is by

$$\vec{F}_{ji} = -\hat{r}_{ij}\frac{dU}{dr_{ij}} \ . \tag{3.26}$$

In order to show this we notice that r_{ij} is in general a function of time, so that

$$\frac{d\dot{r}_{ij}^2}{dr_{ij}} = 2\dot{r}_{ij}\frac{d\dot{r}_{ij}}{dr_{ij}} = 2\dot{r}_{ij}\frac{d\dot{r}_{ij}}{dt}\frac{dt}{dr_{ij}} = 2\ddot{r}_{ij} \ . \tag{3.27}$$

Applying (3.25) and (3.27) in (3.26) yields (3.5).

Another way of getting the force from the potential energy is presented in the next Section, and from the Lagrangian formalism in Section 3.5.

3.4. Conservation of Linear Momentum, of Angular Momentum and of Energy

One of the most important aspects of classical physics is its three basic laws of conservation: linear momentum, angular momentum and energy. Here we show how Weber's electrodynamics, composed of Weber's force and its potential energy, satisfies these three basic laws. First we discuss the conservation of linear momentum.

The conservation of linear momentum follows directly from the fact that Weber's force satisfies the principle of action and reaction. This fact is independent of the form of the force, and it is only based on the equality $\vec{F}_{ji} = -\vec{F}_{ij}$, which is the case of Weber's electrodynamics. Provided that there is action and reaction there will be conservation of linear momentum even if the forces are not central ones.

The total linear momentum of a system of two particles of inertial masses m_i and m_j moving with velocities $\vec{v}_i$ and $\vec{v}_j$ relative to an inertial system is defined by

$$\vec{P} \equiv m_i\vec{v}_i + m_j\vec{v}_j \ . \tag{3.28}$$

Differentiating this expression with respect to time, using Newton's second law of motion (2.5) or (2.7), and the action and reaction law, (2.6), yields immediately $d\vec{P}/dt = 0$, *QED*.

This principle can be generalized to include an arbitrary number of particles interacting via several forces (Weber, elastic, gravitational, etc.), provided that these forces follow the principle of action and reaction.

For instance, if we have 3 interacting particles of electrical charges q_1, q_2 and q_3, inertial masses m_1, m_2 and m_3, moving relative to an inertial frame with velocities $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$, the total linear momentum of the system is defined by

$$\vec{P} \equiv m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 \ . \tag{3.29}$$

Let us suppose that the force acting on a particle i due to another j is composed of two components: Weber's force $\vec{F}^W_{ji}$ and, for instance, a gravitational component $\vec{F}^G_{ji}$. Both components comply with Newton's action and reaction law so that

$$\vec{F}_{ji}^W = -\vec{F}_{ij}^W \, , \; \vec{F}_{ji}^G = -\vec{F}_{ij}^G \; . \tag{3.30}$$

The resultant force acting on particle i is:

$$\vec{F}_{\text{on } i}^R = \sum_{\substack{j=1 \\ j\neq i}}^{3} \vec{F}_{ji}^W + \sum_{\substack{j=1 \\ j\neq i}}^{3} \vec{F}_{ji}^G \; . \tag{3.31}$$

Taking the time derivative of (3.29) and using Newton's second law of motion (2.5) or (2.7) yields, with (3.31):

$$\frac{d\vec{P}}{dt} = \vec{F}_{\text{on } 1}^R + \vec{F}_{\text{on } 2}^R + \vec{F}_{\text{on } 3}^R = \left[(\vec{F}_{21}^W + \vec{F}_{31}^W) + (\vec{F}_{21}^G + \vec{F}_{31}^G)\right]$$

$$+ \left[(\vec{F}_{12}^W + \vec{F}_{32}^W) + (\vec{F}_{12}^G + \vec{F}_{32}^G)\right] + \left[(\vec{F}_{13}^W + \vec{F}_{23}^W) + (\vec{F}_{13}^G + \vec{F}_{23}^G)\right] \; . \tag{3.32}$$

But by (3.30) this is shown to be zero, so that $\vec{P}$ is a constant in time.

This can be easily generalized to N particles.

Now let us prove the conservation of angular momentum. First we consider two charges interacting through Weber's force as seen in an inertial system. The total angular momentum is defined by

$$\vec{L} \equiv \vec{r}_i \times (m_i \vec{v}_i) + \vec{r}_j \times (m_j \vec{v}_j) \; . \tag{3.33}$$

Taking the time derivative of $\vec{L}, d\vec{L}/dt$, using the usual rules of vectorial analysis, remembering that $\vec{v} \times \vec{v} = 0$, using (2.5) and the law of action and reaction (3.30) yields

$$\frac{d\vec{L}}{dt} = (\vec{r}_i - \vec{r}_j) \times \vec{F}_{ji} \; . \tag{3.34}$$

Up to now Weber's force was only utilized as regards its agreement with the law of action and reaction. Now we utilize the fact that it satisfies this principle in the strong form. That is, $\vec{F}_{ji}$ is not only equal to $-\vec{F}_{ij}$, but it is also parallel to $\vec{r}_{ij}$, so that $d\vec{L}/dt = 0$. Again this result did not depend on the explicit form of Weber's force, but only on the fact that it satisfies the action and reaction principle in the strong form.

This result can be generalized to include any number of particles interacting with one another by forces of any nature, provided they all satisfy the action and reaction principle in the strong form.

Let us show this for three particles interacting only through Weber's force (the generalization is immediate). The total angular momentum of the system in an inertial frame is defined by

$$\vec{L} \equiv \sum_{i=1}^{3} \vec{r}_i \times (m_i \vec{v}_i) \ . \tag{3.35}$$

Its time derivative is found to be (by (2.5)):

$$\frac{d\vec{L}}{dt} = \vec{r}_1 \times (\vec{F}_{21} + \vec{F}_{31}) + \vec{r}_2 \times (\vec{F}_{12} + \vec{F}_{32}) + \vec{r}_3 \times (\vec{F}_{13} + \vec{F}_{23}) \ . \tag{3.36}$$

By (3.30) this yields

$$\frac{d\vec{L}}{dt} = \vec{r}_{12} \times \vec{F}_{21} + \vec{r}_{13} \times \vec{F}_{31} + \vec{r}_{23} \times \vec{F}_{32} \ . \tag{3.37}$$

As $\vec{F}_{ij}$ is parallel to $\vec{r}_{ij}$ we obtain $d\vec{L}/dt = 0$, so that $\vec{L}$ is a constant in time.

Once more the explicit form of Weber's force is not important, but only the fact that it follows the law of action and reaction, and is along the straight line connecting the charges. This result may be easily generalized to N particles interacting with one another through several forces (Weber's, elatic, gravitational, etc.) provided all of them comply with (3.30) and are along the line connecting each pair of particles.

We now analyse the conservation of energy (see, for instance, (Wesley, 1987 a)). Supposing once more that we have two charges q_i and q_j, of inertial masses m_i and m_j, interacting with one another through Weber's force, the total energy E of the system is defined by:

$$E \equiv T + U \ , \tag{3.38}$$

where U is Weber's potential energy given by (3.25) and T is the classical kinetic energy defined by

$$T \equiv m_i \frac{\vec{v}_i \cdot \vec{v}_i}{2} + m_j \frac{\vec{v}_j \cdot \vec{v}_j}{2} \ , \tag{3.39}$$

where $\vec{v} \equiv d\vec{r}/dt$ is the velocity of the particle relative to an inertial frame.

Taking the time derivative of E yields, by (2.5) and (3.25):

$$\frac{dE}{dt} = \frac{dT}{dt} + \frac{dU}{dt} = \left(\vec{v}_i \cdot \vec{F}_{ji} + \vec{v}_j \cdot \vec{F}_{ij}\right) - \dot{r}_{ij} \left[\frac{q_i q_j}{4\pi\varepsilon_o} \frac{1}{r_{ij}^2} \left(1 - \frac{\dot{r}_{ij}^2}{c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right)\right] \ . \tag{3.40}$$

By (3.30) and (3.13) this can be written as

$$\frac{dE}{dt} = \vec{v}_{ij} \cdot \vec{F}_{ji} - \vec{v}_{ij} \cdot \left[\frac{q_i q_j}{4\pi\varepsilon_o} \frac{\hat{r}_{ij}}{r_{ij}^2} \left(1 - \frac{\dot{r}_{ij}}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right)\right] \ . \tag{3.41}$$

And this is obviously equal to zero by (3.5), so that E is a constant in time.

If we have 3 particles interacting through Weber's force the total energy is defined by

$$E \equiv \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} + \frac{m_3 v_3^2}{2} + U_{12} + U_{13} + U_{23} \ , \tag{3.42}$$

where U_{ij} is defined by (3.25). The time derivative of this expression yields, by (3.5), (3.25), (3.40) and (3.41):

$$\frac{dE}{dt} = \vec{v}_1 \cdot (\vec{F}_{21} + \vec{F}_{31}) + \vec{v}_2 \cdot (\vec{F}_{12} + \vec{F}_{32}) + \vec{v}_3 \cdot (\vec{F}_{13} + \vec{F}_{23}) - \vec{v}_{12} \cdot \vec{F}_{21} - \vec{v}_{13} \cdot \vec{F}_{31} - \vec{v}_{23} \cdot \vec{F}_{32} \ . \tag{3.43}$$

By (3.30) this is shown to be zero.

This result can be easily generalized by N particles interacting with one another through several forces (Weber, elastic, gravitational, etc.) provided that these forces follow the principle of action and reaction, are along the straight line connecting the particles, and can be derived from a potential energy by an expression like (3.26).

This procedure suggests a new way of determining the force from the potential: Given a potential energy U, the force exerted by j on i, $\vec{F}_{ji}$, can be obtained by

$$\frac{dU}{dt} = -(\vec{v}_i - \vec{v}_j) \cdot \vec{F}_{ji} \ . \tag{3.44}$$

If we define the force by this expression and apply it to (3.25) we obtain Weber's force (3.5), without invoking (3.26).

In this Section we showed that Weber's theory is compatible with the main results of classical physics as we can derive from it the three basic conservation laws of mechanics. These are strong and important results of Weber's electrodynamics.

3.5. Lagrangian and Hamiltonian Formulations of Weber's Electrodynamics

In classical mechanics we can describe and solve most problems using Newton's equations of motion. Equivalently we can solve these problems by the equations of Lagrange or of Hamilton. The same can be done with Weber's electrodynamics.

We will treat the motion of two charges q_i and q_j, of inertial masses m_i and m_j, interacting with one another through Weber's force, without external forces acting on them (the generalization to N charges is straightforward). The classical kinetic energy T of the system is defined by (3.39). We define a function S by

$$S \equiv \frac{q_i q_j}{4\pi\varepsilon_o} \frac{1}{r_{ij}} \left(1 + \frac{\dot{r}_{ij}^2}{2c^2} \right) . \tag{3.45}$$

The Lagrangian L which gives rise to Weber's electrodynamics is defined by

$$L \equiv T - S . \tag{3.46}$$

To our knowledge the first to introduce (3.45) and (3.46) was Carl Neumann, the son of Franz Neumann, in 1868 (Neumann, 1868; Archibald, 1986). He arrived at (3.45) employing the idea of retarded potential. B. Riemann, a student, assistant and friend of Gauss and Weber at Göttingen, had introduced this idea in physics in 1858, but his paper was only published in 1867 (an English translation can be found in (Riemann, 1867)). The procedure followed by C. Neumann, which was inspired by reading Riemann's paper, was criticized by Clausius. An English translation of his paper can be found in (Clausius, 1868). Clausius also discussed the Lagrangian formulation of his own electrodynamics and those of Riemann and Weber in his paper of 1880, which has also been translated to English (Clausius, 1880). For a discussion of the ideas of C. Neumann and Clausius see (Archibald, 1986). We are not aware that Weber himself ever utilized the Lagrangian or Hamiltonian formalisms in connection with his electrodynamics.

Disregarding C. Neumann's procedure to arrive at (3.45) and the related ideas of retarded potential, we can simply postulate (3.45) and (3.46), and work from here. This will be our approach in this book.

Weber's force can be obtained in the normal way from the Lagrangian formulation through S. That is, inasmuch as $\dot{x}_i \equiv dx_i/dt$, where $\vec{r}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$ is the position vector of q_i, we have that the x-component of the force on q_i is given by (O'Rahilly, 1965, Vol. 2, Chapter 11, pp. 525 - 535):

$$F_{ji}^x = \frac{d}{dt}\frac{\partial S}{\partial \dot{x}_i} - \frac{\partial S}{\partial x_i} = \frac{q_i q_j}{4\pi\varepsilon_o}\frac{x_i - x_j}{r_{ij}^3}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right) , \tag{3.47}$$

and similarly for F_{ji}^y and F_{ji}^z.

Alternatively by another set of generalized coordinates we can obtain immediately Weber's force along $\hat{r}_{ij}$, F_{ji} defined by $\vec{F}_{ji} \equiv \hat{r}_{ij}F_{ji}$, by (Whittaker, 1973, Vol. 1, Chapter 7, pp. 201 - 211):

$$F_{ji} = \frac{d}{dt}\frac{\partial S}{\partial \dot{r}_{ij}} - \frac{\partial S}{\partial r_{ij}} = \frac{q_i q_j}{4\pi\varepsilon_o}\frac{1}{r_{ij}^2}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right) . \tag{3.48}$$

The equations of motion are the usual Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = 0; \ k = 1, \ ..., \ 6 . \tag{3.49}$$

In these equations q_k represents one of the coordinates: x_i, y_i, z_i, x_j, y_j, z_j. Perfoming these calculations yields

$$\frac{q_i q_j}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\left(1 - \frac{\dot{r}_{ij}^2}{2c^2} + \frac{r_{ij}\ddot{r}_{ij}}{c^2}\right) = m_i\vec{a}_i , \tag{3.50}$$

and an analogous one for m_j. And this is exactly Newton's second law applied to Weber's force.

On the other hand the Hamiltonian H of the system is defined by

$$H \equiv \left(\sum_{k=1}^{6} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) - L , \tag{3.51}$$

where $\dot{q}_k$, with k ranging from 1 to 6, represents the components of the velocities, namely, $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$, $\dot{x}_j$, $\dot{y}_j$, $\dot{z}_j$.

Hamilton's equations of motion can be obtained from (3.51) by the usual procedures.

Observing that S and T do not depend explicitly on time yields $\partial L/\partial t = 0$ and $\partial H/\partial t = 0$. This means that in Weber's electrodynamics the Hamiltonian H happens to be the same as the total energy E of the system. From (3.45) to (3.51) we obtain:

$$H = T + U = E \ , \tag{3.52}$$

$$\frac{dE}{dt} = \frac{dH}{dt} = 0 \ . \tag{3.53}$$

In (3.52) U is Weber's potential energy defined by (3.25).

From the Lagrangian and Hamiltonian formulations we obtained then another proof of the conservation of energy in Weber's electrodynamics.

It should be emphasized that S is different from U because both differ in the sign in front of $\dot{r}_{ij}^2$. To avoid confusion we call U the potential energy and S the Lagrangian energy. These names distinguish clearly where these functions should be employed (U is the function which added to the kinetic energy T yields the total energy of the system, while S is the function which appears in the Lagrangian together with T). These names are preferred to the often used names "velocity-dependent potentials" or "generalized potentials" because U and S have dimensions of energy (kgm^2s^{-2}) and should not be confused with the electrostatic potential ϕ, (2.16), which has the dimensions of Volt ($1V = 1kgm^2C^{-1}s^{-2}$).

Although the Lagrangian is given by $L = T - S$, the Hamiltonian and the conserved energy are given by $H = E = T + U$, and not by $T + S$. An analogous situation arises in mechanics and classical electrodynamics when there are potential energies which depend not only on the distance between bodies but also on their velocities. We will see an example of this when we discuss Darwin's Lagrangian.

As we are discussing in this Section Lagrangians and Hamiltonians that depend on potential energies which are functions of the velocities of the charges, it is interesting to quote Goldstein in his well known book *Classical Mechanics*:

"**1.5 Velocity-dependent potentials and the dissipation function**. Lagrange's equations can be put in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0 \; , \qquad [1-53]$$

even if the system in not conservative in the usual sense, providing the generalized forces are obtained from a function $U(q_j, \; \dot{q}_j)$ by the prescription

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right) \; . \qquad [1-54]$$

In such case Eqs. [1-53] still follow from Eqs.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \; , \qquad [1-50]$$

with the Lagrangian given by

$$L = T - U \; . \qquad [1-52']$$

U may be called a "generalized potential," or "velocity-dependent potential." (The history of the designation given to such a potential is curious. Apparently spurred by Weber's early (and erroneous) classical electrodynamics, which postulated velocity-dependent forces, the German mathematician E. Schering seems to have been the first to attempt seriously to include such forces in the framework of mechanics, cf. Gött. Abh. **18**, 3 (1873). The first edition of Whittaker's *Analytical Dynamics* (1904) thus refers to the potential as "Schering's potential function," but the name apparently did not stick, for the title was dropped in later editions. We shall preferably use the name "generalized potential," including within this designation also the ordinary potential energy, a function of position only.) The possibility of using such a "potential" is not academic; it applies to one very important type of force field, namely, the electromagnetic forces on moving charges. Considering its importance, a diversion on this subject is well worthwhile. (...)" (Goldstein, 1950, p. 19).

This is one of the few recent books which presents Weber's force and its Lagrangian and Hamiltonian formulations. Despite this fact, this statement that Weber's classical electrodynamics is *erroneous* is very misleading. We can not even reply to it as we do not know the grounds on which the author considers Weber's electrodynamics "erroneous."

Statements strong and emphatic as these in didactic books should be carefully and clearly argued, to avoid a biased formation of students. Unfortunately this was not the case in this particular example in an otherwise excellent book.

3.6. Maxwell and the Electrodynamics of Weber

In this Section we discuss Maxwell's points of view regarding Weber's electrodynamics.

Since his first paper on electromagnetism of 1855 Maxwell always eulogized Weber's theory. For instance, after presenting Faraday's ideas which he was trying to follow, Maxwell said: "There exists however a professedly physical theory of electro-dynamics, which is so elegant, so mathematical, and so entirely different from anything in this paper, that I must state its axioms, at the risk of repeating what ought to be well known. It is contained in M. W. Weber *Electro-dynamic Measurements*, and may be found in the Transactions of the Leibnitz-Society, and of the Royal Society of Sciences in Saxony*. The assumptions are (...) . From these axioms are deducible Ampère's laws of the attraction of conductors, and those of Neumann and others, for the induction of currents. Here then is a really physical theory, satisfying the required conditions better perhaps than any yet invented, and put forth by a philosopher whose experimental researches form an ample foundation for his mathematical investigations" (Maxwell, 1965, pp. 155 - 229, see especially pp. 208 - 209). In the famous paper of 1864 in which Maxwell completed his electromagnetic theory of light he presented a similar view. After pointing out that the most natural theories of electromagnetism are based on forces acting between the interacting bodies without any express consideration of the surrounding medium, he says: "These theories assume, more or less explicitly, the existence of substances the particles of which have the property of acting on one another at a distance by attraction or repulsion. The most complete development of a theory of this kind is that of M. W. Weber, who has made the same theory include electrostatic and electromagnetic phenomena. In doing so, however, he has found it necessary to assume that the force between two electric particles depends on their relative velocity, as well as on their distance. This theory, as developed by MM. W. Weber and C. Neumann, is exceedingly ingenious, and wonderfully

* When this was written, I [Maxwell] was not aware that part of M. Weber's Memoir is translated in Taylor's *Scientific Memoirs*, Vol. V. Art. XIV. The value of his researches, both experimental and theoretical, renders the study of his theory necessary to every electrician.

comprehensive in its application to the phenomena of statical electricity, electromagnetic attractions, induction of currents and diamagnetic phenomena; and it comes to us with the more authority, as it has served to guide the speculations of one who has made so great an advance in the practical part of electric science, both by introducing a consistent system of units in electrical measurement, and by actually determining electrical quantities with an accuracy hitherto unknown" (Maxwell, 1965, pp. 526 - 597, see especially pp. 526 - 527).

But if Maxwell knew so well Weber's electrodynamics and appreciated it so much, why did he not work with it and develop its properties and applications? Only one year after Weber presented his force law in 1846, Helmholtz published his famous and very influential paper on the conservation of energy (in this paper he utilized the name "force" for what we would nowadays call "energy"). An English translation can be found in (Helmholtz, 1847). The principle of conservation of energy had been established by J. R. Meyer (1814 - 1878) in 1842, and also by J. P. Joule (1818 - 1889) in 1843. In his work of 1847 Helmholtz put this principle in a solid theoretical foundation developing the mathematical consequences of central forces. At that time the common name for the quantity mv^2 was *vis viva*, but in this paper Helmholtz explicitly stated that he would call $mv^2/2$ (our kinetic energy) by *vis viva*, as this latter quantity appeared more frequently in mechanics and seemed more useful. What we call nowadays by the name potential energy (like mgh, etc.), he called tension. The main results of his paper were stated as follows:

"The preceding proposition may be collected together as follows:

1. Whenever natural bodies act upon each other by attractive or repulsive forces, which are independent of time and velocity, the sum of their *vires vivae* and tensions must be constant; the maximum quantity of work which can be obtained is therefore a limited quantity.

2. If, on the contrary, natural bodies are possessed of forces which depend upon time and velocity, or which act in other directions than the lines which unite each two separate material points, for example, rotatory forces, then combinations of such bodies would be possible in which force might be either lost or gained *ad infinitum*."

This was understood by Maxwell, among others, as implying that Weber's electrodynamics did not comply with the principle of conservation of energy. The reason was that although Weber's force was a central one (directed along the line joining the charges), it depended on the velocity of the charges. This can be seen in the sequence of Maxwell's statements presented above, where he points out only this problem in Weber's electrodynamics. For instance, the sequence of his paper of 1855 reads: "There are also objections to making any ultimate forces in nature depend on the velocity of the bodies between which they act. If the forces in nature are to be reduced to forces acting between particles, the principle of the Conservation of Force [Energy] requires that these forces should be in the line joining the particles and functions of the distance only" (Maxwell, 1965, pp. 155 - 229, see especially p. 208). The sequence of his paper of 1864 reads (our italics): "*The mechanical difficulties*, however, which are involved in the assumption of particles acting at a distance with forces which depend on their velocities are such at to *prevent me from considering this theory as an ultimate one*, though it may have been, and may yet be useful in leading to the coordination of phenomena" (Maxwell, 1965, pp. 526 - 597, see especially p. 527).

Maxwell was wrong in this regard, as we have seen in this chapter. Although Weber had presented his potential energy in 1848, one year after Helmholtz paper, he did not prove the conservation of energy at this time. It was only in 1869 and 1871 that he proved in detail that his force law followed the principle of the conservation of energy (the important paper of 1871 has already been translated to English, (Weber, 1871)). Maxwell changed his mind only in 1871, after Weber's proof. In (Harmann, 1982, pp. 96 - 97) there is a reproduction of a postcard from Maxwell to Tait, dated 1871, where he informs Tait that Weber was right in stating that his electrodynamics followed the principle of the conservation of energy.

Helmholtz's proof does not apply to Weber's force because it depends not only on the distance and velocity of the charges but also on their accelerations. And this general case had not been considered by Helmholtz.

When Weber discussed the conservation of energy with his force law he said:

"The law of electrical action announced in the First Memoir on Electrodynamics

Measurements (*Elektrodynamische Maasbestimmungen*, Leipzig, 1846) has been tested on various sides and been modified in many ways. It has also been made the subject of observations and speculations on a more general kind; these, however, cannot by any means be regarded as having as yet led to definitive conclusions. The First Part of the following Memoir is limited to a discussion of the relation which this law bears to the *Principle of the Conservation of Energy*, the great importance and high significance of which have been brought specially into prominence in connexion with the Mechanical Theory of Heat. In consequence of its having been asserted that the law referred to is in contradiction with this principle, an endeavour is here made to show that no such contradiction exists. On the contrary, the law enables us to make an addition to the Principle of Conservation of Energy, and to alter it so that its application to each pair of particles is no longer limited solely to the time during which the pair does not undergo either increase or diminution of *vis viva* through the action of other bodies, but always holds good independently of the manifold relations to other bodies into which the two particles can enter.

Besides this, in the Second Part the law is applied to the development of *the equations of motion of two electrical particles subjected only to their mutual action.* Albeit this development does not lead directly to any comparisons or exact control by reference to existing experience (on which account it has hitherto received little attention), it neverthless leads to various results which appear to be of importance as furnishing clues for the investigation of the molecular conditions and motions of bodies which have acquired such special significance in relation to Chemistry and the theory of Heat, and to offer to further investigation interesting relations in these still obscure regions" (Weber, 1871).

When he wrote the *Treatise*, in 1873, Maxwell presented the new point of view that Weber's force was consistent with the principle of energy conservation:

"The formula of Gauss is inconsistent with this principle [of the conservation of energy], and must therefore be abandoned, as it leads to the conclusion that energy might be indefinitely generated in a finite system by physical means. This objection does not apply to the formula of Weber, for he has shewn (*Pogg. Ann.* lxxiii. p. 229 (1848).) that if we assume as the potential energy of a system consisting of two electric particles,

$$\psi = \frac{ee'}{r}\left[1 - \frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right] ,$$

the repulsion between them, which is found by differentiating this quantity with respect to r, and changing the sign, is that given by the formula

$$\frac{ee'}{r^2}\left[1 + \frac{1}{c^2}\left(r\frac{\partial^2 r}{\partial t^2} - \frac{1}{2}\left(\frac{\partial r}{\partial t}\right)^2\right)\right] .$$

Hence the work done on a moving particle by the repulsion of a fixed particle is $\psi_o - \psi_1$, where ψ_o and ψ_1 are the values of ψ at the beginning and at the end of its path. Now ψ depends only on the distance, r, and on the velocity resolved in the direction of r. If, therefore, the particle describes any closed path, so that its position, velocity, and direction of motion are the same at the end as at the beginning, ψ_1 will be equal to ψ_o, and no work will be done on the whole during the cycle of operations.

Hence an indefinite amount of work cannot be generated by a particle moving in a periodic manner under the action of the force assumed by Weber" (Maxwell, 1954, Vol. 2, article [853], p. 484).

It should be emphasized that what Maxwell wrote as $\partial r/\partial t$ would be written today as dr/dt, as is evident from what he wrote in article [847].

Maxwell then presented other criticisms of Helmhotz against Weber's electrodynamics, and we discuss this subject in later Sections of this book. For further discussion of this Section see (Archibald, 1989).

Chapter 4 / Forces of Ampère and Grassmann Between Current Elements

4.1. Ampère's Force Between Current Elements

André-Marie Ampère was born in Pleymieux, near Lyon in France, in 1775. He never went to school because his father, who admired Jean-Jacques Rousseau, wished that Ampère taught himself guided only by his readings and interests. In 1801 he wrote his first important work, on the mathematical theory of games, and the success of this book gave him a position at l'Ecole Polytechnique de Paris (1804). There he became professor of analysis in 1809. He became member of the French Academy of Sciences in 1814. During 1807 to 1816 he worked intensely in chemistry and established experimentally that fluorine, chlorine and iodine are simple chemical elements. Beyond his interests in mathematics and chemistry he also worked with psychology and philosophy. He married twice and had a son and a daughter. He died in 1836.

Until 1820, when he was 45 years old, he had not performed any serious research in electrodynamics (a name he coined later on). And after 1827 he would not return to this subject. His interest was fired by H. C. Oersted discovery of 1820 of the deflection of a compass needle placed parallel to a current carrying wire. This discovery was publicly announced in July 1820, and was described to a meeting of the French Académie des Sciences on the 11th September 1820 by Arago, who had just returned from abroad. Biot and Savart interpreted Oersted's experiment as showing that the electric current had magnetized the wire which then interacted with the magnetic needle as the interaction of two usual magnets. Ampère, on the other hand, looked at the experiment differently. According to him what was basic was the direct interactions between currents, which meant that there should exist microscopic currents within the magnets. As Ampère himself emphasized, this was not obvious because although a bar of soft iron also acts on a magnetized needle, there is no mutual action between two bars of soft iron. To prove his interpretation, Ampère showed at the Academy, one week after Arago's presentation, that two parallel wires carrying currents attract one another if the currents are in the same direction, and repel each other if the currents are in opposite directions. In the next seven years he became completely involved in the experimental research to find the correct mathematical expression describing the force between current elements. His passionate involvement is well described by his own words in a letter to his son of 25 September 1820 where he apologized for not having written earlier, "but all my time has been taken by an important circumstance in my life. Since I heard tell for the first time about the beautiful discovery of M. Oersted ... I've thought about it continuously [and] I've done nothing less than write a grand theory about these phenomena and all those already known about the magnet" (quoted by (Caneva, 1980)).

To arrive at his goal Ampère created the null method of comparing forces between currents. In this method he did not measure the force directly. Instead of this, two forces due to the same source but in different conditions (a straight and a crooked circuit; current in one and in the opposite direction; etc.) are made to act simultaneously on a body already in equilibrium but free to move so that no effect (motion) is produced. This shows that these forces are in equilibrium and in this way important conclusions can be drawn. By this method he discovered four independent cases of equilibrium from which he derived the following laws: (1) The force of a current is reversed when the direction of the current is reversed. (2) The force of a current flowing in a circuit crooked into small sinuosities is the same as if the circuit were smoothed out. (3) The force exerted by a closed circuit of arbitrary form on an element of another circuit is at right angles to the latter. (4) The force between two elements of circuits is unaffected when all the linear dimensions are increased proportionately, the current-strengths remaining unaltered. From these results and from the assumption that the force between current elements is along the line connecting them Ampère arrived at the following force, d^2F^A, between the current elements ids and $i'ds'$ (for a description of his procedure see (Ampère, 1823), (Maxwell, 1954, Vol. 2, Chapter 2, "Ampère's investigation of the mutual action of electric currents," Arts. [502] - [527], pp. 158 - 174), (Whittaker, 1973, Vol. 1, pp. 83 - 88)):

$$d^2F^A = -\frac{ii'dsds'}{r^2}\left(\frac{1}{2}\frac{dr}{ds}\frac{dr}{ds'} - r\frac{d^2r}{dsds'}\right) . \tag{4.1}$$

In this expression i and i' are in the electrodynamic system of units. The force is along the line joining the elements of length ds and ds', respectively. Repulsion or attraction occurs as d^2F^A is positive or negative. The distance between the current elements is r. The distance along the circuit C from ids to a fixed origin A in this circuit is s. On the other hand s' is the distance along the circuit C' from A' to $i'ds'$, see Figure 4.1.

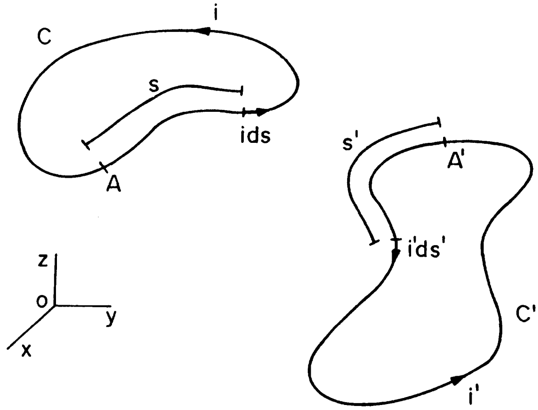

Figure 4.1

Let us denote by x, y, z the rectangular coordinates locating ids from the origin O of a coordinate system S. For $i'ds'$ we have x', y', z'. If $\vec{r}$ is the vector pointing from ids to $i'ds'$ and if $d\vec{s}$ and $d\vec{s}\,'$ point along the direction of the currents in C and C' we have:

$$\vec{r} = (x' - x)\hat{x} + (y' - y)\hat{y} + (z' - z)\hat{z} \ , \tag{4.2}$$

$$d\vec{s} = \hat{x}dx + \hat{y}dy + \hat{z}dz \ , \tag{4.3}$$

$$d\vec{s}\,' = \hat{x}dx' + \hat{y}dy' + \hat{z}dz' \ . \tag{4.4}$$

If θ is the angle between $d\vec{s}$ and $\vec{r}$, if $\pi - \theta'$ is the angle between $d\vec{s}\,'$ and $\vec{r}$, and ε is the angle between $d\vec{s}$ and $d\vec{s}\,'$ (see Figure 4.2) we have, by (4.2) to (4.4) and by (1.1):

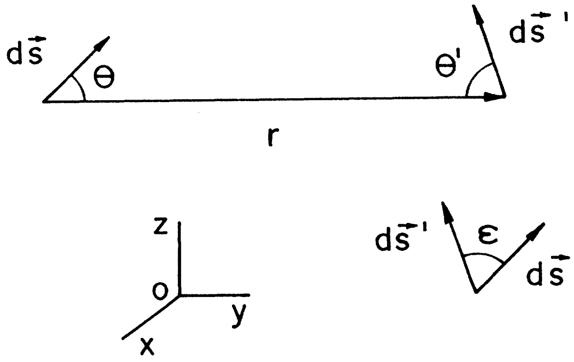

Figure 4.2

$$\vec{r}\cdot d\vec{s} = rds\cos\theta = (x'-x)dx + (y'-y)dy + (z'-z)dz \ , \tag{4.5}$$

$$\vec{r}\cdot d\vec{s}\,' = rds'\cos(\pi-\theta') = -rds'\cos\theta' = (x'-x)dx' + (y'-y)dy' + (z'-z)dz' \ , \tag{4.6}$$

$$d\vec{s}\cdot d\vec{s}\,' = dsds'\cos\varepsilon = dxdx' + dydy' + dzdz' \ . \tag{4.7}$$

From these expressions we obtain

$$\cos\theta = \frac{x'-x}{r}\frac{dx}{ds} + \frac{y'-y}{r}\frac{dy}{ds} + \frac{z'-z}{r}\frac{dz}{ds} \ , \tag{4.8}$$

$$\cos\theta' = -\frac{x'-x}{r}\frac{dx'}{ds'} - \frac{y'-y}{r}\frac{dy'}{ds'} - \frac{z'-z}{r}\frac{dz'}{ds'} \ , \tag{4.9}$$

$$\cos\varepsilon = \frac{dx}{ds}\frac{dx'}{ds'} + \frac{dy}{ds}\frac{dy'}{ds'} + \frac{dz}{ds}\frac{dz'}{ds'} \ . \tag{4.10}$$

Observing that $r = \sqrt{(x'-x)^2 + (y'-y)^2 + (z'-z)^2}$ and from (4.8) to (4.10) yields

$$\frac{dr}{ds} = -\frac{x'-x}{r}\frac{dx}{ds} - \frac{y'-y}{r}\frac{dy}{ds} - \frac{z'-z}{r}\frac{dz}{dr} = -\cos\theta \ , \tag{4.11}$$

$$\frac{dr}{ds'} = \frac{x'-x}{r}\frac{dx'}{ds'} + \frac{y'-y}{r}\frac{dy'}{ds'} + \frac{z'-z}{r}\frac{dz'}{ds'} = -\cos\theta' \ , \tag{4.12}$$

$$\frac{d^2r}{dsds'} = \frac{d}{ds'}\frac{dr}{ds} = -\frac{dx}{ds}\left(\frac{1}{r}\frac{dx'}{ds'} - \frac{x'-x}{r^2}\frac{dr}{ds'}\right)$$

$$-\frac{dy}{ds}\left(\frac{1}{r}\frac{dy'}{ds'} - \frac{y'-y}{r^2}\frac{dr}{ds'}\right) - \frac{dz}{dr}\left(\frac{1}{r}\frac{dz'}{ds'} - \frac{z'-z}{r^2}\frac{dr}{ds'}\right)$$

$$= \frac{1}{r}\frac{dr}{ds'}\cos\theta - \frac{1}{r}\cos\varepsilon = -\frac{1}{r}(\cos\theta\cos\theta' + \cos\varepsilon) \ . \tag{4.13}$$

This means that (4.1) can be put in the form

$$d^2F^A = -\frac{ii'dsds'}{r^2}\left(\frac{3}{2}\cos\theta\cos\theta' + \cos\varepsilon\right)$$

$$= -\frac{ii'}{r^2}\left[(d\vec{s}\cdot d\vec{s}\,') - \frac{3}{2}\frac{(\vec{r}\cdot d\vec{s})(\vec{r}\cdot d\vec{s}\,')}{r^2}\right] \ . \tag{4.14}$$

If i and i' were measured in the electromagnetic system of units d^2F^A would be given by twice this value. If i and i' are in the International System of Units, namely, in Ampères, then we have

$$d^2F^A = -\frac{\mu_o}{4\pi}\frac{ii'dsds'}{r^2}\left(\frac{dr}{ds}\frac{dr}{ds'} - 2r\frac{d^2r}{dsds'}\right)$$

$$= -\frac{\mu_o}{4\pi}\frac{ii'dsds'}{r^2}(2\cos\varepsilon + 3\cos\theta\cos\theta')$$

$$= -\frac{\mu_o}{4\pi}\frac{ii'}{r^2}\left[2(d\vec{s}\cdot d\vec{s}\,') - \frac{3(\vec{r}\cdot d\vec{s})(\vec{r}\cdot d\vec{s}\,')}{r^2}\right] \ . \tag{4.15}$$

Ampère's collected results describing his researches, his main paper, occurs in the *Mémoires de l'Académie de Paris* for 1823. Despite this date this volume was only published in 1827. In the printed version were incorporated communications which happened after 1823. At least part of Ampère's paper was written in 1826, although it contained results which had been obtained previously, as he mentioned in his paper the date of writing, the 30th August 1826. It has since then been published in book form, and there is also a partial English translation. Details of all this can be found in (Ampère, 1823).

According to Whittaker this work by Ampère "is one of the most celebrated memoirs in the history of natural philosophy" (Whittaker, 1973, Vol. 1, p. 83). Maxwell's admiration for Ampère's work and for his force (4.15) were expressed in the following words (our italics): "The experimental investigation by which Ampère established the laws of the mechanical action between electric currents is one of the most brilliant achievements in science. The whole, theory and experiment, seems as if it had leaped, full grown and full armed, from the brain of the 'Newton of electricity.' It is perfect in form, and unassailable in accuracy, and *it is summed up in a formula from which all the phenomena may be deduced, and which must always remain the cardinal formula of electro-dynamics*" (Maxwell, 1954, Vol. 2, Art. [528], p. 175). Unfortunately the modern textbooks dealing with electromagnetism at undergraduate and graduate levels did not follow Maxwell's point of view, as Eq. (4.15) is not found in almost anyone of them.

The best discussion of Ampère's work is undoubtedly Blondel's book (Blondel, 1982). It includes also an extensive bibliography. For a biography of Ampère, a detailed discussion of his work and further references see, for instance, (Moon and Spencer, 1954 a to c), (Tricker, 1962 and 1965), (Williams, 1970), (Graneau, 1985 a, pp. 7 - 22), (Caneva, 1980), (Whittaker, 1973, Vol. 1, pp. 81 - 88), (Maxwell, 1954, Vol. 2, Arts. [502 - 507], pp. 158 - 174) and (O'Rahilly, 1965, Vol. 1, pp. 102 - 113 and Vol. 2, pp. 518 - 523).

4.2. Derivation of Ampère's Force from Weber's Force

In this Section we utilize Weber's force to derive Ampère's force between current elements. This is one of the main results that can be obtained with Weber's electrodynamics. As we saw in the previous Chapter, historically what happened was the opposite of this. The procedure of this Section, however, will highlight some interesting aspects. This procedure was first presented in (Assis, 1990 a) and (Wesley, 1990 a).

Weber's force exerted by an element of charge dq_j on dq_i is given by (3.24), namely

$$d^2\vec{F}_{ji} = \frac{dq_i dq_j}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{1}{c^2}\left(\vec{v}_{ij}\cdot\vec{v}_{ij} - \frac{3}{2}(\hat{r}_{ij}\cdot\vec{v}_{ij})^2 + \vec{r}_{ij}\cdot\vec{a}_{ij}\right)\right] . \qquad (4.16)$$

To derive Ampère's force from this expression we suppose each current element $I_m d\vec{l}_m$ ($m = i,\ j$) to consist of positive and negative charges, dq_{m+} and dq_{m-}, with velocities $\vec{v}_{m+}$ and $\vec{v}_{m-}$, and accelerations $\vec{a}_{m+}$ and $\vec{a}_{m-}$, respectively, relative to a frame of reference S. As the current elements have an infinitesimal size (or are of an atomic size), we can put $\vec{r}_{m+} = \vec{r}_{m-} \equiv \vec{r}_m$. The current element $I_i d\vec{l}_i$ belongs to a circuit C_i while $I_j d\vec{l}_j$ belongs to C_j, see Figure 4.3.

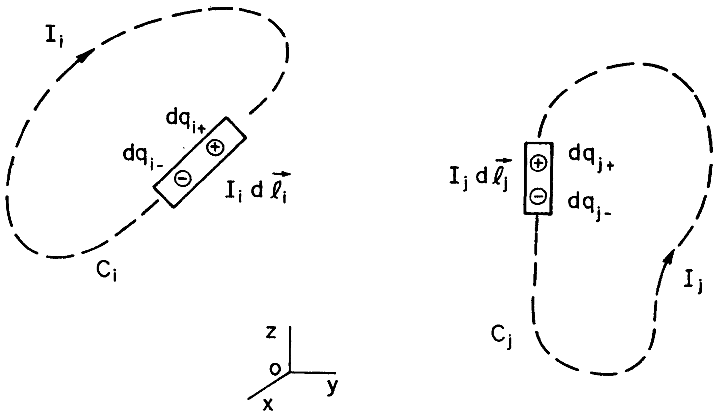

Figure 4.3

To calculate the force exerted by $I_j d\vec{l}_j$ on $I_i d\vec{l}_i$ we need then to add four components of the force, namely, the force exerted by dq_{j+} on dq_{i+} and on dq_{i-}, and the force exerted by dq_{j-} on dq_{i+} and on dq_{i-}:

$$d^2\vec{F} = d^2\vec{F}_{j+,i+} + d^2\vec{F}_{j+,i-} + d^2\vec{F}_{j-,i+} + d^2\vec{F}_{j-,i-} \ . \tag{4.17}$$

To perform this summation we suppose that the current elements are electrically neutral, namely

$$dq_{j-} = -dq_{j+} \ , \quad dq_{i-} = -dq_{i+} \ . \tag{4.18}$$

This was the situation in Ampère's experiments (neutral currents in metallic conductors) and happens in most practical situations (current in wires, in gaseous plasmas, in conducting liquid solutions, etc.)

We now apply (4.17) and (4.18) in (4.16). The Coulombian part yields zero, as is easily seen. We now add the four components of the terms involving $\vec{v}_{ij} \cdot \vec{v}_{ij}$, namely

$$-\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\frac{1}{c^2}\left[(v_{i+}^2 - 2\vec{v}_{i+}\cdot\vec{v}_{j+} + v_{j+}^2) - (v_{i-}^2 - 2\vec{v}_{i-}\cdot\vec{v}_{j+} + v_{j+}^2)\right.$$

$$\left. - (v_{i+}^2 - 2\vec{v}_{i+}\cdot\vec{v}_{j-} + v_{j-}^2) + (v_{i-}^2 - 2\vec{v}_{i-}\cdot\vec{v}_{j-} + v_{j-}^2)\right]$$

$$= -2\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\frac{(\vec{v}_{i+} - \vec{v}_{i-})\cdot(\vec{v}_{j+} - \vec{v}_{j-})}{c^2} \ . \tag{4.19}$$

Adding the four components involving $(\hat{r}_{ij} \cdot \vec{v}_{ij})^2$ yields

$$\frac{3}{2}\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\frac{1}{c^2}\left\{\left[(\hat{r}_{ij}\cdot\vec{v}_{i+})^2 - 2(\hat{r}_{ij}\cdot\vec{v}_{i+})(\hat{r}_{ij}\cdot\vec{v}_{j+}) + (\hat{r}_{ij}\cdot\vec{v}_{j+})^2\right]\right.$$

$$- \left[(\hat{r}_{ij}\cdot\vec{v}_{i-})^2 - 2(\hat{r}_{ij}\cdot\vec{v}_{i-})(\hat{r}_{ij}\cdot\vec{v}_{j+}) + (\hat{r}_{ij}\cdot\vec{v}_{j+})^2\right]$$

$$- \left[(\hat{r}_{ij}\cdot\vec{v}_{i+})^2 - 2(\hat{r}_{ij}\cdot\vec{v}_{i+})(\hat{r}_{ij}\cdot\vec{v}_{j-}) + (\hat{r}_{ij}\cdot\vec{v}_{j-})^2\right]$$

$$\left. + \left[(\hat{r}_{ij}\cdot\vec{v}_{i-})^2 - 2(\hat{r}_{ij}\cdot\vec{v}_{i-})(\hat{r}_{ij}\cdot\vec{v}_{j-}) + (\hat{r}_{ij}\cdot\vec{v}_{j-})^2\right]\right\}$$

$$= 3\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\frac{[\hat{r}_{ij}\cdot(\vec{v}_{i+}-\vec{v}_{i-})]\,[\hat{r}_{ij}\cdot(\vec{v}_{j+}-\vec{v}_{j-})]}{c^2} \ . \tag{4.20}$$

Adding the four components involving $\vec{r}_{ij}\cdot\vec{a}_{ij}$ yields

$$-\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\frac{1}{c^2}\{\vec{r}_{ij}\cdot[(\vec{a}_{i+}-\vec{a}_{j+})-(\vec{a}_{i-}-\vec{a}_{j+})-(\vec{a}_{i+}-\vec{a}_{j-})+(\vec{a}_{i-}-\vec{a}_{j-})]\} = 0 \ . \tag{4.21}$$

Adding all these components yields

$$d^2\vec{F} = -\frac{dq_{i+}dq_{j+}}{4\pi\varepsilon_o}\frac{\hat{r}_{ij}}{r_{ij}^2}\left\{2\frac{(\vec{v}_{i+}-\vec{v}_{i-})\cdot(\vec{v}_{j+}-\vec{v}_{j-})}{c^2}\right.$$

$$\left. - \ 3\frac{[\hat{r}_{ij}\cdot(\vec{v}_{i+}-\vec{v}_{i-})][\hat{r}_{ij}\cdot(\vec{v}_{j+}-\vec{v}_{j-})]}{c^2}\right\} \ . \tag{4.22}$$

In order to obtain the final result we utilize (2.52), $c = (\varepsilon_o\mu_o)^{-1/2}$, (4.18), and the fact that

$$I_i d\vec{l}_i = dq_{i+}(\vec{v}_{i+}-\vec{v}_{i-}) \ , \quad I_j d\vec{l}_j = dq_{j+}(\vec{v}_{j+}-\vec{v}_{j-}) \ . \tag{4.23}$$

This yields Ampère's force exerted by $I_j d\vec{l}_j$ on $I_i d\vec{l}_i$, namely

$$d^2\vec{F}_{ji}^A = -\frac{\mu_o}{4\pi}I_iI_j\frac{\hat{r}_{ij}}{r_{ij}^2}\left[2(d\vec{l}_i\cdot d\vec{l}_j) - 3(\hat{r}_{ij}\cdot d\vec{l}_i)(\hat{r}_{ij}\cdot d\vec{l}_j)\right] = -d^2\vec{F}_{ij}^A \ . \tag{4.24}$$

Now some remarks should be made. We first note that this expression was obtained utilizing neutral current elements, which means that (4.24) does not need to be valid, for instance, for two electron beams. In this case we need to begin with Weber's force, (4.16), without utilizing (4.18). We will not deal with this case here.

The second remark is that the acceleration terms which exist in (4.16) do not appear in (4.24), although we did not impose any conditions on $\vec{a}_{i+}$, $\vec{a}_{i-}$, $\vec{a}_{j+}$ and $\vec{a}_{j-}$. This indicates that Ampère's force remains valid in situations in which the charges are accelerated, not only due to the curvature of the wires (centripetal accelerations), but also when the intensity of the currents are a function of time. So Ampère's force may be applied even in non stationary situations when the currents are changing in time, as is the case in alternating current circuits, or when we turn on or off the current in a circuit.

The third and most important remark is that to arrive at Ampère's force from Weber's force we did not impose any conditions on $\vec{v}_{i+}$, $\vec{v}_{i-}$, $\vec{v}_{j+}$ and $\vec{v}_{j-}$. These four velocities are each one of them arbitrary and independent from one another. This means that (4.24) is still derived from (4.16) even in metallic circuits in which the positive charges are fixed in the lattice ($\vec{v}_{i+} = 0$, $\vec{v}_{j+} = 0$) and only the moving electrons are responsible for the currents. This will also happen when the positive and negative charges move in opposite directions with velocities of different magnitudes (as in situations of electrolysis, or in the usual gaseous plasma where the ratio between the velocities of the positive ions and of the electrons is as the inverse ratio of the masses, namely, $v_{i-} \cong -(m_{i+}/m_{i-})v_{i+}$).

Historically Weber derived his force from Ampère's one utilizing Fechner hypothesis of 1845. According to this hypothesis (Fechner, 1845) the positive and negative charges in metallic wires move in opposite directions with equal velocities, namely

$$\vec{v}_{i-} = -\vec{v}_{i+} \text{ and } \vec{v}_{j-} = -\vec{v}_{j+} \ . \tag{4.25}$$

With the discovery of the Hall effect in 1879 it was soon realized that the current in metallic wires was due to the motion of negative charges only, so that the positive ions were fixed in the lattice (see O'Rahilly, 1965, Vol. 2, pp. 512 - 518; Whittaker, 1973, Vol. 1, pp. 289 - 291). This fact was strengthened by the discovery of the electron in 1897 by J. J. Thomson. This showed that Fechner's hypothesis was untenable. To many this indicated a failure of Weber's electrodynamics because they thought Fechner's hypothesis was intrinsically connected to Weber's force through Ampère's force. But as we have seen, if we assume only Weber's force (4.16) and the neutrality of the current elements we can still derive Ampère's force (4.24) even when Fechner's hypothesis is wrong, as is the case in the usual metallic conductors. The proof of this Section overcomes this limitation pointed out against Weber's electrodynamics as it is based on general assumptions more general than the particular case specified by Fechner's hypothesis.

To give an example of how this misconception regarding Weber's electrodynamics has survived we present here Rohrlich's book (1965) in the only paragraph where he mentions

Weber's theory: "Most of the ideas at that time revolved around electricity as some kind of fluid or at least continuous medium. In 1845, however, Gustav T. Fechner suggested that electric currents might be due to *particles* of opposite charge which move with equal speeds in opposite directions in a wire. From this idea Wilhelm Weber (1804 - 1890) developed the first ***particle electrodynamics*** (1846). It was based on a force law between two particles of charges e_1 and e_2 at a distance r apart,

$$F = \frac{e_1 e_2}{r^2}\left[1 + \frac{r}{c^2}\frac{d^2 r}{dt^2} - \frac{1}{2c^2}\left(\frac{dr}{dt}\right)^2\right] .$$

This force seemed to fit the experiments (Ampère's law, Biot-Savart's law) but ran into theoretical difficulties and eventually had to be discarded when, among other things, the basic assumption of equal speeds in opposite directions was found untenable" (Rohrlich, 1965, p. 9).

As we have just seen, we do not need to discard Weber's law because Fechner's hypothesis for metallic currents is wrong. There is no necessary connexion between Fechner's idea and Weber's electrodynamics, although they were historically linked.

4.3. Grassmann's Force and Biot-Savart's Law

Despite Maxwell's and Whittaker's praises, Ampère's force (4.24) is little known nowadays. It doesn't appear anymore in almost any didactic textbook of undergraduate or graduate levels. Instead of that only Grassmann's force appears, which is given by

$$d^2\vec{F}^G_{ji} = I_i d\vec{l}_i \times d\vec{B}_j \ , \tag{4.26}$$

$$d\vec{B}_j = \frac{\mu_0}{4\pi}\frac{1}{r^2_{ij}}(I_j d\vec{l}_j \times \hat{r}_{ij}) \ . \tag{4.27}$$

In this last expression $d\vec{B}_j$ is the magnetic field generated by the element $I_j d\vec{l}_j$ according to the law of Biot and Savart, (2.23). The expression (4.26) for the force was first given by Grassmann in 1845, and his paper has already been translated to English (Grassmann, 1845).

Grassmann never had a formal education in mathematics or physics (at university he studied only philology and theology). During all his life he was a teacher of mathematics at high schools and never worked in a university, although he wished that. His main scientific work was the development of a generalized algebra in which the commutative property and the law of the existence of an inverse in the multiplication were not necessarily valid. He published his results in a book in 1844 (only one year after the discovery of quaternions by Hamilton), and in a second improved and enlarged version in 1862. It is in his first book that the modern scalar and vector products appear clearly defined for the first time. In 1845 he published his force law between current elements as an important application of his generalized algebra. It seems that he never carried out any experiments in physics, not even related to electrodynamics. For this and other information see: (Crowe, 1985).

Using (1.7) we can put (4.26) and (4.27) into the form

$$d^2\vec{F}^G_{ji} = -\frac{\mu_0}{4\pi}\frac{I_i I_j}{r^2_{ij}}[(d\vec{l}_i \cdot d\vec{l}_j)\hat{r}_{ij} - (d\vec{l}_i \cdot \hat{r}_{ij})d\vec{l}_j] \ . \tag{4.28}$$

Changing the indexes i and j, observing that $\hat{r}_{ij} = -\hat{r}_{ji}$ and that $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$ yields:

$$d^2\vec{F}^G_{ij} = I_j d\vec{l}_j \times d\vec{B}_i = -I_j d\vec{l}_j \times \left[\frac{\mu_0}{4\pi}\left(I_i d\vec{l}_i \times \frac{\hat{r}_{ij}}{r^2_{ij}}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\frac{I_i I_j}{r^2_{ij}}\left[(d\vec{l}_i \cdot d\vec{l}_j)\hat{r}_{ij} - (d\vec{l}_j \cdot \hat{r}_{ij})d\vec{l}_i\right] \neq -d^2\vec{F}^G_{ji} \ . \quad (4.29)$$

The first aspect to be observed in these expressions is that although the first term of the force follows the action and reaction principle in the strong form, the same does not happen with the second term which is parallel to $d\vec{l}_j$ or to $d\vec{l}_i$. So Grassmann's force does not follow the action and reaction principle (not even in the weak form), with the exception of some very particular cases. Here we are restricting our analysis to current elements, but later on we discuss the force between closed circuits.

4.4. Derivation of Grassmann's Force from Lorentz's Force

In this section we show how to arrive at (4.26) using Lorentz's force (2.41). Again the historical happenings were the opposite because Lorentz knew Grassmann's force and arrived at the magnetic part of his force law by substituting $q\vec{v}$ for $Id\vec{l}$ on Grassmann's expression, although he does not mention Grassmann's name (Lorentz, 1895; and Lorentz, 1915, pp. 14 and 15); (O'Rahilly, 1965, Vol. 2, p. 561).

As with Weber's force in Section 4.2, we suppose that there are positive and negative charges in both current elements: dq_{i+}, dq_{i-}, dq_{j+} and dq_{j-}. We suppose that the element $I_j d\vec{l}_j$ generates an electric (if it is not neutral) and magnetic fields, $d\vec{E}_j$ and $d\vec{B}_j$, respectively. Adding the forces (2.41) acting on the positive and negative charges of $I_i d\vec{l}_i$ yields

$$d^2\vec{F}_{ji} = (dq_{i+} + dq_{i-})d\vec{E}_j + (dq_{i+}\vec{v}_{i+} + dq_{i-}\vec{v}_{i-}) \times d\vec{B}_j \ . \qquad (4.30)$$

Imposing the electric neutrality of the current elements ($dq_{i-} = -dq_{i+}$ and $dq_{j-} = -dq_{j+}$) the electric component of the force goes to zero (because $dq_{i+} + dq_{i-} = 0$ and $d\vec{E}_j = 0$). Using then (4.18) and (4.23) in (4.30) yields (4.26), QED. To arrive at (4.28) something extra is necessary such as the law of Biot-Savart (4.27) connecting the currents in the sources with the magnetic field they generate.

Another deduction of Grassmann's force from Lorentz's force and using the retarded potentials of Lienard-Wiechert can be found in Chapter 6.

4.5. Ampère Versus Grassmann

In this Section we compare the expressions of force between current elements obtained by Ampère and Grassmann.

The first aspect to be emphasized is that Ampère's force (4.24), always satisfies Newton's third law (action and reaction) in the strong form. This means that $d^2\vec{F}^A_{ji} = -d^2\vec{F}^A_{ij}$ and that $d^2\vec{F}^A_{ji}$ is parallel to $\vec{r}_{ij}$ for any arbitrary and independent orientation of $I_i d\vec{l}_i$, $I_j d\vec{l}_j$ and $\vec{r}_{ij}$. On the other hand Grassmann's force only satisfies this principle in very special cases. An example of this different property is given in Figure 4.4:

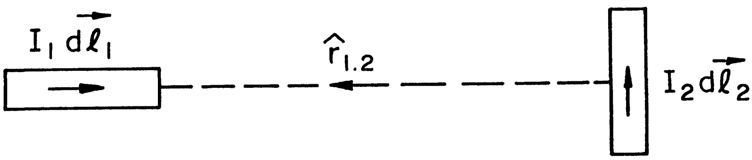

Figure 4.4

From Ampère's force:

$$d^2\vec{F}^A_{21} = d^2\vec{F}^A_{12} = 0 \ . \tag{4.31}$$

On the other hand from Grassmann's force we have

$$d^2\vec{F}^G_{21} = -\frac{\mu_0}{4\pi}\frac{I_1 I_2}{r^2_{12}} dl_1 \ d\vec{l}_2 \neq 0 \ , \tag{4.32}$$

$$d^2\vec{F}^G_{12} = 0 \ . \tag{4.33}$$

That is, according to Grassmann's force the current element $I_1 d\vec{l}_1$ exerts no net force on $I_2 d\vec{l}_2$, but $I_2 d\vec{l}_2$ exerts a net force on $I_1 d\vec{l}_1$.

Some textbooks present this example, but it is usually claimed that current elements do not exist and that we only have closed currents (for interaction between two closed circuits Grassmann's force also predicts equality of action and reaction, as we will see). As we discuss in this Chapter, the real situation can be different from what textbooks say.

We discuss now the situation of Figure 4.5, namely, the force between two current elements which are parallel and collinear:

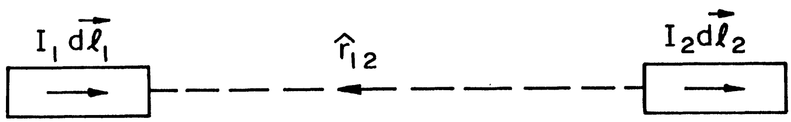

Figure 4.5

Utilizing (4.24) to (4.29) yields

$$d^2\vec{F}^A_{21} = \frac{\mu_o}{4\pi} I_1 I_2 \frac{dl_1 dl_2}{r^2_{12}} \hat{r}_{12} = -d^2\vec{F}^A_{12} \ , \tag{4.34}$$

$$d^2\vec{F}^G_{21} = 0 = d^2\vec{F}^G_{12} \ . \tag{4.35}$$

In this very particular situation we can see that Grassmann's force follows the principle of action and reaction, predicting no net force of one element on the other. On the other hand Ampère's force also follows the principle of action and reaction, but it predicts a repulsive force between the elements. And it is exactly utilizing this fact that many experiments involving a single circuit have been performed recently trying to distinguish these two forces. Before presenting this point we discuss here an important fact which shows why for so long many physicists thought that these two forces were indistinguishable.

This fact can be expressed as: The force exerted by a closed circuit of arbitrary form on a current element of another circuit is the same when calculated by Ampère's force or by Grassmann's force. The main reason explaining this fact is that the difference between (4.24) and (4.28) yields an exact differential, the integral of which along the closed circuit C_j is zero.

We present here a proof of this fact, which has been known since last century. The proof follows (Tricker, 1965, pp. 55 - 58). Let us choose a coordinate system so that the element $I_i d\vec{l}_i$ is situated at the origin and directed along the z-axis. The element $I_j d\vec{l}_j$ is located at $\vec{r}_j = x_j\hat{x} + y_j\hat{y} + z_j\hat{z}$ and belongs to a different closed circuit C_j of arbitrary form (Figure 4.6):

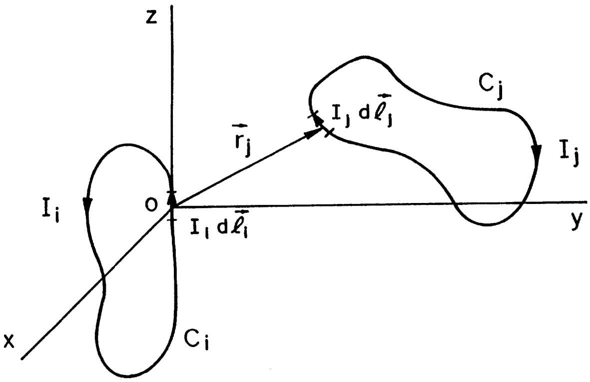

Figure 4.6

With this choice we have

$$\vec{r}_{ij} = -(x_j\hat{x} + y_j\hat{y} + z_j\hat{z}) , \tag{4.36}$$

$$d\vec{l_i} = dz_i \hat{z} \ , \tag{4.37}$$

$$d\vec{l_j} = dx_j \hat{x} + dy_j \hat{y} + dz_j \hat{z} \ . \tag{4.38}$$

Defining

$$dl_i \equiv |d\vec{l_i}| = dz_i \ , \tag{4.39}$$

$$dl_j \equiv |d\vec{l_j}| = \sqrt{(dx_j)^2 + (dy_j)^2 + (dz_j)^2} \ , \tag{4.40}$$

yields

$$d\vec{l_i} \cdot d\vec{l_j} = dz_i dz_j = dz_i dl_j \cos\varepsilon \ , \tag{4.41}$$

$$\hat{r}_{ij} \cdot d\vec{l_i} = -\frac{z_j dz_i}{r_j} = dz_i \cos\theta_i \ , \tag{4.42}$$

$$\hat{r}_{ij} \cdot d\vec{l_j} = -\frac{x_j dx_j + y_j dy_j + z_j dz_j}{r_j} = -dr_j = dl_j \cos\theta_j \ . \tag{4.43}$$

In these expressions we have called ε the angle between $d\vec{l_i}$ and $d\vec{l_j}$, θ_i the angle between $d\vec{l_i}$ and $\hat{r}_{ij}$ and θ_j the angle between $d\vec{l_j}$ and $\hat{r}_{ij}$.

From (4.41) to (4.43) we obtain

$$\cos\varepsilon = \frac{dz_j}{dl_j} \ , \tag{4.44}$$

$$\cos\theta_i = -\frac{z_j}{r_j} \ , \tag{4.45}$$

$$\cos\theta_j = -\frac{dr_j}{dl_j} \ . \tag{4.46}$$

Applying these expressions in Ampère's force (4.24) yields

$$d^2\vec{F}_{ji}^A = \frac{\mu_o}{4\pi} I_i I_j dl_i dl_j \frac{x_j\hat{x} + y_j\hat{y} + z_j\hat{z}}{r_j^3} \left(2\frac{dz_j}{dl_j} - 3\frac{z_j}{r_j}\frac{dr_j}{dl_j} \right) . \quad (4.47)$$

The z-component can be written as

$$\frac{\mu_o}{4\pi} I_i I_j dl_i dl_j \left(2\frac{z_j}{r_j^3}\frac{dz_j}{dl_j} - 3\frac{z_j^2}{r_j^4}\frac{dr_j}{dl_j} \right)$$

$$= \frac{\mu_o}{4\pi} I_i I_j dl_i dl_j \left(\frac{d}{dl_j}\frac{z_j^2}{r_j^3} \right) = \frac{\mu_o}{4\pi} I_i I_j dl_i d\left(\frac{z_j^2}{r_j^3} \right) . \quad (4.48)$$

As this is an exact differential, this vanishes when integrated along the closed circuit C_j of which $d\vec{l}_j$ belongs. This proves for the general case that the force exerted by a closed circuit of arbitrary form on a current element of another circuit is at right angles to the element, according to Ampère's force (4.24).

The x-component of (4.47) can be written as

$$\frac{\mu_o}{4\pi} I_i I_j dl_i dl_j \left(2\frac{x_j}{r_j^3} - 3\frac{x_j z_j}{r_j^4}\frac{dr_j}{dl_j} \right) = \frac{\mu_o}{4\pi} I_i I_j dl_i dl_j \left(\frac{d}{dl_j}\frac{x_j z_j}{r_j^3} + \frac{x_j}{r_j^3}\frac{dz_j}{dl_j} - \frac{z_j}{r_j^3}\frac{dx_j}{dl_j} \right)$$

$$= \frac{\mu_o}{4\pi} I_i I_j dl_i d\left(\frac{x_j z_j}{r_j^3} \right) + \frac{\mu_o}{4\pi}\frac{I_i I_j}{r_j^3} dl_i dl_j \left(x_j \frac{dz_j}{dl_j} - z_j \frac{dx_j}{dl_j} \right) . \quad (4.49)$$

On integration round the closed circuit C_j, the first term vanishes.

So the final expression for the force exerted by a closed circuit C_j on $I_i d\vec{l}_i$ localized at the origin along the z-axis according to Ampère's force (4.24) is given by (after performing a similar calculation for the y-component):

$$d\vec{F}_{C_j \text{ on } I_i d\vec{l}_i}^A = \frac{\mu_o}{4\pi} I_i I_j dl_i \oint_{C_j} \left[\hat{x}\left(x_j\frac{dz_j}{dl_j} - z_j\frac{dx_j}{dl_j} \right) + \hat{y}\left(y_j\frac{dz_j}{dl_j} - z_j\frac{dy_j}{dl_j} \right) \right] \frac{dl_j}{r_j^3} . \quad (4.50)$$

We now calculate the force with Grassmann's expression (4.28). With (4.36) to (4.46) this yields

$$d^2\vec{F}_{ji}^G = \frac{\mu_o}{4\pi}\frac{I_i I_j}{r_j^2} dl_i dl_j \left[\frac{dz_j}{dl_j}\frac{(x_j\hat{x} + y_j\hat{y} + z_j\hat{z})}{r_j} - \frac{z_j}{r_j}\frac{(dx_j\hat{x} + dy_j\hat{y} + dz_j\hat{z})}{dl_j} \right]$$

$$= \frac{\mu_o}{4\pi} I_i I_j dl_i \left[\hat{x} \left(x_j \frac{dz_j}{dl_j} - z_j \frac{dx_j}{dl_j} \right) + \hat{y} \left(y_j \frac{dz_j}{dl_j} - z_j \frac{dy_j}{dl_j} \right) \right] \frac{dl_j}{r_j^3} \ . \qquad (4.51)$$

Integrating this over C_j yields the same result as (4.50). This completes the proof.

What we have just shown is that the force of a *closed* circuit C_j of arbitrary form acting on a current element $I_i d\vec{l}_i$ of another circuit has the same value according to Ampère's force and to Grassmann's one. This means that in this case we can write

$$d\vec{F}^A_{C_j \text{ on } I_i d\vec{l}_i} = d\vec{F}^G_{C_j \text{ on } I_i d\vec{l}_i} = I_i d\vec{l}_i \times \left(\frac{\mu_o}{4\pi} \oint_{C_j} \frac{I_j d\vec{l}_j \times \hat{r}_{ij}}{r_{ij}^2} \right) \ . \qquad (4.52)$$

In this last form it is easily seen that the force of a closed circuit on a current element is orthogonal to this element, because if $\vec{C}$ is given by $\vec{A} \times \vec{B}$ then $\vec{C}$ is orthogonal to $\vec{A}$ and to $\vec{B}$. This remarkable fact was first obtained experimentally by Ampère himself and represents his third case of equilibrium from which he began in order to arrive at (4.24). In his own words: "The general conclusion may therefore be drawn that the action of a closed circuit, or of an assembly of closed circuits, on an infinitesimal element of an electric current is perpendicular to this element" (Ampère, 1823; English translation in Tricker, 1965, p. 170).

4.6. Force Between Circuits from the Coefficient of Mutual Inductance

In practice Ampère's force (4.24) and Grassmann's force (4.26) to (4.28) are not directly utilized to calculate the force between closed circuits. Even (4.52) is hardly utilized except in some extremely symmetrical situations (an infinite straight wire, a long solenoid, etc.) Instead of these expressions it is employed the coefficient of mutual inductance M which has been calculated and tabulated for many important practical situations (see, for instance, Grover, 1946). Let us show how this is done.

From (4.52) the force exerted by a closed circuit C_j on another closed circuit C_i is given by, according to Ampère's and Grassmann's forces:

$$\vec{F}_{C_jC_i} = \frac{\mu_o}{4\pi} I_i I_j \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \times (d\vec{l}_j \times \hat{r}_{ij})}{r_{ij}^2} \ . \tag{4.53}$$

The integrand can be written by (1.7) as

$$\frac{d\vec{l}_j(d\vec{l}_i \cdot \hat{r}_{ij})}{r_{ij}^2} - \frac{\hat{r}_{ij}(d\vec{l}_i \cdot d\vec{l}_j)}{r_{ij}^2} \ . \tag{4.54}$$

Integrating the first term of (4.54) around C_i yields, by Stokes's theorem (1.33):

$$\frac{\mu_o}{4\pi} I_i I_j \oint_{C_j} d\vec{l}_j \oint_{C_i} \frac{\hat{r}_{ij}}{r_{ij}^2} \cdot d\vec{l}_i = \frac{\mu_o}{4\pi} I_i I_j \oint_{C_j} d\vec{l}_j \left\{ \int\int_{S_j} \left[\nabla_j \times \left(\frac{\hat{r}_{ij}}{r_{ij}^2} \right) \right] \cdot d\vec{a}_j \right\} \ . \tag{4.55}$$

Observing that

$$\nabla_i \frac{1}{r_{ij}} = -\nabla_j \frac{1}{r_{ij}} = -\frac{\hat{r}_{ij}}{r_{ij}^2} \ , \tag{4.56}$$

and that $\nabla \times (\nabla\phi) = 0$, (1.26), shows that the integrand of (4.55) is identically zero. So the first term of (4.54) gives no contribution to (4.53). The net force between the two closed circuits is then given by

$$\vec{F}_{C_jC_i} = -\frac{\mu_o}{4\pi} I_i I_j \oint_{C_i} \oint_{C_j} \frac{(d\vec{l}_i \cdot d\vec{l}_j)\hat{r}_{ij}}{r_{ij}^2} \ . \tag{4.57}$$

In this symmetrical form we can see that when we have two or more *closed* circuits Grassmann's force will satisfy the principle of action and reaction, although it is not valid in general for Grassmann's force between two current elements. Ampère's force (4.24) always complies with this principle for any arbitrary orientation of the two current elements, as we have seen.

Suppose we have two rigid circuits whose orientations in space are fixed, but whose relative separation can be changed, in which flow constant currents. We choose two points located at $\vec{R}_i$ and $\vec{R}_j$, P_i and P_j, respectively, rigidly connected to the circuits C_i and C_j (Figure 4.7):

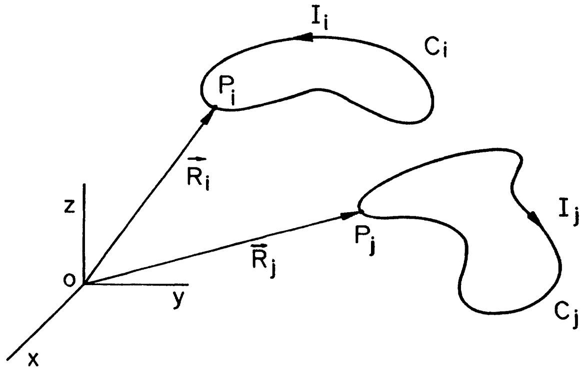

Figure 4.7

Let the current elements $I_i d\vec{l}_i$ and $I_j d\vec{l}_j$ be located at $\vec{r}_i$ and $\vec{r}_j$ from O, or at $\vec{r}_i\,'$ and $\vec{r}_j\,'$ from P_i and P_j, respectively, so that

$$\vec{r}_{ij} \equiv \vec{r}_i - \vec{r}_j = (\vec{R}_i + \vec{r}_i\,') - (\vec{R}_j + \vec{r}_j\,') \equiv \vec{r}_{ij}\,' + \vec{R}_{ij}\,' \ . \tag{4.58}$$

We then have

$$\nabla_{R_{ij}} \frac{1}{r_{ij}} = \nabla_{R_{ij}} \frac{1}{|\vec{r}_{ij}|} = \nabla_{R_{ij}} \frac{1}{|\vec{r}_{ij}\,' + \vec{R}_{ij}|} = -\frac{\vec{r}_{ij}\,' + \vec{R}_{ij}}{|\vec{r}_{ij}\,' + \vec{R}_{ij}|^3} = -\frac{\hat{r}_{ij}}{r_{ij}^2} \ . \tag{4.59}$$

Applying (4.59) in (4.57) yields

$$\vec{F}_{C_j C_i} = \nabla_{R_{ij}}(I_i I_j M_{ij}) = I_i I_j \nabla_{R_{ij}} M_{ij} \ , \qquad (4.60)$$

where

$$M_{ij} \equiv \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{r_{ij}} = \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{r}_{ij}|} = \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{|\vec{r}_{ij}^{\,\prime} + \vec{R}_{ij}|} \ . \qquad (4.61)$$

The quantity M_{ij} defined by (4.61) was first introduced by Franz Neumann in 1845. It is called the coefficient of mutual inductance between the closed circuits C_i and C_j. It is a purely geometrical factor which depends on the form, orientation and separation of the circuits, but which is independent of the electric currents. Eq. (4.60) is the usual way of calculating the force between closed circuits. The scalar function M_{ij} has been calculated for many geometries. The force is then obtained through a gradient and this is much simpler than a direct vectorial integration of (4.53). In Grover's words: "The calculation of the magnetic attraction between two coils, carrying current, is a subject closely related to the calculation of their mutual inductance. Since their mutual energy is equal to the product of their mutual inductance by the currents in the coils, the component of the magnetic force (attraction or repulsion) in any direction is equal to the differential coefficient of the mutual inductance taken with respect to that coordinate, and multiplied by the product of the currents" (Grover, 1946, Chapter 23: Formulas for the calculation of the magnetic force between coils, p. 248).

Let us now relate (4.60) to Weber's electrodynamics. The most direct connection is through Weber's force, from which Ampère's force (4.24) can be derived. And here we have seen that (4.60) follows from Ampère's force between current elements. But there is another link between (4.60) and Weber's electrodynamics. It is through Weber's potential energy (3.25). Let us suppose the current elements to be composed of positive and negative charges as in Figure 4.3. Neglecting, for the time being, the energy to form each of the current elements, the energy to bring them from an infinite distance from one another to the final separation r_{ij} is given by

$$d^2U = d^2U_{j+,i+} + d^2U_{j+,i-} + d^2U_{j-,i+} + d^2U_{j-,i-} \ . \qquad (4.62)$$

In this expression $d^2U_{m,n}$ is Weber's potential energy (3.25), namely

$$d^2U_{m,n} = \frac{dq_m dq_n}{4\pi\varepsilon_o}\frac{1}{r_{mn}}\left(1 - \frac{\dot{r}_{mn}^2}{2c^2}\right) . \tag{4.63}$$

Adding the four terms of (4.62) utilizing only the charge neutrality of the current elements (4.18) and the definition of the current elements (4.23) yields (it should be emphasized that we are **not** imposing here Fechner's hypothesis $\vec{v}_{m-} = -\vec{v}_{m+}$):

$$d^2U = \frac{\mu_o}{4\pi} I_i I_j \frac{(\hat{r}_{ij}\cdot d\vec{l}_i)(\hat{r}_{ij}\cdot d\vec{l}_j)}{r_{ij}} . \tag{4.64}$$

The potential energy (which may also be called the magnetic energy) between two closed circuits is then given by

$$U = I_i I_j N_{ij} , \tag{4.65}$$

where

$$N_{ij} \equiv \frac{\mu_o}{4\pi}\oint_{C_i}\oint_{C_j}\frac{(\hat{r}_{ij}\cdot d\vec{l}_i)(\hat{r}_{ij}\cdot d\vec{l}_j)}{r_{ij}} . \tag{4.66}$$

Utilizing Stoke's theorem (1.33) this can be written as

$$N_{ij} = \frac{\mu_o}{4\pi}\oint_{C_i}\int\int_{S_j}\left\{\nabla_j \times \left[\frac{(\hat{r}_{ij}\cdot d\vec{l}_i)}{r_{ij}}\hat{r}_{ij}\right]\right\}\cdot d\vec{a}_j . \tag{4.67}$$

Observing that

$$\nabla_i \times \hat{r}_{ij} = \nabla_j \times \hat{r}_{ij} = 0 , \tag{4.68}$$

and utilizing (1.21) this yields

$$N_{ij} = -\frac{\mu_o}{4\pi}\oint_{C_i}\int\int_{S_j}\left\{\hat{r}_{ij}\times\nabla_j\left[\frac{(\hat{r}_{ij}\cdot d\vec{l}_i)}{r_{ij}}\right]\right\}\cdot d\vec{a}_j . \tag{4.69}$$

We now observe that

$$\nabla_j(\hat{r}_{ij}\cdot d\vec{l}_i) = -\frac{d\vec{l}_i}{r_{ij}} + (\hat{r}_{ij}\cdot d\vec{l}_i)\frac{\hat{r}_{ij}}{r_{ij}} , \tag{4.70}$$

$$\nabla_i(\hat{r}_{ij} \cdot d\vec{l}_j) = \frac{d\vec{l}_j}{r_{ij}} - (\hat{r}_{ij} \cdot d\vec{l}_j)\frac{\hat{r}_{ij}}{r_{ij}} \ . \tag{4.71}$$

With (4.56), (4.70) and (1.14) in (4.69) yields

$$N_{ij} = \frac{\mu_o}{4\pi} \oint_{C_i} \int\int_{S_j} \left(\frac{\hat{r}_{ij}}{r_{ij}^2} \times d\vec{l}_i \right) \cdot d\vec{a}_j \ . \tag{4.72}$$

With (4.56) and (1.21) this can be written as

$$N_{ij} = \frac{\mu_o}{4\pi} \oint_{C_i} \int\int_{S_j} \left[\nabla_j \times \left(\frac{d\vec{l}_i}{r_{ij}} \right) \right] \cdot d\vec{a}_j \ . \tag{4.73}$$

Applying Stoke's theorem once more and comparing N_{ij} with M_{ij} shows that the interaction energy between two closed circuits is given in Weber's electrodynamics by

$$U = I_i I_j M_{ij} \ . \tag{4.74}$$

Expression (4.60) indicates then that the force between two closed circuits is given by the gradient of the mutual potential energy. Eq. (4.74) can also be called the mutual magnetic energy of the two circuits. An analogous expression is obtained in classical electromagnetism.

4.7. Derivation of the Magnetic Circuital Law and of the Law of Nonexistence of Magnetic Monopoles

In this Section we show how to derive the magnetic circuital law which, as we have seen in Section 2.5, is ususally called "Ampère's" circuital law. This is the name given to equation (2.57), sometimes to this equation without the term in $d\Phi_E/dt$. We will also derive another of Maxwell's equations, namely, the law of the nonexistence of magnetic monopoles, (2.58). To derive these two laws we will follow the procedure presented by Jackson (Jackson, 1975, Section 5.3).

As we saw in Section 2.5, the magnetic circuital law (2.57) can be derived directly from (2.49). So we concentrate on the derivation of this law and of (2.50) from the force between current elements. The main result obtained by Ampère in his extensive experimental researches is his force between current elements given by (4.24). Everything else that he discovered had its origins in this force. We see here how to derive (2.49) and (2.50) from this force, provided that we also assume the equation for the conservation of charges (2.53).

In Maxwell's formulation there are only closed currents (magnets interacting with magnets, magnets interacting with closed currents, closed currents interacting with closed currents, etc.) We will then deal with (4.52), which can be derived from Ampère's force (4.24) or from Grassmann's one, (4.28). Eq. (4.52) can be written as

$$d\vec{F}_{C_2 \text{ on } I_1 d\vec{l}_1} = I_1 d\vec{l}_1 \times \vec{B}_2 \ , \tag{4.75}$$

where we defined the magnetic field $\vec{B}_2$ due to a closed circuit C_2 by

$$\vec{B}_2 \equiv \frac{\mu_o}{4\pi} \oint_{C_2} I_2 d\vec{l}_2 \times \frac{\hat{r}_{12}}{r_{12}^2} \ . \tag{4.76}$$

In general we will have many circuits interacting with $I_1 d\vec{l}_1$, and in many situations we will have currents distributed over a volume as in real circuits, instead of filiform currents.

Substituting $\vec{J} dV_2$ for $I_2 \vec{l}_2$ and integrating over all space the expression (4.76) for the magnetic field generated by a circuit 2 yields the magnetic field where $I_1 d\vec{l}_1$ (or $\vec{J}_1 dV_1$) is located, namely

$$\vec{B}(\vec{r}_1,\ t) = \frac{\mu_0}{4\pi}\int\int\int \vec{J}(\vec{r}_2,\ t) \times \frac{\hat{r}_{12}}{r_{12}^2} dV_2\ . \tag{4.77}$$

In this expression we let $\vec{J}$ depend not only on the position in the circuit but also on time because we want to treat the general situation in which the intensity of the current can vary explicitly in time. Despite this fact we will suppose the circuits fixed in the laboratory such that $\hat{r}_{12}$, $\vec{r}_{12}$ and r_{12} are not functions of time.

From (4.56) we can write $\hat{r}_{12}/r_{12}^2$ as $-\nabla_1(1/r_{12})$. As ∇_1 operates only on the variables with label 1, it can be removed to outside the triple integral. Utilizing (1.4) and (1.21) yields

$$\vec{B}(\vec{r}_1,\ t) = \frac{\mu_0}{4\pi}\ \nabla_1 \times \left(\int\int\int \frac{\vec{J}(\vec{r}_2,\ t)}{r_{12}} dV_2\right)\ . \tag{4.78}$$

Applying the divergence operator in both sides of this equation, $\nabla_1 \cdot$, and using (1.27) yields equation (2.50) for the nonexistence of magnetic monopoles. It should be observed that to arrive at this result it was not necessary to utilize the equation for the conservation of charges (2.53).

We now follow this line of reasoning to derive the second of Maxwell's equations, the magnetic circuital law, from Ampère's force.

Applying the curl operator, $\nabla_1 \times$, to both sides of (4.78) and utilizing (1.29) and (1.18) yields the following result

$$\nabla_1 \times \vec{B}(\vec{r}_1,\ t) = \frac{\mu_0}{4\pi}\nabla_1\left(\int\int\int \vec{J}\cdot\nabla_1\frac{1}{r_{12}}dV_2\right)$$

$$-\frac{\mu_0}{4\pi}\int\int\int \vec{J}(\vec{r}_2,\ t)\nabla_1^2\frac{1}{r_{12}}dV_2\ . \tag{4.79}$$

To solve this second integral we need to utilize (1.38). Utilizing also (4.56) in the first integral of (4.79) yields:

$$\nabla_1 \times \vec{B}(\vec{r}_1,\ t) = \mu_0\vec{J}(\vec{r}_1,\ t) - \frac{\mu_0}{4\pi}\nabla_1\int\int\int \vec{J}(\vec{r}_2,\ t)\cdot\nabla_2\frac{1}{r_{12}}dV_2\ . \tag{4.80}$$

We only need to solve the last integral of (4.80). Using (1.18) once more, together with Gauss's theorem (1.32) yields

$$\int\int\int \vec{J}(\vec{r}_2,\ t)\cdot\nabla_2\frac{1}{r_{12}}dV_2 = \int\int \frac{\vec{J}(\vec{r}_2,\ t)}{r_{12}}\cdot d\vec{a}_2$$

$$-\int\int\int \frac{1}{r_{12}}\nabla_2\cdot\vec{J}(\vec{r}_2,\ t)dV_2\ . \qquad (4.81)$$

Remembering that we are integrating over all space, the surface integral which appears in (4.81) is calculated at infinity. Supposing that the circuit 2 is limited in space and that it does not extend to infinity, yields zero for this surface integral.

Eq. (4.80) then takes the form

$$\nabla_1\times\vec{B}(\vec{r}_1,\ t) = \mu_o\vec{J}(\vec{r}_1,\ t) + \frac{\mu_o}{4\pi}\nabla_1\int\int\int\frac{1}{r_{12}}\nabla_2\cdot\vec{J}(\vec{r}_2,\ t)dV_2\ . \qquad (4.82)$$

At this point Jackson says: "But for steady-state magnetic phenomena $\nabla\cdot\vec{J}=0$, so that we obtain $\nabla\times\vec{B} = 4\pi\vec{J}/c$. This is the second equation of magnetostatics, corresponding to $\nabla\cdot\vec{E} = 4\pi\rho$ in electrostatics" (Jackson, 1975, p. 174; he utilizes the Gaussian system of units). Then in Section 6.3 Jackson says that the system of Maxwell's equations with "Ampère's" law in the form $\nabla\times\vec{B} = 4\pi\vec{J}/c$ is inconsistent (as regards the equation for the conservation of charges). He claims that "the faulty equation is Ampère's law" (Jackson, 1975, p. 217). He says that "it required the genius of J. C. Maxwell, (...), to see the inconsistency of these equations and to modify them into a consistent set." According to Jackson, Maxwell replaced $\vec{J}$ in Ampère's law by $\vec{J}+(\partial\vec{D}/\partial t)/4\pi$ so that it became mathematically consistent with the continuity equation $\nabla\cdot\vec{J}+\partial\rho/\partial t=0$.

The impression we get from Jackson's statements (similar statements are found in most textbooks) is that Ampère arrived at $\nabla\times\vec{B} = \mu_o\vec{J}$ and that Maxwell modified this expression to $\nabla\times\vec{B} = \mu_o\vec{J}+(\partial\vec{E}/\partial t)/c^2$ so that it could become consistent with the equation of the conservation of charges $\nabla\cdot\vec{J}+\partial\rho/\partial t = 0$ through Gauss's law $\nabla\cdot\vec{E} = \rho/\varepsilon_o$.

But this is not the real situation. First of all Ampère never wrote $\nabla\times\vec{B} = \mu_o\vec{J}$. The first to arrive at this equation was Maxwell himself, in 1855, twenty years after Ampère's death (Maxwell, 1965, Vol. 1, p. 155; Whittaker, 1973, Vol. 1, pp. 242 - 245). Then Maxwell corrected himself in 1861 and 1864 writing $\nabla \times \vec{B} = \mu_o \vec{J} + (\partial \vec{E}/\partial t)/c^2$.

Let us then return to where Jackson stopped, in (4.82), and let us utilize the equation of the conservation of charges. As we have seen, Kirchhoff in 1857, previous to the main papers of Maxwell, had already worked with the equation of the conservation of charges (2.53), (Kirchhoff, 1857 a and b). Utilizing that $\vec{J} = \rho \vec{v}$ and that $\partial/\partial t$ can come outside the integral because it does not operate on r_{12}, as the circuits are fixed in space and this is only a partial derivative yields:

$$\nabla_1 \times \vec{B} = \mu_0 \vec{J} - \frac{\mu_0}{4\pi}\frac{\partial}{\partial t}\nabla_1 \int\int\int \frac{\rho(\vec{r}_2,\ t)}{r_{12}} dV_2 \ . \tag{4.83}$$

From (2.18), (2.20) and (2.52) we obtain

$$\nabla_1 \times \vec{B}(\vec{r}_1,\ t) = \mu_0 \vec{J}(\vec{r}_1,\ t) + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}(\vec{r}_1,\ t) \ . \tag{4.84}$$

And this is exactly the equation (2.49) that we wanted to derive.

This derivation assumed stationary circuits. We do not intend to discuss here the case of moving sources and detectors. For a general discussion of the theme of this Section including many references see the important papers (Weber and Macomb, 1989; Jefimenko, 1990; Griffiths and Heald, 1991).

The fact to be emphasized in this derivation is that to arrive at (2.49) we utilized, besides Ampère's force between current elements, also the equation of conservation of charges. This shows that the magnetic circuital law can be derived from the magnetic field of Biot -Savart or from Ampère's force in the form (4.24), (4.52) and (4.76), provided that we also assume the continuity equation for electric charges. It is important to emphasize that the displacement current, the term in $\partial \vec{E}/\partial t$ in (4.84), appears naturally in this derivation. And we saw in Chapter 2 how to arrive at the magnetic circuital law in an integrated form, (2.57), from this differential form.

In Chapters 3 and 4 we saw how from Weber's law we can arrive at Coulomb's force, and then at Gauss's law, which is the first of Maxwell's equations. In this Chapter we first saw how from Weber's law we can arrive at Ampère's force between current elements. Then we saw how to arrive at the second and third of Maxwell's equations from this expression. To complete the proof of the compatibility of Weber's force with Maxwell's equations we need only to derive Faraday's law of induction from Weber's one. This is the subject of the next Chapter.

4.8. Modern Experiments Related to the Controversy Ampère Versus Grassmann

In this Chapter we saw that the force of a closed circuit on a current element of another circuit is the same according to Ampère and to Grassmann, eq. (4.52). This means that if we are considering the interactions between two or more closed circuits, between two or more magnets, or between closed circuits and magnets, we can not distinguish between Ampère and Grassmann. The similarity of these two laws in these situations caused many physicists to think that in all situations both expressions would always agree. Adding to this the fact that Grassmann's force is sometimes easier to integrate than Ampère's force, caused in course of time the replacement of Ampère's force by Grassmann's one. Another reason for the neglect of Ampère's force is that Einstein's special theory of relativity is based in Maxwell's equations plus Lorentz's force. But it happens that Grassmann's force is compatible with Lorentz's one (we only need to substitute $q\vec{v}$ for $Id\vec{l}$ in Lorentz's expression), while Ampère's force is not compatible with Lorentz. Due to the success and popularity of the relativity theory, all models that were not compatible with Lorentz's force were abandoned. Only in the last few years the laws of Ampère and Weber returned to be considered seriously for experimental reasons that we are discussing in this book. As regards Ampère's force, the situation changed with an experimental paper published in Nature in 1982 which demonstrated jet-propulsion in the direction of current flow between liquid and solid conductors (Graneau, 1982 a). This paper showing experiments which could be explained in terms of longitudinal forces (which should not exist according to Grassmann's force) renewed the interest in Ampère's force and stimulated a number of new experiments in this extremely important subject.

The existence of longitudinal forces had been stressed by Ampère himself, who devised the so called Ampère's bridge experiment to show the existence of these longitudinal forces. This experiment has been discussed by Maxwell (Maxwell, 1954, Vol. 2, Arts. [686 - 688], pp. 318 - 320) and a number of authors. A schematic diagram is represented in Figure 4.8.

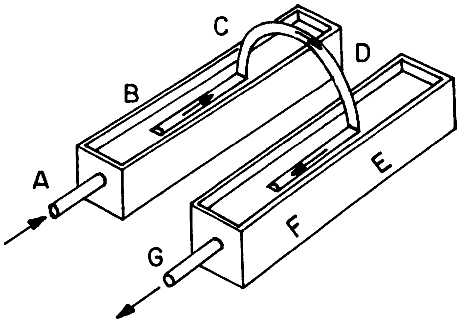

Figure 4.8

Here we have a closed circuit where flows a constant current I. There are two mercury troughs and the metallic bridge $BCDEF$ floats on them. When the current flows along the circuit the bridge moves forward increasing its distance from the battery. This motion of the bridge is due to an electromagnetic force, but Ampère and Grassmann's expressions differ on where this force is located. According to Ampère's expression the main component of the forward force is along the pieces BC and EF due to the repulsion from the pieces AB and FG, respectively (see Figure 4.5 and eq. (4.34)). On the other hand according to Grassmann's expression there can not exist any longitudinal force, that is, there can not exist any component of the force parallel to the current as the force is always in a plane normal to the current element, no matter the direction of the magnetic field (as is easily seen from (4.26) and the properties of a vectorial product). So the forward force on BC and EF according to Grassmann's expression is zero and the forward motion of the bridge is explained by the force acting on the arch CDE. Maxwell himself did not estimate the force on the bridge by any of these expressions, but through an approximate calculus utilizing the coefficient of self inductance of the circuit.

The interest in these experiments decreased with the increasing utilization of Grassmann's force and the neglect of Ampère's expression. In the beginning of this century Hering, who discovered and coined the expression "pinch effect," made many experiments to show the existence of longitudinal forces. He devised and built many electrical furnaces based on these forces. Discussions of his work can be found in (Northrup, 1907; Hering, 1911 and 1921). He wrote a review paper in 1923 which is followed by a very interesting discussion of his ideas by a number of authors (Hering, 1923). After this period the interest in this subject declined again, having another short burst of life in the 1940's, only to be forgotten again for some forty years. In the last ten years there have been many publications on this subject again due to new experimental techniques, numerical calculations and theoretical reasonings. Here we will mention only a few topics. For a detailed discussion of many of these experiments see the important book and review paper by P. Graneau: (Graneau, 1985 a, and 1986). The majority of his own experiments were performed in his laboratory at the Massachusetts Institute of Technology (MIT) in Cambridge, USA.

The result that the force exerted by a circuit on a current element is the same according to Ampère and Grassmann may be valid (this is not yet a settled question) only if the circuit is closed and if the current element which experiences the force does not belong to this same circuit. If we are calculating the force of the remainder of a circuit on a part of itself this result may be not valid anymore. The important fact is that this can be realized in the laboratory, namely, we can detect and measure the force on a part of a circuit due to the remaining portion of the circuit. A typical experimental technique utilized is to join the two parts of a single metallic and solid circuit by liquid mercury or by electric arcs, as represented in Figure 4.9.

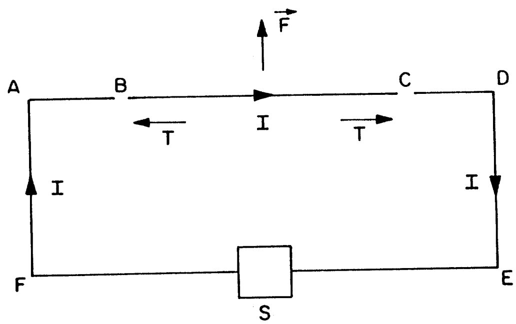

Figure 4.9

Here we have a closed circuit $ABCDEFA$ where flows a current I due to a battery or capacitor bank S. The piece of the circuit BC is disconnected mechanically from the remainder $CDEFAB$. At B and C we may have electric arcs (sparks) or liquid mercury in a trough, which close the current but allow the piece BC to move separately from the remainder of the circuit. We can calculate the net force $\vec{F}$ on this piece BC with both laws, Ampère and Grassmann, and we can also measure this force in the laboratory. We could, for instance, balance this electromagnetic force by elastic forces due to dielectric springs, which would keep the piece BC at rest, and so measuring the force. An idea of this kind has been utilized by Moyssides and Pappas to measure the force (Moyssides and Pappas, 1986). The circuit which they utilized is represented in Figure 4.10.

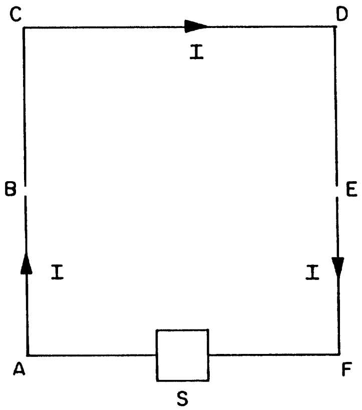

Figure 4.10

Here we have a bridge $BCDE$ which is mechanically disconnected from the remainder $EFAB$. At B and E there is a liquid mercury trough which connects the current in the two parts of the closed circuit. In this case we can measure the net force on the bridge and also calculate this force with the expressions of Ampère and Grassmann.

The main theoretical problems involved are the divergences which appear with both laws when we try to integrate. As the distance between the current elements goes to zero the forces tend to diverge. There are two main ways of overcoming this limitation. The first one is to utilize numerical integration in which we divide the cross sections of each wire in many sections and each filament in many small pieces, so that the divergences do not appear any more. This technique of finite current element analysis has been utilized by P. Graneau, N. Graneau, Moyssides, etc. (P. Graneau, 1992 b, 1983 a and b, 1985 a, 1986; N. Graneau, 1990; Moyssides, 1989 a, b and c).

The second way of overcoming this limitation is to utilize the force expressions in terms of the current densities. In general we work with filamentary currents, but in reality we have an electric current flowing upon a surface or through a certain volume or cross section of a wire, and not along a line. If σ is the surface charge density and ρ the volume charge density all previous results can be maintained substituting $\vec{K}da$ or $\vec{J}dV$ for $Id\vec{l}$, where $\vec{K}$ is the surface density of current and $\vec{J}$ is the volume current density. When the current elements are electrically neutral we have ($i = 1,\ 2$ and $j = 2,\ 1$):

$$\vec{K}_i da_i = \sigma_{i+} da_i (\vec{v}_{i+} - \vec{v}_{i-}) \ , \tag{4.85}$$

$$\vec{J}_i dV_i = \rho_{i+} dV_i (\vec{v}_{i+} - \vec{v}_{i-}) \ . \tag{4.86}$$

The forces of Ampère and Grassmann in these cases take the form

$$d^4 \vec{F}_{ji}^{A\ sur} = -\frac{\mu_o}{4\pi} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2(\vec{K}_i \cdot \vec{K}_j) - 3(\hat{r}_{ij} \cdot \vec{K}_i)(\hat{r}_{ij} \cdot \vec{K}_j) \right] da_i da_j \ , \tag{4.87}$$

$$d^4 \vec{F}_{ji}^{G\ sur} = -\frac{\mu_o}{4\pi} \frac{1}{r_{ij}^2} \left[(\vec{K}_i \cdot \vec{K}_j)\hat{r}_{ij} - (\hat{r}_{ij} \cdot \vec{K}_i)\vec{K}_j \right] da_i da_j \ , \tag{4.88}$$

$$d^6 \vec{F}_{ji}^{A\ vol} = -\frac{\mu_o}{4\pi} \frac{\hat{r}_{ij}}{r_{ij}^2} \left[2(\vec{J}_i \cdot \vec{J}_j) - 3(\hat{r}_{ij} \cdot \vec{J}_i)(\hat{r}_{ij} \cdot \vec{J}_j) \right] dV_i dV_j \ , \tag{4.89}$$

$$d^6 \vec{F}_{ji}^{G\ vol} = -\frac{\mu_o}{4\pi} \frac{1}{r_{ij}^2} \left[(\vec{J}_i \cdot \vec{J}_j)\hat{r}_{ij} - (\hat{r}_{ij} \cdot \vec{J}_i)\vec{J}_j \right] dV_i dV_j \ . \tag{4.90}$$

The main person utilizing this technique to calculate the forces of Ampère and Grassmann has been Wesley: (Wesley, 1987 a and b, 1990 a and b, 1991, Chapter 6). In this way he obtains finite results because the numerator of these expressions goes faster to zero than the denominator when $r_{ij} \to 0$.

Beyond calculating and measuring the net force on Ampère's bridge or in pieces BC or $BCDE$ of Figures 4.9 and 4.10, there is also another very important subject which is being discussed nowadays. It is related with Ampère's tension and can be visualized in Figure 4.9. Beyond the net force $\vec{F}$ which acts on the piece BC there may also exist a tension T acting on this piece. Obviously Grassmann's force can never predict any such tension in a straight wire like BC because there are no longitudinal forces in this expression. On the other hand there is the possibility of this force according to Ampère's expression, as can be seen in Figure 4.5 and eq. (4.34). If this tension exists it can cause the rupture, breaking and explosion of the wire. As a matter of fact this fact has been known to happen since the 1840's (Riess, 1966). Once more these experiments seem to have been forgotten and were rediscovered in this century by Nasilowski in the 1960's, see (Nasilowski, 1989) for references. When we discharge a capacitor bank in the circuit of Figure 4.9, with a small air gap at B and C, the wire breaks in many pieces. The order of breaking is more or less the following: First the wire BC breaks approximately in the middle, then each half breaks approximately in the middle again, and so on. This sequence does not go on indefinitely until the pulverization of the wire, but stops when the length of the fragments is of the same order of magnitude as the length of their cross sections. Some thought this breaking was due to fusion or melting of the wire due to the Joule effect. However this is not the case and this fact has been known since 1845, as is evident from these words written by Riess describing his experiments performed at this date: "The aspect of the ends of the broken wires shows that what takes place is simple breakage and not fusion, in confirmation of which further proof will be given below." (...) "If wires are exposed to stronger discharges than such as are necessary to break them, a flash of light appears, and they are shivered into a greater or lesser number of pieces, which are scattered about to some distance. It is perceptible on the examination of the collected fragments that the division of the wire into small pieces is caused by a slitting and shivering action, and that fusion, where it has taken place, is a secondary phaenomenon" (Riess, 1966, see especially pp. 453 - 458). Much more complete confirmation of this remarkable fact has been given recently by the detailed experiments of P. Graneau on this subject: (Graneau, 1983 a, 1984, 1985 a, 1986, 1987 a). He presented metallurgical evidence indicating that the wire parted in the solid state under the action of tension, and not due to melting. This was also confirmed by optical micrograph and scanning electron micrograph of fracture face. There is clearly no more doubt that the breaking of the wire was due to longitudinal tension, the question is to know if this electromagnetic force is really Ampère's force or some other unknown mechanism.

Another kind of very interesting experiment is related to the railgun. A schematic diagram is represented in Figure 4.11.

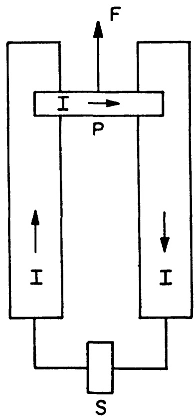

Figure 4.11

In this situation we have a battery or capacitor bank connected to two thick metallic rails which are fixed in the laboratory. We close the circuit with a metallic projectile which is free to move along the rails. When the current is flowing in the circuit there appears a net force on the projectile forcing it to move in the forward direction (supposing the projectile to be made of a non magnetic material). The net force on the projectile when calculated with the expressions of Ampère and Grassmann is the same in this case. The interesting question is to know where is the reaction force. Where is it acting? A possible answer is that the force on the projectile is due to the magnetic field, so that the reaction should be in the field in the form of a backward electromagnetic radiation. This explanation is untenable due to questions of conservation of linear momentum and energy, as has been shown by Pappas: (Pappas, 1983). See also (Graneau, 1987 b) and

(Assis, 1992 b). Another possible answer is utilizing Grassmann's force. As it predicts no longitudinal force there can be no longitudinal reaction forces in the rails, as Grassmann's force is always perpendicular to the direction of the current. So the only alternative with Grassmann's force is to suppose that all the back reaction force happens in the source or in the straight wire passing through the source and connecting the two rails. According to Ampère's force, on the other hand, the reaction will be mainly in the rails. This distinction is represented in Figure 4.12.

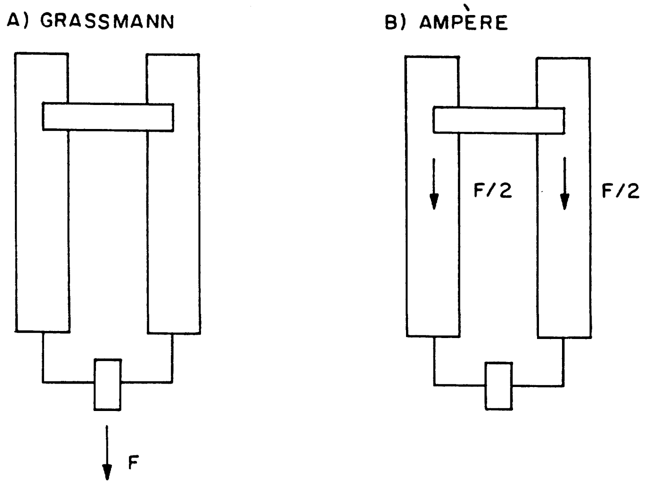

Figure 4.12

In order to discover where is the reaction force, P. Graneau performed the experiment represented in Figure 4.13 (Graneau, 1987 b and c).

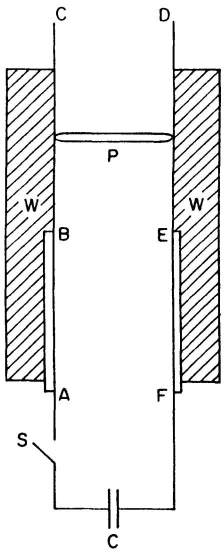

Figure 4.13

In this experiment AB and EF are thick rails which do not bend under compression, while BC and DE are thin rails which can bend if compressed. He placed wooden side boards W at the external sides of the rails to support the lateral forces which should exist according to the expressions of Ampère or Grassmann. When closing the switch S the capacitor C was discharged through this closed circuit and the projectile was impelled forward. Prior to the current the thin rails were straight. The buckling of the thin rails was very clear after the recoil experiment, proving that the reaction force acted on them. For a further discussion of Ampère's force and its relation to railgun accelerators see also (P. Graneau, 1982 b, 1985 a, 1986), (Robson and Sethian, 1992), (Phipps, 1993).

There are other experiments on this subject but we will not discuss them here. For interested readers we mention the following literature for experiments dealing with: electromagnetic explosion in liquids (Graneau and Graneau, 1985; Azevedo et al., 1986; Aspden, 1986); electromagnetic impulse pendulum (Pappas, 1983; Graneau and Graneau, 1986; Pappas and Moyssides, 1985); Ampère forces in gaseous conductors (Nasilowski, 1985); Ampère forces in liquid mercury (Phipps, 1990 a; Phipps and Phipps, 1990).

Although these experiments are very remarkable, there is still a great controversy surrounding this whole subject. One of the reasons is that many people think that even in these cases of a single circuit in which we perform experiments with part of this circuit, Ampère's force will always give the same answer as Grassmann's one. If this is the case the longitudinal forces demonstrated by these experiments would need to have a different origin, as yet unknown. Beyond the references already mentioned we would like to point out some more indicating similar and discordant points of view, showing the great actuality of this stimulating subject: (Jolly, 1985), (Ternan, 1985 a and b), (Graneau, 1985 b), (Christodoulides, 1987 and 1989), (Peoglos, 1988), (Wesley, 1989), (Strnad, 1989), (Cornille, 1989), (Whitney, 1988), (Graneau, Thompson and Morrill, 1990), (Pappas, 1990), (Rambaut, 1991), (Saumont, 1991 and 1992), etc.

We ourselves consider that the controversy Ampère versus Grassmann is still an open question and that more experiments and theoretical analysis is necessary before a final conclusion can be drawn.

We close this Chapter with Maxwell's judgement of this whole subject. He knew not only Ampère's force (4.24) but also Grassmann's one (4.26) to (4.28), which is from 1845 (see Maxwell, 1954, Vol. 2, Art. [526], p. 174). He also knew the result (4.52) that the force of a closed circuit on a current element of another circuit is the same according to both laws. There are many other forces between current elements which differ from Ampère's one by exact differentials so that they will yield the same result as (4.52) for closed circuits. Maxwell himself presented two other force laws of his own between current elements which gave the same result as Ampère's one for the force of a closed circuit on a current element of another circuit. After discussing the forces of Ampère, Grassmann and his two expressions for the force between current elements, Maxwell made the following analysis: "Of these four different assumptions that of Ampère is undoubtedly the best, since it is the only one which makes the forces on the two elements not only equal and opposite but in the straight line which joins them" (Maxwell, 1954, Vol. 2, Art. [527], p. 174).

Chapter 5 / Faraday's Law of Induction

5.1. Faraday's Law

Usually an electric current is produced through a voltage or a difference of electrostatic potential, as when we connect the terminals of a battery by a wire or metallic conductor. A completely independent way of generating an electric current, not related to the previous one, was discovered by Michael Faraday (1791 - 1867) in 1831. This is the subject of this Chapter.

He never had a formal education in science and was always a self-taught person. He went to some public seminars given by the chemist and physicist Humphry Davy (1778-1829) and when he was 21 years old he began to be Davy's assistent in the Chemical Laboratory of the Royal Institution in London. There he worked all his life. After Davy's death he became the director of the laboratory. He was essentially an experimental scientist, and his mathematical knowledge was very meagre. He was greatly influenced by Davy and during some ten years he occupied himself mainly with chemistry (electrolysis, etc). After Oersted's fundamental discovery in 1820 he turned to electromagnetism.

The goal of his first experimental researches in this field was to find, in electrodynamics, phenomena analogous to those in electrostatics. He knew that when an electric charge is approached to a neutral conductor (for instance a metal), the charge induces an opposite charge in the conductor (Figure 5.1).

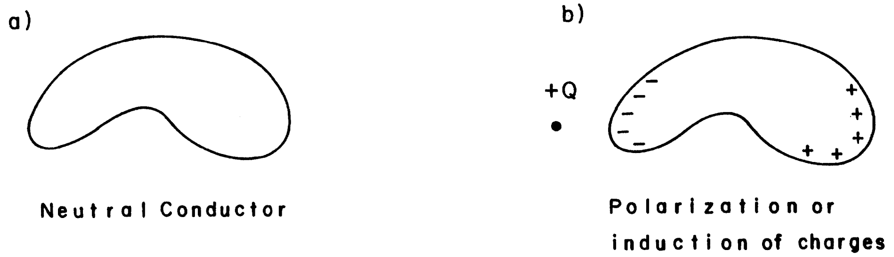

Figure 5.1

His first idea was that if he held an electric current near another closed circuit or metallic plate, the first current would induce another opposite current in the neighbour bodies, and this induced current would remain while the first current existed. Experimentally he observed that this idea did not work, but in 1831 he made his great discovery that an electric current is induced in the secondary circuit provided that the current in the primary circuit is varied in intensity. Next he observed that even with a constant current in the primary circuit he could induce a current in the secondary one provided that there were a relative motion between them. Also if the area of one of the closed circuits were varied a current would be induced in the other while there was such a variation. These three cases are represented in simple situations in the next Figure.

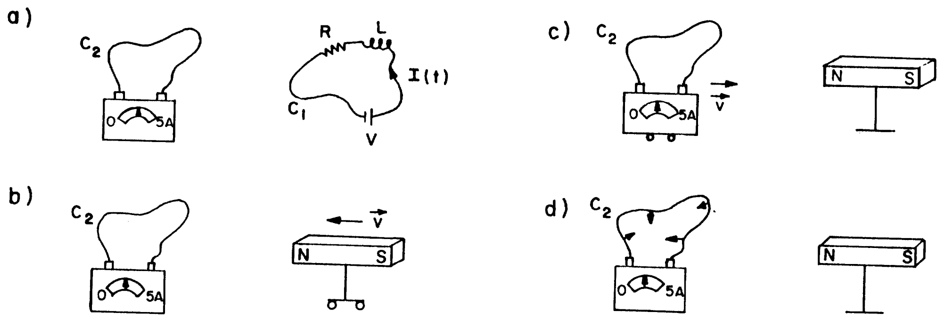

Figure 5.2: Induction of currents

a) Changing the intensity of the current in the primary circuit C_1,

b) A magnet approaching the circuit,

c) A circuit approaching the magnet,

d) Changing the area of a circuit in a region containing a magnetic field.

In case (a) the circuits lie in parallel planes. In cases (b) to (d) the north-south axis of the magnet is normal to the plane of C_2.

Faraday expressed his findings by saying that the induced current I_2 is due to an electromotive force, emf_{12}, and this emf_{12} appears when there is a variation of the magnetic flux over the area of the secondary circuit where the current is being induced. It should be emphasized that the emf, although being called a force, is a non-electrostatic voltage. Its unit is the volt ($1V = 1kgm^2C^{-1}s^{-2}$). Analogously to Ohm's law ($I = V/R$) we can write Faraday's law as (when there are no batteries connected to the circuit):

$$I_2 = \frac{emf_{12}}{R_2} \ , \tag{5.1}$$

$$emf_{12} \equiv -\frac{d}{dt}\Phi_B \ , \tag{5.2}$$

$$\Phi_B \equiv \int\int_{S_2} \vec{B} \cdot d\vec{a}_2 \ . \tag{5.3}$$

In (5.2) the minus sign was introduced in order to make this law compatible with Lenz's discovery of 1834, which is related to the direction of the induced current (and which Faraday had not determined). Lenz's law states that when there is a variation of the magnetic flux upon a circuit the induced current flows in such a direction that the resultant force acting on it is such as to oppose the variation of the flux. For instance, consider a circular loop of radius r centered on the origin, located in the xy plane, without current. If a magnet aligned with the z axis in the region $z > 0$, with north pole downwards and south pole upwards, approaches the loop the induced current will flow in the opposite sense of the clock. That is, it happens as if the loop had transformed itself into a small magnet with north pole upwards such as to repel the magnet which is approaching. It can be said that Lenz's law expresses the fact that, for induction, nature behaves in such a way as to avoid instabilities. In the previous example if the induced current flowed in the opposite direction it would produce an attraction between the loop and the magnet. This would mean an instable situation, because any perturbation in the position of the magnet would grow larger and larger indefinitely.

Maxwell stated Lenz's law as follows: "If a constant current flows in the primary circuit A, and if, by the motion of A, or of the secondary circuit B, a current is induced in B, the direction of this induced current will be such that, by its electromagnetic action on A, it tends to oppose the relative motion of the circuits" (Maxwell, 1954, Vol. 2, article [542], p. 190).

Faraday's works can be found in (Faraday, 1952). A good discussion of his work can be found in (Tricker, 1966) and (Whittaker, 1973, Vol. 1, Chapter 6: Faraday).

5.2. Franz Neumann

Besides Faraday and Lenz there is yet another important person related to the law of induction, F. Neumann. His goal was to deduce Faraday's law (5.1) to (5.3) from Ampère's force (4.24). During his researches he introduced for the first time the magnetic vector potential $\vec{A}$ defined by:

$$\vec{A}(\vec{r}_2) \equiv \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} I_1 \frac{d\vec{l}_1}{r_{12}} . \tag{5.4}$$

This is the vector potential at $\vec{r}_2$ due to the circuit $\mathcal{C}_1$. Applying the curl operator, $\nabla_2 \times$, to both sides of (5.4) yields (with (1.21), (4.56) and remembering that the operator ∇_2 can go inside $\oint_{\mathcal{C}_1}$ and does not operate on $I_1 d\vec{l}_1$):

$$\nabla_2 \times \vec{A} = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} \frac{\hat{r}_{12}}{r_{12}^2} \times I_1 d\vec{l}_1 . \tag{5.5}$$

But this is exactly the magnetic field due to the first circuit ((4.76) changing the indexes 1 and 2, and using that $\hat{r}_{21} = -\hat{r}_{12}$), namely:

$$\vec{B}(\vec{r}_2) = \nabla_2 \times \vec{A} , \tag{5.6}$$

with $\vec{A}$ given by (5.4). If we apply this result in (5.2) and utilize Stokes' theorem (1.33), we can write Faraday's law (5.1) with an emf_{12} given by:

$$emf_{12} = -\frac{d}{dt}\left(\oint_{\mathcal{C}_2} \vec{A} \cdot d\vec{l}_2\right) = \oint_{\mathcal{C}_2} \left(-\frac{\partial \vec{A}}{\partial t}\right) \cdot d\vec{l}_2 . \tag{5.7}$$

That is, Neumann suceeded in expressing the induction law only in terms of his vector potential $\vec{A}$, without invoking the magnetic field $\vec{B}$.

As we have seen, Neumann also introduced what is called the coefficient of mutual inductance M given by

$$M \equiv \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_1} \oint_{\mathcal{C}_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}} . \tag{5.8}$$

This coefficient M is independent of the intensity of the currents I_1 and I_2, and so it is only a geometrical factor relating the two circuits. In terms of M the induction law can be written as (with (5.1), (5.7), (5.4) and (5.8)):

$$I_2 = \frac{emf_{12}}{R_2} \ , \tag{5.9}$$

$$emf_{12} = -\frac{d}{dt}(I_1 M) \ . \tag{5.10}$$

There is still another way of looking at the induction law. An electric dipole is made of two charges of the same magnitude q but of opposite signs, separated by the small distance l. The dipole moment of the two charges is defined by

$$\vec{p} \equiv q\vec{l} \ , \tag{5.11}$$

where q is the positive charge and $\vec{l}$ is the vector pointing from the negative to the positive charge, with a magnitude given by l. The potential energy of this dipole in a region with an electric field $\vec{E}$ (that is, the energy spent to bring slowly this dipole from infinity to the final position, without changing the value of l and supposing that $\vec{E}$ does not depend on time) is given by

$$W = -\vec{p} \cdot \vec{E} \ . \tag{5.12}$$

Analogously to all of this it is possible to define the magnetic moment of a small loop of area a and current I as

$$\vec{m} \equiv Ia\hat{u} \ , \tag{5.13}$$

where $\hat{u}$ is the unit vector normal to the area a and pointing according to the right hand rule. The potential energy of this dipole in a region with magnetic field $\vec{B}$ (that is, the energy spent to bring slowly this magnetic dipole from infinity to this region supposing that $\vec{B}$ does not depend on time and that I and a remain constant during the process) is given by

$$W = -\vec{m} \cdot \vec{B} \ . \tag{5.14}$$

We can generalize this result for the situation of a macroscopic circuit C_1 in the presence of a magnetic field $\vec{B}$ due to a current I_2 flowing in a circuit C_2. In this case the potential energy is given by

$$W = -I_1 \int\int_{S_1} \vec{B} \cdot d\vec{a}_1 = -I_1 I_2 M \ . \tag{5.15}$$

In this way the induced electromotive force is given by

$$emf_{12} = \frac{d}{dt}\left(\frac{W}{I_2}\right) \ . \tag{5.16}$$

In (5.15) W can be seen as the work which needs to be done against the force between the circuits C_1 and C_2 to separate them to an infinite distance, supposing that the magnitudes of the currents remain constant.

Before going on it should be emphasized that according to Ohm's law the emf is a voltage and then can be written as $\oint \vec{E} \cdot d\vec{l}$. Using this result in (5.2) and (5.3) yields

$$emf_{12} = \oint_{C_2} \vec{E} \cdot d\vec{l}_2 = \int\int_{S_2} \left(-\frac{\partial \vec{B}}{\partial t}\right) \cdot d\vec{a}_2 \ . \tag{5.17}$$

From Stokes' theorem (1.33) we have

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \ , \tag{5.18}$$

which is Faraday's law in a differential form, (2.51). Moreover, comparing (5.17) with (5.7) shows that the component of the electric field responsible for the law of induction is not the Poisson component $(-\nabla\phi)$, but $-\partial\vec{A}/\partial t$.

We close this Section with Maxwell's words regarding the work of F. Neumann (our emphasis): "On this law [Lenz's law] F. E. Neumann founded his mathematical theory of induction, in which he established the mathematical laws of the induced currents due to the motion of the primary or secondary conductor. He shewed that the quantity M, which we have called the potential of one circuit on the other, is the same as the electromagnetic potential of the one circuit on the other, which we have already investigated in connection with Ampère's formula. **We may regard F. E. Neumann, therefore, as having completed for the induction of currents the mathematical treatment which Ampère had applied to their mechanical action**" (Maxwell, 1954, Vol. 2, article [542], p. 190).

5.3. Derivation of Faraday's Law from Weber's Force

We now show how to derive the law of induction beginning with Weber's force. There are several precedures to do that, each one of them with its own peculiarities. As examples we cite: (Whittaker, 1973, Vol. 1, Chapt. 7), (O'Rahilly, 1965, Vol. 2, Chapt. 11), (Wesley, 1987 a and 1990 a), (Maxwell, 1954, Vol. 2, Chapt. 23). Here we follow Maxwell's procedure, but utilizing modern vectorial language.

We begin with Maxwell's words:

"After deducing from Ampère's formula for the action [force] between the elements of currents, his own formula for the action [force] between moving electric particles, Weber proceeded to apply his formula to the explanation of the production of electric currents by magneto-electric induction. In this he was eminently successful, and we shall indicate the method by which the laws of induced currents may be deduced from Weber's formula" (Maxwell, 1954, article [856], p. 486).

We want then to calculate the induced electromotive force in circuit 2 due to the current in circuit 1 in the two situations studied by Faraday: When the circuit 1 translates as a whole relative to the circuit 2 (approaching one another or going away); and when the current in the first circuit changes in time, $I_1(t)$. We present once more Weber's force exerted by dq_1 on dq_2, (3.24) and $\hat{r}_{21} = -\hat{r}_{12}$:

$$d^2\vec{F}_{12} = -\frac{dq_1 dq_2}{4\pi\varepsilon_0}\frac{\hat{r}_{12}}{r_{12}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{12}\cdot\vec{v}_{12} - \frac{3}{2}(\hat{r}_{12}\cdot\vec{v}_{12})^2 + \vec{r}_{12}\cdot\vec{a}_{12}\right)\right] . \qquad (5.19)$$

The electromotive force, emf, is a voltage which gives rise to a current. We can think of a voltage as being due to an electric field. When we have free charges in space we know that the positive ones move from the higher to the lower potential, that is, in the same direction as the electric field, while the negative ones move in the opposite direction. If this happens then both contribute with the increase of the current.

In many situations the force on a charge can be expressed as $\vec{F} = q\vec{E}$. If we have positive and negative charges, dq_{2+} and dq_{2-}, located at $\vec{r}_2$, in the presence of the same electric field $d\vec{E}_1(\vec{r}_2)$, we get

$$d\vec{E}_1(\vec{r}_2) = \frac{d^2\vec{F}_{2+}}{dq_{2+}} = \frac{d^2\vec{F}_{2-}}{dq_{2-}} = \frac{1}{2}\left(\frac{d^2\vec{F}_{2+}}{dq_{2+}} + \frac{d^2\vec{F}_{2-}}{dq_{2-}}\right) . \tag{5.20}$$

In this expression $d^2\vec{F}_{2+}$ ($d^2\vec{F}_{2-}$) is the resultant force acting on dq_{2+} (dq_{2-}). If these forces are due to positive and negative charges dq_{1+} and dq_{1-} we have $d^2\vec{F}_{2+} = d^2\vec{F}_{1+,2+} + d^2\vec{F}_{1-,2+}$ and $d^2\vec{F}_{2-} = d^2\vec{F}_{1+,2-} + d^2\vec{F}_{1-,2-}$. With these expressions in (5.20) and utilizing also the charge neutrality (4.18) yields

$$d\vec{E}_1(\vec{r}_2) = \frac{(d^2\vec{F}_{1+,2+} + d^2\vec{F}_{1-,2+}) - (d^2\vec{F}_{1+,2-} + d^2\vec{F}_{1-,2-})}{2dq_{2+}} . \tag{5.21}$$

For induced currents the only component of the current that matters is the one parallel to the wire at each point, that is, to $d\vec{l}_2$. Putting all of this together yields the emf_{12} in $d\vec{l}_2$ due to $d\vec{l}_1$ as (Maxwell, 1954, Vol. 2, Chapt. 23):

$$d^2\ emf_{12} \equiv \frac{(d^2\vec{F}_{1+,\ 2+} + d^2\vec{F}_{1-,\ 2+}) - (d^2\vec{F}_{1+,\ 2-} + d^2\vec{F}_{1-,\ 2-})}{2dq_{2+}} \cdot d\vec{l}_2 . \tag{5.22}$$

The velocities of the positive and negative charges in each current element are given by

$$\vec{v}_{1+} \equiv \frac{d\vec{r}_{1+}}{dt} = \vec{v}_{1+d} + \vec{V}_1 , \tag{5.23}$$

$$\vec{v}_{1-} \equiv \frac{d\vec{r}_{1-}}{dt} = \vec{v}_{1-d} + \vec{V}_1 , \tag{5.24}$$

$$\vec{v}_{2+} \equiv \frac{d\vec{r}_{2+}}{dt} = \vec{v}_{1+d} + \vec{V}_2 , \tag{5.25}$$

$$\vec{v}_{2-} \equiv \frac{d\vec{r}_{2-}}{dt} = \vec{v}_{2-d} + \vec{V}_2 . \tag{5.26}$$

In these expressions the symbol d means the velocity of the charges relative to the wire, namely, the drifting velocity (the velocity responsible for the electric currents). The velocities $\vec{V}_1$ and $\vec{V}_2$ are the translational velocities of the circuits C_1 and C_2 (considered rigid here for simplicity) relative to a frame of reference S (Figure 5.3). We also define their relative velocity $\vec{V}_{12}$ by $\vec{V}_{12} \equiv \vec{V}_1 - \vec{V}_2$.

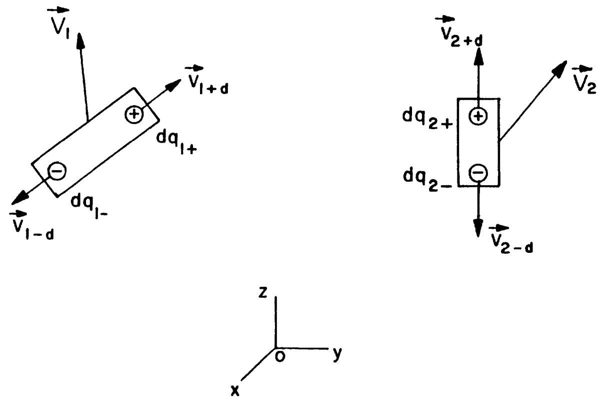

Figure 5.3

In this Section we will assume Fechner's hypothesis, namely, $\vec{v}_{1-d} = -\vec{v}_{1+d}$ and $\vec{v}_{2-d} = -\vec{v}_{2+d}$. This was a common hypothesis in the second half of the last century and was utilized by Weber, and by Maxwell in this chapter of his book (Chapt. 23, Vol. 2). It is worth while to remember that the electron was only discovered in 1897. Despite this fact we will not impose any relation between $\vec{v}_{1+d}$ and $\vec{v}_{2+d}$.

With Fechner's hypothesis, Weber's force (5.19) and relations (5.23) to (5.26) in (5.22) we obtain

$$d^2\ emf_{12} = -\frac{dq_{1+}}{4\pi\varepsilon_0}\frac{\hat{r}_{12}\cdot d\vec{l}_2}{r_{12}^2c^2}\Big\{2\vec{V}_{12}\cdot(\vec{v}_{1+d}-\vec{v}_{1-d})$$

$$-\ 3(\hat{r}_{12}\cdot\vec{V}_{12})[\hat{r}_{12}\cdot(\vec{v}_{1+d}-\vec{v}_{1-d})] + \vec{r}_{12}\cdot(\vec{a}_{1+}-\vec{a}_{1-})\Big\}\ . \qquad (5.27)$$

Following Maxwell once more, we will consider all magnitudes of the system, as r_{12} for instance, as functions of only three independent variables: l_1, l_2 and t. That is, l_1 is a length measured over the circuit C_1 from a certain origin O_1 arbitrarily chosen in this circuit, with positive sense along the direction of the current I_1. The same is to be understood from l_2 relative to O_2 in C_2 and I_2; and t is the time (Figure 5.4).

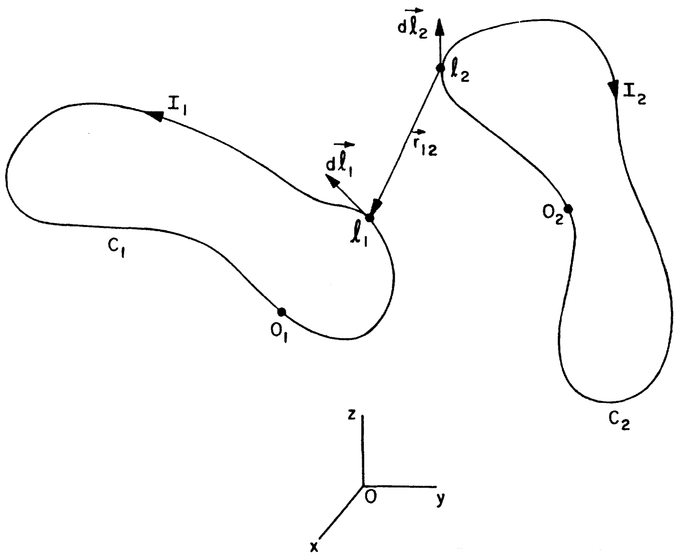

Figure 5.4

This procedure is obviously correct and yields, for instance, $r_{12} = r_{12}(l_1,\ l_2,\ t)$ and $dr_{12} = dl_1(\partial r_{12}/\partial l_1) + dl_2(\partial r_{12}/\partial l_2) + dt(\partial r_{12}/\partial t)$.

From (4.23) we have

$$I_1 d\vec{l_1} = dq_{1+}(\vec{v}_{1+} - \vec{v}_{1-}) = dq_{1+}(\vec{v}_{1+d} - \vec{v}_{1-d}) \ . \tag{5.28}$$

In general when the magnitude of a current is varied in a metallic wire what changes is the drift velocity of the charges, but not the number or density of the free charges. Applying this idea in (5.28) yields

$$dq_{1+}(\vec{a}_{1+} - \vec{a}_{1-}) = \frac{d}{dt}\left[dq_{1+}(\vec{v}_{1+} - \vec{v}_{1-})\right] = \frac{d}{dt}(I_1 d\vec{l_1}) = \frac{dI_1}{dt} d\vec{l_1} = \frac{\partial I_1}{\partial t} d\vec{l_1} \ . \tag{5.29}$$

In order to arrive at this result we are assuming that the circuits have only a translational velocity but no rotational one, which means that we can put $\partial(d\vec{l_1})/\partial t = 0$. We are also assuming that the current is uniform over the length of the circuit so that $\partial I_1/\partial l_1 = 0$.

With (5.28) and (5.29) in (5.27), using that $c^2 = (\mu_0\varepsilon_0)^{-1}$, and integrating over both closed circuits yields:

$$emf_{12} = -\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(\hat{r}_{12}\cdot d\vec{l}_2)}{r_{12}^2}\left[2I_1(\vec{V}\cdot d\vec{l}_1) - 3I_1(\hat{r}_{12}\cdot\vec{V})(\hat{r}_{12}\cdot d\vec{l}_1)\right.$$
$$\left. + r_{12}\frac{\partial I_1}{\partial t}(\hat{r}_{12}\cdot d\vec{l}_1)\right] . \qquad (5.30)$$

Utilizing the chain rule for the derivatives (Leibniz's rule) in the last term yields

$$emf_{12} = -\frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\left\{2I_1\frac{(\vec{V}\cdot d\vec{l}_1)(\hat{r}_{12}\cdot d\vec{l}_2)}{r_{12}^2} - 3I_1\frac{(\hat{r}_{12}\cdot\vec{V})(\hat{r}_{12}\cdot d\vec{l}_1)(\hat{r}_{12}\cdot d\vec{l}_2)}{r_{12}^2}\right.$$
$$+ \frac{\partial}{\partial t}\left[I_1\frac{(\hat{r}_{12}\cdot d\vec{l}_1)(\hat{r}_{12}\cdot d\vec{l}_2)}{r_{12}}\right] - I_1\frac{(\hat{r}\cdot d\vec{l}_2)}{r_{12}}\frac{\partial}{\partial t}(\hat{r}_{12}\cdot d\vec{l}_1)$$
$$\left. - I_1\frac{(\hat{r}_{12}\cdot d\vec{l}_1)}{r_{12}}\frac{\partial}{\partial t}(\hat{r}_{12}\cdot d\vec{l}_2) + I_1\frac{(\hat{r}_{12}\cdot d\vec{l}_1)(\hat{r}_{12}\cdot d\vec{l}_2)}{r_{12}^2}\frac{\partial r_{12}}{\partial t}\right\} . \qquad (5.31)$$

Let us now show some relations which follow in Maxwell's derivation from the idea of considering the relevant magnitudes of the problem as a function of l_1, l_2 and t. If $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the current elements $I_1 d\vec{l}_1$ and $I_2 d\vec{l}_2$ relative to the origin O of the frame of reference S (Figure 5.4) we then have

$$\vec{r}_1 - \vec{r}_2 \equiv \vec{r}_{12} = \vec{r}_{12}(l_1,\ l_2,\ t) . \qquad (5.32)$$

It then follows that

$$\frac{d\vec{r}_{12}}{dt} = \frac{\partial\vec{r}_{12}}{\partial l_1}\frac{dl_1}{dt} + \frac{\partial\vec{r}_{12}}{\partial l_2}\frac{dl_2}{dt} + \frac{\partial\vec{r}_{12}}{\partial t} . \qquad (5.33)$$

As we are utilizing Fechner's hypothesis we can define $|\vec{v}_{1+d}| = |\vec{v}_{1-d}| \equiv v_{1d}$, $|\vec{v}_{2+d}| = |\vec{v}_{2-d}| \equiv v_{2d}$. We then have

$$\frac{dl_1}{dt} = v_{1d} , \quad \frac{dl_2}{dt} = v_{2d} . \qquad (5.34)$$

We also have

$$\vec{v}_{1\pm d} \equiv \pm v_{1d}\hat{l}_1, \ \vec{v}_{2\pm d} = \pm v_{2d}\hat{l}_2 \ , \tag{5.35}$$

where $\hat{l}_1$ and $\hat{l}_2$ are the unit vectors parallel to $d\vec{l}_1$ and $d\vec{l}_2$, namely: $\hat{l}_1 \equiv d\vec{l}_1/|d\vec{l}_1|$ and $\hat{l}_2 \equiv d\vec{l}_2/|d\vec{l}_2|$.

But from (5.23) to (5.26) we can also write

$$\frac{d\vec{r}_{12}}{dt} = \vec{v}_{12} = \vec{v}_{1d} - \vec{v}_{2d} + \vec{V}_{12} \ . \tag{5.36}$$

Comparing (5.33) to (5.36) yields

$$\frac{\partial \vec{r}_{12}}{\partial l_1} = \hat{l}_1 \ , \tag{5.37}$$

$$\frac{\partial \vec{r}_{12}}{\partial l_2} = -\hat{l}_2 \ , \tag{5.38}$$

$$\frac{\partial \vec{r}_{12}}{\partial t} = \vec{V}_{12} \ . \tag{5.39}$$

Analogously we have $r_{12} = r_{12}(l_1, \ l_2, \ t)$ so that

$$\frac{dr_{12}}{dt} = \frac{\partial r_{12}}{\partial l_1}\frac{dl_1}{dt} + \frac{\partial r_{12}}{\partial l_2}\frac{dl_2}{dt} + \frac{\partial r_{12}}{\partial t} \ . \tag{5.40}$$

But from (3.13) and (5.23) to (5.26) we also have

$$\frac{dr_{12}}{dt} = \dot{r}_{12} = \hat{r}_{12} \cdot \vec{v}_{12} = \hat{r}_{12} \cdot \vec{v}_{1d} - \hat{r}_{12} \cdot \vec{v}_{2d} + \hat{r}_{12} \cdot \vec{V}_{12} \ . \tag{5.41}$$

Comparing these two expressions yields (with (5.34) and (5.35)):

$$\frac{\partial r_{12}}{\partial l_1} = \hat{r}_{12} \cdot \hat{l}_1 \ , \tag{5.42}$$

$$\frac{\partial r_{12}}{\partial l_2} = -\hat{r}_{12} \cdot \hat{l}_2 \ , \tag{5.43}$$

$$\frac{\partial r_{12}}{\partial t} = \hat{r}_{12} \cdot \vec{V}_{12} \ . \tag{5.44}$$

As the circuits do not rotate but only translate we also have

$$\frac{\partial}{\partial t}(d\vec{l}_1) = \frac{\partial}{\partial t}(d\vec{l}_2) = 0 \ . \tag{5.45}$$

With (5.39), (5.44) and (5.45) we get

$$\frac{\partial}{\partial t}(\hat{r}_{12} \cdot d\vec{l}_1) = \frac{\vec{V}_{12} \cdot d\vec{l}_1}{r_{12}} - \frac{(\hat{r}_{12} \cdot \vec{V}_{12})(\hat{r}_{12} \cdot d\vec{l}_1)}{r_{12}} \ , \tag{5.46}$$

$$\frac{\partial}{\partial t}(\hat{r}_{12} \cdot d\vec{l}_2) = \frac{\vec{V}_{12} \cdot d\vec{l}_2}{r_{12}} - \frac{(\hat{r}_{12} \cdot \vec{V}_{12})(\hat{r}_{12} \cdot d\vec{l}_2)}{r_{12}} \ . \tag{5.47}$$

These values in (5.31) yield

$$emf_{12} = -\frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \left\{ \frac{\partial}{\partial t} \left[I_1 \frac{(\hat{r}_{12} \cdot d\vec{l}_1)(\hat{r}_{12} \cdot d\vec{l}_2)}{r_{12}} \right] + I_1 \frac{(\vec{V}_{12} \cdot d\vec{l}_1)(\hat{r}_{12} \cdot d\vec{l}_2)}{r_{12}^2} \right.$$
$$\left. - I_1 \frac{(\vec{V}_{12} \cdot d\vec{l}_2)(\hat{r}_{12} \cdot d\vec{l}_1)}{r_{12}^2} \right\} \ . \tag{5.48}$$

Remembering that $I_1(\vec{V}_{12} \cdot d\vec{l}_1)$ can be taken out of the integral in C_2, it can be shown that the second integral goes to zero as we have seen in Section 4.6. Analogously it can be shown that the third integral goes to zero integrating first in C_1. In this way (5.48) takes the form

$$emf_{12} = -\frac{\mu_0}{4\pi}\frac{d}{dt}\left[I_1 \oint_{C_1} \oint_{C_2} \frac{(\hat{r}_{12} \cdot d\vec{l}_1)(\hat{r}_{12} \cdot d\vec{l}_2)}{r_{12}} \right] \ . \tag{5.49}$$

In Section 4.6 we saw that this double integral is equal to $\oint_{C_1} \oint_{C_2} (d\vec{l}_1 \cdot d\vec{l}_2)/r_{12}$. With the definition of the coefficient of mutual inductance M by (4.61) we then obtain that (5.49) can be written as

$$emf_{12} = -\frac{d}{dt}(I_1 M) \ , \tag{5.50}$$

which is exactly one of the ways to express Faraday's law.

This completes the proof that Faraday's law of induction can be derived from Weber's force. In the next Section we generalize this result so that it will be seen to remain valid even when Fechner's hypothesis is not utilized.

5.4. Derivation of Faraday's Law from Weber's Electrodynamics Without Utilizing Fechner's Hypothesis

In this Section we generalize our previous result. First let us consider induction in metallic wires in which only the electrons move relative to the wire ($\vec{v}_{1+d} = \vec{v}_{2+d} = 0$, $\vec{a}_{1+d} = \vec{a}_{2+d} = 0$). As the positive ions are always fixed in the lattice we need to consider only the forces in the electrons of the current element $I_2 d\vec{l}_2$ due to the positive and negative charges of $I_1 d\vec{l}_1$. In analogy with (5.22) we now have

$$d^2 emf_{12} = \frac{d^2\vec{F}_{1+,2-} + d^2\vec{F}_{1-,2-}}{dq_{2-}} \cdot d\vec{l}_2 \ . \tag{5.51}$$

With (5.19) and (5.23) to (5.26) we get (with $dq_{2-} = -dq_{2+}$):

$$emf_{12} = -\oint_{C_1}\oint_{C_2} \frac{dq_{1+}}{4\pi\varepsilon_o}\frac{\hat{r}_{12}\cdot d\vec{l}_2}{r_{12}^2 c^2}\left\{\left[3(\hat{r}_{12}\cdot\vec{V}_{12})(\hat{r}_{12}\cdot\vec{v}_{1-d}) - 2\vec{V}_{12}\cdot\vec{v}_{1-d} - \vec{r}_{12}\cdot\vec{a}_{1-d}\right]\right.$$

$$\left. + \left[2\vec{v}_{1-d}\cdot\vec{v}_{2-d} - v_{1-d}^2 + \frac{3}{2}(\hat{r}_{12}\cdot\vec{v}_{1-d})^2 - 3(\hat{r}_{12}\cdot\vec{v}_{1-d})(\hat{r}_{12}\cdot\vec{v}_{2-d})\right]\right\} \ . \tag{5.52}$$

As we have seen in the previous Section, the integrand with the first square bracket gives Faraday's law (5.50). What would need to be shown is that the double integral of the integrand with the second square bracket in (5.52) gives exactly zero.

We have been able to show this in the three cases represented in Figure 5.5. For a detailed proof see (Thober, 1993).

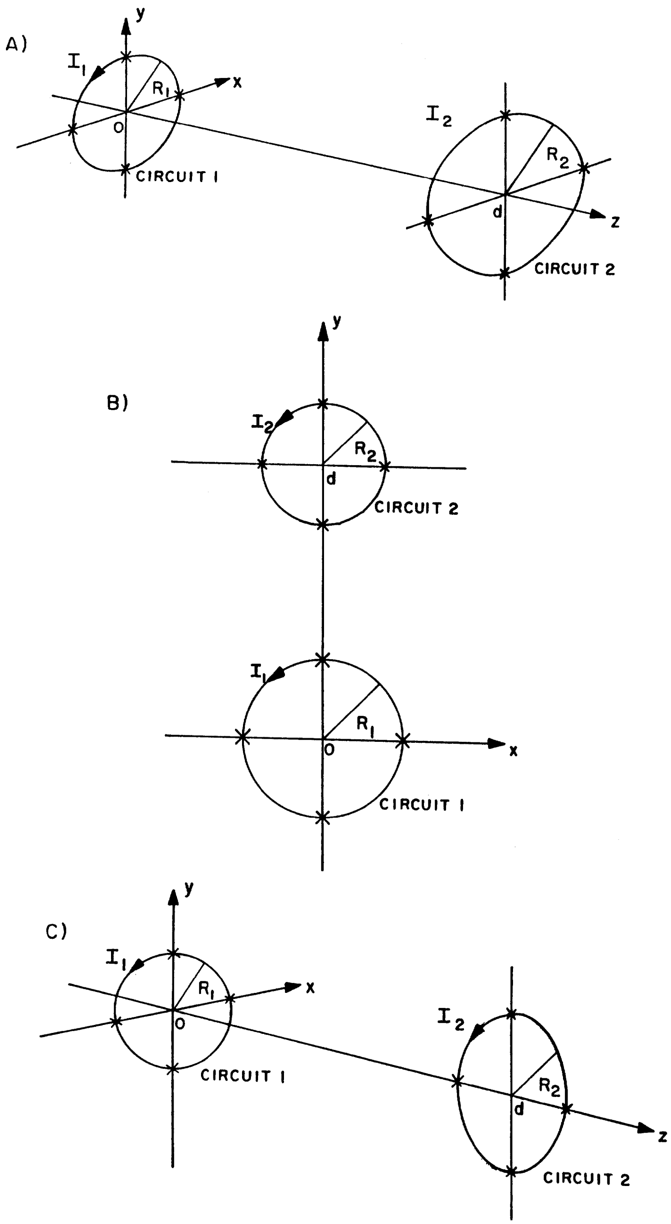

Figure 5.5

In this Figure we have two interacting circular loops of radius R_1 and R_2 where are flowing currents I_1 and I_2, respectively. In case (A) the two loops are in planes xy and $x'y'$ parallel to one another, and are centered along the same z axis. In case (B) they are in the same plane. In case (C) their planes are orthogonal to one another (xy and $x'z$) and their centers are along the z axis.

In circuits of arbitrary form and orientation as in Figure 5.6 it is very difficult to solve the integrals (5.52) for the second square bracket.

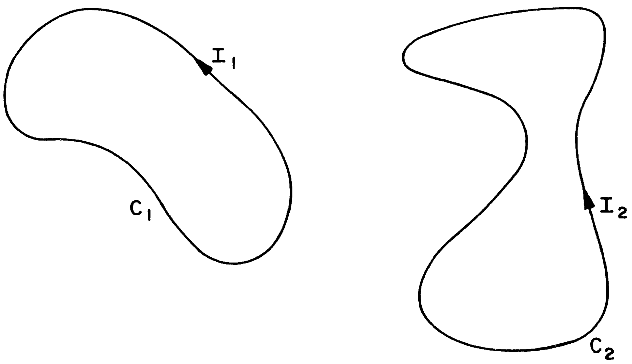

Figure 5.6

Instead of solving directly these integrals in the general situation, we present a different line of reasoning which proves that they need to be exactly zero.

As we have seen, the interaction magnetic energy between two circuits C_1 and C_2 according to Weber's electrodynamics is given by (4.74), namely

$$U_{12} = I_1 I_2 M_{12} \ , \tag{5.53}$$

$$M_{12} = \frac{\mu_o}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\vec{l}_1 \cdot d\vec{l}_2}{r_{12}} \ . \tag{5.54}$$

We could similarly have calculated the energy to form each circuit integrating (4.62) for each of them and the result would be

$$U_i = \frac{I_i^2 L_i}{2} \ , \ i = 1, \ 2 \ , \tag{5.55}$$

$$L_i \equiv \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_i} \frac{d\vec{l}_i \cdot d\vec{l}_i{}'}{r_{ii'}} \ , \ i = 1, \ 2 \ , \tag{5.56}$$

taking care with the divergences in (5.56).

The factor 1/2 in (5.55) is due to the fact that in (5.56) each pair ii' contributes twice to L_i as we integrate twice around the same circuit. The geometrical factor L_i is called the coefficient of self induction of the circuit C_i.

So the total magnetic energy to form the two circuits according to Weber's electrodynamics is given by

$$U = \frac{L_1 I_1^2}{2} + M_{12} I_1 I_2 + \frac{L_2 I_2^2}{2} \ . \tag{5.57}$$

This expression agrees with the one given in all textbooks dealing with classical electrodynamics.

We now follow Maxwell to show that this expression leads to Faraday's law of induction (see Maxwell, 1954, Vol. 2, Chapters 6 to 7, articles [568] to [584], pp. 211 to 228).

According to the Lagrangian formulation we may say that U in (5.57) is a function of I_1, I_2 and y_i ($i = 1, \ ..., \ N$), where the y_i's are geometrical variables (distance between the circuits, an angle representing their relative orientation, etc.) The generalized force Y in the Lagrangian formulation is given by

$$Y = - \ \frac{d}{dt} \frac{\partial U}{\partial \dot{y}_i} + \frac{\partial U}{\partial y_i} \ , \tag{5.58}$$

where $\dot{y}_i \equiv dy_i/dt$ and y_i are generalized coordinates.

For instance, if $y_i = x$ is any one of the geometrical variables on which the form and relative position of the circuits depend, the electromagnetic force (Ampère's force) tending to increase x is (from (5.57) and (5.58) observing that U does not depend on $\dot{x}$ and that I_1 and I_2 are independent of x):

$$F = \frac{I_1^2}{2} \frac{\partial L_1}{\partial x} + I_1 I_2 \frac{\partial M_{12}}{\partial x} + \frac{I_2^2}{2} \frac{\partial L_2}{\partial x} \ . \tag{5.59}$$

If the circuits are rigid then L_1 and L_2 will be independent of x so that the force is given by

$$F = I_1 I_2 \frac{\partial M_{12}}{\partial x} \ . \tag{5.60}$$

We had obtained this result in (4.60) and (4.74).

If $y_i = \theta$ is any angle describing the relative orientation of the two rigid circuits the torque T tending to increase θ is similarly given by

$$T = I_1 I_2 \frac{\partial M_{12}}{\partial \theta} \ . \tag{5.61}$$

Let us now consider $\dot{y}_i = I_2$ so that Y is then the electromotive force acting on circuit 2, emf_2. As L_1, M_{12} and L_2 are independent of the currents and U depends on the currents but not on the charge or amount of electricity $y_i = q_i$ which crossed a given cross section of the conductor C_i since the beginning of the time t ($q_i = \int_0^t I_i dt$; $i = 1, \ 2$) we obtain from (5.58) and (5.57):

$$emf_2 = -\frac{d}{dt}\left(\frac{\partial U}{\partial I_2}\right) = -\frac{d}{dt}(L_2 I_2 + M_{12} I_1) \ . \tag{5.62}$$

And this is exactly Faraday's law of induction taking also into account the self induction of the circuit.

Now it is important to realize that to arrive at this result from Weber's electrodynamics we did not utilize Fechner's hypothesis. The only result we employed was (5.53), which was derived from (4.74) and (4.64). And to arrive at (4.64) we utilized only Weber's potential energy (4.63) and the charge neutrality of the current elements. This means that (4.64) and (5.62) are valid for any value of the velocity of the positive ions, even when they are at rest relative to the lattice and only the electrons move in the wire generating the current.

There is another proof which is partly due to Helmholtz in 1847, to W. Thomson in 1848 - 1853, and has been generalized by J. J. Thomson in 1891. This proof can be found in (Maxwell, 1954, Vol. 2, articles [543 - 4], pp. 190 - 193). What we present here follows from J. J. Thomson's proof of 1891, after Maxwell's death, which appeared as a footnote of article [544], p. 192, of (Maxwell, 1954).

Let L_1, L_2 be the coefficient of self-induction of the first and second circuit, (5.56), and M_{12} the coefficient of mutual induction of the two circuits, (5.54). The magnetic energy of the two circuits according to Weber's electrodynamics and to the usual classical electromagnetism is given by (5.57). We can then write the variation of U as:

$$dU = \frac{\partial U}{\partial I_1} dI_1 + \frac{\partial U}{\partial I_2} dI_2 + \sum \frac{\partial U}{\partial x} dx \ , \tag{5.63}$$

where x is a coordinate of any type helping to fix the position of the circuits.

From (5.57) we get

$$\frac{\partial U}{\partial I_1} = L_1 I_1 + M_{12} I_2 \ , \tag{5.64}$$

$$\frac{\partial U}{\partial I_2} = L_2 I_2 + M_{12} I_1 \ . \tag{5.65}$$

From these expressions we get

$$I_1 \frac{\partial U}{\partial I_1} + I_2 \frac{\partial U}{\partial I_2} = 2U \ . \tag{5.66}$$

The variation of (5.66) yields:

$$2dU = (dI_1) \frac{\partial U}{\partial I_1} + I_1 d\frac{\partial U}{\partial I_1} + (dI_2) \frac{\partial U}{\partial I_2} + I_2 d\frac{\partial U}{\partial I_2} \ . \tag{5.67}$$

Subtracting (5.63) from (5.67) yields

$$dU = I_1 d\frac{\partial U}{\partial I_1} + I_2 d\frac{\partial U}{\partial I_2} - \sum \frac{\partial U}{\partial x} dx \ . \tag{5.68}$$

The force F_x of type x acting on the system is given by $\partial U / \partial x$. Since we suppose no external force acts on the system, $\sum \frac{\partial U}{\partial x} dx$ will be the increase in kinetic energy T due to the motion of the system, dT. Eq. (5.68) can then be written as

$$d(U + T) = I_1 d\frac{\partial U}{\partial I_1} + I_2 d\frac{\partial U}{\partial I_2} \ . \tag{5.69}$$

Let A_1 and A_2 be the electromotive forces acting on circuits 1 and 2, respectively, due to voltaic batteries (the usual chemical batteries). If these circuits 1 and 2 have resistances R_1 and R_2, the power dissipated in these circuits due to internal frictions are given according to Joule's law by $R_1 I_1^2$ and $R_2 I_2^2$, respectively. The work dW done by these batteries in a time dt is

$$dW = A_1 I_1 dt + A_2 I_2 dt \ . \tag{5.70}$$

The heat dQ produced in the same time by Joule's law is given by

$$dQ = R_1 I_1^2 dt + R_2 I_2^2 dt \ . \tag{5.71}$$

By the conservation of energy the work done by the batteries must equal the heat produced in the circuit plus the increase in the energy of the system, hence

$$A_1 I_1 dt + A_2 I_2 dt = (R_1 I_1^2 + R_2 I_2^2) dt + d(U + T) \ . \tag{5.72}$$

Substituting for $d(U + T)$ from (5.69) we get

$$I_1 \left(A_1 - R_1 I_1 - \frac{d}{dt} \frac{\partial U}{\partial I_1} \right) + I_2 \left(A_2 - R_2 I_2 - \frac{d}{dt} \frac{\partial U}{\partial I_2} \right) = 0 \ . \tag{5.73}$$

By (5.64) and (5.65) this yields

$$I_1 \left[A_1 - R_1 I_1 - \frac{d}{dt}(L_1 I_1 + M_{12} I_2) \right] + I_2 \left[A_2 - R_2 I_2 - \frac{d}{dt}(L_2 I_2 + M_{12} I_1) \right] = 0 \ . \tag{5.74}$$

J. J. Thomson then concludes his derivations saying that the principle of the conservation of energy gives us only this equation.

In order to arrive at the laws of induction (the two square brackets equated to zero) we need something else. The best principle to utilize is Kirchhoff's second rule for electrical circuits. According to this second rule (the first one stating that the sum of the algebraic currents into any node must be zero) the sum of the voltage drops around a closed loop is zero, (Feynman, 1964, Vol. 2, Section 22.3, pp. 22-7 to 22-10). Kirchhoff's rules were derived by him in 1844 - 1845, independently of Weber's own simultaneous derivation of these laws in a simplified form (Jungnickel and McCormmach, 1986, Vol. 1, pp. 87, 125 and 152 - 155). The sum of the voltage drops arond circuit 1 (2) is given by the first (second) square bracket in (5.74). With Kirchhoff's second rule we then get:

$$A_1 - R_1 I_1 - \frac{d}{dt}(L_1 I_1 + M_{12} I_2) = 0 \ , \tag{5.75}$$

$$A_2 - R_2 I_2 - \frac{d}{dt}(L_2 I_2 + M_{12} I_1) = 0 \ . \tag{5.76}$$

And this is the law of induction as applied to each circuit which we wanted to derive. This completes the proof that from Weber's electrodynamics we derive Faraday's law of induction even when Fechner's hypothesis is not valid.

Chapter 6 / Forces of Weber and of Lorentz

6.1. Introduction

As we saw in the previous Chapters, Weber's law follows all the principles of conservation of classical physics: linear momentum, angular momentum and energy. With Weber's force we derived Ampère's force between current elements. Lastly we showed how to derive from Weber's law the set of Maxwell's equations (Gauss's law, the magnetic circuital law, the law for the absence of magnetic monopoles, and Faraday's induction law). In particular, to derive the magnetic circuital law it was necessary, besides Weber's force, to introduce the continuity equation for electric charges in order to obtain Maxwell's displacement current. The converse was shown in Section 2.5, namely, from the law of Gauss and the magnetic circuital law, with displacement current, it is possible to derive the equation of continuity. The important aspect to be emphasized here is that Weber's force is compatible with Maxwell's equations. Later on we will discuss the limitations of this derivation of Maxwell's equations from Weber's electrodynamics.

So one of the main differences between Weber's electrodynamics and classical electromagnetism concerns the force which is exerted on the charges. We remember here that this is not furnished by Maxwell's equations, which give only the fields generated by the charges but do not tell us how the charges react in the presence of the external fields. In classical electromagnetism this is furnished by Lorentz's force, while in Weber's electrodynamics we have Weber's force itself. Another difference is that Weber's electrodynamics does not need "fields" at all - an improvement in logical economy. Also it holds in general coordinate systems, even for non-inertial ones.

In this Chapter we compare these two forces.

6.2. Retarded and Liénard-Wiechert's Potentials

The most direct way of comparing both expressions for the force is through the potentials of Liénard-Wiechert. A short historical context: The idea that the interaction between the bodies at a distance is not instantaneous, so that it should take time for its propagation from one body to another, is an old one. But in electromagnetism the first to clearly express this idea seems to have been Gauss, in 1845, in a letter addressed to Weber (Gauss, 1877, Vol. 5, pp. 627 - 629), (Whittaker, 1973, Vol. 1, p. 240), (Maxwell, 1954, Vol. 2, article 861, p. 489), (O'Rahilly, 1965, Vol. 1, p. 226). In 1858 Riemann (1826-1866), a student and collaborator of Weber and Gauss at Göttingen University, introduced the idea of retarded time in physics. This idea can be expressed by saying that the force exerted on a charge q_1 localized at $\vec{r}_1$ at time t due to another charge q_2 depends on the position, velocity and acceleration of q_2 at the earlier moment $t - r_{12}/c$. In this expression r_{12} is the distance between the two charges and c is the velocity of the interaction, assumed to be the light velocity. Riemann's work was only published in 1867 (Riemann, 1867), (Jungnickel and McCormmach, 1986, Vol. 1, pp. 179 - 181). In the same year Ludwig Lorenz (1829-1891), a Danish physicist (who should not be confused with the Dutch H. A. Lorentz of the Lorentz's force), published a paper where he developed independently the idea of the retarded time (Lorenz, 1867). In this way it can be said that Lorenz and Riemann introduced the retarded time into physics.

In 1867 the German physicist Clausius obtained a force law analogous to that which H. A. Lorentz would introduce twenty years later, and showed how to derive Grassmann's force from his law (Clausius, 1880), (Whittaker, 1973, Vol. 1, pp. 234 - 236). H. A. Lorentz introduced his force law in 1895 (Lorentz, 1895), (Pais, 1982, p. 125; and Pais, 1986, p. 76). His expression differs from that of Clausius in that Lorentz included the retarded time in the fields (Whittaker, 1973, Vol. 1, pp. 392 - 396). In 1898 A. Liénard provided a great advance relative to Lorentz's work because he worked with the retarded potentials due to corpuscular charges. This work was followed by another on the same lines written by E. Wiechert in 1900. For this reason the potentials which we will present receive usually the name Liénard-Wiechert. It should be emphasized that also K. Schwarzschild presented important advances along these lines (the calculus of the electrodynamic potential, etc.) in 1903. For a discussion of the works of Liénard, Wiechert and Schwarzschild, see: (Whittaker, 1973, Vol. 1, pp. 392 - 410), (O'Rahilly, 1965, Vol. 1, pp. 212 - 223).

After this short account we give here the corresponding formulas. Lorentz's force (2.41) expressed in terms of the potentials ϕ and $\vec{A}$ through (2.39) and (2.33) takes the form:

$$\vec{F}_{21} = -q_1\left(\nabla_1\phi_2 + \frac{\partial\vec{A}_2}{\partial t}\right) + q_1\vec{v}_1 \times (\nabla_1 \times \vec{A}_2) \ . \tag{6.1}$$

In this expression ∇_1 and $\nabla_1 \times$ are to be applied at the position of charge 1, while ϕ_2 and $\vec{A}_2$ are the potentials due to the charges in volume V_2.

The retarded potentials are the analogous to (2.20) and (2.32), changing the indexes 1 and 2, and substituting $\rho\vec{v}dV$ for $Id\vec{l}$, namely

$$\phi_2(\vec{r}_1, t) \equiv \frac{1}{4\pi\varepsilon_0}\int\int\int \frac{\rho_2(\vec{r}_2{}^*, t^*)}{r_{12}} dV_2^* \ , \tag{6.2}$$

$$\vec{A}_2(\vec{r}_1, t) \equiv \frac{\mu_0}{4\pi}\int\int\int \frac{\rho_2(\vec{r}_2{}^*, t^*)\vec{v}_2(t^*)}{r_{12}} dV_2^* \ . \tag{6.3}$$

In these expressions the great modification is that now the potentials at $\vec{r}_1$ in time t are obtained as a function of where $dq_2 = \rho_2 dV_2$ was localized at the retarded time $t^* = t - r_{12}/c$. All magnitudes with an asterisk are to be understood as being obtained at the retarded time, namely $\vec{r}_2{}^* \equiv \vec{r}_2(t^*)$, etc. These retarded potentials were first introduced by Ludwig Lorenz in 1867 (Lorenz, 1867).

Through the works of Liénard, Wiechert and Schwarzschild it is possible to obtain directly the force between two point charges q_1 and q_2, utilizing these expressions. To do this correctly we cannot simply substitute q_2 for $\rho_2 dV_2$, see: (Whittaker, 1973, Vol. 1, pp. 407 to 409), (O'Rahilly, 1965, Vol. 1, pp. 212 - 223), (Griffiths, 1989, pp. 416 - 419), (Feynmann, 1964, Vol. 2, Section 21-5, pp. 21-9 to 21-11). In these works it is detailed how to arrive at the potentials due to a point charge q_2 beginning with Lorenz's retarded potentials (6.2) and (6.3). The final result is given by

$$\phi(\vec{r}_1,\ t) = \frac{1}{4\pi\varepsilon_o}\frac{q_2}{[r_{12} - (\vec{v}_2 \cdot \vec{r}_{12})/c]^*}\ , \tag{6.4}$$

$$\vec{A}_2(\vec{r}_1,\ t) = \frac{\mu_o}{4\pi}\frac{q_2\vec{v}_2}{[r_{12} - (\vec{v}_2 \cdot \vec{r}_{12})/c]^*}\ . \tag{6.5}$$

These are the scalar and vector potentials due to a point charge q_2. Once more the asterisk indicates retarded time. These potentials were first deduced in this form by Liénard and Wiechert. They are called the Liénard-Wiechert potentials. With (6.1), (6.4) and (6.5) we can obtain the complete form for the force exerted by q_2 on q_1, namely (Griffiths, 1989, pp. 421 - 425):

$$\vec{F}_{21} = q_1\vec{E}_2(\vec{r}_1,\ t) + q_1\vec{v}_1 \times \vec{B}_2(\vec{r}_1,\ t)\ , \tag{6.6}$$

where

$$\begin{aligned} \vec{E}_2(\vec{r}_1,\ t) &= \frac{q_2}{4\pi\varepsilon_o}\frac{\vec{r}_{12}}{[r_{12} - (\vec{r}_{12} \cdot \vec{v}_2)/c]^3}\ , \\ \vec{B}_2(\vec{r}_1,\ t) &= \frac{\hat{r}_{12} \times \vec{E}_2(\vec{r}_1,\ t)}{c}\ . \end{aligned} \tag{6.7}$$

In (6.7) the quantities in the right hand sides are calculated at the retarded time.

The correct way of arriving at the final result for the force involving only the present time t is a complicated one and is beyond the scope of this book. What matters to us here is the final result, which is obtained by a Taylor expansion of all variables which contain t^* around $t^* = t$.

The final result for the force, which is independent of the gauge, can be obtained by (6.1), (6.4) and (6.5). The final value of the force exerted by q_2 on q_1 valid until the second order in $1/c$ can be found in many places. See, for instance: (O'Rahilly, 1965, Vol. 1, pp. 215 - 223), (Pearson and Kilambi, 1974), (Edwards, Kenyon and Lemon, 1976). It is given by

$$\vec{F}_{21} = q_1\vec{E}_2{}^* + q_1\vec{v}_1 \times \vec{B}_2{}^* \simeq q_1\left\{\frac{q_2}{4\pi\varepsilon_0}\ \frac{1}{r_{12}^2}\left[\hat{r}_{12}\left(1 + \frac{\vec{v}_2 \cdot \vec{v}_2}{2c^2}\right.\right.\right.$$

$$- \frac{3}{2}\frac{(\hat{r}_{12} \cdot \vec{v}_2)^2}{c^2} - \frac{\vec{r}_{12} \cdot \vec{a}_2}{2c^2}\bigg) - \frac{r_{12}\vec{a}_2}{2c^2}\bigg]\bigg\} + q_1\vec{v}_1 \times \left\{\frac{q_2}{4\pi\varepsilon_0}\frac{1}{r_{12}^2}\frac{\vec{v}_2 \times \hat{r}_{12}}{c^2}\right\} . \qquad (6.8)$$

An extremely important fact to be emphasized here is that only $\vec{E}_2{}^*$ and $\vec{B}_2{}^*$ in (6.8) are calculated at the retarded time, because **all magnitudes on the right hand side of the last equality (including $r_{12}, \hat{r}_{12}, \vec{v}_2$ and $\vec{a}_2$) are calculated and measured at the present time t and not at the retarded time t^*** (remember that there was a Taylor expansion around $t^* = t$ to arrive at this result). Although the general expression for the force has terms of the infinite orders in powers of $1/c$, we present here terms only up to the second order because practically all the phenomena studied in electromagnetism (as Coulomb's force, the magnetic circuital law, Biot-Savart's magnetic field and Faraday's law of induction) are included in this approximation.

Changing the indexes 1 and 2, and remembering that $\vec{r}_{21} = -\vec{r}_{12}$ yields

$$\vec{F}_{12} = q_2\vec{E}_1{}^* + q_2\vec{v}_2 \times \vec{B}_1{}^* \simeq q_2\left\{\frac{-q_1}{4\pi\varepsilon_0}\frac{1}{r_{12}^2}\left[\hat{r}_{12}\left(1 + \frac{\vec{v}_1 \cdot \vec{v}_1}{2c^2}\right.\right.\right.$$

$$- \frac{3}{2}\frac{(\hat{r}_{12} \cdot \vec{v}_1)^2}{c^2} + \frac{\vec{r}_{12} \cdot \vec{a}_1}{2c^2}\bigg) + \frac{r_{12}\vec{a}_1}{2c^2}\bigg]\bigg\} + q_2\vec{v}_2 \times \left\{\frac{-q_1}{4\pi\varepsilon_0}\frac{1}{r_{12}^2}\frac{\vec{v}_1 \times \hat{r}_{12}}{c^2}\right\} . \qquad (6.9)$$

These are the fundamental expressions which will be compared with Weber's force. The equations (6.8) and (6.9) are the basic force laws of classical electromagnetism.

As we will see, Eqs. (6.8) and (6.9) include not only time retardation and radiation phenomena, but also relativistic corrections. So they are really complete as regards classical electrodynamics, bearing in mind that they are valid only until the second order in $1/c$, inclusive. Later on we will derive these forces from Darwin's Lagrangian.

Following O'Rahilly we will call (6.8) and (6.9) the Liénard-Schwarzschild's force-formula (O'Rahilly, 1965, Vol. 1, p. 218). Although he gave this name to the expression involving retarded time, we will keep this name to (6.8) and (6.9) as they follow naturally from the original Liénard-Schwarzschild's force. Eqs. (6.8) and (6.9) are analogous to Eq. (7.17) of O'Rahilly's book.

One aspect to be observed in (6.8) and (6.9) is that the velocities and accelerations of the charges are relative to a frame of reference (to an observer). When these equations are utilized together with Newton's second law of motion without "fictitious" forces, as in the form (2.5), or together with the analogous relativistic equation of motion, then this frame of reference needs to be an inertial one.

When the charges are at rest in this frame, (6.8) reduces to Coulomb's force (2.13). In Section 2.6 we saw how to derive Gauss's law from (2.13), which means that the first of Maxwell's equation can be derived from (6.8). As we have seen, the same happens with Weber's force.

Faraday's law of induction can also be derived from Lorentz's force or from Liénard-Schwarzschild's one, as is the case with Weber's force (O'Rahilly, 1965, Vol. 2, pp. 572 - 581).

6.3. Derivation of Grassmann's Force from the Liénard-Schwarzschild's Force

From (6.8) and $c^2 = 1/\mu_o\varepsilon_o$ we have the magnetic field due to a moving charge, namely

$$d\vec{B}_2(\vec{r}_1) = \frac{\mu_o}{4\pi}\frac{dq_2\vec{v}_2 \times \hat{r}_{12}}{r_{12}^2} \ . \tag{6.10}$$

If we have a current element composed of positive and negative charges, dq_{2+} and dq_{2-}, moving with velocities $\vec{v}_{2+}$ and $\vec{v}_{2-}$ relative to a frame of reference, respectively, the magnetic field generated by these charges will be given by (adding (6.10) due to both charges):

$$d\vec{B}_{I_2d\vec{l}_2}(\vec{r}_1) = \frac{\mu_o}{4\pi}\frac{dq_{2+}(\vec{v}_{2+} - \vec{v}_{2-}) \times \hat{r}_{12}}{r_{12}^2} = \frac{\mu_o}{4\pi}\frac{I_2d\vec{l}_2 \times \hat{r}_{12}}{r_{12}^2} \ . \tag{6.11}$$

To arrive at this expression we utilized the charge neutrality of the current element, (4.18), $\vec{r}_{i+} = \vec{r}_{i-} \equiv \vec{r}_i$ (due to the infinitesimal size of the current element), and the definition of a current element, (4.23). Eq. (6.11) is exactly Biot-Savart's expression for the magnetic field, (2.23).

We now utilize (6.8) to calculate directly the force exerted by a neutral current element $I_2d\vec{l}_2$ composed of the charges dq_{2+} and dq_{2-}, on another current element $I_1d\vec{l}_1$ composed of dq_{1+} and dq_{1-}. Adding the four components of the force (6.8) and utilizing (4.18), (4.23) and (6.11) yields:

$$d^2\vec{F}_{21} = d^2\vec{F}_{2+,1+} + d^2\vec{F}_{2+,1-} + d^2\vec{F}_{2-,1+} + d^2\vec{F}_{2-,1-}$$

$$= I_1d\vec{l}_1 \times \left(\frac{\mu_o}{4\pi}\frac{I_2d\vec{l}_2 \times \hat{r}_{12}}{r_{12}^2}\right) = I_1d\vec{l}_1 \times d\vec{B}_2 \ . \tag{6.12}$$

And this is exactly Grassmann's force (4.26) to (4.29).

As we saw in Section 4.7, the second and the third of Maxwell's equations can be derived from the integrated form of Grassmann's force, the same happening with the integrated form of Ampère's force (4.24).

In Section 4.4 we presented a derivation of Grassmann's force directly from Lorentz's force. Here we presented a different derivation utilizing (6.8). We also derived here Biot-Savart's magnetic field. These proofs are very simple and straightforward.

Rambaut and Vigier claimed to have derived Ampère's force between current elements, (4.24), from Lorentz's force, utilizing the Liénard-Wiechert potentials, (Rambaut and Vigier, 1990). In one of our previous publications we said that: "Although we can derive Ampère's force from Weber's one performing a statistical summation over all interacting charges of the neutral current elements, this can also be done from other approaches and different force laws. For instance, recently Rambaut and Vigier suceeded in deriving Ampère's force from the relativistic limit on the macroscopic level" (Assis and Caluzi, 1991). Now we can not support their claim anymore. Their calculation involves many assumptions and different approximations. It is complex and full of subtleties. On the other hand, the extremely simple and direct proofs presented in Section 4.4 and in this one, as well as the analogous ones presented in many textbooks, convince us that from Lorentz's force we derive only Grassmann's force between current elements, but not Ampère's force (4.24). On the other hand from Weber's force we derive only Ampère's expression (4.24), but not Grassmann's force (4.28).

However, the reverse way is not unique. Ampère's force (4.24) can be derived not only from Weber's force but also from Gauss's force (Gauss, 1877, Vol. 5, pp. 616 - 617), which differs from Weber's force (4.16) in that it does not contain the acceleration term in $\vec{a}_{ij}$. Grassmann's force (4.28) can be derived not only from Lorentz's force but also from Clausius's force (Clausius, 1880). And Clausius's force is different from (6.8). We show these other forces in Appendix B.

As the four Maxwell's equations can be derived from Weber's force as well as from Lorentz's force (or from Liénard-Schwarzschild's force), these forces can not be distinguished in this way. In the following Sections we show distinct procedures to compare and distinguish Weber's force from Lorentz's one (or from the Liénard-Schwarzschild's force).

6.4. Comparison Between Weber's Force and Liénard-Schwarzschild's Force

Weber's force is given by (3.24), namely

$$\vec{F}_{21} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{r}_{12}}{r_{12}^2}\left[1 + \frac{1}{c^2}\left(\vec{v}_{12}\cdot\vec{v}_{12} - \frac{3}{2}(\hat{r}_{12}\cdot\vec{v}_{12})^2 + \vec{r}_{12}\cdot\vec{a}_{12}\right)\right] = -\vec{F}_{12} \ . \tag{6.13}$$

The first aspect to be observed in a comparison between these two expressions is that Weber's force always satisfies the action and reaction principle in the strong form, while Lorentz's force and the Liénard-Schwarzschild's one do not satisfy this principle even in the weak form, with the exception of some very specific cases. This can be seen adding (6.8) and (6.9), and observing that the remaining terms do not necessarily cancel out to zero.

The Coulombian part is the same in Weber and in Liénard-Schwarzschild. Let us now analyse the components which depend on the velocities and accelerations. We will concentrate on $\vec{F}_{21}$. We will call q_2 and everything else which has the index 2 in (6.8) and in the first equality of (6.13) as sources. We will call q_1 and everything else which has the index 1 in (6.8) and the first equality of (6.13) as test charges (that is, the charges which experience or feel the force).

In terms of the sources it can be observed that Liénard-Schwarzschild's force has terms which depend linearly on the velocity, $\vec{v}_1 \times (\vec{v}_2 \times \hat{r}_{12})$, (this arises from the charges which generate the magnetic field of Biot-Savart and of Grassmann's force), and terms which depend on the square of the velocity, namely, $[\vec{v}_2\cdot\vec{v}_2/2 - 3(\hat{r}_{12}\cdot\vec{v}_2)^2/2]\hat{r}_{12}$. It has also terms which depend on the acceleration, namely $-[(\vec{r}_{12}\cdot\vec{a}_2)\hat{r}_{12} + r_{12}\vec{a}_2]/2$. These are the terms which will give Faraday's induction with dI_2/dt. Weber's force has also terms with these general characteristics, but with their own peculiarities. The terms proportional to the source velocities are $[-2\vec{v}_1\cdot\vec{v}_2 + 3(\hat{r}_{12}\cdot\vec{v}_1)(\hat{r}_{12}\cdot\vec{v}_2)]\hat{r}_{12}$, which as we saw are responsible for the magnetic effects and for Ampère's force between current elements. The terms proportional to the square of the velocity are $[\vec{v}_2\cdot\vec{v}_2 - 3(\hat{r}_{12}\cdot\vec{v}_2)^2/2]\hat{r}_{12}$. The term proportional to the acceleration is $-(\vec{r}_{12}\cdot\vec{a}_2)\hat{r}_{12}$. As we saw this is the term responsible for Faraday's induction with dI_2/dt.

Although the two forces are not exactly the same in these aspects, the general behaviour is similar. In general they will give the same results, in particular in most cases when we have closed circuits. An exception to this fact is in the terms proportional to the square of the velocity of the sources. We will study a particular example of this exception in Section 6.6.

Relative to the test charges, it can be observed that Liénard-Schwarzschild's force has terms proportional to $\vec{v}_1$ given by $-\vec{v}_1 \times (\vec{v}_2 \times \hat{r}_{12})$. As we have seen, the terms proportional to $\vec{v}_1$ in Weber's force are $[-2\vec{v}_1 \cdot \vec{v}_2 + 3(\hat{r}_{12} \cdot \vec{v}_1)(\hat{r}_{12} \cdot \vec{v}_2)]\hat{r}_{12}$. In the general situation of closed circuits these terms will be usually equivalent. On the other hand Weber's force has terms proportional to the square of the velocities of the test charge given by $[\vec{v}_1 \cdot \vec{v}_1 - (3/2)(\hat{r}_{12} \cdot \vec{v}_1)^2]\hat{r}_{12}$. There are no such terms in Liénard-Schwarzschild's force or in Lorentz's force (2.41). The possible relevance of these terms is discussed in Section 6.7.

A fundamental distinction between the forces of Weber and Lorentz is that there are no terms which depend on the acceleration of the test charge in Lorentz's force (2.41) or in Liénard-Schwarzschild's force (6.8). On the other hand there are such terms in Weber's force (6.13) given by $(\hat{r}_{12} \cdot \vec{a}_1)\hat{r}_{12}$. The relevance of these terms will be discussed in the next Chapter. Here we would like only to mention that Przeborski discussed force laws which depend on the acceleration of the test charge and their connection with the principle of superposition of forces (Przeborski, 1933). Waldron also discussed this topic and concluded that forces of this kind were incompatible with Newton's second law of motion (Waldron, 1991). As we have shown, Waldron's arguments are incorrect and resulted from a simple mathematical mistake (Assis, 1992 c).

Another way of analysing Weber's force is through (3.5). There we can see that it is a sum of three terms. The first one is the usual Coulombian force, responsible for electrostatics and for Gauss's law. The second one is what gives rise to the magnetic effects of Ampère's force and the magnetic circuital law. The third one given by $q_i q_j \ddot{r}_{ij} \hat{r}_{ij}/(4\pi\varepsilon_0 c^2 r_{ij})$ is the one responsible for Faraday's induction law and for the inertial effects when applied to gravitation (Assis, 1989 a). This term is also one of those which give the effects of electromagnetic radiation from Weber's law (remember that the intensity of the dipole radiation, or of an antenna, falls as $1/r$ at great distances). See also Chapter 8.

There is another aspect which can be clearly seen regarding the forces of Weber or of Lorentz. It is related to the orders of magnitude between electric forces, magnetic forces and the effects of induction. From (6.8) and (6.13) it can be seen that all terms, except that of Coulomb, have c^2 in the denominator. But in the numerator there are terms of the order v_1^2, $v_1 v_2$, v_2^2, $r_{12} a_1$ and $r_{12} a_2$. But these are exactly the terms responsible for the magnetic field, for Ampère's force, for the magnetic circuital law, and for the induction effects. This means that for small velocities and accelerations as usually happens (electric currents of a few Ampères, magnetic fields of a few Gauss, etc.) the effects of magnetism and of induction are of second order relative to the electrostatic ones. That is, if two systems interact with one another and there is a net electric charge in both of them, usually the Coulombian force will supplant the magnetic and inductive effects. Although these effects also exist in this case, they are then negligible. For these effects to appear clearly it is necessary that the interacting systems have a zero net charge, or that this charge be extremely small. In this case the magnetic and inductive effects will be the main components. Examples of this last situation are when a magnet (electrically neutral) interacts with an electric current in a wire or with another magnet, or when two current carrying wires interact with one another.

In the next Sections we will present and discuss in greater detail the distinction between the force of Lorentz (or of Liénard-Schwarzschild) and that of Weber.

6.5. Two Charges in Uniform Rectilinear Motion

To show the difference of emphasis of these two force laws we discuss here some simple situations involving only two charges.

A) Charges at rest

In this first case we have two charges q_1 and q_2 separated by the distance $\vec{r}_{12} = r_{12}\hat{y}$ and which are at rest in an inertial frame S. To this end we can think that the Coulombian force is counteracted by some other force as, for instance, by an elastic force (supposing that the two charges are connected by a spring made of a non-conducting material). In this case as there is no motion of both charges Weber and Liénard-Schwarzschild's forces take the same form, namely (see Figure 6.1):

$$\vec{F}_{21}^{LS} = \vec{F}_{21}^{W} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{y}}{r_{12}^2} = -\vec{F}_{12}^{LS} = -\vec{F}_{12}^{W} \quad . \tag{6.14}$$

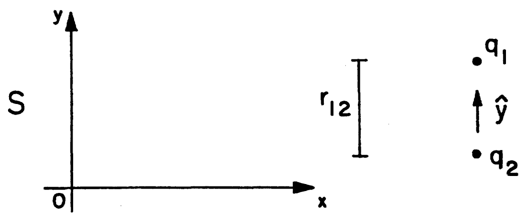

Figure 6.1

B) Charges in uniform rectilinear parallel motion

We now suppose the same previous situation as seen in another inertial frame S' which moves itself relative to S with a constant velocity - $V\hat{x}$, where we suppose $V << c$. The situation seen in this frame is represented in Figure 6.2.

Naturally we have $q_1' = q_1, q_2' = q_2, \hat{x}' = \hat{x}, \hat{y}' = \hat{y}$ and $r_{12}' = r_{12}$. As we are supposing a constant velocity we have $\vec{a}\,'_1 = \vec{a}\,'_2 = O$. From (6.13) we get (we represent by $'$ the forces in the frame S')

$$\vec{F}_{21}^{W'} = \vec{F}_{21}^{W} = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{\hat{y}}{r_{12}^2} = -\vec{F}_{12}^{W'} = -\vec{F}_{12}^{W} \ . \tag{6.15}$$

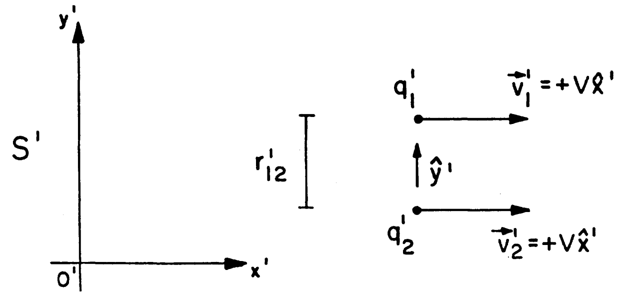

Figure 6.2

According to Weber's force there is no difference between the two frames of reference because there is no relative motion between the charges in both cases. This illustrates once more the relational character of Weber's force, because it has the same value in all frames of reference.

As we are supposing $V^2 << c^2$ we can utilize (6.8) and (6.9) as correct expressions of Liénard-Schwarzschild's force, valid up to the second order in $1/c$. From (6.8) and (6.9) we get

$$\vec{F}_{21}^{LS'} = -\vec{F}_{12}^{LS'} = q_1 \left[\frac{q_2}{4\pi\varepsilon_0} \left(1 + \frac{V^2}{2c^2} \right) \frac{\hat{y}}{r_{12}^2} \right] + \ q_1 V \hat{x} \times \left[\frac{q_2}{4\pi\varepsilon_0} \frac{V\hat{z}}{c^2 r_{12}^2} \right]$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_0} \left(1 - \frac{V^2}{2c^2} \right) \frac{\hat{y}}{r_{12}^2} \ . \tag{6.16}$$

There are several aspects to be observed in this equation. The first one is that Liénard-Schwarzschild's force in this specific example follows the action and reaction principle in the frame S'. In the first square bracket of (6.16) we have the electric field of q_2 as seen in S', which is larger than the electric field in S by the factor $(1 + V^2/2c^2)$. In the second square bracket we have the magnetic field due to the motion of q_2, and this magnetic field did not exist in frame S. The combined result of these two modifications is that the resulting electromagnetic force is decreased relative to the Liénard-Schwarzschild's force in frame S by a factor $(1 - V^2/2c^2)$.

And this is the reason why it is said in classical electromagnetism that the electric and magnetic fields transform one another in different inertial frames. This is also the reason why it is claimed that there is no physical reality in each one of them separately, but only in the electromagnetic force as a whole.

C) Charges in uniform rectilinear parallel motion aligned with their separation

Another typical situation is represented in Figure 6.3.

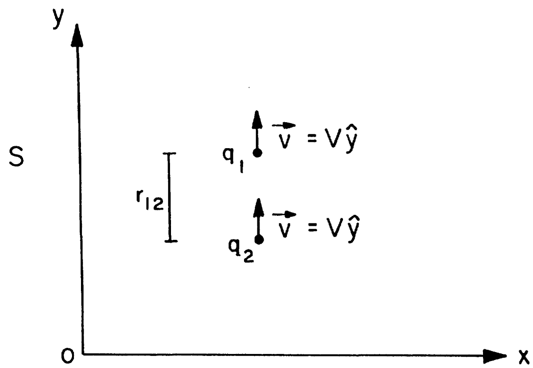

Figure 6.3

Here two charges q_1 and q_2 are aligned parallel to the y axis and move along this axis relative to the inertial frame S with a constant velocity $\vec{v}_1 = \vec{v}_2 = V\hat{y}$, so that $\vec{a}_1 = \vec{a}_2 = 0$. If their separation is r_{12} then their mutual forces according to Weber's expression (6.13) are given by

$$\vec{F}_{21}^W = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{\hat{y}}{r_{12}^2} = -\vec{F}_{12}^W \ . \tag{6.17}$$

Once more we are supposing $V << c$ so that according to (6.8) and (6.9) we get

$$\vec{F}_{21}^{LS} = \frac{q_1 q_2}{4\pi\varepsilon_o}\frac{\hat{y}}{r_{12}^2}\left(1 - \frac{V^2}{c^2}\right) = -\vec{F}_{12}^{LS} \ . \tag{6.18}$$

So the expressions of Weber and Liénard-Schwarzschild predict different values for the mutual forces.

D) Charges in uniform rectilinear parallel motion inclined relative to their separation

Another situation is represented in Figure 6.4.

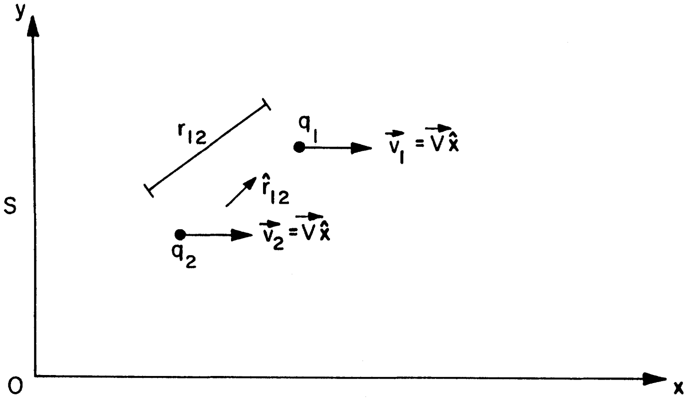

Figure 6.4

Two charges q_1 and q_2 are separated by a distance r_{12}. The line connecting them is inclined relative to the x axis by an angle θ ($\hat{r}_{12} \cdot \hat{x} = \cos\theta$). And they move relative to the inertial frame S with a constant velocity $\vec{v}_1 = \vec{v}_2 = V\hat{x}$, so that $\vec{a}_1 = \vec{a}_2 = 0$.

According to Weber's expression their forces are given by

$$\vec{F}_{21}^{W} = \frac{q_1 q_2}{4\pi\varepsilon_o}\frac{\hat{r}_{12}}{r_{12}^2} = -\vec{F}_{12}^{W} \ . \tag{6.19}$$

On the other hand Liénard-Schwarzschild's forces (6.8) and (6.9) yield, with $V << c$ and $\hat{r}_{12} = \hat{x}\cos\theta + \hat{y}\sin\theta$:

$$\vec{F}_{21}^{LS} = q_1\left[\frac{q_2}{4\pi\varepsilon_o}\frac{\hat{r}_{12}}{r_{12}^2}\left(1 + \frac{V^2}{2c^2} - \frac{3}{2}\frac{V^2\cos^2\theta}{c^2}\right)\right] + q_1 V\hat{x} \times \left(\frac{q_2}{4\pi\varepsilon_o}\frac{1}{r_{12}^2}\frac{V\sin\theta\hat{z}}{c^2}\right)$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_o}\left[\frac{\hat{r}_{12}}{r_{12}^2}\left(1+\frac{V^2}{2c^2}-\frac{3}{2}\frac{V^2\cos^2\theta}{c^2}\right)-\frac{\hat{y}}{r_{12}^2}\frac{V^2\sin\theta}{c^2}\right] = -\vec{F}_{12}^{LS} . \quad (6.20)$$

Once more Weber's force yields a different result compared to Liénard-Schwarzschild's one. What is even more interesting now is that according to Liénard-Schwarzschild's result (6.20) there should exist an electromagnetic torque $\vec{T}$ acting on this system of two charges due to internal forces! This is easily seen calculating the torque with (6.20) which yields

$$\vec{T} = \vec{r}_1 \times \vec{F}_{21} + \vec{r}_2 \times \vec{F}_{12} = \vec{r}_{12} \times \vec{F}_{21}$$

$$= -\frac{q_1 q_2}{4\pi\varepsilon_o}\frac{V^2\sin\theta}{c^2}\frac{\hat{r}_{12}\times\hat{y}}{r_{12}} = -\frac{q_1 q_2}{4\pi\varepsilon_o}\frac{V^2\sin\theta\cos\theta}{c^2}\frac{\hat{z}}{r_{12}} . \quad (6.21)$$

This is a very strange result as the situation of Figure 6.4 is the same as that of Figure 6.1 only seen in an inertial frame S' which is inclined relative to S (of Figure 6.1) by an angle θ and moving relative to it with a constant velocity $\vec{V} = -V(\hat{x}\cos\theta + \hat{y}\sin\theta)$, Figure 6.5.

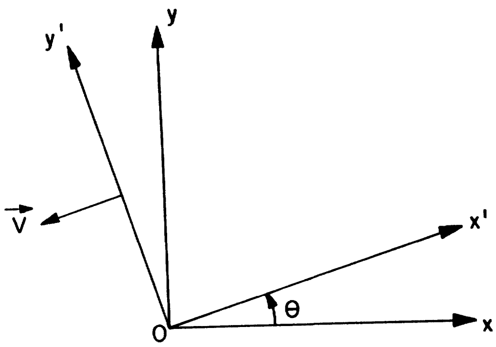

Figure 6.5

As there is no torque in the situation of Figure 6.1 there should be no torque also in the situation of Figure 6.4. And yet Liénard-Schwarzschild's force predicts this torque in the second situation but not in the first. The existence of this torque in classical electromagnetism can also be easily seen utilizing Lorentz's force (2.41).

Another way of visualizing this problem is represented in Figure 6.6 (this problem was first presented to us in this way by F. M. Peixoto):

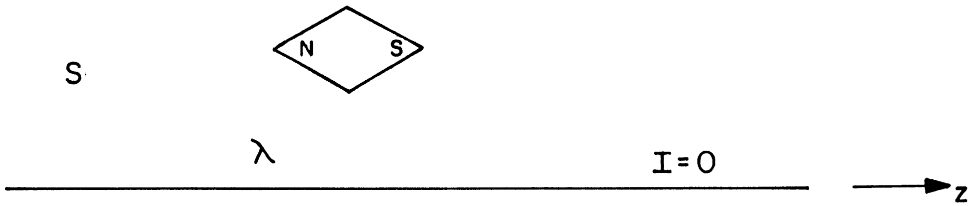

Figure 6.6

Here an infinite straight wire is charged uniformly with a linear charge density λ. Aligned with this wire there is a small magnetized needle NS. Everything is at rest in this inertial frame (the needle, the wire and the charges in the wire). Obviously there is no torque in the needle due to this stationary charged wire. Let us now analyse this situation in another inertial frame of reference S' which moves relative to the needle with a constant velocity $-V\hat{z}$, where the z axis is chosen parallel to the wire. In this frame S' the needle and the charged wire are moving to the right with a constant velocity V, Figure 6.7.

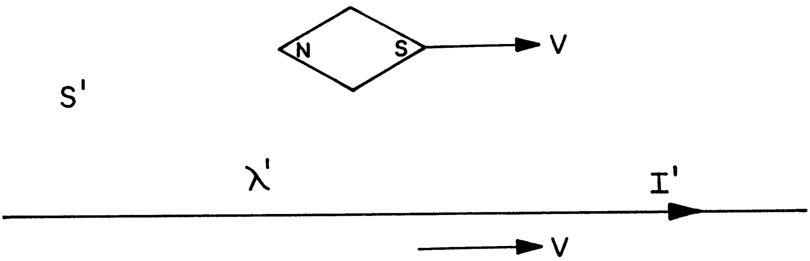

Figure 6.7

In this inertial frame S' the moving charged wire generates, according to classical electromagnetism, not only a radial electric field but also a poloidal magnetic field. According to classical electromagnetism this poloidal magnetic field will exert a torque on the magnetized needle to let it orthogonal to the wire and to the line connecting them (after all this is somewhat similar to Oersted's fundamental experimental discovery of 1820). If this happened the magnetized needle would become orthogonal to the wire and parallel to the poloidal magnetic field. But obviously the needle can not stay parallel to the wire in one frame of reference and orthogonal to it in another frame because both situations are the same only seen in different frames. To avoid this paradoxical situation people utilizing the Lorentz's force need to find an opposite torque to cancel this one which appears in the frame S'. This is known as the missing torque problem. For references and discussions see: (Bedford and Krumm, 1986), (Namias, 1989) and (Spavieri, 1990).

Obviously none of these problems and paradoxes arise in Weber's electrodynamics. If there is no torque in one frame there will be no torque in any other frame due to the relational character of Weber's force.

E) Charges in orthogonal uniform rectilinear motion

Here we analyse a different physical situation. We have two charges q_1 and q_2 moving at a constant speed relative to the inertial frame S (this can be obtained by external forces or constraints), $\vec{a}_1 = \vec{a}_2 = 0$, so that at time t they are found in the situation of Figure 6.8:

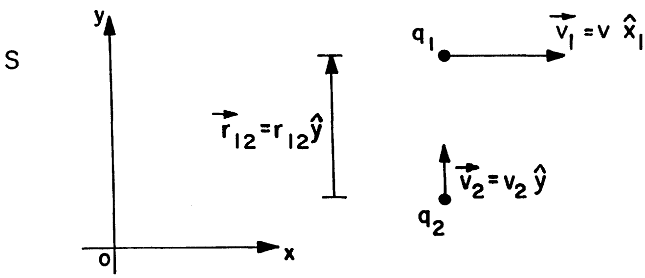

Figure 6.8

Weber's force (6.13) yields in this case

$$\vec{F}_{21}^W = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{y}}{r_{12}^2}\left[1+\frac{1}{c^2}\left(v_1^2-\frac{v_2^2}{2}\right)\right] = -\vec{F}_{12}^W \ . \tag{6.22}$$

On the other hand Liénard-Schwarzschild's force (6.8) and (6.9) yield

$$\vec{F}_{21}^{LS} = \frac{q_1 q_2}{4\pi\varepsilon_0}\frac{\hat{y}}{r_{12}^2}\left(1-\frac{v_2^2}{c^2}\right) \ , \tag{6.23}$$

$$\vec{F}_{12}^{LS} = -\frac{q_1 q_2}{4\pi\varepsilon_0}\frac{1}{r_{12}^2}\left[\left(1+\frac{v_1^2}{2c^2}\right)\hat{y}+\frac{v_1 v_2}{c^2}\hat{x}\right] \neq -\vec{F}_{21}^{LS} \ . \tag{6.24}$$

This example shows once more that Weber's law always satisfies the action and reaction principle in the strong form, while Liénard-Schwarzschild's force does not satisfy this principle even in the weak form in certain situations. This example of Figure 6.8 is the analog for charges of the situation of Figure 4.4 for neutral current elements.

6.6. Electric Field Due to a Stationary, Neutral and Constant Current

In this Section we discuss a specific difference between Weber and Lorentz's forces, which could be tested in the laboratory. This difference is related to the component of the force proportional to the square of the velocity of the sources. We first discussed this situation in (Assis, 1991 b).

Suppose an infinite straight wire at rest along the z axis. In this wire we have a stationary current I_2 which is constant in time and electrically neutral. Supposing that this wire is an ordinary metallic conductor, only the electrons will move. From all of this: $\vec{v}_{2+} = 0, \vec{a}_{2+} = 0, \vec{v}_{2-} \equiv v_{2-}\hat{z} = -V_D\hat{z}$, and $\vec{a}_{2-} = 0$ (we designate by the index 2 the charges of the wire, and $|\vec{v}_{2-}| = V_D$ is the drifting velocity of the electrons). All velocities and accelerations are relative to the inertial frame of reference in which the wire is stationary. As usually happens in the laboratory, we suppose this wire to be electrically neutral which means $\lambda_{2-} = -\lambda_{2+}$, where λ is the linear charge density. We can then calculate the force exerted by this wire on a charge q_1 situated at $\vec{r}_1$ with velocity $\vec{v}_1$ and acceleration $\vec{a}_1$, Figure 6.9:

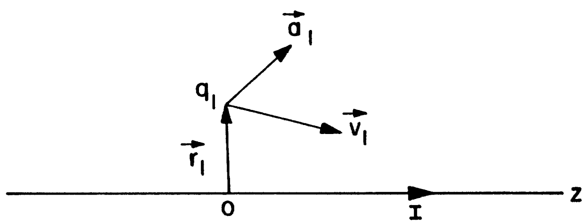

Figure 6.9

Let us calculate the force with Lorentz's force (2.41). The wire is electrically neutral so that $\nabla\phi = 0$. As the wire is at rest in this frame and the current is constant in time this is a stationary situation, so that $\partial\vec{A}/\partial t = 0$. According to (2.39) the electric field outside the wire is identically zero. So the net force on the test charge q_1 is then given by

$$\vec{F}^L = q_1\vec{v}_1 \times \vec{B}(\vec{r}_1) . \tag{6.25}$$

The magnetic field can be calculated by the direct integration of (4.76) or more easily by the magnetic circuital law (2.57). As this is a stationary situation, $d\Phi_E/dt = 0$. Choosing a circular path of integration of radius ρ centered on the wire yields, in cylindrical coordinates:

$$\vec{B} = \frac{\mu_o I}{2\pi\rho}\hat{\varphi} . \tag{6.26}$$

This is the usual poloidal magnetic field which decreases as $1/\rho$. Utilizing (1.45) and (6.26) in (6.25) yields

$$\vec{F}^L = -\frac{q_1\mu_o I}{2\pi\rho_1}\left[\hat{x}v_{1z}\cos\varphi_1 + \hat{y}v_{1z}\sin\varphi_1 - \hat{z}(v_{1x}\cos\varphi_1 + v_{1y}\sin\theta_1)\right] . \tag{6.27}$$

We now calculate the same expression from Liénard-Schwarzschild's force (6.8). We choose the coordinate system so that $z_1 = 0$ ($\vec{r}_1 = x_1\hat{x} + y_1\hat{y} = \vec{\rho}_1$) and with the z axis along the wire ($\vec{r}_{2+} = \vec{r}_{2-} = z_2\hat{z}$). With this choice and the previous consideration we have $\vec{r}_{12} = x_1\hat{x} + y_1\hat{y} - z_2\hat{z}$, $\vec{v}_1 = v_{1x}\hat{x} + v_{1y}\hat{y} + v_{1z}\hat{z}$, $\vec{v}_{2+} = 0$, $\vec{v}_{2-} = -V_D\hat{z}$, $\vec{a}_1 = \vec{a}_{1x}\hat{x} + \vec{a}_{1y}\hat{y} + \vec{a}_{1z}\hat{z}$, $\vec{a}_{2+} = \vec{a}_{2-} = 0$.

We also have $dq_{2\pm} = \pm\lambda_{2+}dz_2$, $I = \lambda_{2-}v_{2-} = \lambda_{2+}V_D$.

We apply these expressions in (6.8) and add the forces on q_1 due to dq_{2+} and dq_{2-}. Then we integrate from $z_2 = -l$ to $+l$. This yields

$$\vec{F}_{21} = -\frac{q_1\lambda_{2+}}{4\pi\varepsilon_o}\frac{V_D^2}{c^2}\frac{x_1\hat{x} + y_1\hat{y}}{x_1^2 + y_1^2}\left[\frac{l}{\sqrt{x_1^2 + y_1^2 + l^2}} - \frac{l^3}{(x_1^2 + y_1^2 + l^2)^{3/2}}\right]$$

$$+ q_1\vec{v}_1 \times \frac{\lambda_{2+}V_D}{2\pi\varepsilon_o}\frac{\hat{z} \times (x_1\hat{x} + y_1\hat{y})}{(x_1^2 + y_1^2)c^2}\frac{l}{\sqrt{x_1^2 + y_1^2 + l^2}} . \tag{6.28}$$

Taking the limit when l goes to infinity yields

$$\vec{F}_{21} = q_1\vec{v}_1 \times \frac{\mu_o I}{2\pi}\frac{-y_1\hat{x} + x_1\hat{y}}{x_1^2 + y_1^2} = q_1\vec{v}_1 \times \frac{\mu_o I\hat{\varphi}_1}{2\pi\rho_1} , \tag{6.29}$$

This final result obtained from a direct integration of Liénard-Schwarzschild's force (6.8) is then the same as the result obtained directly from Lorentz's force (6.25) and (6.26), as expected. Nevertheless it is interesting to perform this direct integration beginning with (6.8).

Let us now calculate the force on q_1 utilizing Weber's force (6.13). Utilizing the previous values of $\vec{r}_{12}$, $\vec{v}_{12}$, $\vec{a}_{12}$, $dq_{2\pm}$, adding the forces on q_1 due to dq_{2+} and dq_{2-}, and integrating from $z_2 = -l$ to l yields

$$\vec{F} = -\frac{q_1\lambda_{2+}}{2\pi\varepsilon_o c^2}\left\{\frac{(x_1\hat{x}+y_1\hat{y})}{(x_1^2+y_1^2)}\left[\frac{(2v_{1z}V_D+V_D^2)l}{\sqrt{x_1^2+y_1^2+l^2}} - \frac{v_{1z}V_D l^3}{(x_1^2+y_1^2+l^2)^{3/2}}\right.\right.$$

$$\left.\left.-\frac{1}{2}\frac{V_D^2 l^3}{(x_1^2+y_1^2+l^2)^{3/2}}\right] - \hat{z}\left[\frac{(x_1v_{1x}+y_1v_{1y})V_D}{(x_1^2+y_1^2)}\frac{l^3}{(x_1^2+y_1^2+l^2)^{3/2}}\right]\right\} . \qquad (6.30)$$

Taking the limit when l goes to infinity this can be written as, with $I = \lambda_{2+}V_D$:

$$\vec{F} = q_1\left[\frac{\mu_o I}{2\pi\rho_1}\left(\frac{x_1v_{1z}}{\sqrt{x_1^2+y_1^2}}\hat{x} + \frac{y_1v_{1z}}{\sqrt{x_1^2+y_1^2}}\hat{y} - \frac{x_1v_{1x}+y_1v_{1y}}{\sqrt{x_1^2+y_1^2}}\hat{z}\right)\right]$$

$$- q_1\left(\frac{\lambda_{2+}V_D^2}{4\pi\varepsilon_o c^2}\frac{x_1\hat{x}+y_1\hat{y}}{(x_1^2+y_1^2)}\right) . \qquad (6.31)$$

With (1.40) and (1.45) we can write

$$\vec{v}_1\times\hat{\varphi}_1 = \vec{v}_1\times\left(-\frac{y_1}{\rho_1}\hat{x}+\frac{x_1}{\rho_1}\hat{y}\right) = -\left(\frac{x_1v_{1z}}{\rho_1}\hat{x}+\frac{y_1v_{1z}}{\rho_1}\hat{y}-\frac{x_1v_{1x}+y_1v_{1y}}{\rho_1}\hat{z}\right) . \qquad (6.32)$$

This means that (6.31) can be written as (Wesley, 1987 a, 1990 a to c, 1991 Chapter 6), (Assis, 1991 b):

$$\vec{F} = q_1\vec{v}_1\times\vec{B} + q_1\vec{E}_M , \qquad (6.33)$$

with $\vec{B}$ given by (6.26) and $\vec{E}_M$ defined by

$$\vec{E}_M \equiv -\frac{\lambda_{2+}\hat{\rho}_1}{4\pi\varepsilon_o\rho_1^2}\frac{V_D^2}{c^2} = -|\vec{B}V_D|\hat{\rho}_1 . \qquad (6.34)$$

Let us now discuss this result. The first term on the right hand side of (6.33) has the same value of the force calculated from Lorentz's force, (6.25) and (6.26). It is interesting that although Weber's force (6.13) has no vectorial product in it, the final result of the force can be cast in the form (6.33). This happens also in other situations (Assis, 1989 b). The reason is that in the usual definitions of the magnetic field, (2.23) or (2.24), there is another vectorial product. By (1.7) we can see that a double vectorial product of three vectors is a real vector, although if $\vec{A}$ and $\vec{B}$ are vectors the procuct $\vec{A} \times \vec{B}$ is not a vector but a pseudo vector.

The only difference between Weber and Lorentz in this case is that Weber's expression predicts an additional force on q_1 given by $q_1 \vec{E}_M$. This force is independent of the velocity of q_1 and so $\vec{E}_M$ can be called an electric field. But this is not an ordinary electric field because the wire is supposed to be electrically neutral, and it only appears due to the fact that the electrons in the wire are moving while the positive ions are fixed in the lattice. This means that Weber's force on q_1 due to each one of these components (electrons and positive charges in the lattice) has a different value. As this field arises from the different motions of the source charges it is called a "motional electric field," and it is proportional to the square of the drifting velocity. So it is also proportional to the square of the current, and points always in the same direction irrespective of the current's direction.

Although this force $q_1 \vec{E}_M$ has no analogous one in classical electromagnetism, it is not easy to be tested experimentally because it is very small, of second order (proportional to V_D^2/c^2). For instance, for a current of $10^3 A$ and an electric charge q_1 typical of laboratory conditions, $q_1 \simeq 10^{-10} C$, this force is of the order $10^{-13} N$ for a separation $\rho_1 \simeq 10$ cm. This is extremely small and hard to detect.

Let us calculate now the force on a charge q_1 at rest relative to a circular filiform loop of radius a where flows a constant current I. The test charge q_1 is considered along the axis of the loop. For simplicity we put the loop in the xy plane centered on the origin, so that q_1 is in the z axis. To perform the integrals we utilize cylindrical coordinates, Figure 6.10.

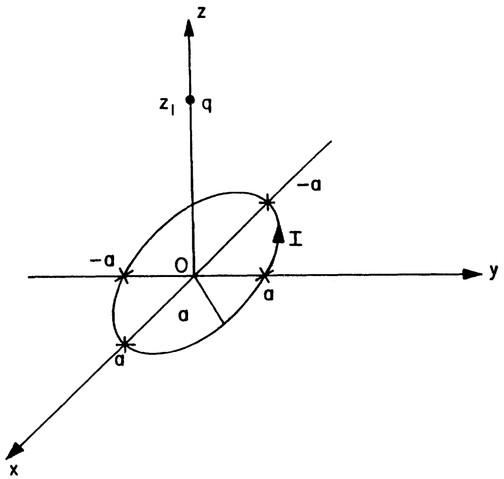

Figure 6.10

In this case we have $\vec{r}_{12} = -a\cos\varphi_2\hat{x} - a\sin\varphi_2\hat{y} + z_1\hat{z}$, $\vec{v}_1 = 0$, $\vec{a}_1 = 0$, $\vec{v}_{2+} = 0$, $\vec{a}_{2+} = 0$, $\vec{v}_{2-} = -V_D\hat{\varphi}_2 = V_D\sin\varphi_2\hat{x} - V_D\cos\varphi_2\hat{y}$, $\vec{a}_{2-} = -V_D^2\hat{\rho}_2/a = -(V_D^2\cos\varphi_2\hat{x} + V_D^2\sin\varphi_2\hat{y})/a$, $dq_{2\pm} = \pm\lambda_{2+}ad\varphi_2$, $I = \lambda_{2-}v_{2-} = \lambda_{2+}V_D$.

In this case it is easy to integrate directly (6.8) and (6.13). The final result is that Liénard-Schwarzschild and Weber's forces both predict a zero net force on q_1 anywhere along the axis of the loop.

Now we suppose the charge is along the x axis in the plane of the loop, Figure 6.11.

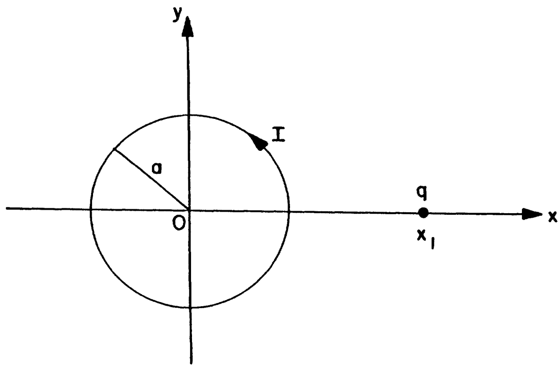

Figure 6.11

We have in this case $\vec{r}_{12} = (x_1 - a\cos\varphi_2)\hat{x} - a\sin\varphi_2\hat{y}$. The previous values of $\vec{v}_{12}$ and

$\vec{a}_{12}$ are still valid in this case.

Integrating (6.8) yields again a zero net force for any position of q_1 in the plane of the loop. O'Rahilly had discussed this case and the previous one and found a net force different from zero after integrating the Liénard-Schwarzschild's force in both cases (O'Rahilly, 1965, Vol. 2, pp. 588 - 590). In both cases he was mistaken in his calculations as he did not consider the centripetal acceleration of the electrons due to the curvature of the loop, $\vec{a}_{2-} = -V_D^2\hat{\rho}_2/a$.

When we calculate the force on q_1 with Weber's expression (6.13) we find (Assis, 1991 b):

$$\vec{F} \simeq q\frac{\mu_o I V_D \rho_1}{4a^2}\hat{\rho}_1 \ , \ \text{if } \rho_1 << a \ , \tag{6.35}$$

$$\vec{F} \simeq q\frac{\mu_o I V_D a}{8\rho_1^2}\hat{\rho}_1 \ , \ \text{if } \rho_1 >> a \ . \tag{6.36}$$

In these expressions $\vec{r}_1 = x_1\hat{x} + y_1\hat{y} = \rho_1\hat{\rho}_1$, which means that if $q_1 > 0$ the stationary test charge will be attracted by the loop as if it had become negatively charged. The order of magnitude of these motional electric fields is equivalent to what we had obtained previously for a straight wire, (6.34).

Edwards et al. presented an important proof that there is no electric field due to a closed loop *of arbitrary form* carrying a constant current I_2 according to Liénard-Schwarzschild's force (6.8). Let us reproduce here their proof (Edwards, Kenyon and Lemon, 1976).

The electric field $\vec{E}_2(\vec{r}_1)$ at the point $\vec{r}_1$ due to the source charges 2 is given according to (6.8) by

$$\vec{E}_2(\vec{r}_1) = \frac{1}{4\pi\varepsilon_o}\oint_{C_2} dq_2 \left[\frac{\hat{r}_{12}}{r_{12}^2} + \frac{v_2^2\hat{r}_{12}}{2r_{12}^2c^2} - \frac{3}{2}\frac{(\hat{r}_{12}\cdot\vec{v}_2)^2\hat{r}_{12}}{r_{12}^2c^2} - \frac{(\hat{r}_{12}\cdot\vec{a}_2)\hat{r}_{12}}{2r_{12}c^2} - \frac{\vec{a}_2}{2r_{12}c^2}\right] \ . \tag{6.37}$$

In this equation the line integral is over the closed circuit C_2. With a differential operator d_2 which varies only the source quantities we have

$$d_2\vec{r}_{12} = -d\vec{r}_2 = -\vec{v}_2 dt \ , \tag{6.38}$$

$$d_2\left(\frac{\vec{v}_2}{r_{12}}\right) = \left[\frac{\vec{a}_2}{r_{12}} + \frac{\vec{v}_2(\hat{r}_{12}\cdot\vec{v}_2)}{r_{12}^2}\right] dt \ , \tag{6.39}$$

$$d_2\left[\frac{(\hat{r}_{12}\cdot\vec{v}_2)\hat{r}_{12}}{r_{12}}\right] = \left[\frac{(\hat{r}_{12}\cdot\vec{a}_2)\hat{r}_{12}}{r_{12}} + \frac{3(\hat{r}_{12}\cdot\vec{v}_2)^2\hat{r}_{12}}{r_{12}^2} - \frac{(\hat{r}_{12}\cdot\vec{v}_2)\vec{v}_2}{r_{12}^2} - \frac{v_2^2\hat{r}_{12}}{r_{12}^2}\right] dt \ . \tag{6.40}$$

We have $I_2 = dq_2/dt$. We suppose that the current is uniform along the loop. With (6.38) to (6.40) we can then write (6.37) in the form

$$\vec{E}_2(\vec{r}_1) = \frac{1}{4\pi\varepsilon_o}\oint_{C_2} dq_2\frac{\hat{r}_{12}}{r_{12}^2} - \frac{I_2}{8\pi\varepsilon_o c^2}\oint_{C_2} d_2\left[\frac{(\hat{r}_{12}\cdot\vec{v}_2)\hat{r}_{12}}{r_{12}} + \frac{\vec{v}_2}{r_{12}}\right] \ . \tag{6.41}$$

This last integral goes to zero when integrated around any closed loop because the integrand is an exact differential. If the conductor is electrically neutral at all points so that the Coulomb term is zero then $\vec{E}_2(\vec{r}_1) = 0$. And this completes the proof.

So we have a distinguishing feature between theories which predict a force on a stationary charge due to a stationary and neutral conductor carrying a constant current, and theories which do not predict this force. In the first class we have, for instance, the forces of Weber, Riemann and Gauss. All these theories predict a second-order force of the order V_D^2/c^2 where usually we have $V_D^2/c^2 \simeq 10^{-20}$, which is a very small effect. In the second class we have, for instance, the forces of Lorentz (or Liénard-Schwarzschild) and Clausius. A good discussion of this topic has been given by (O'Rahilly, 1965, Vol. 2, pp. 288 - 290), (Edwards, Kenyon and Lemon, 1976), (Whittaker, 1973, Vol. 1, pp. 205 - 206 and 234 - 236), (Bush, 1926).

The question naturally is to know if this force exists or not. Historically the prediction of this force of a current carrying conductor on a stationary charge has been considered a flaw of Weber's electrodynamics, as people thought no such force could exist. Maxwell, for instance, said that "such an action [force] has never been observed" (Maxwell, 1965, Vol. 2, article [848], p. 482). Despite this statement he did not quote a single experiment where they tried to find this force. Another example: Pearson and Kilambi, in a very interesting paper where they show that Weber's force constitutes a far better classical analog of the velocity-dependent forces of nuclear physics than Liénard-Schwarzschild's force, have a section in their paper entitled "Invalidity of Weber's electrodynamics." In this section they discuss the motional electric field and conclude: "The fact that F_y^W [Weber's force exerted on an isolated stationary charge q by a long straight wire carrying a current I] is non-vanishing means that Weber's electrodynamics give rise to spurious induction effects. This is probably the most obvious defect of the theory, and the only way of avoiding it is to suppose that the positive charges in the wire move with an equal velocity v_2 [drift velocity of the electrons] in the opposite direction, which of course they do not" (Pearson and Kilambi, 1974). Once more they did not mention a single experiment which tried to detect these "spurious" induction effects.

However no experiments were performed until the 1970's trying to measure this effect. As we have seen the force even if it exists is very small, of the order of 10^{-13} N for typical currents. So it is very difficult to detect.

The best experiment known to us devised to detect such an effect is due to Edwards et al. (Edwards, Kenyon and Lemon, 1976). They measured a potential difference associated to this motional electric field and found a value compatible to the order of magnitude predicted by Weber's electrodynamics. Moreover, they concluded that the field was radial and pointing to the current, irrespective of the direction of the current, and found it proportional to the square of the current. Despite all these positive evidences in agreement with Weber's electrodynamics the experiment cannot be said to be decisive and more experimental researches are necessary before reaching a conclusion.

There are some recent articles which appeared in the literature dealing with this stimulating subject. Some experimental papers: (Bartlett and Ward, 1977), (Sansbury, 1985), (Bartlett and Maglic, 1990), (Kenyon and Edwards, 1991), (Lemon, Edwards and Kenyon, 1992). Some theoretical works: (Wesley, 1987 a, 1990 c, 1991 Chapter 6), (Bonnet, 1981), (Curé, 1982), (Gray, 1988, pp. 1-4 and 1-5), (Hayden, 1990), (Bartlett and Edwards, 1990), (Ivezić, 1990 and 1991), (Assis, 1991 b), (Bilić, 1992), (Singal, 1992), (Strel'tsov, 1992).

6.7. Weber's Law and Mass Variation

Another component of Weber's force which has no analogous one in classical electromagnetism is that which depends on the square of the velocity of the test charge, but which does not depend on the velocity of the source charges. Here we present a specific situation where this component appears explicitly.

The situation discussed here has been analysed utilizing Weber's electrodynamics and similar forces by many authors. For instance: (Bush, 1926), (Moon and Spencer, 1955), (O'Rahilly, 1965, Vol. 2, pp. 613 - 622), (Assis, 1989 b), (Wesley, 1990 c and d, 1991 Chapter 6), (Assis and Caluzi, 1991).

We suppose a capacitor of parallel plates of linear dimensions L much larger than the separation d of the plates, $d << L$, so that the calculations can be performed using infinite plates. Supposing the plates situated on the planes $z = z_0$ and $z = -z_0$, with surface charge densities σ_A and $-\sigma_A$, respectively, the classical electric field inside the capacitor is given by (as is easily calculated utilizing Gauss's law):

$$\vec{E}_c = -\frac{\sigma_A}{\varepsilon_0}\hat{z} \ , \tag{6.42}$$

where $\hat{z}$ is an unit vector pointing from the negative to the positive plate:

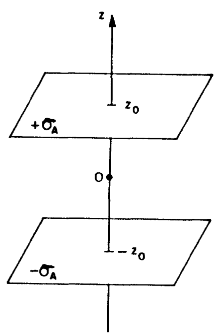

Figure 6.12

As the capacitor does not generate any magnetic field and this is a static situation, the only force which can act on q_1 inside the capacitor according to Lorentz's force law is given by $q_1\vec{E}_c$, no matter how fast q_1 is moving.

Outside the ideal capacitor there are no electric or magnetic fields classically. So the net force on q_1 according to Lorentz's or Liénard-Schwarzschild's forces are given by:

$$\vec{F} = -q_1\frac{\sigma_A}{\varepsilon_o}\hat{z} \ , \text{ if } \ -z_o < z_1 < z_o \ , \tag{6.43}$$

$$\vec{F} = 0 \ , \text{ if } z_1 < -z_o \text{ or } z_1 > z_o \ . \tag{6.44}$$

Let us now calculate the force on q_1 utilizing Weber's electrodynamics. We choose a coordinate system so that at the time t the charge q_1 is located along the z axis but moving in any direction relative to the plates of the capacitor. We then have, utilizing cylindrical coordinates (the indexes $2+$ or $2-$ indicate the positive or negative plates): $\vec{r}_{12\pm} = -\rho_2\cos\varphi_2\hat{x} - \rho_2\sin\varphi_2\hat{y} + (z_1 \mp z_o)\hat{z}$, $\vec{v}_1 = v_{1x}\hat{x} + v_{1y}\hat{y} + v_{1z}\hat{z}$, $\vec{a}_1 = a_{1x}\hat{x} + a_{1y}\hat{y} + a_{1z}\hat{z}$, $\vec{v}_{2+} = \vec{v}_{2-} = 0$, $\vec{a}_{2+} = \vec{a}_{2-} = 0$, $dq_{2\pm} = \pm\sigma_A\rho_2 d\varphi_2 d\rho_2$.

We then add the forces of dq_{2+} and dq_{2-} on q_1 by (6.13) and integrate from $\varphi_2 = 0$ to 2π and $\rho_2 = 0$ to R. It is easier to integrate first in φ_2 then in ρ_2. We then take the limit when $R >> z_o$, taking care with the value of $\sqrt{(z_1 \pm z_o)^2}$ above, inside and below the capacitor. This yields (Assis, 1989 b; Assis and Caluzi, 1991):

$$\vec{F}(z_1 > z_o) = \frac{q_1\sigma_A z_o}{\varepsilon_o c^2}(\vec{a}_1 - 2a_{1z}\hat{z}) \ , \tag{6.45}$$

$$\vec{F}(-z_o < z_1 < z_o) = -q_1\frac{\sigma_A}{\varepsilon_o}\left[\hat{z} + \frac{v_1^2}{2c^2}\hat{z} - \frac{v_{1z}(v_{1x}\hat{x} + v_{1y}\hat{y})}{c^2} - \frac{z_1\vec{a}_1}{c^2} + \frac{2z_1 a_{1z}}{c^2}\hat{z}\right] \ , \tag{6.46}$$

$$\vec{F}(z_1 < -z_o) = -\frac{q_1\sigma_A z_o}{\varepsilon_o c^2}(\vec{a}_1 - 2a_{1z}\hat{z}) \ . \tag{6.47}$$

There are many differences between (6.45) to (6.47) and (6.43) to (6.44). First of all when q_1 is outside the capacitor. Even in this case the capacitor will exert a force on q_1 whenever q_1 is accelerated relative to the plates by other forces, according to Weber's expression. This will be discussed in the next Chapter.

Now when the test charge is inside the capacitor. In this case (6.46) predicts forces on the test charge which depend on its velocity and acceleration relative to the capacitor. There are no such components in (6.43). After all the Liénard-Schwarzschild force (6.8) has no components which depend on the acceleration or on the square of the velocity of the test charge.

Once more it is difficult to know if these extra terms in (6.46) exist or not. A possible relevance of these extra terms is connected with the experiments of Kaufmann and Bucherer which were realized in the beginning of this century. Nowadays it is generally accepted that these experiments confirm the variation of mass with velocity according to the formula

$$m = \frac{m_o}{\sqrt{1 - v^2/c^2}} , \tag{6.48}$$

where m_o is the constant rest mass of the particle and $\vec{v}$ its velocity relative to an inertial frame. It is curious to observe, however, that when these experiments were carried out they were considered as disproving this relation! For interesting discussions see (Cushing, 1981) and (Miller, 1981, pp. 345 - 352 and 418).

These experiments involved crossed electric and magnetic fields. We already calculated the force on a test charge in a region of uniform electric field, (6.42) to (6.47). We now need to calculate the force in a region of uniform magnetic field. In classical electromagnetism we generate a uniform magnetic field when there is a constant poloidal current in a straight cylinder of circular cross section. In this case classically there will be a uniform magnetic field inside the cylinder and no magnetic field outside. The value of the internal magnetic field is easily calculated utilizing the magnetic circuital law (2.57) in this steady situation ($d\Phi_E/dt = 0$).

To calculate Weber's force in this case we suppose an infinitely long cylindric shell of radius ρ, composed of surface charge densities $\pm\sigma_B$. The positive one, σ_B, is at rest in the laboratory (positive ions fixed in the lattice). The negative one, $-\sigma_B$, circulates uniformly around the axis of the cylinder, which we call the z axis, with constant velocity $-\rho\omega\hat{\varphi}$, where $\hat{\varphi}$ is the unit azimuthal vector, Figure 6.13.

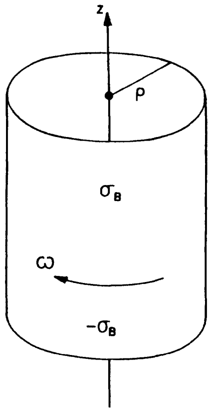

Figure 6.13

According to classical electromagnetism this distribution of charges generates no electric field but only a uniform magnetic field inside the cylinder with a value given by

$$\vec{B} = \mu_o \rho \omega \sigma_B \hat{z} = \mu_o n I \hat{z} \equiv B \hat{z} \ . \tag{6.49}$$

In this expression we substituted $\rho\omega\sigma_B$ for nI, because Figure 6.13 is analogous to the case of an ideal solenoid with n turns per unit length, where a current I flows in each turn (if there are N turns or coils in a length L then $n = N/L$).

With Lorentz's force we get

$$\vec{F} = q_1 \vec{v}_1 \times \vec{B} = q_1 B (v_{1y} \hat{x} - v_{1x} \hat{y}) \ . \tag{6.50}$$

We now calculate the force with Weber's expression (6.13). We now have $dq_{2\pm} = \pm\sigma_B \rho d\varphi_2 dz_2$, $\vec{r}_2 = \rho\cos\varphi_2\hat{x} + \rho\sin\varphi_2\hat{y} + z_2\hat{z}$, $\vec{v}_{2+} = 0$, $\vec{a}_{2+} = 0$, $\vec{v}_{2-} = -\rho\omega\hat{\varphi}_2 = \rho\omega\sin\varphi_2\hat{x} - \rho\omega\cos\varphi_2\hat{y}$, $\vec{a}_{2-} = -\rho\omega^2\hat{\rho}_2 = -\rho\omega^2\cos\varphi_2\hat{x} - \rho\omega^2\sin\varphi_2\hat{y}$. We suppose q_1 at the time t on the axis of the cylinder in the xy plane, $\vec{r}_1 = z_1\hat{z}$, but with any velocity and acceleration relative to the laboratory or to the positive charges σ_B: $\vec{v}_1 = v_{1x}\hat{x} + v_{1y}\hat{y} + v_{1z}\hat{z}$, $\vec{a}_1 = a_{1x}\hat{x} + a_{1y}\hat{y} + a_{1z}\hat{z}$. Adding the forces of dq_{2+} and dq_{2-} on q_1 and integrating from $\varphi_2 = 0$ to 2π and from $z_2 = -\infty$ to ∞ we get (Assis, 1989 b):

$$\vec{F} = q_1 \mu_o \rho \omega \sigma_B (v_{1y}\hat{x} - v_{1x}\hat{y}) \ . \tag{6.51}$$

With (6.49) and (6.50) this can be written as $q_1\vec{v}_1 \times \vec{B}$. Once more we see that Weber's electrodynamics can reproduce the magnetic component of Lorentz's force, although it has no vectorial product in it.

We next turn our attention to the Kaufmann and Bucherer's experiments. A simplified diagram can be found in (Rosser, 1964, p. 193), and is represented in Figure 6.14:

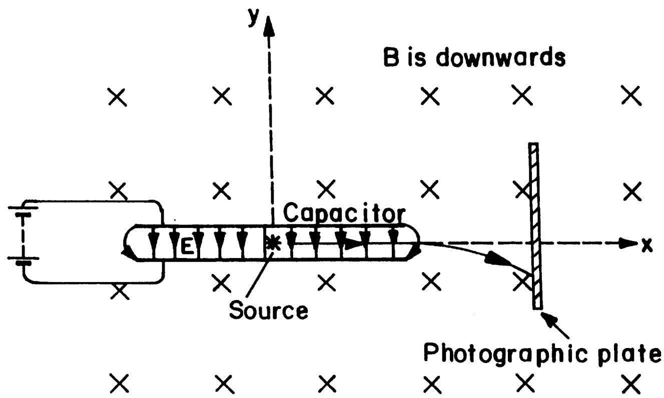

Figure 6.14

A source of electrons (β-rays emitted from a radium source) is located in the center of a large capacitor (diameters of the plates much larger than their separation). It generates inside itself (classically) a uniform electric field in the negative y direction,

$$\vec{E}_c = -E\hat{y} = -\frac{\sigma_A}{\varepsilon_o}\hat{y} \ . \tag{6.52}$$

Orthogonal to it there is a uniform magnetic field in the negative z direction,

$$\vec{B} = -B\hat{z} \ . \tag{6.53}$$

As the separation between the plates is small compared with its size, the capacitor acts as a velocity selector. The only electrons which can leave the capacitor are the ones on which a zero resultant force acts in the y direction and which also have a zero initial velocity in this direction, otherwise they would collide with the plates. Equating the y component of the Lorentz's force to zero yields the velocity of the electrons which left the capacitor, namely:

$$v_{1x} = \frac{E_c}{B} = \frac{\sigma_A}{\varepsilon_o B} \ . \tag{6.54}$$

After they left the capacitor, the electrons are only under the influence of the magnetic field. They follow then a circular path with a Larmor's radius given by

$$|-e\vec{v}_1 \times \vec{B}| = m_1|\vec{a}_1| = m_1\frac{v_1^2}{r} \ , \tag{6.55}$$

or

$$r = \frac{m_1 v_1}{eB} \ , \tag{6.56}$$

where $q_1 = -e$ is the electron's charge and m_1 its mass. Applying (6.54) yields

$$r = \frac{m_1 \sigma_A}{\varepsilon_o e B^2} \ . \tag{6.57}$$

This expression with m_1 being the rest mass of the electron ($m_1 = 9.1 \times 10^{-31}\ kg$) does not fit the experiments of Kaufmann and Bucherer. However, utilizing (6.48) in (6.57) and expanding the square root in powers of v_1/c yields the relativistic radius r_R. With (6.54) it is given by

$$r_R = \frac{m_o \sigma_A}{\varepsilon_o e B^2}\left[1 + \frac{1}{2}\left(\frac{\sigma_A}{\varepsilon_o c B}\right)^2 + \frac{3}{8}\left(\frac{\sigma_A}{\varepsilon_o c B}\right)^4 + ...\right] \ . \tag{6.58}$$

This expression agrees with the experimental results.

Let us now analyse this problem with Weber's electrodynamics. As the source of electrons in their experiment was in the middle of the plates of the capacitor, $\vec{r}_1 = 0$.

From (6.46), (6.50) and the requirement that no noticeable force acts along the y direction yields

$$e\frac{\sigma_A}{\varepsilon_o}\left(1+\frac{v_{1x}^{W2}}{2c^2}\right) - eBv_{1x}^W = 0 \ , \tag{6.59}$$

or

$$v_{1x}^W = \frac{\varepsilon_o c^2 B}{\sigma_A}\left(1-\sqrt{1-\frac{2\sigma_A^2}{\varepsilon_o^2 c^2 B^2}}\right) . \tag{6.60}$$

Outside the capacitor we have only (6.50), (6.55) and (6.56), neglecting (6.45) and (6.47). Applying (6.60) in (6.56) and expanding the square root yields the Weberian radius r_W, namely (Wesley, 1990 c and d; Assis and Caluzi, 1991):

$$r_W = \frac{m_1\sigma_A}{\varepsilon_o e B^2}\left[1+\frac{1}{2}\left(\frac{\sigma_A}{\varepsilon_o cB}\right)^2+\frac{1}{2}\left(\frac{\sigma_A}{\varepsilon_o cB}\right)^4+...\right] . \tag{6.61}$$

We now compare (6.58) with (6.61). Both expressions yield an infinite series in powers of $\sigma_A/\varepsilon_o cB = E_c/cB$, or v/c. They agree exactly until second order in v/c, inclusive. The difference is a small one in the fourth order, 3/8 instead of 1/2.

Nowadays we know that the precision of the experiments of Kaufmann and Bucherer was not beyond the second order in v/c, (Zahn and Spees, 1938; Faragó and Jánossy, 1957). So there are at least two equally valid explanations of these specific experiments: (A) The force inside the capacitor is given by (6.43) no matter how fast the test charge is moving relative to the plates, and mass changes with velocity according to (6.48); and (B) the force inside the capacitor is given by (6.46) and mass does not change with velocity. Both models are in quantitative agreement with the experimental findings. Obviously there may be models different from these two which will also fit the data.

Some other critical remarks: (A) In the previous analysis of both models we considered the charges in the capacitor to be fixed while the test charge is moving through it. In practice this is only an approximation as the test electrons should lose energy by inducing currents in the plates of the capacitor as they move through it. (B) We did not include as well the border effects when the electrons leave the capacitor. These effects may be large and should be taken into consideration in a complete analysis of the problem. This is not easy to do because the electric forces are different according to Lorentz and Weber, (6.43) compared with (6.46). So the effect due to the borders of the capacitor should be different in both models. (C) We utilized (6.51) in Weber's model but this is only exact when the test charge is along the axis of the cylinder. When the charge is not in this symmetrical location there are other components of Weber's force which were not taken into account here. (D) In the relativistic calculation we did not consider the loss of energy by the electrons when they are being accelerated outside the capacitor, in the form of electromagnetic radiation.

These remarks show that we must be careful when interpreting experiments and stating their conclusions and significance. Other experiments related to Weber's electrodynamics were discussed in (Assis, 1990 b), (Wesley, 1990 a to c).

6.8. Darwin's Lagrangian

In this Section we present two different approaches to derive Liénard-Schwarzschild's force (6.8) from a Lagrangian L given by

$$L = L_f - U \ . \tag{6.62}$$

In this expression L_f is the Lagrangian of free particles. It may be given by the classical kinetic energy T for N particles:

$$L_f = T = \sum_{i=1}^{N} m_i \frac{\vec{v}_i \cdot \vec{v}_i}{2} = \sum_{i=1}^{N} \frac{m_i v_i^2}{2} \ , \tag{6.63}$$

or by the relativistic expression, namely

$$L_f = -\sum_{i=1}^{N} m_{oi} c^2 \sqrt{1 - \frac{v_i^2}{c^2}} \ , \tag{6.64}$$

where m_{oi} is the rest mass of particle i and $\vec{v}_i$ its velocity relative to an inertial frame of reference.

The function U represents electromagnetic interactions. The force can be obtained by the standard Lagrangian formulation. For instance, the x component of the force acting on particle 1 is given by

$$F_x = -\frac{\partial U}{\partial x_1} + \frac{d}{dt} \frac{\partial U}{\partial v_{x1}} \ . \tag{6.65}$$

There are two functions U which generate Liénard-Schwarzschild's force from this procedure. Both of them can be written as

$$U_{1,2*} = q_1 \varphi_{2*}(\vec{r}_1) - q_1 \vec{v}_1 \cdot \vec{A}_{2*}(\vec{r}_1) \ . \tag{6.66}$$

In this expression $\varphi_{2*}(\vec{r}_1)$ is the scalar electric potential at the point $\vec{r}_1$ due to the sources 2 calculated at the retarded time $t - r_{12}/c$. Similarly for $\vec{A}_{2*}$, the magnetic vector potential.

Now let us suppose that this external field is due to a point charge q_2. We perform a series expansion of φ_{2*} and $\vec{A}_{2*}$ around the present time time t. We go only up to second order in $1/c$, inclusive.

The first expression for U is presented in (O'Rahilly, 1965, Vol. 1, pp. 220 - 221), namely

$$U_{12} = q_1 \left\{ \frac{q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left[1 + \frac{\vec{v}_2 \cdot \vec{v}_2}{2c^2} - \frac{(\hat{r}_{12} \cdot \vec{v}_2)^2}{2c^2} - \frac{\vec{r}_{12} \cdot \vec{a}_2}{2c^2} \right] \right\} - q_1 \vec{v}_1 \cdot \left(\frac{\mu_o}{4\pi} \frac{q_2 \vec{v}_2}{r_{12}} \right) . \quad (6.67)$$

The second expression for U was first given by (Darwin, 1920):

$$U_{12}^D = q_1 \left(\frac{q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \right) - q_1 \vec{v}_1 \cdot \left\{ \frac{\mu_o}{4\pi} \frac{q_2}{2r_{12}} [\vec{v}_2 + \hat{r}_{12}(\vec{v}_2 \cdot \hat{r}_{12})] \right\} . \quad (6.68)$$

All quantities ($\vec{r}_1$, $\vec{r}_2$, $\vec{v}_1$, $\vec{v}_2$, $\vec{a}_1$, $\vec{a}_2$, ...) which appear in these two last expressions for U are calculated at the present time t and not at the retarded time $t - r_{12}/c$. Both yield Liénard-Schwarzschild's force (6.8) with the procedure (6.65), as is easily verified. However, the first one is not symmetric ($U_{12} \neq U_{21}$) and involves the acceleration of the test charge. So it is not completely certain that we can apply (6.65) to calculate the force, as in this case we might need also to include a term containing $\partial U/\partial a_{x1}$.

Darwin's potential is completely symmetric and can be written as

$$U_{12}^D = U_{21}^D = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left[1 - \frac{\vec{v}_1 \cdot \vec{v}_2 + (\vec{v}_1 \cdot \hat{r}_{12})(\vec{v}_2 \cdot \hat{r}_{12})}{2c^2} \right] . \quad (6.69)$$

It is relativistically correct up to second order in $1/c$, inclusive, and is much utilized nowadays (Batygin and Toptygin, 1964, pp. 150 - 151; Jackson, 1975, Section 12.7, pp. 593 - 595). As we have just seen, Liénard-Schwarzschild's force can be derived from it by the standard Lagrangian formulation. This shows that Liénard-Schwarzschild's force (6.8) involves not only retardation and radiation phenomena but is also relativistically correct up to second order in v/c. Once more this demonstrates that up to second order it is the complete expression for the force between two point charges in classical electromagnetism.

Eq. (6.69) may be compared with the Lagrangian of Weber's electrodynamics given in Chapter 3, which has a Lagrangian energy given by

$$U_{12}^W = U_{21}^W = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 + \frac{\dot{r}_{12}^2}{2c^2}\right)$$

$$= \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left[1 + \frac{(\vec{v}_1 \cdot \hat{r}_{12})^2 - 2(\vec{v}_1 \cdot \hat{r}_{12})(\vec{v}_2 \cdot \hat{r}_{12}) + (\vec{v}_2 \cdot \hat{r}_{12})^2}{2c^2}\right] , \tag{6.70}$$

Both are symmetric ($U_{12} = U_{21}$), yield Coulomb's potential in zeroth-order and do not depend on the acceleration of the charges. Eq. (6.69) depends only on the product of the two velocities, while (6.70) depends also on the square of each velocity.

The Hamiltonian H is given by

$$H = \left(\sum_{k=1}^{6N} \dot{q}_k \frac{\partial L}{\partial \dot{q}_k}\right) - L . \tag{6.71}$$

Applying this in (6.62) and (6.69) yields, for two particles

$$H = E_m + V^D , \tag{6.72}$$

where

$$V^D = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left[1 + \frac{\vec{v}_1 \cdot \vec{v}_2 + (\vec{v}_1 \cdot \hat{r}_{12})(\vec{v}_2 \cdot \hat{r}_{12})}{2c^2}\right] \neq U^D . \tag{6.73}$$

The difference between U^D and V^D is in the sign in front of $1/c^2$. The same had happened with Weber's Lagrangian formulation. In (6.72) E_m is the mechanical energy. Classically it is given by

$$E_m = T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} , \tag{6.74}$$

while relativistically it is given by

$$E_m = \frac{m_{o1} c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_{o2} c^2}{\sqrt{1 - v_2^2/c^2}} . \tag{6.75}$$

Chapter 7 / Important Topics Related to Weber's Law

7.1. Two Body Problem According to Weber's Law

(A) Radial Motion

In this Section we study the motion of two point charges interacting with one another through Weber's force.

Let us begin once more with Maxwell. After showing that Weber's law is consistent with the law of the conservation of energy, Maxwell says: "But Helmholtz, in his very powerful memoir on the 'Equations of Motion of Electricity in Conductors at Rest'†, while he shews that Weber's formula is not inconsistent with the principle of the conservation of energy, as regards only the work done during a complete cyclical operation, points out that it leads to the conclusion, that two electrified particles, which move according to Weber's law, may have at first finite velocities, and yet, while still at a finite distance from each other, they may acquire an infinite kinetic energy, and may perform an infinite amount of work.

To this Weber* replies, that the initial relative velocity of the particles in Helmholtz's example, though finite, is greater than the velocity of light; and that the distance at which the kinetic energy becomes infinite, though finite, is smaller than any magnitude which we can perceive, so that it may be physically impossible to bring two molecules so near together. The example, therefore, cannot be tested by any experimental method" (Maxwell, 1954, Vol. 2, article [854], pp. 484 - 485).

This paper by Weber which Maxwell mentioned has already been translated to English: (Weber, 1871).

We follow here our discussion of this topic to clarify these arguments: (Assis and Clemente, 1992).

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† *Crelle's Journal,* 72. pp. 57-129 (1870).

* *Elektr. Maasb. inbesondere über das Princip der Erhaltung der Energie.*

Two charges q_1 and q_2, of inertial masses m_1 and m_2, interact with one another through Weber's law. If they move with velocities $\vec{v}_1$ and $\vec{v}_2$ relative to an inertial frame S, the conserved energy E of the system is given classically by the sum of the kinetic and potential energies, T and U, namely:

$$E = T + U = m_1 \frac{\vec{v}_1 \cdot \vec{v}_1}{2} + m_2 \frac{\vec{v}_2 \cdot \vec{v}_2}{2} + \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 - \frac{\dot{r}_{12}^2}{2c^2} \right) , \tag{7.1}$$

With Weber's electrodynamics there is conservation of linear momentum when there are no external forces applied to the system, as in this case. It is easier to analyse this problem in the inertial frame of reference in which the center of mass is at rest. The position of the center of mass, $\vec{R}$, is defined by $\vec{R} \equiv (m_1\vec{r}_1 + m_2\vec{r}_2)/(m_1 + m_2)$, where $\vec{r}_1$ ($\vec{r}_2$) is the position of q_1 (q_2) at time t relative to S. The velocity of the center of mass, $\vec{V}$, is defined by $\vec{V} \equiv (m_1\vec{v}_1 + m_2\vec{v}_2)/(m_1 + m_2)$. The reduced mass μ is defined by $\mu \equiv m_1 m_2/(m_1 + m_2)$. In the center of mass rest frame, $\vec{V} = 0$. So the kinetic energy T will be given simply by $T = \mu v^2/2$, where $v \equiv |\vec{v}_{12}|$. In this particular example there is only radial motion so that $\vec{v}_{12}$ is parallel to $\hat{r}_{12}$. This means that $v^2 = \dot{r}_{12}^2$. Writing r and $\dot{r}$ instead of r_{12} and $\dot{r}_{12}$ to simplify the notation, we can write the conserved energy in this rest frame as

$$E = \frac{\mu \dot{r}^2}{2} + \frac{\alpha}{r} \left(1 - \frac{\dot{r}^2}{2c^2} \right) , \tag{7.2}$$

where $\alpha \equiv q_1 q_2/4\pi\varepsilon_o$.

Solving this equation yields

$$\frac{\dot{r}}{c} = \pm \sqrt{2 \frac{rE - \alpha}{r\mu c^2 - \alpha}} . \tag{7.3}$$

If $E = \mu c^2$ the two charges will approach or move away from one another at a constant relative speed $\dot{r} = \pm\sqrt{2}c$, for any r, as if they did not feel one another. This is a characteristic energy of the problem, which resembles the relativistic energy, but which arose naturally in Weber's electrodynamics much earlier.

We now define $r_o \equiv r(t = 0)$, $\dot{r}_o \equiv \dot{r}(t = 0)$, $A \equiv \pm\sqrt{2E/\mu}$, $r_1 \equiv \alpha/E$, $r_2 \equiv \alpha/\mu c^2$. We can then write

$$\dot{r} = A\sqrt{\frac{r - r_1}{r - r_2}} , \tag{7.4}$$

$$\ddot{r} = \frac{\alpha}{\mu}\frac{1 - r_2/r_1}{(r - r_2)^2} . \tag{7.5}$$

With $r_2 = 0$ we recover the classical problem of two charges interacting through Coulomb's force. So a characteristic distance where Weber's law brings a significant departure from Coulomb is $r \simeq r_2$.

An analysis of this equation for other values of the energy $E \neq \mu c^2$ is presented in Figure 7.1 (A) to (F), (Assis and Clemente, 1992). We are plotting $\dot{r}$ against r.

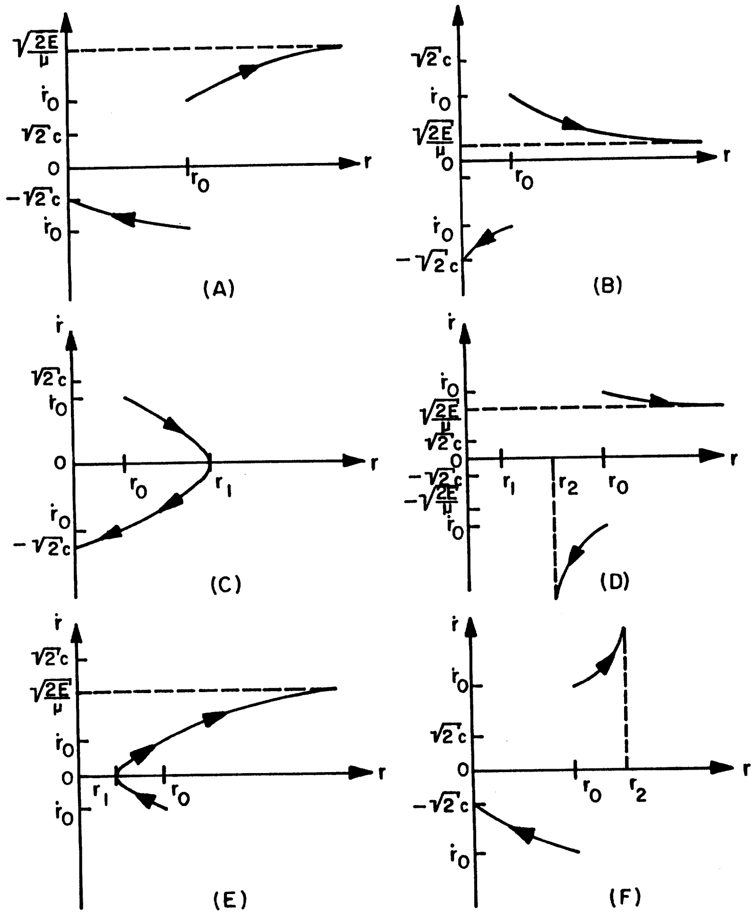

Figure 7.1. (A) $\alpha < 0$ and $E > \mu c^2$; (B) $\alpha < 0$ and $0 \leq E < \mu c^2$; (C) $\alpha < 0$, $E < 0$, $r_o \leq r_1$ and $\alpha > 0$, $E > \mu c^2$, $r_2 > r_1 \geq r_o$; (D) $\alpha > 0$, $E > \mu c^2$ and $r_o > r_2 > r_1$; (E) $\alpha > 0$, $0 < E < \mu c^2$, and $r_2 < r_1 < r_o$; (F) $\alpha > 0$, $0 < E < \mu c^2$, $r_o < r_2 < r_1$, and $\alpha > 0$, $E \leq 0$, $r_o < r_2$.

In Fig. 7.1 (A) to (C) we have attraction ($\alpha < 0$) and we can see that the relative radial velocity $\dot{r}$ is always smaller than $\sqrt{2}c$ or than $\sqrt{2E/\mu}$, no matter if the charges are initially approaching ($\dot{r}_o < 0$) or moving away ($\dot{r}_o > 0$) from one another. Fig. 7.1 (C) also represents the repulsive case $\alpha > 0,\ E > \mu c^2,\ r_2 > r_1 \geq r_o$.

In Fig. 7.1 (D) to (F) we have repulsion ($\alpha > 0$). In cases (D) and (F) we can see that the relative velocity can go to infinity at $r = r_2$, as the denominator of Eq. (7.4) goes to zero. As $r \geq 0$ this only happens for $r_2 > 0$ or $\alpha > 0$.

It is now easy to understand Weber's answer to Helmholtz criticism. In Fig. 7.1 (B), (C) and (E) we have $|\dot{r}_o| < \sqrt{2}c$. In all these cases $|\dot{r}|$ remains smaller or equal to $\sqrt{2}c$, no matter if it is attraction or repulsion, or if the charges are initially approaching or moving away from one another. On the other hand if $|\dot{r}_o| > \sqrt{2}c$, then $|\dot{r}|$ will always remain greater than $\sqrt{2}c$, as is represented in Fig. 7.1 (A), (D) and (F). The relative radial velocity $\dot{r}$ would only go to infinity in this model (cases (D) to (F)) if $|\dot{r}_o| > \sqrt{2}c$. Moreover, this would only happen when they were very close to one another. For instance, r_2 would be twice the classical radius of the electron ($r_2 = 2r_e$) if $q_1 = q_2 = \pm e$ and $m_1 = m_2 = m_e$. If $q_1 = q_2 = \pm e$, $m_1 = m_e$ and $m_2 \gg m_e$ (like a positron and a proton) then $r_2 = r_e$, where the classical radius of the electron r_e is defined by

$$r_e \equiv \frac{e^2}{4\pi\varepsilon_o}\frac{1}{mc^2} = 2.8 \times 10^{-15} m \ . \tag{7.6}$$

At the time of Helmholtz and Weber the electron had not yet been discovered, anyhow it is amazing that the distance characterized by the classical radius of the electron appears naturally in Weber's electrodynamics. Obviously the reason is that the constant c appeared here for the first time and when we couple it with the mass and charge of the electron a characteristic length may naturally be constructed playing with these constants.

We agree with Maxwell that this prediction of an infinite velocity can not be tested experimentally. The reason is not only due to this short distance but also because the initial relative velocity of the charges would need to be greater than the light velocity. And up to now we never succeeded in accelerating any particle to a velocity larger than c, so that we can not perform the experiment. Even if this were possible, at short distances like r_e other forces come into play, like nuclear forces, and this would change the predictions and outcome of the experiment.

Another remark: To arrive at these results we utilized not only Weber's potential energy but also the classical kinetic energy $mv^2/2$. Nowadays it seems that this last expression is only valid for small velocities compared to c. For $v \sim c$ we might need to utilize

$$T = \frac{mc^2}{\sqrt{1 - v^2/c^2}} - mc^2 \ , \tag{7.7}$$

or something similar. Once more this would change the predictions of this example. We will discuss this kinetic energy when analysing a Weber's law applied to gravitation and its connection to Mach's principle.

In (Assis and Clemente, 1992) we also studied the two body problem with Phipps's potential energy, namely (Phipps, 1990 b and c, 1992):

$$U = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 - \frac{\dot{r}_{12}^2}{c^2} \right)^{1/2} \ . \tag{7.8}$$

It reduces to Weber's potential energy in second order of $\dot{r}/c$. In this case the limiting relative speed $\dot{r}$ is given by c instead of $\sqrt{2}c$. Here we will not go into any details of this analysis.

Weber's reply to Helmholtz can be found in Section 21 of his paper of 1871. See especially pages 146 to 149 of the English translation (Weber, 1871).

(B) Arbitrary Motion

We now deal with the two body problem where there is angular rotation relative to an inertial frame or relative to the frame of distant galaxies. The classical analog is the case of central forces depending on the inverse square of the distance between the two bodies, from which Kepler's laws and Rutherford's differential scattering cross section are widely known results.

This problem with Weber's potential instead of the Newtonian or Coulombian one seems to have been first solved in terms of elliptic integrals by Seegers in 1864 (North, 1965, p. 46). Here we follow (Clemente and Assis, 1991).

As Weber's force is along the line connecting the two charges, the total angular momentum of the system will be conserved. Introducing in the plane of motion a polar coordinate system (r, φ), with origin at the center of mass, the conserved angular momentum L can be expressed in terms of the reduced mass μ as $L = \mu r^2 \dot{\varphi}$. As we already dealt with radial motion ($L = 0$), we now only consider the case $L \neq 0$. Instead of (7.2) we have in this case

$$\frac{\mu}{2}(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{\alpha}{r}\left(1 - \frac{\dot{r}^2}{2c^2}\right) = E \ . \qquad (7.9)$$

We define $x^2 \equiv 1 - r_2/r$, where $r_2 = \alpha/\mu c^2$. In order to keep $x^2 > 0$ we restrict the energy E to be smaller than $\mu c^2(1 + c^2L^2/2\alpha^2)$. With this definition we can write (7.9) as

$$\frac{dx}{d\varphi} = \pm\frac{1}{2x^2}\sqrt{(x_A^2 - x^2)(x^2 - x_B^2)} \ , \qquad (7.10)$$

where

$$x_{A,B}^2 = 1 \pm \frac{\mu r_2 \alpha}{L^2}\left[1 \pm \sqrt{1 + \frac{2EL^2}{\mu\alpha^2}}\right] \ . \qquad (7.11)$$

In (7.10) x_A and x_B represent possible turning points for x. If we assume that at least one of them exists, we need to have $E \geq -\mu\alpha^2/2L^2$.

In the attractive case, $\alpha < 0$, x_A^2 represents the radius of closest approach. Taking $\varphi = 0$ when $x^2 = x_A^2$ it is possible to find (Clemente and Assis, 1991):

$$\varphi^A = \pm 2|x_A|E(\phi,\ k) \ , \qquad (7.12)$$

where $E(\phi,\ k)$ is the incomplete elliptic integral of the second kind. Its argument ϕ ($0 \leq \phi \leq \pi/2$) and parameter k ($0 \leq k^2 \leq 1$) are given by

$$\phi = arcsin\sqrt{\frac{x_A^2 - x^2}{x_A^2 - x_B^2}} \ , \quad k = \sqrt{\frac{x_A^2 - x_B^2}{x_A^2}} \ . \qquad (7.13)$$

In the repulsive case, $\alpha > 0$, x_B^2 represents the radius of closest approach. Taking $\varphi = 0$ when $x^2 = x_B^2$ it is possible to find

$$\varphi^R = \pm 2|x_A|[E(k) - E(\phi,\ k)] \ , \tag{7.14}$$

where $E(k)$ is the complete elliptic integral of the second kind.

The Coulombian case can be recovered in these expressions by properly taking the limit $c \to \infty$. In this case $|x_A| \to 1$ and $k \to 0$ in such a way that $E(\phi,\ k) \to \phi$ and $E(k) \to \pi/2$.

Let us analyse first the limited trajectories. This occurs when the force is attractive and $E < 0$. The orbit will be comprised between the turning points characterized by x_A^2 and x_B^2, but it will not be a closed ellipse. After a complete turn the major axis of the ellipse precessed by an angle $\Delta\varphi = 4|x_A|E(k) - 2\pi$. For small r_2 this yields $\Delta\varphi \simeq \pi|r_2|/a(1-\varepsilon^2)$, where $r_{A,B} = a(1 \mp \varepsilon)$ are the perihelion and aphelion radii, a the semimajor axis and ε the eccentricity of the ellipse approximating the orbit.

For open trajectories we have $E \geq 0$. In the classical Rutherford scattering problem we would have the angle of deflection Φ^C given by $2arctan(S^2)^{1/2}$, where $S^2 = 4E^2s^2/\alpha^2$, s being the impact parameter such that $E = \mu v_o^2/2$ and $L = \mu v_o s$, v_o being the initial velocity (Symon, 1971, pp. 137 - 140). With Weber's potential the corresponding deflection angles for the attractive and repulsive cases, Φ^A and Φ^R, respectively, are given by (Clemente and Assis, 1991):

$$\Phi^A = \frac{4E(\phi^*,\ k)}{\sqrt{1 - k^2 \sin^2 \phi^2}} - \pi \ , \tag{7.15}$$

$$\Phi^R = \pi - 4\frac{E(k) - E(\phi^*,\ k)}{\sqrt{1 - k^2 \sin^2 \phi^*}} \ , \tag{7.16}$$

where $\sin^2 \phi^* = (x_A^2 - 1)/(x_A^2 - x_B^2)$.

In general $\Phi^A \neq \Phi^R$. This did not happen in the classical Rutherford scattering. Moreover, while $\Phi^R \leq \pi$, as is the case for Φ^C, Φ^A has no upper limit. The scattered charge might, for instance, give two turns around the attracting center before moving away from it. The difference $\Phi^A - \Phi^R$ is an increasing function of E or of v_o^2/c^2.

For a graphical analysis of these scattering angles and of the scattering differential cross sections with Weber's potential energy see (Clemente and Assis, 1991). Once more

Weber's law predicts a difference in the attractive and repulsive cross sections, which is an increasing function of v_o^2/c^2. These differences did not happen classically.

We are not aware of any experiment which tried to test these predictions.

Once more it should be remarked that this analysis employed the classical kinetic energy $mv^2/2$ which may not be applicable for velocities of the order of c. And again for short distances nuclear forces should also be included in the analysis.

For an analysis of this problem with Ritz's electrodynamics see (O'Rahilly, 1965, Vol. 2, pp. 536 - 545).

7.2. Motion of a Charge Orthogonal to the Plates of a Capacitor

In Section 6.7 we calculated the force inside and outside an ideal capacitor according to Weber's expression. We might as well obtain the energy integrating Weber's potential energy (3.25). While in Section 6.7 we analysed the motion of a charge parallel to the plates of the capacitor, here we want to study the motion orthogonal to the plates, or parallel to the electric field. The situation we analyse here is that of Figure 6.12. Following a procedure similar to that of Section 6.7 we obtain the energy of a test charge q_1 interacting with an ideal capacitor integrating (3.25), namely

$$U = -q_1 \frac{\sigma_A}{\varepsilon_o} z_o \left(1 - \frac{v_1^2 - 2v_{1z}^2}{2c^2}\right) , \text{ if } z_1 < -z_o , \tag{7.17}$$

$$U = q_1 \frac{\sigma_A}{\varepsilon_o} z \left(1 - \frac{v_1^2 - 2v_{1z}^2}{2c^2}\right) , \text{ if } -z_o \leq z_1 \leq z_o , \tag{7.18}$$

$$U = q_1 \frac{\sigma_A}{\varepsilon_o} z_o \left(1 - \frac{v_1^2 - 2v_{1z}^2}{2c^2}\right) , \text{ if } z_1 > z_o . \tag{7.19}$$

We define the zero of the potential as being at $z = 0$, the middle of the plates, and the voltage difference between the two plates by $V_o = 2\sigma_A z_o/\varepsilon_o$. This means that the potential $\phi(z)$ will be given by $\phi(z_1 \leq -z_o) = -V_o/2 = -\sigma_A z_o/\varepsilon_o$, $\phi(-z_o \leq z_1 \leq z_o) = V_o z/2z_o = \sigma_A z/\varepsilon_o$, $\phi(z_1 \geq z_o) = V_o/2 = \sigma_A z_o/\varepsilon_o$. Here we restrict the analysis only to motion orthogonal to the plates, namely, $\vec{v}_1 = v_{1z}\hat{z}$. With these definitions and restrictions, adding the kinetic energy $mv_{1z}^2/2$ to the previous results yields the total conserved energy E as:

$$E = \frac{mv_{1z}^2}{2} + q\phi\left(1 + \frac{v_{1z}^2}{2c^2}\right) = q\phi + \frac{(m + m_W)v_{1z}^2}{2} , \tag{7.20}$$

where $m_W \equiv q\phi/c^2$ is what we call Weber's inertial mass in this case. If we had chosen $\phi = 0$ for $z \leq -z_o$ then this last equation would also hold but with $m_W = q\phi/c^2 - q\sigma_A z_o/\varepsilon_o c^2$.

From this equation we can see that the charge will behave as if it had an effective inertial mass given by $m + q\phi/c^2$. This is equivalent to an inertial mass which depends on the electrostatic potential where the charge is located. Moreover, this effective inertial mass is anisotropic as it depends on the direction in which the charge is moving relative to the plates. This can be seen from (7.18) where the coefficient in front of v_{1x}^2 or v_{1y}^2 has an opposite sign than that in front of v_{1z}^2. This can also be seen in the coefficients in front of a_{1x} or a_{1y} with that of a_{1z} in (6.46).

This effective inertial mass depends not only on the direction of motion and value of the electrostatic potential but also on the geometry of the problem. For instance, if a charge q were moving inside a hollow charged spherical shell of radius R and charge Q, it would behave as if it had an effective inertial mass given by $m + q\phi/3c^2$, where $\phi = Q/4\pi\varepsilon_o R$ is the electrostatic potential of the shell relative to infinity (Assis, 1992 d; and Assis, 1993 a). And this is different from the previous result $m + q\phi/c^2$ due to the factor 1/3.

It should also be emphasized that this effective inertial mass is independent of the velocity of the test charge. This means that it is conceptually different from the relativistic inertial mass (6.48).

It is also completely different from the classical electromagnetic mass given by $m = q^2/6\pi\varepsilon_o ac^2$ (Feynman, 1964, Vol. 2, pp. 28-1 to 28-4). In this last expression we have a particle which has a charge q uniformly distributed over the surface of a sphere of radius a. This last expression is independent of the potential where the test charge is located and also of its velocity.

We do not know any experiment designed specifically to test the existence of this effective inertial mass which depends on the electrostatic potential where the charge is located. Recently we proposed some experiments to test this prediction for a charge inside a charged spherical shell, instead of inside a capacitor (Assis, 1993 a).

Let us return to our problem. A charge q_1 coming from $z < -z_o$ moves toward the capacitor along the z axis with the initial velocity $v_{1z} = v_i$ before entering the capacitor. It is accelerated between the plates and leaves the capacitor at $z = z_o$ with final velocity $v_{1z} = v_f$ which will remain constant after the capacitor if the charge does not interact with other bodies. Equating the total energy E before and after the capacitor with (7.20) yields (Assis and Caluzi, 1991):

$$v_f = \sqrt{\frac{(m + m_{Wi})v_i^2 - 2qV_o}{m + m_{Wf}}} . \tag{7.21}$$

In this expression m_{Wi} and m_{Wf} are the initial and final Weberian masses, respectively.

If the test charge is an electron ($q = -e$) and it entered the capacitor with $v_i \simeq 0$ this yields ($eV_o > 0$):

$$v_f = \sqrt{\frac{2eV_o}{m - eV_o/2c^2}} . \tag{7.22}$$

This shows that there is no limit for the final velocity of the accelerated electron. In particular $v_f \to \infty$ when the effective inertial mass goes to zero, namely, when $V_o = 2mc^2/e \simeq 10^6 V$.

In reality we know there is a limiting velocity c whenever we try to accelerate any charge. This voltage difference of $1MV$ has been obtained in laboratories and the electrons were never observed to move faster than light. In the previous Section we had obtained a limiting velocity for two charges interacting through Weber's law. Now we have seen that in a many body system (the test charge and the charges belonging to the capacitor) this is not valid anymore, although the interaction energy is still Weber's one.

As we never observed any electron moving faster than light after being accelerated in electrostatic accelerators we can conclude that the ensemble Newtonian mechanics ($T = mv^2/2$ or $\vec{F} = m\vec{a}$) plus Weber's electrodynamics is not valid for velocities near the light velocity. This example is stronger than the one presented by Helmholtz (see previous Section) because now the infinite velocity is obtained for a charge beginning at a very small velocity (instead of an initial velocity larger than c), and this infinite velocity is predicted to happen at macroscopic distances (the separation of the plates of the capacitor, instead of at the classical radius of the electron).

One way of overcoming this limitation is to modify Weber's potential energy and Weber's force. A proposal in this direction has been made by Phipps in the form of eq. (7.8), (Phipps, 1990 b and c, 1992).

Another alternative is to modify the classical kinetic energy. If we had (7.7) or something similar for the kinetic energy instead of $mv^2/2$, we would not obtain any divergence in the velocity of the test charge. In this way the limitation pointed out by (7.22) would not be connected to Weber's electrodynamics itself but to Newtonian mechanics. This modification of the classical kinetic energy might sound strange, like a mixing of Weber's potential energy and Einstein's special theory of relativity. However, when we analyse Weber's law applied to gravitation we will see that it is possible to **derive** an analogue to (7.7) from a relational law similar to Weber's one, as was first performed by Erwin Schrödinger (Schrödinger, 1925).

It is also possible that we need to modify both expressions ($mv^2/2$ and Weber's potential energy) for $v \simeq c$. Only further research will indicate the correct way to follow from now on.

Once more it should be remembered that we did not include the losses of energy in the form of radiation when the test charge is accelerated. We did not take into account as well the losses of energy of the test charge due to the currents it should induce in the plates of the capacitor as it moves through it. Border effects were also neglected in this analysis.

7.3. Charged Spherical Shell

One of the most important situations discussed in this book is that of a charged spherical shell interacting with a point charge. Let us suppose a spherical shell of radius R, made of a dielectric (non conducting material), charged uniformly with a net charge Q and spinning with an angular velocity $\vec{\omega}(t)$ relative to an inertial frame S. The center of the shell is located at the origin O of S. A point charge q is located at the time t at $\vec{r}$, and moves with velocity $\vec{v} = d\vec{r}/dt$ and acceleration $\vec{a} = d\vec{v}/dt = d^2\vec{r}/dt^2$ relative to the origin O of S, see Figure 7.2.

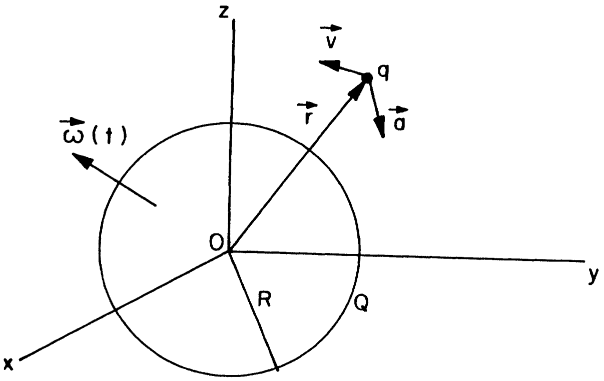

Figure 7.2

We can integrate Weber's potential energy (3.25) utilizing spherical coordinates. To this end we employ $dq_2 = \sigma da = (Q/4\pi R^2)R^2 \sin\theta d\theta d\varphi$, $\vec{r}_2 = R\hat{r}_2 = R(\sin\theta\cos\varphi\hat{x} + \sin\theta\sin\varphi\hat{y} + \cos\theta\hat{z})$, $\vec{v}_2 = \vec{\omega}\times\vec{r}_2$, $\vec{a}_2 = \vec{\omega}\times(\vec{\omega}\times\vec{r}_2) + (d\vec{\omega}/dt)\times\vec{r}_2$. After the integration we get

$$U(r<R) = \frac{qQ}{4\pi\varepsilon_o}\frac{1}{R}\left[1 - \frac{v^2 - 2\vec{v}\cdot(\vec{\omega}\times\vec{r}) + (\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})}{6c^2}\right] , \tag{7.23}$$

$$U(r>R) = \frac{qQ}{4\pi\varepsilon_o}\frac{1}{r}\left\{1 - \frac{[\hat{r}\cdot(\vec{v}-\vec{\omega}\times\vec{r})]^2}{2c^2}\right.$$

$$\left. - \frac{1}{6c^2}\frac{R^2}{r^2}\left[(\vec{v}-\vec{\omega}\times\vec{r})\cdot(\vec{v}-\vec{\omega}\times\vec{r}) - 3[\hat{r}\cdot(\vec{v}-\vec{\omega}\times\vec{r})]^2\right]\right\} , \tag{7.24}$$

where $\hat{r} = \vec{r}/r$.

Integrating Weber's force exerted by the shell on q, instead of Weber's energy, yields (Assis, 1989 a and 1992 d):

$$\vec{F}(r < R) = \frac{\mu_o q Q}{12\pi R}\left[\vec{a} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + 2\vec{v}\times\vec{\omega} + \vec{r}\times\frac{d\vec{\omega}}{dt}\right] , \quad (7.25)$$

$$\vec{F}(r > R) = \frac{qQ}{4\pi\varepsilon_o r^2}\left\{\left[1 + \frac{1}{c^2}\left(v^2 - \frac{3}{2}(\hat{r}\cdot\vec{v})^2 + \vec{r}\cdot\vec{a}\right)\right]\hat{r}\right.$$

$$+\frac{1}{c^2}\frac{R^2}{r^2}\left[\frac{r}{3}\vec{a} - (\hat{r}\cdot\vec{v})\vec{v} - \frac{v^2}{2}\hat{r} + \frac{5}{2}(\hat{r}\cdot\vec{v})^2\hat{r} - (\vec{r}\cdot\vec{a})\hat{r} + (\hat{r}\cdot\vec{v})(\vec{\omega}\times\vec{r})\right.$$

$$\left.\left.+\frac{2}{3}r(\vec{v}\times\vec{\omega}) + \frac{r}{3}(\vec{\omega}\cdot\vec{r})\vec{\omega} + \frac{r^2\omega^2}{6}\hat{r} - \frac{(\hat{r}\cdot\vec{\omega})^2}{2}\hat{r} + [\vec{r}\cdot(\vec{\omega}\times\vec{v})]\hat{r} + \frac{r}{3}\left(\vec{r}\times\frac{d\vec{\omega}}{dt}\right)\right]\right\} . \quad (7.26)$$

If the center of the shell were localized at $\vec{R}_o$ and were moving with velocity $\vec{V}_o$ and acceleration $\vec{A}_o$ relative to S, eqs. (7.23) to (7.26) would still be valid with the replacements $\vec{r} \to \vec{r} - \vec{R}_o$, $\vec{v} \to \vec{v} - \vec{V}_o$, $\vec{a} \to \vec{a} - \vec{A}_o$. Eqs. (7.23) and (7.25) would be valid for $|\vec{r} - \vec{R}_o| < R$, while (7.24) and (7.26) for $|\vec{r} - \vec{R}_o| > R$.

The first to obtain Eq. (7.23), with $\vec{\omega} = 0$, was Helmholtz in 1872. In Section 7.1 we saw how he criticized Weber's electrodynamics in the two body problem. After Weber's reply he found this result and utilized it as a new criticism against Weber's theory. Let us see how Maxwell presented it, immediately after the discussion of the two body problem: "Helmholtz† has therefore stated a case in which the distances are not too small, nor the velocities too great, for experimental verification. A fixed non-conducting spherical surface, of radius a, is uniformly charged with electricity to the surface-density σ. A particle, of mass m and carrying a charge e of electricity, moves within the sphere with velocity v. The electrodynamic potential calculated from the formula

† *Berlin Monatsbericht*, April 1872, pp. 247-256; *Phil. Mag.*, Dec. 1872, Supp., pp. 530-537.

$$\psi = \frac{ee'}{r}\left[1 - \frac{1}{2c^2}\left(\frac{\partial r}{\partial t}\right)^2\right] ,$$

is

$$4\pi a\sigma e\left(1 - \frac{v^2}{6c^2}\right) ,$$

and is independent of the position of the particle within the sphere. Adding to this V, the remainder of the potential energy arising from the action of other forces, and $mv^2/2$, the kinetic energy of the particle, we find as the equation of energy

$$\frac{1}{2}\left(m - \frac{4}{3}\frac{\pi a\sigma e}{c^2}\right)v^2 + 4\pi a\sigma e + V = const.$$

Since the second term of the coefficient of v^2 may be increased indefinitely by increasing a, the radius of the sphere, while the surface-density σ remains constant, the coefficient of v^2 may be made negative. Acceleration of the motion of the particle would then correspond to diminution of its *vis viva*, and a body moving in a closed path and acted on by a force like friction, always opposite in direction of its motion, would continually increase in velocity, and that without limit. This impossible result is a necessary consequence of assuming any formula for the potential which introduces negative terms into the coefficient of v^2" (Maxwell, 1954, Vol. 2, article [854], p. 485).

It is easier to understand Helmholtz's criticism (Helmholtz, 1872) working with the forces instead of the energies. Let us suppose a point charge q moving inside a stationary and non-spinning charged spherical shell. If it is accelerated relative to the shell by other forces, eq. (7.25) predicts that the shell will exert a force on q given by

$$\vec{F}_{\text{shell on } q} = \frac{\mu_o qQ}{12\pi R}\vec{a} \equiv m_W\vec{a} , \tag{7.27}$$

where $m_W \equiv \mu_o qQ/12\pi R$ is what we call Weber's inertial mass for this geometry.

Applying Newton's mechanics to this problem, (2.7), yields:

$$\sum_{j=1}^{N}\vec{F}_{jq} + \vec{F}_{\text{shell on } q} = m\vec{a} . \tag{7.28}$$

In this equation $\vec{F}_{jq}$ is the symbolic representation of the force exerted by body j on q and $\sum_{j=1}^{N} \vec{F}_{jq}$ is the resultant force acting on q, with the exception of the force exerted by the shell on q. Inserting (7.27) in (7.28) yields

$$\sum_{j=1}^{N} \vec{F}_{jq} = (m - m_W)\vec{a} \ . \tag{7.29}$$

This shows that the test charge will behave according to Weber's electrodynamics as if it had an effective inertial mass given by $m - m_W$. If q and Q are of the same sign then $m_W > 0$. In principle we might increase m_W by increasing Q/R or $R\sigma$, so that it might become eventually larger than m. In this case the effective inertial mass of the particle would become negative, $m - m_W < 0$. Suppose the particle were acted on by a force like friction, which is in general of the type $\vec{F}_f = -|b(v)|\vec{v}/v$, where $v \equiv |\vec{v}|$ and $b(v)$ is the coefficient of friction. Then instead of decreasing the velocity of the particle as usual, the particle velocity would be increase by this force. If the particle were moving in a closed path inside the shell this increase of velocity might go on indefinitely, Helmholtz argued. This unusual prediction is the core of his criticism.

Several remarks can be made on this problem. (A) We do not know any comment of Weber on this criticism. Maybe because he was ceasing research and retired from teaching (1873) just at this time.

(B) The most important remark: Before considering this a failure of Weber's electrodynamics, the situation should be analysed experimentally. We do not know of any experiment which has ever been performed in order to test this prediction. It is possible that some charges may behave as if they had a negative inertial mass in some regions of high electrostatic potential. We can not rule out this possibility only because it is unusual. Only a carefully designed experiment can decide the matter. Maxwell himself had doubts that an experiment could be performed in his days to test this prediction. This is evident from the next sentence after this quoted section, namely: "But we have now to consider the application of Weber's theory to phenomena which can be realized." We believe that if an experiment be performed Weber's electrodynamics will be vindicated, at least for low velocities (see (G) below).

(C) None of these results follow from Lorentz or Liénard-Schwarzschild's force laws. If $\vec{\omega} = 0$ the integration of the force of a stationary charged shell on an internal test charge yields zero instead of $m_W \vec{a}$, as there is no electric or magnetic field inside the shell. In the next Section we will see that only the term with $2\vec{v} \times \vec{\omega}$ in (7.25) is derived with Lorentz or Liénard-Schwarzschild's force laws.

(D) The conclusion that the velocity might increase indefinitely was obtained disregarding many aspects. For instance, the accelerated test charge may lose energy due to electromagnetic radiation. Also Maxwell's statement that we may increase indefinitely Q/R or $R\sigma$ is not true in practice due to the corona effect.

(E) We only know the behaviour of frictional forces acting on neutral bodies at low velocities ($v \ll c$). We do not have enough experimental knowledge to predict how friction will behave when acting on a charged particle approaching light velocity.

(F) The tendency of the particle is to move in a straight line. It may be impossible in practice to have a constraint strong enough to keep the particle moving in a closed path inside the shell as its velocity is increased and approaches the light velocity.

(G) This prediction of a negative or zero effective inertial mass was based not only on Weber's electrodynamics but also on Newtonian mechanics ($\vec{F} = m\vec{a}$ like here, or $T = mv^2/2$ as in Helmholtz's original analysis). It is reasonable to suppose that for velocities near the light velocity both expressions should be modified. As we will see, it is possible to derive an energy like $mc^2/\sqrt{1 - v^2/c^2}$ or a dynamics like $\vec{F} = md(\vec{v}/\sqrt{1 - v^2/c^2})/dt$ with a potential of Weber's type applied to gravitation, as was done by Schrödinger in 1925. With this new mechanics we do not get anymore an infinite velocity for any electrostatic potential.

Despite all these facts let us explore a little more (7.29). Let us suppose the test charge is an electron ($m = 9.1 \times 10^{-31} kg$, $q = -e = -1.6 \times 10^{-19} C$). Choosing the zero of the potential of the shell at infinity, Weber's inertial mass for this geometry can be written as $m_W = q\phi/3c^2$, where $\phi = Q/4\pi\varepsilon_o R$ is the potential of the shell. In order to double its effective inertial mass (or to make it go to zero), the electron would need to be inside a spherical shell charged to a potential of $1.5 \times 10^6 V$. It is possible to obtain potentials of this order of magnitude. So an experiment might in principle be performed. We suggested some experiments of this kind in (Assis, 1993 a).

The effective inertial mass in this case is isotropic due to the symmetry of the shell. This did not happen inside a capacitor, as we have seen.

Helmholtz never accepted Weber's electrodynamics. He was always opposed to force laws which depended on the velocities of the particles. He thought Weber's theory was against the principle of the conservation of energy but was later shown to be wrong. He always tried to find inconsistencies in Weber's theory. He discussed the radial two body problem with Weber's model and found unusual results. Weber replied correctly indicating the unphysical initial conditions which were required in order to get these unusual results, and that these results could not be tested experimentally. Then Helmholtz obtained eq. (7.23) when $\vec{\omega} = 0$. As he had a negative attitude towards Weber's theory, he tried to find reasons to reject this result. At this time Ernst Mach was presenting his first criticisms against Newton's formulation of mechanics and suggesting that the inertia of the bodies might be due to some kind of interaction with the distant matter of the universe. And this result obtained by Helmholtz, (7.23), is the key to implement Mach's principle with a relational Weberian potential energy applied to gravitation, as was realized by E. Schrödinger fifty years later (Schrödinger, 1925). In this way Schrödinger derived $mv^2/2$ as a gravitational interaction energy of any body with the remainder of the universe. He also derived the precession of the perihelion of the planets, the proportionality between inertial and gravitational masses, explained the fact that the best inertial frame is the frame of the distant universe, etc. All of these remarkable results might had been derived by Helmholtz if he only had an open mind towards Weber's theory and tried to explore its consequences constructively.

7.4. Centrifugal Electrical Force

Let us analyse this problem classically. We will consider here only the case when the test charge is inside the charged spherical shell. First when the shell is not spinning. In this case it generates no magnetic field inside or outside. Outside the electric field is radial and falls as $1/r^2$. However, there is no electric field anywhere inside the shell as it is a region of constant electrostatic potential. These results are easily obtained with Gauss's law (2.56). As there are no electric or magnetic fields inside the shell, Lorentz's expression predicts no force on the internal test charge no matter its velocity and acceleration relative to the shell. This result is also easily obtained integrating Liénard-Schwarzschild's force (6.8).

As we have seen, Weber's theory predicts a force in this case given by $m_W\vec{a}$, where $\vec{a}$ is the acceleration of the test charge relative to the center of the shell. This effect will only appear when the test charge is accelerated by other forces, and then the charge should behave as if it had changed its inertial mass.

Now when the shell is spinning with a constant angular velocity $\vec{\omega}$ relative to an inertial frame. Due to the uniform distribution of charges in the shell the electric field will remain zero inside. The electric potential is still constant and $\partial\vec{A}/\partial t = 0$ because $d\vec{\omega}/dt = 0$. So Lorentz's force is given simply by $q\vec{v} \times \vec{B}$. Now it is well known that this uniformly charged spherical shell spinning with a constant angular velocity generates a uniform magnetic field anywhere inside the shell given by (Griffiths, 1989, pp. 229 - 230; Batygin and Toptygin, 1964, p. 61):

$$\vec{B}(r < R) = \frac{\mu_o Q\vec{\omega}}{6\pi R} . \tag{7.30}$$

So Lorentz's force predicts in this case the result

$$\vec{F} = q\vec{v} \times \vec{B} = \frac{\mu_o qQ}{6\pi R}\vec{v} \times \vec{\omega} . \tag{7.31}$$

This can also be obtained integrating (6.8) directly.

Comparing with Weber's electrodynamics, (7.25) with $d\vec{\omega}/dt = 0$, we can see that Lorentz's $q\vec{v} \times \vec{B}$ is the same as Weber's "Coriolis electrical force" $2m_W\vec{v} \times \vec{\omega}$. However, there are two components in (7.25) which have no analog classically: Weber's electrical inertial force $m_W\vec{a}$ and Weber's centrifugal electrical force $m_W\vec{\omega}\times(\vec{\omega}\times\vec{r})$.

We discussed the first component in the previous Section. We here show only one example of where the second component might be relevant or tested experimentally (for further discussion of this whole subject see (Assis, 1992 d)). The charged spherical shell is spinning with a constant angular velocity around the z axis, $\vec{\omega}=\omega\hat{z}$. Two charges q_1 and q_2 are inside the shell, in the xy plane, at the distances ρ_1 and ρ_2 of the axis of rotation ($\rho_1>0,\ \rho_2>0$). They are connected by a non-conducting spring, Figure 7.3.

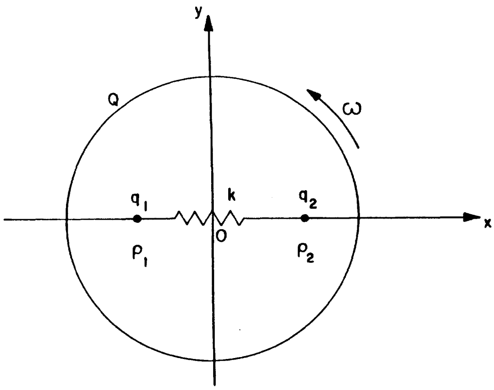

Figure 7.3

With (2.11), (3.5), (7.27) and (7.28) this yields (being l_o the relaxed length of the spring and $\rho\equiv\rho_1+\rho_2$)

$$k(\rho_1+\rho_2-l_o)\hat{x}-\frac{q_1q_2}{4\pi\varepsilon_o}\frac{\hat{x}}{(\rho_1+\rho_2)^2}\left(1-\frac{\dot{\rho}^2}{2c^2}+\frac{\rho\ddot{\rho}}{c^2}\right)$$

$$+\,m_{W1}\left[\vec{a}_1+\vec{\omega}\times(\vec{\omega}\times\vec{r}_1)+2\vec{v}_1\times\vec{\omega}\right]=m_1\vec{a}_1\ , \qquad (7.32)$$

$$-k(\rho_1+\rho_2-l_o)\hat{x}+\frac{q_1q_2}{4\pi\varepsilon_o}\frac{\hat{x}}{(\rho_1+\rho_2)^2}\left(1-\frac{\dot{\rho}^2}{2c^2}+\frac{\rho\ddot{\rho}}{c^2}\right)$$

$$+\,m_{W2}\left[\vec{a}_2+\vec{\omega}\times(\vec{\omega}\times\vec{r}_2)+2\vec{v}_2\times\vec{\omega}\right]=m_2\vec{a}_2\ . \qquad (7.33)$$

Equilibrium ($\vec{v}_1 = \vec{v}_2 = 0$, $\vec{a}_1 = \vec{a}_2 = 0$) will only be maintained in this case if

$$k(\rho_1 + \rho_2 - l_o) - \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{(\rho_1 + \rho_2)^2} + m_{W1}\omega^2 \rho_1 = 0 \ , \tag{7.34}$$

$$-k(\rho_1 + \rho_2 - l_o) + \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{(\rho_1 + \rho_2)^2} - m_{W2}\omega^2 \rho_2 = 0 \ . \tag{7.35}$$

Adding these two equations and utilizing $m_W = \mu_o q Q/12\pi R$ yields another equilibrium condition:

$$q_1 \rho_1 = q_2 \rho_2 \ . \tag{7.36}$$

This remarkable result only appears in Weber's electrodynamics because the centrifugal force is not present in Lorentz's force law. It is somewhat similar to Archimedes's law of the lever with the charges replacing the masses. Eq. (7.36) could be tested in the laboratory if the centrifugal force were large enough to overcome random fluctuations due to air impurities and that the system can arrive at the equilibrium situation in a reasonable time. However usually the centrifugal electrical force $\vec{F}_c$ is extremely small. For instance, for $q_1 = q_2 = 10^{-10}C$, $\phi = 1.5 \times 10^6 V$ ($m_{W1} = m_{W2} \simeq 5 \times 10^{-22} kg$), $\rho_1 = \rho_2 = 1m$ and $\omega = 10^3 s^{-1}$ we obtain $F_c \simeq 10^{-15} N$.

It is of interest to know the value of the parameters to counterbalance Coulomb's force. That is, two charges of the same sign repelling each other can be maintained at relative rest inside the spinning charged shell, even without the spring or other external forces, but only through the centrifugal electrical force. To this end we need to satisfy Eqs. (7.34) to (7.36) with $k = 0$. Supposing $q_1 = q_2$ then $\rho_1 = \rho_2$ by (7.36). In (7.35), with $\phi = Q/4\pi\varepsilon_o R$ being the electrostatic potential of the shell relative to infinity, we get

$$\phi\omega^2 = \frac{3c^2 q_1}{16\pi\varepsilon_o \rho_1^3} \ . \tag{7.37}$$

Usually we want to minimize ϕ and ω^2 as it is difficult to generate a voltage much higher than some mega-volts and even more difficult to rotate this high-voltage system. This means that q_1 should be small and ρ_1 large. To estimate the order of magnitude we suppose $q_1 \simeq Q$ and $\rho_1 \simeq R$, R just a little bigger. From (7.37) this yields $\rho_1\omega \simeq c$. This means that this could only be realized microscopically ($q_1 \simeq$ electron charge, $\omega \simeq 10^{21}s^{-1}$) and with these values in (7.37) we obtain $\rho_1 \simeq \lambda_c/2\pi = 3.7 \times 10^{-13}m$, where λ_c is the Compton wavelength of the electron.

7.5. Weber's Law Applied to Gravitation

Newton's law of gravitation of 1687 and its tremendous success when applied to celestial mechanics has exerted a lasting influence in physics. It seems that Coulomb arrived at his force law a century later (1785) more by analogy with Newton's expression than by his doubtful measurements with the torsion balance (Heering, 1992). However, the limitations of Coulomb's force were soon realized in electromagnetism with Oersted and Faraday's discoveries of 1820 and 1831. In order to explain these findings with a force between point charges, Weber needed to introduce in 1846 a generalization of Coulomb's force including terms which depended on the velocity and acceleration between the charges. Then it was the success of Weber's electrodynamics (from a single force we could derive the forces of Coulomb and Ampère, as well as Faraday's law of induction) which prompted some people to modify Newton's law including terms dependent on the velocity and acceleration between gravitational masses.

The idea is that the potential energy between two particles of gravitational masses m_i and m_j should be given by (with the previous definitions and numerical values of r_{ij}, $\dot{r}_{ij}$, G and c):

$$U_{ij} = -G\frac{m_i m_j}{r_{ij}}\left(1 - \xi\frac{\dot{r}_{ij}^2}{2c^2}\right) . \tag{7.38}$$

In this expression ξ is a dimensionless constant. With $\xi = 0$ or $c \to \infty$ we recover the usual potential energy for gravitation. The force exerted by m_j on m_i is then given by

$$\vec{F}_{ji} = -Gm_i m_j\frac{\hat{r}_{ij}}{r_{ij}^2}\left[1 - \frac{\xi}{c^2}\left(\frac{\dot{r}_{ij}^2}{2} - r_{ij}\ddot{r}_{ij}\right)\right] , \tag{7.39}$$

With these expressions we would still have the conservation of linear momentum, angular momentum and energy for gravitational interactions, as we have seen.

The first to propose a Weber's law to gravitation seems to have been G. Holzmuller in 1870, almost two centuries after Newton's law (North, 1965, p. 46). Then Tisserand in 1872 studied Weber's force (7.39) applied to gravitation and its application to the precession of the perihelion of the planets (Tisserand, 1872 and 1895; Whittaker, 1973, Vol. 1, pp. 207

- 208). As we have seen, the general two body problem had already been solved in terms of elliptic integrals by Seegers in 1864. This included the precession of the perihelion as a special case (see also Clemente and Assis, 1991). Tisserand, however, preferred to solve the problem approximately. Recently we followed a similar procedure and arrived at the same solution of Tisserand (Assis, 1989 a). Applying Newton's second law in (7.39), $\vec{F} = m\vec{a}$, yields conservation of linear and angular momentum. This means that $\rho^2\dot{\varphi} \equiv H$ will be a constant (two body problem in plane polar coordinates, ρ being the distance between m_i and m_j). The radial component of the equation of motion becomes (with $M \equiv m_i + m_j$):

$$\ddot{\rho} - \rho\dot{\varphi}^2 = -\frac{GM}{\rho^2}\left[1 - \frac{\xi}{c^2}\left(\frac{\dot{\rho}^2}{2} - \rho\ddot{\rho}\right)\right] . \tag{7.40}$$

The orbit equation is obtained with the substitution $u \equiv 1/\rho$. This yields

$$\frac{d^2u}{d\varphi^2} + u = \frac{GM}{H^2} - \frac{\xi}{c^2}GM\left[\frac{1}{2}\left(\frac{du}{d\varphi}\right)^2 + u\frac{d^2u}{d\varphi^2}\right] . \tag{7.41}$$

This equation may be solved iteratively observing that the second and third terms in the right hand side are much smaller than the first one. The solution yields a precession of the perihelion of the planet. After one revolution it is given by (Assis, 1989 a):

$$\Delta\varphi = \pi\frac{\xi}{c^2}\frac{G^2M^2}{H^2} = \pi\frac{\xi}{c^2}\frac{GM}{a(1-\varepsilon^2)} , \tag{7.42}$$

where a is the semimajor axis and ε the eccentricity of the orbit. With $\xi = 6$ we arrive at exactly the same algebraic result as the one obtained with general relativity, although through a different orbit equation. And this result agrees reasonably well with the observational data for the planets.

This discovery that a Weber's law applied to gravitation leads to a precession of the perihelion of the planets has been rediscovered from time to time by many people. As examples we can cite Paul Gerber in 1898 and 1917, Erwin Schrödinger in 1925 (working with Weber's energy (7.38) instead of Weber's force (7.39), and also fitting $\xi = 6$ or $\xi/2 = 3$ in order to agree with the observations) and Eby in 1977 (Gerber, 1898 and 1917; Schrödinger, 1925; Eby, 1977).

It is curious to observe that none of these three authors mentioned Weber's electrodynamics or Weber's name. Gerber was working with ideas of retarded time and arrived at a Lagrangian energy given by (until the second order in $1/c$):

$$S = -\frac{Gm_i m_j}{r_{ij}} \left(1 + \frac{\xi}{c^2} \frac{\dot{r}_{ij}^2}{2} \right) . \tag{7.43}$$

The change of sign in front of $\dot{r}_{ij}^2$ is analogous to the electromagnetic case. Schrödinger said that he arrived at (7.38) heuristically (heuristisch), in order to implement Mach's principle. It is amazing that with his vast knowledge of physics and being a German speaking person he would not know Weber's electrodynamics. The dictionary defines the adjective heuristic as of the theory in education that a learner should discover things for himself. The noun heuristics is the method of solving problems by inductive reasoning, by evaluating past experience and moving by trial and error to a solution. Eby was following the work of Barbour and Bertotti, to be discussed later, also connected with Mach's principle.

The work of Tisserand of applying a Weber's law to gravitation in celestial mechanics was discussed by Poincaré in a course which he delivered at the Faculté des Sciences de Paris during 1906 - 1907 (Poincaré, 1953; see especially p. 125 and Chapter IX, pp. 201 - 203, "Loi de Weber"). Gerber's works were criticized by Seeliger, who was aware of Weber's electrodynamics (Seeliger, 1917).

Treder, Borzeszkowski, van der Merwe, Yourgrau and collaborators have worked with and discussed Weber's law applied to gravitation. References to their original works and to other authors can be found in (Treder, 1971 and 1975), (Treder, Borzeszkowski, van der Merwe and Yourgrau, 1980). They are among a group os scientists responsible for updating the research of gravitation with Weber's law, discussing at length a great variety of topics, including the velocity of propagation of gravitational interactions, the bending of light in a gravitational field (on this topic see also (Ragusa, 1992)), the absorption of gravity, etc.

Connected with the idea of the absorption of gravity, there is a potential energy given by (Assis, 1992 e):

$$U = -G\frac{m_i m_j}{r_{ij}}\left(1 - \xi\frac{\dot{r}_{ij}^2}{2c^2}\right)e^{-\alpha r_{ij}} \ . \tag{7.44}$$

In this equation α gives the characteristic length for gravitational interactions.

The force exerted by m_j on m_i can be obtained utilizing $\vec{F}_{ji} = -\hat{r}_{ij}dU_{ij}/dr_{ij}$. For a constant α this yields

$$\vec{F}_{ji} = -G\frac{m_i m_j}{r_{ij}^2}\hat{r}_{ij}\left[1 - \frac{\xi}{2}\frac{\dot{r}_{ij}^2}{c^2} + \xi\frac{r_{ij}\ddot{r}_{ij}}{c^2} + \alpha r_{ij}\left(1 - \frac{\xi}{2}\frac{\dot{r}_{ij}^2}{c^2}\right)\right]e^{-\alpha r_{ij}} \ . \tag{7.45}$$

The first to propose an exponential decay in the Newtonian gravitational potential energy were H. Seeliger and C. Neumann, mainly in 1895 - 1896. What they proposed would be equivalent to (7.44) with $\xi = 0$. An exponential decay multiplying Newton's gravitational force (but not in the potential energy) had been proposed much earlier by Laplace, in 1825. In this century there is a remarkable paper by W. Nernst proposing an exponential decay in gravitation. For references and further discussion see (Laplace, 1969; Seeliger, 1895; Nernst, 1937; North, 1965, pp. 16 - 18; Steenbeck and Treder, 1984; Jaki, 1990, Chapter 8). To our knowledge the best laboratory experiments on the absorption of gravity are those due to Quirino Majorana (Majorana, 1920 and 1930; Dragoni, 1988). We were the first to propose an exponential in a Weberian potential (Assis, 1992 e and 1993 c).

These exponential decays in gravitation have been proposed following two main lines of reasoning. The first one as an analogy with the propagation and absorption of light. In this case α would depend on the amount and distribution of matter in the straight line between m_i and m_j. This could be called an absorption of gravity. The absorption of light coupled to the energy of the photon led to the so called tired light mechanism to explain Hubble's law of redshift in a stationary (non expanding) and boundless universe. Ideas of this kind and related ones led even to a prediction of the characteristic temperature of the cosmic background radiation of 2.7 K prior to Gamow and collaborators. For references and further discussion see (Eddington, 1988; Regener, 1933; Nernst, 1937 and 1938; Finlay-Freundlich, 1953, 1954 a and b; Born, 1953 and 1954; de Broglie, 1966; Assis, 1992 e, f and 1993 c).

The second line of reasoning leading to an exponential decay in gravitation is due to the gravitational paradoxes arising in an infinite and homogeneous universe (infinite value of the potential, indefinite value of the gravitational force). In this last situation α in (7.44) may be considered as a universal constant irrespective of the medium between m_i and m_j. This was the point of departure of Seeliger and C. Neumann.

A completely different study of a Weber's law applied to gravitation has been performed recently by: (Sokol'skii and Sadovnikov, 1987). They analysed the stability of planetary orbits with a law like (7.38), (7.39) and (7.43). To our knowledge they were the first to apply the modern techniques of dynamical systems (chaotic motions, Lyapunov coefficients, etc.) to Weber's non-linear law. This is a new field of research related to Weber's law which is still in its infancy.

We close this Section with a brief mention that Weber himself considered his law as applied to gravitation. Essentially he and F. Zollner were working with an idea developed by Thomas Young in 1807 and Mossotti in 1836 according to which the electric attractive force between unlike charges is slightly larger than the electric repulsive forces between like charges of the same absolute magnitude. Nature behaving like this, there would remain a resultant attractive force between neutral atoms, which would be what we call gravitation (Mossotti, 1966; Whittaker, 1973, Vol. 1, pp. 51 - 52 and Vol. 2, p. 150). Although Whittaker has claimed that the first model proposing this was due to the German physicist Aepinus in 1759, he never made such a suggestion. This was discussed by R. W. Home in his introductory monograph to the first English translation of Aepinus important work, where Home also mentioned Young's work (Aepinus, 1979, pp. 119 - 120 and 223 - 224). The idea of Weber and Zollner in the 1870's and 1880's was to apply the idea of Young and Mossotti to Weber's force (3.5) instead of applying it to Coulomb's force. So the final result was something similar to (7.39) instead of simply Newton's law of gravitation (Woodruff, 1976; Wise, 1981, see especially pp. 282 - 283).

7.6. Mach's Principle

Newtonian mechanics is based on the concepts of absolute space, absolute time and absolute motion. In Newton's second law of motion, (2.5) or (2.7), we have velocities and accelerations of the test body relative to absolute space according to Newton. These concepts were criticized by Berkeley, Leibniz and especially by Ernst Mach. According to these authors there is no philosophical or practical meaning in referring motion to space. Whenever there is motion it is of one body relative to another or to many other bodies. Implicit in the works of these authors is the idea that it should be possible to construct a purely relational mechanics based only on r_{ij}, $\dot{r}_{ij}$, $\ddot{r}_{ij}$ etc. without resort to absolute space. Only the distances, velocities and accelerations between material bodies would matter. Mach expressed this idea clearly in the following words: "*Relatively*, not considering the unknown and neglected medium of space, the motions of the universe are the same whether we adopt the Ptolemaic or the Copernican mode of view. Both views are, indeed, equally *correct*; only the latter is more simple and more *practical*." (...) "The principles of mechanics can, indeed, be so conceived, that even for relative rotations centrifugal forces arise" (Mach, 1960, pp. 283 - 284).

The idea that the inertial properties of a body (its inertial mass, inertial forces acting on it, etc.) are due to its interaction with the material universe has been called Mach's principle.

To prove the existence of absolute space and its influence upon accelerated matter Newton presented the famous bucket experiment (Newton, 1952 a, pp. 11 - 12). The concave figure of the water when it is revolving relative to the earth does not depend on the relative rotation between the water and the bucket. On this ground Newton concluded that the concave figure was due to an absolute rotation of the water relative to absolute space. According to Mach this was not the case. The concave figure was due to rotation of the water relative to the earth and the other celestial bodies, the so called fixed stars (Mach, 1960, pp. 279 and 283 - 284). According to Mach if we could keep the water at rest relative to the earth and rotate the heaven of fixed stars in the opposite direction, centrifugal forces would arise and the surface of the water would become concave, ascending to the sides of the bucket: "Try to fix Newton's bucket and rotate the heaven of fixed stars and then prove the absence of centrifugal forces" (Mach, 1960, p. 279). According to Newton nothing would happen and the surface of the water would remain plain, as it remained at rest relative to his absolute space, which to Newton had no relations whatsoever with the heaven of fixed stars.

We agree with Mach and not with Newton on this point. Although we can not perform this thought experiment we might try a similar one. We keep the bucket and water at rest relative to the earth. We could then surround both with a hollow spherical shell made of a heavy material like a metal. If we spin only the spherical shell relative to the earth, keeping the water and bucket at rest, there should appear according to Mach a small centrifugal force on any molecule of water inside the shell not along the axis of rotation, Figure 7.4. This force should not be there according to Newton.

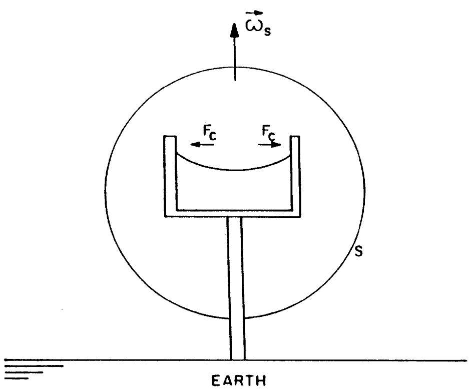

Figure 7.4

Unfortunately this force would be too small to be detected even if it were there. An estimate can be obtained in (Assis, 1989 a). With a Weber's force for gravitation (7.39) it would be given by $8\pi G m \rho r dr \omega^2 l/c^2$, where m is the mass of the test particle, r and dr are the radius and thickness of the shell of density ρ spinning with an angular velocity ω, and l is the distance of m to the axis of rotation. With $m = 1kg$, $\rho = 8 \times 10^3 kg/m^3$ (iron), $r = 1m$, $dr = 0.1m$, $f = \omega/2\pi = 100Hz$, $l = 0.5m$ the centrifugal force would be given by $3 \times 10^{-18} N$. And this is negligible compared with the downward force due to the weight of the test particle, mg. In this case the centrifugal force per unit mass is given by $3 \times 10^{-18} ms^{-2} \ll 9.8 ms^{-2}$, which is the gravitational field or acceleration of a free falling body. Although we can not test this prediction in the laboratory due to the small value of the force, this example shows where Weber's law for gravitation differs from Newton's law. Moreover, it indicates clearly how Mach's ideas clash with Newton's absolute space and has dynamical consequences.

Nowadays we speak of inertial frames instead of absolute space. We might say that an inertial frame is a reference system where Newton's second law of motion (2.5) or (2.7) holds (is valid) without the introduction of "fictitious" forces (centrifugal forces, Coriolis forces, etc.) One of the strongest empirical evidences in favour of Mach's principle is that the best inertial frame we have is the frame of distant galaxies, namely, the frame in which they are not spinning as a whole and in which they have no linear translational acceleration as a whole. This is a coincidence in classical mechanics, which can not explain this fact. The earth spins around its axis relative to the sun with a period of 24 hours ($\omega = 7 \times 10^{-5} s^{-1}$). The planet revolves around the sun relative to the heaven of stars with a period of 365 days ($\omega = 2 \times 10^{-7} s^{-1}$). The Milky Way as a whole rotates relative to the distant galaxies with a period of 2.5×10^8 *years* ($\omega = 8 \times 10^{-16} s^{-1}$). The universe as a whole might be spinning relative to absolute space without violating any principle of mechanics. As a matter of fact it does not. If there is a rotation between the material universe and absolute space (or an inertial frame) it is smaller than $2 \times 10^{-8} rad/yr$ and has never been detected (Schiff, 1964). This coincidence of classical physics has a simple explanation according to Mach's principle, namely, the distant universe is what defines and creates what is called an inertial frame. In other words, the "fictitious" forces are in fact real forces which arise in any frame in which the universe as a whole is spinning or has a translational acceleration. The centrifugal force is then a real force between the test body and the remainder of the universe, which arises when the latter is spinning as a whole.

Instead of Newton's three laws of motion Mach proposed a set of alternative propositions of his own, (Mach, 1960, pp. 264 - 271 and pp. 303 - 304). Although in his key definition of inertial mass ("The mass-ratio of any two bodies is the negative inverse ratio of the mutually induced accelerations of those bodies") he did not specify clearly the frame of reference with respect to which the accelerations in this definition should be measured, it is evident from his writings that he had in mind the frame of fixed stars. This has been shown conclusively in an important paper by Yourgrau and van der Merwe (Yourgrau and van der Merwe, 1968). This is confirmed by the following quotation from Mach: "I have remained to the present day the only one who insists upon referring the law of inertia to the earth, and in the case of motions of great spatial and temporal extent, to the fixed stars" (Mach, 1960, p. 336).

Related with this inertial mass there is another strong empirical evidence in favour of Mach's principle. There are two concepts of mass in Newtonian mechanics. The first one is the gravitational mass which appears in Newton's law of gravitation, (2.8), in the weight of a body, (2.10), and in the gravitational potential energy, (7.38) with $\xi = 0$. It is analogous to the electric charge: (2.8) and (2.13), (2.10) and (2.14), (7.38) and (3.25). A gravitational mass (an electric charge) exerts and feels a gravitational (electric) force due to another gravitational mass (electric charge). We might also say that a gravitational mass (an electric charge) generates a gravitational (electric) field and reacts to the presence of a gravitational (electric) field. The other concept of mass in Newtonian mechanics is the inertial mass. It is the mass which appears in linear and angular momentum, in the kinetic energy, and in the right hand side of Newton's second law of motion, (2.5) or (2.7).

Conceptually these two masses have no relation whatsoever with one another, although both are called "masses." One is related to a fundamental interaction, gravitation, while the other is a measure of the resistance of a body to being accelerated relative to absolute space or relative to an inertial frame by external forces of any origin (gravitational forces, elastic forces, electric and magnetic forces, frictional forces, etc.) However, since the time of Galileo and Newton we know from experiments that these two masses are proportional or equal to one another. For instance, a coin and a feather fall with the same acceleration in the same gravitational field of the earth (neglecting air resistance), but a proton and an alpha particle do not move with the same acceleration in the same electric field, Figure 7.5.

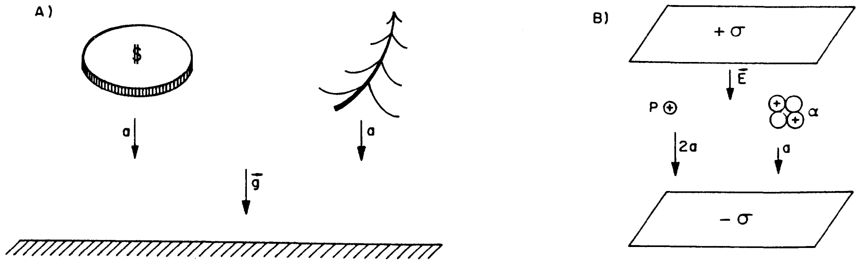

Figure 7.5

The first experiment shows that the inertial mass is proportional to the gravitational mass, while the second one shows that it is not proportinal to the electric charge. There is no explanation for this fact in classical mechanics and nature might very well behave in the opposite way.

Another example: Two pendulums of the same length but filled with different substances oscillate with the same period in a specific location of the earth, no matter the weight (neglecting air resistance) or chemical composition of the substances, as was first experimentally shown by Newton (Newton, 1952 a, Book III, Prop. 6, Theor. 6, p. 279). On the other hand two springs with the same elastic constant k will oscillate at different frequencies and periods in a frictionless table if connected to different masses,

Figure 7.6.

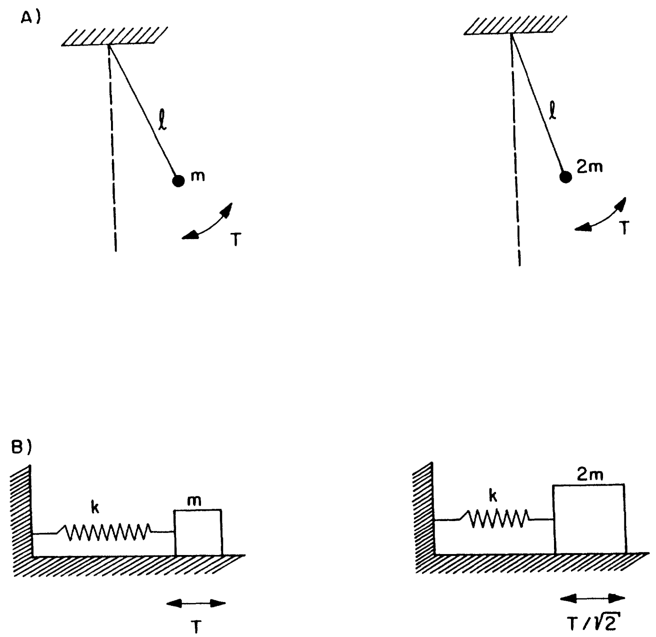

Figure 7.6

The first of these experiments shows once more that inertial and gravitational masses are proportional to one another, while the second one shows that the inertial mass is not proportional to any elastic property of the body or of the intervening medium (the spring).

In all experiments like these it is always found that the inertial mass of a body is proportional to its gravitational mass. The inertial mass is not proportional to any other property of the body like: its electric charge, the magnitude of its magnetic pole, any elastic or nuclear property, etc. Accepting Mach's principle we are then naturally led to suspect that the interaction of any body with the remainder of the universe responsible for the inertial mass of the body is of **gravitational** origin. The first to propose this idea seems to have been the Friedlander brothers in 1896 (Friedlander and Friedlander, 1896).

Later on it was taken up by Einstein as the basis of his general theory of relativity. With his pendulum experiments Newton showed that the proportionality between inertial and gravitational masses (or between inertia and weight, as he expressed it), was established to within one part in 10^3. With Eötvos experiments at the turn of the century the precision improved to one part in 10^8. Nowadays it is known to be true to within one part in 10^{12}. It is difficult to accept this remarkable result as a simple coincidence. Although by itself it does not prove anything, this proportionality is highly suggestive in favour of Mach's principle and in indicating the nature of the interaction responsible for inertia.

There is a general program to implement quantitatively Mach's principle. First of all several concepts should not be introduced in the beginning: absolute space, absolute time, absolute motion, inertial frame or inertial reference system, inertial mass, inertial force, etc. Only the primitive concepts of electrical charge and gravitational mass should appear. To describe locations and motions, only relational quantities like r_{ij}, $\dot{r}_{ij}$, $\ddot{r}_{ij}$, ..., $d^n r_{ij}/dt^n$, ... should be introduced. Only a relational equation of motion should appear. Due to the enormous success of Newtonian mechanics we should arrive at results similar to his, and specially similar to his three laws of motion. When we perform this analogy we should be able to identify the previous concepts (inertial mass, inertial frame, ...) in the new mechanics. We should also be able to explain several facts like: the proportionality between inertial and gravitational masses, the fact that the frame of the distant universe is the best inertial frame we have, the observation that centrifugal and Coriolis forces appear in any frame in which the universe as a whole is spinning, etc.

For an excellent discussion of Mach's principle in a historical perspective see (Barbour, 1989). Good discussions of Mach's principle can be found in (Sciama, 1953; Phipps, 1978).

In the next Section we describe briefly how a Weber's law applied to gravitation can implement all of these features. This is one of the main reasons why Weber's law has been brought once more to the forefront of modern science.

7.7. The Mach-Weber Model

Although Mach developed the key ideas of the previous Sections, he did not propose any specific model of how to implement them quantitatively. Although he dealt with and published in many branches of physics (mechanics and gravitation, optics, thermodynamics) we are not aware that he ever mentioned Weber's electrodynamics. We do not know, either, any reference of Einstein to Weber's force or potential energy. The first to propose a Weber's law for gravitation in order to implement Mach's principle seems to have been Friedlander in 1896 (Friedlander and Friedlander, 1896, p. 17, footnote). Then W. Hofmann in 1904 proposed to replace the kinetic energy $mv^2/2$ by a symmetric law $L = kMmf(r)v^2$, where k is a constant and $f(r)$ some function to be determined. In this last expression v is the relative speed between masses m and M. The usual result $mv^2/2$ should be recovered after integrating L over all the masses in the universe (Norton, 1993). In this century we have Reissner and especially E. Schrödinger considering relational quantities in gravitation to implement Mach's principle, (Reissner, 1914 and 1915; Schrödinger, 1925). They arrived independently at a potential energy very similar to Weber's, (7.38), without apparently being aware of Weber's electrodynamics or of Weber's work. Edwards worked explicitly with relational quantities and with analogies between electromagnetism and gravitation (Edwards, 1974). Once more Weber's electrodynamics is not considered. Barbour; and Barbour and Bertotti opened new lines of research working not only with relational quantities but with intrinsic derivatives and with the relative configuration space of the universe, RCS (Barbour, 1974; Barbour and Bertotti, 1977 and 1982). Eby followed their work and dealt with a Lagrangian energy like (7.43) to implement Mach's principle, although he did not mention Weber's work (Eby, 1977). Ghosh worked with closely related ideas, although not being aware of Weber's force (Ghosh, 1984, 1986 and 1991). His force law is similar to Weber's, although it has a new velocity dragging term which leads to some interesting and reasonable results. More recently we have Wesley and a direct use of Weber's law to implement Mach's principle (Wesley, 1990 c and 1991, Chapter 5). He also worked with Schrödinger's potential energy without being aware of his work. Our own work in gravitation is along these lines (Assis, 1989 a, 1992 e).

There are two main postulates in the Mach-Weber model. The first one can be expressed in two ways: (A) The sum of all interaction energies (gravitational, electromagnetic, elastic, nuclear, etc.) in the universe is a constant (the same) in all frames of reference; (B) The sum of all forces of any nature acting on any body is always zero in all frames of reference. The first way may be called the principle of the conservation of energy, while the second way may be called the principle of dynamical equilibrium. The second postulate is related with the gravitational interaction, stating that it is given by (7.38) and (7.39), or by (7.44) and (7.45).

Let us suppose a body 1 of gravitational mass m_{g1} and charge q_1 inside a spherical shell of radius R, thickness dR, with an isotropic matter density $\rho(R)$, spinning with an angular velocity $\vec{\omega}(t)$ relative to an arbitrary frame of reference S. The center of the stationary (but spinning) shell is at the origin O of S. The point mass 1 is located at $\vec{r}_1$ and moves with velocity $\vec{v}_1 = d\vec{r}_1/dt$ and acceleration $\vec{a}_1 = d\vec{v}_1/dt = d^2\vec{r}_1/dt^2$ relative to the origin O of S. Following Section 7.3 we can integrate the gravitational potential energy (7.38) and the force (7.39) exerted by the shell on body 1. When body 1 is inside the shell this yields (Assis, 1989 a):

$$dU = -4\pi G m_{g1}\rho(R)RdR\left(1 - \frac{\xi}{6}\frac{v_1^2 - 2\vec{v}_1\cdot(\vec{\omega}\times\vec{r}_1) + (\vec{\omega}\times\vec{r}_1)\cdot(\vec{\omega}\times\vec{r}_1)}{c^2}\right) , \quad (7.46)$$

$$d\vec{F} = -\frac{4\pi}{3}G\frac{\xi}{c^2}m_{g1}\rho(R)RdR\left[\vec{a}_1 + \vec{\omega}\times(\vec{\omega}\times\vec{r}_1) + 2\vec{v}_1\times\vec{\omega} + \vec{r}_1\times\frac{d\vec{\omega}}{dt}\right] . \quad (7.47)$$

If the center of the shell were localized at $\vec{R}_o$ and were moving with velocity $\vec{V}_o$ and acceleration $\vec{A}_o$ relative to S these relations would remain valid with the replacements $\vec{r}_1 - \vec{R}_o$, $\vec{v}_1 - \vec{V}_o$, $\vec{a}_1 - \vec{A}_o$ instead of $\vec{r}_1$, $\vec{v}_1$, $\vec{a}_1$, respectively, supposing that the test particle is still inside the shell ($|\vec{r}_1 - \vec{R}_o| < R$, the radius of the shell).

In order to obtain the equation for the conservation of energy and the equation of motion for body 1 we need to include its interaction with all the bodies in the universe. We can divide these interactions in two parts. (A) The first part is its interaction with local bodies (springs, charges, magnets, contact forces like friction, the gravitational force of the earth, etc.) and with anisotropic distributions of bodies surrounding it (the moon and the sun, the center of our galaxy, etc.) The energy of body 1 interacting with all these N bodies will be represented by $U_{A1} = \sum_{j=2}^{N} U_{j1}$, where U_{j1} is the energy of body j interacting with body 1. The force of all these bodies on body 1 will be represented by $\vec{F}_{A1} = \sum_{j=2}^{N} \vec{F}_{j1}$, where $\vec{F}_{j1}$ is the force exerted by body j on body 1.

(B) The second part is the interaction of body 1 with isotropic distributions of bodies which surround it. The energy and force of this second part will be represented by U_{I1} and $\vec{F}_{I1}$. It is a known fact that the universe is remarkably isotropic when measured by the integrated microwave and X-ray backgrounds, or by radio source counts and deep galaxy counts. As the earth does not occupy a central position with respect to the universe, this fact suggests homogeneity on a very large scale ($\rho(R) = \rho_o =$ constant). Due to the great distance between the galaxies and to their charge neutrality, they can only interact significantly with any distant body through gravitation. From (7.46) and (7.47) we find U_{I1} and $\vec{F}_{I1}$ (force on m_{g1} due to this isotropic and homogeneous distribution of galaxies which is rotating with angular velocity $\vec{\omega}_U$ relative to the frame of reference S):

$$U_{I1} = \Phi \left[\frac{-3}{\xi} m_{g1} c^2 + m_{g1} \frac{v_1^2 - 2\vec{v}_1 \cdot (\vec{\omega}_U \times \vec{r}) + (\vec{\omega}_U \times \vec{r}_1) \cdot (\vec{\omega}_U \times \vec{r}_1)}{2} \right] , \qquad (7.48)$$

$$\vec{F}_{I1} = -\Phi m_{g1} \left[\vec{a}_1 + \vec{\omega}_U \times (\vec{\omega}_U \times \vec{r}_1) + 2\vec{v}_1 \times \vec{\omega}_U + \vec{r}_1 \times \frac{d\vec{\omega}_U}{dt} \right] , \qquad (7.49)$$

where

$$\Phi = \frac{4\pi}{3} G \frac{\xi}{c^2} \int_0^{c/H_o} \rho(R) R dR = \frac{2\pi}{3} \xi \frac{G\rho_o}{H_o^2} . \qquad (7.50)$$

In this last equation H_o is Hubble's constant and c/H_o is the radius of the known and observable universe.

If we had utilized (7.44) and (7.45) we could have integrated R from zero to infinity, without divergences. Then we would obtain eqs. (7.48) and (7.49) with $A \equiv 4\pi\xi G\rho_o / 3H_o^2$ instead of Φ, if in (7.44) and (7.45) $\alpha = H_o/c$ (Assis, 1992 e).

Application of the first postulate of the Mach-Weber model yields

$$U_{A1} + U_{I1} = \sum_{j=2}^{N} U_{j1} + \Phi \left[\frac{-3}{\xi} m_{g1} c^2 \right.$$

$$\left. + \, m_{g1} \frac{v_1^2 - 2\vec{v}_1 \cdot (\vec{\omega}_U \times \vec{r}_1) + (\vec{\omega}_U \times \vec{r}_1) \cdot (\vec{\omega}_U \times \vec{r}_1)}{2} \right] = \text{constant} \; , \qquad (7.51)$$

$$\vec{F}_{A1} + \vec{F}_{I1} = \sum_{j=2}^{N} \vec{F}_{j1} - \Phi m_{g1} \left[\vec{a}_1 + \vec{\omega}_U \times (\vec{\omega}_U \times \vec{r}_1) + 2\vec{v}_1 \times \vec{\omega}_U + \vec{r}_1 \times \frac{d\vec{\omega}_U}{dt} \right] = 0 \; . \; (7.52)$$

Eq. (7.51) is analogous to the classical equation for the conservation of kinetic plus potential energy in a non inertial frame of reference. Eq. (7.52) is analogous to Newton's second law of motion in a non inertial frame of reference. This identification will be complete if $\Phi = 1$ or $3H_o^2 = 2\pi\xi G\rho_o$. This remarkable relation connecting three independent magnitudes of physics (G, H_o and ρ_o) is a necessary consequence of any model trying to implement Mach's principle. It is then an important result of the Mach-Weber model. And it has been known to be approximately true (with ξ between 1 and 20) since the 1930's with Dirac's great numbers. The value of G is $6.67 \times 10^{-11} Nm^2/kg^2$ while $\rho_o/H_o^2 \simeq 4.5 \times 10^8 kgs^2/m^3$ (Börner, 1988, Sections 2.2 and 2.3, pp. 44 - 74). The greatest uncertainty is in the value of ρ_o/H_o^2, which is not yet accurately known. From now on we will take $\Phi = 1$ (or $A = 1$, if we had chosen (7.44), (7.45), $\alpha = H_o/c$, and had integrated to infinity).

If we are in a frame of reference in which the universe as a whole (the frame of distant galaxies) is stationary and not rotating, (7.51) and (7.52) yield

$$\sum_{j=2}^{N} U_{j1} + \frac{m_{g1} v_1^2}{2} = \text{constant} \; , \qquad (7.53)$$

$$\sum_{j=2}^{N} \vec{F}_{j1} - m_{g1} \vec{a}_1 = 0 \; . \qquad (7.54)$$

These are analogous to the equations of Newtonian mechanics. But now we have **derived** the kinetic energy and Newton's second law of motion. The kinetic energy is seen in the Mach-Weber model as another interaction energy comparable to any of the

U_{j1}'s. It is an energy of gravitational origin arising from the relative motion between m_{g1} and the universe as a whole. It is not frame-dependent anymore as it has the same numerical value (although not the same form) in all frames of reference. If we were in another frame of reference in which the universe as a whole were translating with velocity $\vec{V}_U$ and translational acceleration $\vec{A}_U$ we would get $|\vec{v}_1{}' - \vec{V}_U|^2$ instead of v_1^2 in (7.53), and $\vec{a}_1{}' - \vec{A}_U$ instead of $\vec{a}_1$ in (7.54). Here $\vec{v}_1{}'$ and $\vec{a}_1{}'$ are the velocity and acceleration of body 1 in this new frame. For instance, in the rest frame of the particle we would have $\vec{v}_1{}' = 0$, but $\vec{V}_U = -\vec{v}_1$, so that $|\vec{v}_1{}' - \vec{V}_U|^2 = v_1^2$, as before. From (7.54) we see that $-m\vec{a}$ is a real gravitational force which arises when there is a relative acceleration between m_{g1} and the universe as a whole.

These simplified equations, (7.53) and (7.54), have this form only in the rest frame of the universe. Identifying (7.54) with Newton's second law of motion explains at once why the best inertial frame we have is the frame of distant galaxies.

This identification of (7.54) with Newton's second law, or of (7.53) with the classical equation for the conservation of kinetic plus potential energy, explains the proportionality between inertial and gravitational masses of Newtonian mechanics. The inertial mass concept was never introduced in the Mach-Weber model. The second terms in the left hand side of (7.53) and (7.54) arose from gravitational interactions, so that m_{g1} is still the gravitational mass of body 1. Only when we identify these terms with Newtonian mechanics, where we have $m_{i1}v_1^2/2$ and $m_{i1}\vec{a}_1$, m_{i1} being the inertial mass of body 1, does it become clear that these "kinetic" expressions of Newtonian mechanics have a gravitational origin. Newtonian mechanics gains a new meaning and clear understanding in the Mach-Weber model. In this model we do not need to postulate the proportionality or equality between m_{g1} and m_{i1}, as is necessary to do in Einstein's general theory of relativity. Here this result is a consequence of the model.

In a frame in which the universe as a whole is spinning with $\vec{\omega}_U(t)$ (7.52) yields

$$\sum_{j=2}^{N} \vec{F}_{j1} = m_{g1} \left[\vec{a}_1 + \vec{\omega}_U \times (\vec{\omega}_U \times \vec{r}_1) + 2\vec{v}_1 \times \vec{\omega}_U + \vec{r}_1 \times \frac{d\vec{\omega}_U}{dt} \right] . \qquad (7.55)$$

This shows that the centrifugal and Coriolis forces are not fictitious forces, but real ones arising from the gravitational interaction of m_{g1} with the spinning universe. This is in complete agreement with Mach's ideas, as we have shown that 'rotating the heaven of fixed stars, centrifugal forces arise!'

Schrödinger obtained (7.53) in 1925. We obtained (7.51) and (7.52) in 1989 and 1992. In his work of 1925 Schrödinger obtained another important result which has been rediscovered independently by Wesley in 1990. What they proposed was a gravitational potential energy given by (Schrödinger, 1925; Wesley, 1990 c):

$$U = \beta\frac{m_1m_2}{r_{12}} + \gamma\frac{m_1m_2}{r_{12}}\frac{1}{(1-\dot{r}_{12}^2/c^2)^{3/2}} \ . \qquad (7.56)$$

Schrödinger proposed $\beta = -3G$ and $\gamma = 2G$, while Wesley took $\beta = -4G/3$ and $\gamma = G/3$.

The force is obtained by $\vec{F}_{21} = -\hat{r}_{12}dU/dr_{12}$ or by $dU/dt = -\vec{v}_{12}\cdot\vec{F}_{21}$. This yields

$$\vec{F}_{21} = \beta m_1m_2\frac{\hat{r}_{12}}{r_{12}^2} + \gamma m_1m_2\frac{\hat{r}_{12}}{r_{12}^2}\left(1-\frac{\dot{r}_{12}^2}{c^2} - 3\frac{r_{12}\ddot{r}_{12}}{c^2}\right)\left(1-\frac{\dot{r}_{12}^2}{c^2}\right)^{-5/2} \ . \qquad (7.57)$$

Integrating for the whole stationary and non rotating universe interacting with m_{g1} as above yields:

$$U_{I1} = 2\pi\frac{m_{g1}\rho_oc^2}{H_o^2}\left(\beta + \frac{\gamma}{\sqrt{1-v_1^2/c^2}}\right) \ , \qquad (7.58)$$

$$\vec{F}_{I1} = -2\pi\gamma\frac{\rho_o}{H_o^2}\left[\frac{m_{g1}\vec{a}_1}{\sqrt{1-v_1^2/c^2}} + \frac{m_{g1}\vec{v}_1(\vec{v}_1\cdot\vec{a}_1)}{c^2(1-v_1^2/c^2)^{3/2}}\right]$$

$$= -2\pi\gamma\frac{\rho_o}{H_o^2}\frac{d}{dt}\left(\frac{m_{g1}\vec{v}_1}{\sqrt{1-v_1^2/c^2}}\right) \ . \qquad (7.59)$$

If we wanted to integrate to infinity it would only be needed to include an exponential in both terms on the right hand side of (7.56).

With the first postulate of the Mach-Weber model we reproduce now the relativistic dynamics, instead of Newtonian mechanics. But obviously the velocity and accelerations which appear here are not relative to an arbitrary inertial frame. They are relational quantities, that is, the velocity and acceleration of body 1 relative to the universe as a whole. That is, as we performed these calculations in the frame of reference in which the universe as a whole (the set of distant galaxies) has no translational velocity in any direction nor any rotation, the velocity and acceleration which appear in (7.58) and (7.59) are relative to this universal frame of reference. In a frame of reference in which the universe as a whole is moving with a constant velocity $\vec{V}_U$ and the test particle is moving with $\vec{v}_1'$, we would have $\vec{v}_1' - \vec{V}_U$ instead of $\vec{v}_1$ in (7.58) and (7.59). We also do not have any variation of mass with velocity. These results only indicate that the gravitational force of the universe on any body depends not only on its acceleration relative to the universe, but also on its velocity. This identification of the Mach-Weber mechanics with the Newtonian or relativistic ones will be complete provided that (see the reasoning in the paragraph below (7.52))

$$-2\pi \frac{\gamma}{\beta + \gamma} \frac{G\rho_o}{H_o^2} = 1 \ . \tag{7.60}$$

As we have seen, this will be approximately true with the choices of β and γ made by Schrödinger and Wesley.

It is remarkable that with a relational law modelled on Weber's one we can derive a dynamics analogous to the Newtonian or relativistic one. These results of Schrödinger and Wesley are beginning to be explored only now, so that many new results should appear in the near future. But certainly they indicate a very fruitful line of research which is leading in the right direction.

One last remark. Einstein in 1922 pointed out some consequences which any model of interaction satisfying Mach's principle should lead to, namely (Einstein, 1980, pp. 95 - 96; Reinhardt, 1973):

1) The inertial mass of a body should increase with the agglomeration of masses in its neighborhood.

2) A body in an otherwise empty universe should have no inertia.

3) A body should experience an acceleration if nearby bodies are accelerated. The accelerating force should be in the same direction as the acceleration of the latter.

4) A rotating body should generate inside it a Coriolis force.

As is well known, these four consequences do not follow completely from Einstein's general theory of relativity, as he himself discovered later on (Reinhardt, 1973; Raine, 1981). On the other hand all four consequences follow completely from the Mach-Weber model, as is easily seen with the results of this Section. We discussed this in details in (Assis, 1993 b).

Chapter 8 / General Discussion

8.1. Weber's Electrodynamics and Maxwell's Equations

We have seen in Chapters 3 to 5 that from Weber's force we can derive the set of Maxwell's equations, namely: (2.48) to (2.51). Here we want to present some critical remarks on these derivations.

First as regards Gauss's law (2.48). In Section 3.2 we showed that when there is no motion between the interacting charges then Weber's force reduces to Coulomb's one. And in Section 2.6 we had shown how to derive Gauss's law from Coulomb's force after defining an electric field by (2.15). Everything is all right here. However, this is a limited proof valid only when there is no motion between the charges. As we saw in Section 6.6 a neutral and stationary current carrying wire will exert a force on a stationary charge nearby according to Weber's force. This force may be expressed in terms of a motional electric field arising from the stationary positive ions in the lattice of the metal and from the drifting electrons responsible for the current. This motional electric field in the case of a long filiform straight wire along the z axis was found to be given by:

$$\vec{E}_M(\vec{\rho}) = -\frac{\lambda_{2+}\vec{\rho}}{4\pi\varepsilon_o\rho^2}\frac{V_D^2}{c^2} , \tag{8.1}$$

where ρ is the distance to the z axis, Figure 8.1.

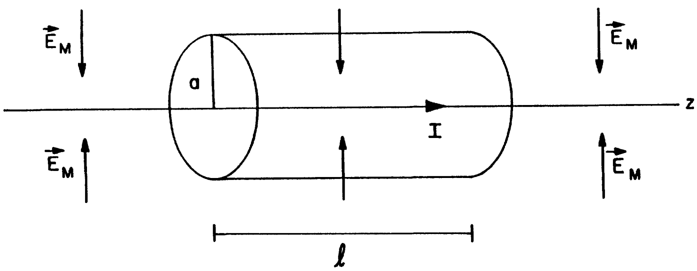

Figure 8.1

Integrating this motional electric field over a closed cylindrical surface centered on the z axis of radius a and length l, Figure 8.1, yields

$$\oiint_S \vec{E}_M \cdot d\vec{a} = -\frac{\lambda_{2+} l}{2\varepsilon_o} \frac{V_D^2}{c^2} \neq 0 \ . \tag{8.2}$$

This shows that despite the net charge neutrality of the wire there will be a non zero net flux of the motional electric field. This is obviously a violation of Gauss's law (2.48).

This shows that the first of Maxwell's equations can be derived from Weber's electrodynamics only in very stringent conditions, namely, when all the charges are at rest relative to one another. If this condition is not fulfilled Gauss's law should not be valid anymore according to Weber's theory. To our knowledge Gauss himself never said that his law (2.48) should be valid when the charges have relative motion. As a matter of fact if we apply Gauss's own force, see Appendix B, the result will be the same as with Weber's force for this situation of a straight wire carrying a constant current, namely, (8.2).

We now discuss the law for the non existence of magnetic monopoles, (2.50). We derived it in Section 4.7 utilizing the integrated form of Ampère's force, (4.52). The first remark is that to derive Ampère's force between current elements from Weber's force we supposed charge neutrality of the current elements. This is the first restriction which is involved in the derivation of (2.50) from Weber's electrodynamics. This means that we have shown that (2.50) can be derived from Weber's theory only when there is charge neutrality of the currents. The general case of current carrying wires which have a net charge has not been treated here. A more relevant limitation is that (4.52) was derived only for **closed** material circuits. We did not discuss here the case of open material circuits. When we talk of open circuits we mean a circuit which is not closed by matter. Examples (Figure 8.2): (A) We charge a small glass ball by friction and throw it in space; (B) Charging or discharging a capacitor through an external circuit when there is a large separation d between its plates; (C) Polarizing alternately a linear antenna of finite length by external sources; etc.

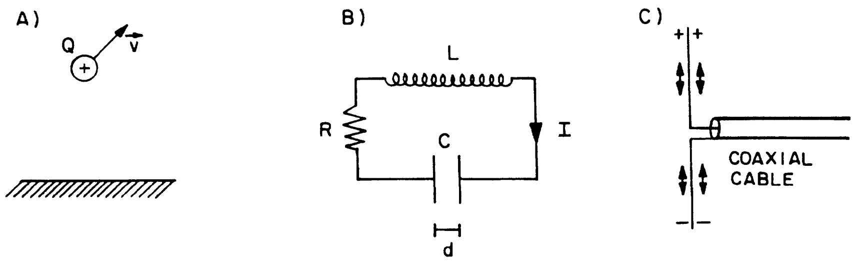

Figure 8.2

In classical electromagnetism these are also examples of closed circuits due to the displacement current. But with Weber's electrodynamics the displacement current does not appear naturally so that we can not state that (4.52) will remain valid in all these cases. This means that the derivation of Section 4.7, which utilized (4.75) and (4.76), may not follow from Ampère's force (and then from Weber's electrodynamics) in all these cases. A more careful analysis is necessary in these cases.

Now the derivation of the magnetic circuital law, (2.49). The restrictions pointed out in the previous paragraph are also valid here. But there is a further remark now. In order to derive the magnetic circuital law with the term $\partial\vec{E}/\partial t$ we supposed stationary circuits so that r_{12} did not depend on time. If this were not the case then it would not be possible to go from (4.82) to (4.84) as we did, because then $\partial/\partial t$ would also operate on r_{12}. This general case will not be discussed here, but we think it is important to point out this restriction. Other aspects related with this derivation, even classically beginning with Biot-Savart's law, can be found in the important papers: (Weber and Macomb, 1989), (Jefimenko, 1989), (Griffiths and Heald, 1991).

As regards the derivation of Faraday's law of induction, once more we utilized the charge neutrality of the current elements. We also worked with closed circuits, so that as we showed in Section 4.6:

$$N_{ij} = \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{(\hat{r}_{ij} \cdot d\vec{l}_i)(\hat{r}_{ij} \cdot d\vec{l}_j)}{r_{ij}} = \frac{\mu_o}{4\pi} \oint_{C_i} \oint_{C_j} \frac{d\vec{l}_i \cdot d\vec{l}_j}{r_{ij}} = M_{ij} \ . \tag{8.3}$$

This relation was utilized in Sections 5.3 and 5.4 to derive Faraday's law of induction from Weber's force or potential energy. But it is valid only for closed mechanical circuits. For open circuits it may not be valid anymore. These cases will not be discussed here, but once more it is important to stress this point.

These aspects indicate where we need to be careful as regards the utilization of Maxwell's equations in Weber's electrodynamics.

8.2. Action at a Distance Versus Contact Action

Conceptually there is an enormous distinction between classical electromagnetism and Weber's electrodynamics. It is based on the mechanism by which the charges exert and feel the forces.

In Weber's electrodynamics what matter are the charges, their distances, relative velocities and accelerations. We speak of a direct force between each pair of particles, no matter how far apart they are. Moreover, this interaction is considered to be instantaneous. Suppose we have two charges which are initially at rest in the laboratory, or one in the laboratory and the other in the moon. If we move only one of them relative to the laboratory, increasing its distance to the second one, Weber's force and potential energy will be modified instantaneously. This is because they depend on the distance between them (r_{ij}) and if this changes, the force and energy will change automatically as there is no retardation in Weber's original expressions. This is what characterizes an action at a distance theory. Newton's original law of gravitation is of this kind as well.

According to Faraday and Maxwell's picture, one charge does not interact directly with any other, but only through a medium which was called the ether. Each charge would then interact only with this ether. Nowadays we do not speak of the ether anymore, but we have the field taking its place. So the classical picture is that each charge generates electric and magnetic fields, these fields are propagated at a finite speed c from each charge, and they interact with the other charges when they reach them. This is called contact action.

These are the two basic mechanisms which have always been proposed to explain the motion of bodies. (For a good discussion of these two mechanisms see (Graneau and Graneau, 1993)). Philosophically it is difficult to accept action at a distance. For instance, take Newton's law of gravitation (2.8). How can one of the masses know how much matter there is in the other body and what is its distance and direction from itself if they are far apart? If there is nothing between the masses how can the information reach one of them so that it can react to the presence of the other? As Newton said in his famous letter to Bentley of 1693: "The idea that one body may act upon another at a distance through a vacuum without the mediation of anything else by or through which their action or force may be conveyed from one to another is to me so great an absurdity that I believe no man who has in philosophical matters any competent faculty of thinking can ever fall into it." These letters are reproduced in (Newton, 1978, pp. 269 - 312).

On the other hand it is much easier to deal with material points and their distances and velocities, without worrying about the intervening medium, if there is such a medium. We only need to specify the masses, charges, locations and motions and then everything else follows from the formulas for the forces and energies.

It pleases the mind to speak of mechanisms, connections between the bodies, etc. However, when we speak of abstract entities like fields it is often difficult to form mental pictures of their nature. Let us take the magnetic field of a bar magnet, for instance. If we translate the magnet relative to the laboratory at $1m/s$ does the magnetic field follow the magnet or stay stationary relative to the earth? Can the magnetic field travel at such a low velocity or only at c? And if we spin the magnet around its axis, does the magnetic field rotate with it? We have also other natural questions. For example, how can something immaterial interact with matter?

In theories of action at a distance the most natural hypothesis is that of action and reaction between the bodies. This great principle required the genius of Newton to be spelled out and applied so cleverly in physics. When the interaction is mediated by the field we do not need to have anymore action and reaction between the bodies, although action and reaction between each body and the local field interacting with it is often required. Feynmann discussed these ideas clearly in Volume 1, Sections 10.1 and 10.5; as well as in Volume 2, Sections 26.2 and 27.6 of his famous book (Feynmann, Leighton and Sands, 1964).

Which one of these two basic mechanisms is the correct one? We do not know and maybe we will never know. From a practical point of view what is important is the correct prediction of experiments. From this point of view we have two completely different theories which have been able to explain most of the electromagnetic phenomena with an enormous success as has been known since Maxwell's time. This was the reason of his wonder in the Preface of his book from which we began this work.

Let us discuss some specific details. Although Weber's model is an action at a distance theory, the quantity $c = 1/\sqrt{\mu_o \varepsilon_o}$ appeared for the first time in his force law. He was also the first to measure this quantity and found its value equal to the light velocity. He and Kirchhoff were also the first to derive the wave equation for a signal propagating along a conducting circuit with the velocity c. And they arrived at this result working with Weber's action at a distance theory. These are historical facts which can not be changed. Later on these results could be connected and derived in a field theory, but this was a later development. How could Weber and Kirchhoff arrive at their result? There are two main reasons. Although the interaction is supposed to be instantaneous, each of the interacting charges is supposed to have inertia, so that their reaction to the applied force (acquired velocities and accelerations, etc.) is a function of their inertial masses. The second reason is that they were dealing with a many body system, a wire composed of an enormous number of charges. The wire in this case would behave as an intermediary medium or carrier of information. Newton derived that sound would propagate at a finite velocity in air utilizing his action at a distance mechanics. The same can be said of d'Alembert and the propagation of perturbation along a stretched string. These results were obtained without the use of time retardation, of an ether, of a displacement current, or of a field propagating at a finite velocity. The reason seems to be the same in all cases: It is a many body system (the wire, the air, the string) in which the simultaneously interacting bodies have inertia. Although the interaction of any two particles may be considered to be instantaneous, the collective behaviour (macroscopic wave, etc.) has a finite characteristic velocity.

Then the question naturally arises: Is it possible to derive the finite velocity of light with an action at a distance theory? We do not know a precise answer, but we believe that this is possible in principle. Newton believed in the corpuscular theory of light and talked of a direct interaction between these corpuscles and matter, as is evident from many passages in the *Optics*. For instance: Book Two, Part III, Prop. VIII, "*The cause of reflexion is not the impinging of light on the solid or impervious parts of bodies, as is commonly believed.*" (...) "So then it remains a problem, how glass polished by fretting substances can reflect light so regularly as it does. And this problem is scarce otherwise to be solved, than by saying, that the reflexion of a ray is effected, not by a single point of the reflecting body, but by some power of the body which is evenly diffused all over its surface, and by which it acts upon the ray without immediate contact. For that the parts of bodies do act upon light at a distance shall be shewn hereafter." Book Three, *Query I*: "Do not bodies act upon light at a distance, and by their action bend its rays; and is not this action (*caeteris paribus*) strongest at the least distance?" (Newton, 1952 b, pp. 485 - 488 and p. 516).

Nowadays we know that light interacts with charges and vice-versa (Faraday rotation, Compton scattering, etc.) Although the photon has no net charge it is natural to suppose that it is composed of charges. There is evidence in this direction from the fact that it has spin and also in pair production (a gamma ray giving rise to a pair electron-positron), etc. If this is the case then when we accelerate charges in a wire the intervening medium (a gas of photons) will respond electromagnetically to these motions and a signal will propagate along this medium, in analogy with what Weber and Kirchhoff showed for a signal propagating along a wire. The difference is that now the medium is composed of mobile corpuscles, so that it would be more analogous to an electromagnetic signal propagating in a gaseous plasma. Obviously these ideas are only qualitative but we want here only to indicate a possible program on how to follow from now on. One of the main tasks of those working with Weber's electrodynamics will be to explain radiation phenomena (antennae, radio communication, etc.) from this approach.

The analogy here is with sound in air. The molecules in air move with a root mean square velocity given by $\sqrt{3k_BT/m}$, where k_B is Boltzmann's constant, T the temperature of the gas and m the mass of the molecules. Sound waves, on the other hand, move with a velocity given by $\sqrt{\gamma k_BT/m}$, where $\gamma = C_P/C_V$ is the ratio of specific heats at constant pressure and volume, respectively. As $1 < \gamma < 5/3$ we can see that these two velocities are essentially the same, although they represent velocities of entities which are completely different from one another (individual molecules and a collective behaviour of these molecules). Our idea is then that the photons move with velocity c and when they are absorbed or emitted by atoms and molecules this happens discretely, one by one. There would be electromagnetic waves moving also at c in this gas of photons, as there are sound waves moving in a gas of molecules. These electromagnetic waves would represent a collective behaviour of this gas of photons and would be responsible for the wave properties of light (interference, etc.) The main difference between electromagnetic and sound waves in this picture is that in sound we have longitudinal waves while in light we have transverse waves. But the main idea is the same. The main point we want to stress here is that there is always a medium between macroscopic charges, magnets and current carrying circuits: a gas of photons, instead of a complete vacuum. This is the reason why we believe it will be possible to derive electromagnetic waves propagating at a finite velocity between two circuits or antennae beginning only with Weber's action at a distance theory.

8.3. Weber's Electrodynamics in Terms of Fields and Retarded Time

Although the fields are not an essential part of Weber's electrodynamics, it is possible to write Weber's force in terms of fields. This has been done by Wesley, (Wesley, 1987 a, 1990 a). The first idea is to write Weber's force (3.24) replacing q by ρdV and $\rho\vec{v}$ by $\vec{J}$. Neglecting the velocity squared forces (the terms with v^2 and $(\hat{r}\cdot\vec{v})^2$) this yields:

$$\frac{d^6\vec{F}_{ji}}{dV_i dV_j} = \frac{\hat{r}_{ij}}{4\pi\varepsilon_o r_{ij}^2}\left[\rho_i\rho_j - \frac{2\vec{J}_i\cdot\vec{J}_j}{c^2}\right.$$

$$\left. + \frac{3(\hat{r}_{ij}\cdot\vec{J}_i)(\hat{r}_{ij}\cdot\vec{J}_j)}{c^2} + \frac{\rho_j\vec{r}_{ij}}{c^2}\cdot\frac{\partial\vec{J}_i}{\partial t} - \frac{\rho_i\vec{r}_{ij}}{c^2}\cdot\frac{\partial\vec{J}_j}{\partial t}\right] . \tag{8.4}$$

After integrating over a fixed volume V_j Wesley obtained:

$$\frac{d^3\vec{F}_{ji}}{dV_i} = -\rho_i\nabla\phi + \vec{J}_i\times(\nabla\times\vec{A}) - \rho_i\frac{\partial\vec{A}}{\partial t} - \vec{J}_i\nabla\cdot\vec{A}$$

$$+ \frac{\phi}{c^2}\frac{\partial\vec{J}_i}{\partial t} + (\vec{J}\cdot\nabla)\nabla\Gamma + \rho_i\nabla\frac{\partial\Gamma}{\partial t} - \left[\left(\frac{\partial\vec{J}_i}{\partial t}\right)\cdot\nabla\right]\frac{\vec{G}}{c^2} , \tag{8.5}$$

where

$$\phi \equiv \frac{1}{4\pi\varepsilon_o}\int\int\int_{V_j}\frac{\rho_j(\vec{r}_j,\ t)dV_j}{r_{ij}} , \tag{8.6}$$

$$\vec{A} \equiv \frac{\mu_o}{4\pi}\int\int\int_{V_j}\frac{\vec{J}_j(\vec{r}_j,\ t)dV_j}{r_{ij}} , \tag{8.7}$$

$$\Gamma \equiv \frac{\mu_o}{4\pi}\int\int\int_{V_j}\hat{r}_{ij}\cdot\vec{J}_j(\vec{r}_j,\ t)dV_j , \tag{8.8}$$

$$\vec{G} \equiv \frac{1}{4\pi\varepsilon_o}\int\int\int_{V_j}\hat{r}_{ij}\rho_j(\vec{r}_j,\ t)dV_j . \tag{8.9}$$

Here ϕ and $\vec{A}$ are the usual electric and magnetic potentials and Γ and $\vec{G}$ are two new potentials. It should be emphasized that to arrive at this result the velocity squared terms were neglected.

Wesley even extended this result introducing time retardation into the fields. To this end he replaced the time t in equations (8.6) to (8.9) by $t^* = t - r_{ij}/c$. It should be remarked that the first to propose the introduction of the retarded time in Weber's electrodynamics were Moon and Spencer, although they introduced it directly in Weber's force instead of introducing it in the potentials, as has been done by Wesley (see Moon and Spencer, 1954 c).

Wesley has also obtained the wave equations satisfied by these fields but here we will not go into these details as they are beyond the original form of Weber's electrodynamics. These are new lines of research which must be explored and analysed carefully before stating their validity. Neverthless they are very important as they show another possible way of overcoming the negative stigma of Weber's electrodynamics related of its being an action-at-a-distance theory.

An alternative procedure of arriving at time delays in an action-at-a-distance theory has been presented in (Graneau, 1987 d).

It should be mentioned here that Weber's law by itself, with its dependences on velocity and acceleration, already models a delay in the propagation of interactions. This was discussed in (Sokol'skii and Sadovnikov, 1987).

The reality of the finite velocity of propagation of electromagnetic effects between two current carrying circuits (two antennae) was shown by Hertz, the former student of Helmholtz, in his famous experiments of 1885-89 (Hertz, 1962; Mulligan, 1987 and 1989). Although they have usually been regarded as the definitive confirmation of Maxwell's theory, this is not true. For instance, Ritz's ballistic theory has been proved to be equally consistent with them (O'Rahilly, 1965, Vol. 1, pp. 230 - 233; and Vol. 2, pp. 499 - 512). To our knowledge these experiments were never analysed from a point of view based on Weber's law. So there is the possibility that Weber's electrodynamics will also prove to be compatible with them. This is an open question for the time being.

This is a fascinating subject and we should always keep an open mind to explore all possibilities in order to discover new and important results.

8.4. Weber's Law and Plasma Physics, Quantum Mechanics, Nuclear Physics, Etc.

In this book we discussed Weber's law as applied to electromagnetism and gravitation. Here we want to indicate a few other connections of Weber's law.

Weber's original work was devoted to electrodynamics. It would be very important to develop the equivalent to the magnetohydrodynamic equations (MHD equations) of plasma physics beginning with Weber's force instead of Lorentz's one. If this is done some interesting comparisons between these two theories will be possible.

In our work we applied the motional electric field of Section 6.6 as a possible model to explain the anomalous diffusion of electrons in tokamaks (Assis, 1991 b). Ampère's force between current elements has been discussed more directly as regards plasma physics, in particular in connection with disruptions in tokamaks: (Nasilowski, 1985; Rambaut and Vigier, 1990). The exploding wire phenomenon has also been discussed in this analogy. In particular due to the fact that if we have deuterium in the solid, liquid or gaseous material where current is flowing, when the current breaks in many pieces there is a simultaneous emission of neutrons. This happens in dense Z-pinch experiments, capillary fusion, and in plasma focus devices, (Graneau and Graneau, 1992). This may indicate a possible connection between electromagnetic and nuclear forces.

We already mentioned the important paper by Pearson and Kilambi, where they discuss the analogy between Weber's force and the velocity dependent nuclear forces (Pearson and Kilambi, 1974). More results may appear in the near future.

If we have modifications of Newton's law of gravitation with velocity and acceleration terms like Weber's ones, then there may exist a connection between gravitation and temperature. If we heat a body its molecules will vibrate at a higher rate so that the average gravitational force between this body and another one may be a function of temperature. We discussed this possibility in (Assis and Clemente, 1993).

Weber himself pointed out a possible explanation of the thermoelectric effects (thermomagnetism, Peltier and Seebeck's experiments), with his force law between point charges (Weber, 1871). This important idea should be explored in more detail. He discussed these ideas in Section 20 of this paper, see especially pages 144 to 146 of the English translation.

Weber also discussed a possible explanation of the catalytic forces in chemistry with his force law: (Weber, 1871). A proper development of these ideas is also lacking.

A discussion of Weber's law in connection with quantum mechanics has been made recently by Wesley: (Wesley, 1990 d). In particular, he applied Weber theory for the hydrogen atom, inserting Weber's potential energy in Schrödinger's equation. Using the usual Schrödinger perturbation method to solve the equation he obtained the usual energy levels, but with their fine structure splitting, without the need of mass change with velocity. A similar result had been obtained by Bush using the old Sommerfeld quantum theory with elliptical electrom orbits (Bush, 1926).

These are only some links between Weber's law and certain areas of knowledge. With the renewed interest in Weber's electrodynamics we may expect that all of these branches will be explored anew in the near future.

8.5. Limitations of Weber's Electrodynamics

In this book we analysed several positive aspects of Weber's electrodynamics: It is a completely relational theory (has the same value to all observers), it satisfies the principle of action and reaction in the strong form, and also the principles of conservation of linear momentum, angular momentum and energy. Moreover, we can derive from Weber's law the forces of Coulomb for point charges and of Ampère for current elements. We saw how from Weber's law we can derive for many cases Maxwell's equations: Gauss's law, law for the nonexistence of magnetic monopoles, the magnetic circuital law and Faraday's law of induction.

Despite these positive aspects it should be emphasized here that Weber's electrodynamics is only a model of interaction between charges which describes certain classes of phenomena. As such it can have limitations and its range of validity should be searched. For instance, a more complete model of interaction can include time derivatives of r_{ij} of all orders, like dr_{ij}/dt, d^2r_{ij}/dt^2, d^3r_{ij}/dt^3, etc. It may include as well powers of these derivatives, like $\dot{r}_{ij}^m$, $\ddot{r}_{ij}^n$, etc. (with m, n, ... integers). Only a careful analysis of many experimental results will determine if these terms should be included or not. If this happens to be the case then the validity of Weber's law would go only up to second order in $\dot{r}/c$, inclusive. This would mean that for charges moving at velocities comparable to that of light Weber's law should not be applied as such without warnings. As we have seen, Schrödinger proposed modifications for Weber's potential as applied to gravitation. Phipps proposed modifications of Weber's potential and force as applied to electrodynamics. Other modifications may be necessary as well.

The introduction of time retardation in Weber's electrodynamics has been mentioned in the previous Section.

Recently we have shown a possible way to derive Newton's law of gravitation from a generalized Weber's force for electromagnetism including terms of fourth and higher orders in $\dot{r}/c$, (Assis, 1992 g). We studied the interaction between two neutral dipoles in which the negative charges oscillate around the positions of equilibrium. We showed that these extra terms yield an attractive force between the neutral dipoles which can be interpreted as the usual Newtonian gravitational interaction for the following reasons: It is of the correct order of magnitude, it is along the line connecting the dipoles, it follows Newton's action and reaction law, and falls off as the inverse square of the distance. This idea only works supposing higher order terms in Weber's force between electrical charges.

Ideas of this kind may suggest how to investigate theoretically extensions of Weber's electrodynamics. But only careful experiments will give the final answer.

We began this work with Maxwell's words. We will finish it with O'Rahilly's statement, written in his masterpiece, *Electromagnetic Theory - A Critical Examination of Fundamentals* (in square brackets are our words):

"If any one man deserves credit for the synthetic idea which unifies the various branches of magnetic and electrical science, that man is Wilhelm Weber. Today even those who uphold the aether-theory or profess to be relativists accept these principles introduced or developed by him: that Ampère's idea of magnetism as due to micro-currents can account for the relevant phenomena; that electricity has an atomic structure [electric charges as particles or corpuscules]; that currents are streams of electrical particles; that Ampère's forces [between current elements] act directly between these particles and not between the conductors; that Coulomb's law must be modified for charges in motion; that, as Gauss said, action is not instantaneous; that the laws of electrodynamics [Ampère's force between current elements] and induction must be derived, by statistical summation, from a force formula for electrical particles [charges]. Even his ballistic principle, submerged for so long by aetherists and relativists, seems likely to challenge physicists once more in the developed form given to it by Walther Ritz" (O'Rahilly, 1965, Vol. 2, p. 535).

Appendix A

The Origins and Meanings of the Magnetic Force $\vec{F} = q\vec{v} \times \vec{B}$

In Chapter 2 we showed that the electric component of Lorentz's force ($\vec{F}_E = q\vec{E}$, with $\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t$) had been utilized by Kirchhoff in 1857, although Kirchhoff did not speak in terms of electric fields. The magnetic vector potential $\vec{A}$ had been introduced by Franz Neumann in 1845. We also saw that $\vec{B} = \nabla \times \vec{A}$, where $\vec{B}$ is the magnetic field. On the other hand the expression for the magnetic force had a later and more tortuous origin, and this is the subject of this appendix. We first discussed this topic in (Assis and Peixoto, 1992).

The first relevant information to be stressed is that the expression for the magnetic force appeared after Maxwell's death (1879). According to Whittaker the first to arrive at the magnetic force were J. J. Thomson (1856-1940) and O. Heaviside (1850-1925), in 1881 and 1889, respectively (Whittaker, 1973, Vol. 1, pp. 306 to 310).

One of the goals of Thomson's theoretical paper (Thomson, 1881) was to know how an electrified body is affected by a magnet. Thomson followed Maxwell's theory and in particular he utilized the idea that a displacement current ($\varepsilon\partial\vec{E}/\partial t$) produces the same effect as an ordinary conduction current $\vec{J}$, namely, a magnetic field $\vec{B}$. He supposed a uniformly charged sphere moving in a certain medium with dielectric constant ε and magnetic permeability μ, and calculated the displacement current at an external point Q. Then he calculated at another external point P the value of the vector potential $\vec{A}$ due to this displacement current in Q, and integrated over all points Q in space. But he then observed that the value of $\nabla \cdot \vec{A}$ at this point P was different from zero. Maxwell, however, always assumed $\nabla \cdot \vec{A} = 0$. Then to satisfy this condition Thomson supposed the existence of another component in $\vec{A}$, adding this component to what he had obtained for $\vec{A}$ (he did not justify the physical origin of this extra component of $\vec{A}$). Through $\vec{B} = \nabla \times \vec{A}$ he obtained the value of $\vec{B}$ at P. He then calculated the value of $\vec{H}$ in this medium, $\vec{B}/\mu$. Next he calculated the force exerted by a magnet (which generates $\vec{B}$) on a charged body which moves in this medium. To this end he calculated the interacting energy $E = \int\int\int(\vec{B}\cdot\vec{H}/2)dV$, and utilized Lagrange's equations to get the force. His final result was

$$\vec{F} = q\frac{\vec{v}\times\vec{B}}{2} . \qquad (A1)$$

This is half the present day value of classical electromagnetism. The most important aspect which we want to emphasize here is the meaning of the velocity which appears in (A1). Here Thomson was careful. He called this the "actual velocity" of the charge. On page 248 of his article he says: "It must be remarked that what we have for convenience called the actual velocity of the particle is, in fact, the velocity of the particle relative to the medium through which it is moving " ..., "medium whose magnetic permeability is μ." This shows that to Thomson the velocity $\vec{v}$ in (A1) was not the velocity of the charge relative to the ether (as this medium and the earth might be moving relative to the ether without dragging it), nor relative to the magnet, and not either the velocity relative to the observer.

Thomson extended his researches in this direction with a paper published in 1889 (Thomson, 1889).

In 1889, in another theoretical paper, Heaviside obtained (Heaviside, 1889):

$$\vec{F} = q\vec{v}\times\vec{B} . \qquad (A2)$$

The main difference of his work from Thomson's is that he included, following Fitzgerald in 1881, the convection current as a source of magnetic field. In the other aspects he followed Thomson's work. He did not make any additional comment on the velocity $\vec{v}$ in (A2). So it can be assumed that for him as well this is the velocity of the charge q relative to the medium of magnetic permeability μ and dielectric constant ε. This is even more evident from the title of his paper: On the electromagnetic effects due to the motion of electrification through a dielectric. A discussion of the works of Thomson and Heaviside can be found in (Buchwald, 1985, Appendix I: Maxwellian Analysis of Charge Convection, pp. 269 - 277).

In 1895, the theoretical physicist Lorentz presented the known expression (Lorentz, 1895; Pais, 1982, p. 125; and Pais, 1986, p. 76):

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B} \ . \tag{A3}$$

Lorentz did not comment on the works of Thomson and Heaviside, and arrived at the magnetic part of this expression from Grassmann's force, substituting $q\vec{v}$ for $Id\vec{l}$, although he did not mention Grassmann's work as well. This can be seen in Lorentz most famous book, *The Theory of Electrons* (Lorentz, 1915, pp. 14 and 15). This book is based on a course he delivered in 1906 at Columbia University, which was edited for the first time in 1909. Unfortunately Lorentz did not specify in (A3) what is the object, medium or system relative to which the velocity $\vec{v}$ of the charge q is to be understood. As Lorentz still accepted Maxwell's ether (that is, a medium in a state of absolute rest relative to the frame of the fixed stars; see (Pais, 1982, p. 111)), it is natural to suppose that for him this was the velocity of the charge q relative to this ether, and not relative to any other medium or observer. In support of this statement we have Lorentz's own words on this same page 14: "Now, in accordance with the general principles of Maxwell's theory, we shall consider this force as caused by the state of the ether, and even, since this medium pervades the electrons, as exerted by the ether on all internal points of these particles where there is a charge" (Lorentz, 1915, p. 14). A conclusive proof of this interpretation can be found in another work of Lorentz: *Lectures on Theoretical Physics* (Lorentz, 1931, Vol. 3, p. 306; se also O'Rahilly, 1965, Vol. 2, p. 566). Here Lorentz says that if a wire carrying an electric current (and thus generating $\vec{B}$), and a charge are at rest relative to the ether, then there will be no magnetic force. On the other hand if both share a common translation with velocity $\vec{v}$ relative to the ether (while the observer and the laboratory also translate with this same velocity $\vec{v}$, because he gives as an example of this velocity that of the whole earth relative to the ether), then the says that there will be a magnetic force on the test charge. In this example there is no relative motion between the test charge and the current carrying wire, the observer, nor the laboratory, but only relative to the ether.

On the other hand nowadays we utilize expression (A3) with $\vec{v}$ being the velocity of the charge q relative to an observer. This change of meaning happened after Einstein's work of 1905 on the special theory of relativity (Einstein, Lorentz, Minkowski and Weyl, 1952). In this work, after obtaining Lorentz's transformations of coordinates, Einstein applies them for the force (A3) and begins to utilize the velocity as being relative to the observer. For instance, on page 54 he gives (between square brackets are our words) the difference between the old paradigm of electromagnetism and the new one based on his theory of relativity:

"Consequently the first three equations above [for the transformation of the field components in different inertial systems] allow themselves to be clothed in words in the two following ways:

1. If a unit electric point charge [$q = 1$] is in motion in an electromotive field, there acts upon it, in addition to the electric force [$\vec{F}_E = q\vec{E}$], an "electromotive force" which, if we neglect the terms multiplied by the second and higher powers of v/c, is equal to the vector-product of the velocity of the charge and the magnetic force [$\vec{B}$], divided by the velocity of light [that is, $\vec{F}_M = q\vec{v} \times \vec{B}/c$, so that the resultant force is $\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}/c$]. (Old manner of expression).

2. If a unit electric point charge is in motion in an electromagnetic field, the force acting upon it is equal to the electric force [$\vec{F}' = q\vec{E}'$] which is present at the locality of the charge, and which we ascertain by transformation of the field [$O \rightarrow O'$] to a system of co-ordinates [O'] at rest relatively to the electrical charge [$\vec{v}\,' = 0$, so that $q\vec{E} + q\vec{v} \times \vec{B}/c = q\vec{E}'$, where all magnitudes with $'$ refer to the fields in the system O' which moves with velocity $\vec{v}$ relative to O]. (New manner of expression)."

It is instructive to see this conceptual change in one of the most utilized expressions of physics. It should be stressed that all of these works (Thomson, Heaviside, Lorentz, Einstein) were of a theoretical nature and realized after Maxwell's death. This change in the meaning of $\vec{v}$ in (A3) is very strange, confusing and unusual in physics.

Appendix B

Alternative Formulations of Electrodynamics

In this book we dealt with mainly Weber's theory and classical electrodynamics (Maxwell's equations, Lorentz's force, retarded and Liénard-Wiechert potentials, etc.) In this Appendix we present some other approaches which have been proposed by some important scientists like Gauss, Riemann, Clausius and Ritz. We will utilize the standard notation of Chapters 2 and 3. All velocities and accelerations are relative to an inertial frame S.

Gauss discovered in 1835 a force from which he could derive the forces of Coulomb and Ampère as special cases. It is given by (Gauss, 1877, Vol. 5, [13]: Grundgesetz für alle Wechselwirkungen galvanischer Ströme, pp. 616 - 617; Maxwell, 1954, Vol. 2, article [851], p. 483; O'Rahilly, 1965, Vol. 1, p. 226; Jungnickel and McCormmach, 1986, Vol. 1, pp. 130 and 140):

$$\vec{F}_{21}^G = \frac{q_1 q_2}{4\pi\varepsilon_o}\frac{\hat{r}_{12}}{r_{12}^2}\left[1 + \frac{\vec{v}_{12}\cdot\vec{v}_{12}}{c^2} - \frac{3}{2}\frac{(\hat{r}_{12}\cdot\vec{v}_{12})^2}{c^2}\right] = -\vec{F}_{12}^G \ . \qquad (B1)$$

If $\vec{v}_{12} = 0$ we recover Coulomb's force. Adding the forces of the positive and negative charges of a current element on the positive and negative charges of the other current element yields Ampère's force (4.24).

Gauss's force was only published in 1877, in his Collected Works. It looks like Weber's force (3.24) but it lacks the acceleration term in $\vec{a}_{ij}$. For this reason we can not derive Faraday's complete law of induction from it as it will not generate the term with dI/dt. It is also inconsistent with the principle of conservation of energy as it can not be derived from a potential energy. Despite these facts it complies with the principle of action and reaction. Moreover, the force is along the line connecting the charges. In this form the force involves no time retardation as it depends on the mutual and instantaneous distances and velocities of the interacting charges. But Gauss believed that the real keystone of electrodynamics would be an interaction which depended on the relative distances and velocities of the charges, and which were also retarded in time. Nowadays some of the main followers of this general Gaussian program for electrodynamics are Moon, Spencer and collaborators (Moon and Spencer, 1954 c; Moon, Spencer, Uma and Mann, 1991; Spencer and Uma, 1991).

Riemann proposed the following force, in a text which was published posthumously in 1876 (see (Riemann, 1977), for an English translation; O'Rahilly, 1965, Vol. 2, p. 527):

$$\vec{F}_{21}^R = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}^2} \left[\left(1 + \frac{\vec{v}_{12} \cdot \vec{v}_{12}}{2c^2} \right) \hat{r}_{12} - \frac{\dot{r}_{12}(\vec{v}_1 - \vec{v}_2)}{c^2} + \frac{r_{12}(\vec{a}_1 - \vec{a}_2)}{c^2} \right] = -\vec{F}_{12}^R \ . \qquad (B2)$$

As it follows the principle of action and reaction there is conservation of linear momentum. However, as this is not a central force angular momentum is not conserved. There is conservation of energy as the force can be derived from a potential energy given by

$$U^R = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 - \frac{\vec{v}_{12} \cdot \vec{v}_{12}}{2c^2} \right) \ . \qquad (B3)$$

Riemann's force can also be derived from the standard procedure by a Lagrangian energy given by

$$S^R = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 + \frac{\vec{v}_{12} \cdot \vec{v}_{12}}{2c^2} \right) \ . \qquad (B4)$$

As usual there is a change of sign in front of $\vec{v}_{12}$.

Beginning with Riemann's force and following the usual procedure we can derive his force between current elements which is given by

$$d^2\vec{F}_{21}^R = -\frac{\mu_o}{4\pi} \frac{I_1 I_2}{r_{12}^2} \left[(d\vec{l}_1 \cdot d\vec{l}_2)\hat{r}_{12} - (\hat{r}_{12} \cdot d\vec{l}_1)d\vec{l}_2 - (\hat{r}_{12} \cdot d\vec{l}_2)d\vec{l}_1 \right] = -d^2\vec{F}_{12}^R \ . \qquad (B5)$$

The force of a closed circuit on a current element of another circuit by this expression is the same as Ampère and Grassmann, namely, (4.52).

Nowadays some of the main followers of the general Riemannian program for electrodynamics including the existence of the fields and potentials, with time retardation, are White and collaborators (White, 1977).

Clausius force was proposed in 1876 and it is given by (Clausius, 1880; O'Rahilly, Vol. 1, p. 222):

$$\vec{F}_{21}^{C} = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}^2} \left[\left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \hat{r}_{12} + \frac{\dot{r}_{12}\vec{v}_2}{c^2} - \frac{r_{12}\vec{a}_2}{c^2} \right] \neq -\vec{F}_{12}^{C} \ , \qquad (B6)$$

$$\vec{F}_{12}^{C} = -\frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}^2} \left[\left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \hat{r}_{12} - \frac{\dot{r}_{12}\vec{v}_1}{c^2} + \frac{r_{12}\vec{a}_1}{c^2} \right] \ . \qquad (B7)$$

It does not comply with action and reaction. So there is no conservation of linear and angular momentum. However, there is a Clausius's potential energy given by

$$U^C = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 + \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \ . \qquad (B8)$$

Clausius's force can be derived from a Lagrangian energy S^C by the usual procedure. It is given by

$$S^C = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}} \left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) \ . \qquad (B9)$$

Calculating by the usual procedure the force between current elements beginning with (B6) yields Grassmann's force (4.28).

It is curious to observe that Clausius arrived at this result for the force between current elements unware of Grassmann's earlier and identical result of 1845. In 1877 Grassmann had to publish a paper claiming correctly his priority over Clausius's work. Only then Grassmann's work in electrodynamics began to be widely known (Crowe, 1985, pp. 80, 93 - 94 and 152 - 155).

We are not aware of anyone following Clausius's approach nowadays.

Ritz published his ballistic theory in 1908. See (O'Rahilly, 1965) for a general presentation of Ritz's theory. The force law, in particular, is presented in (O'Rahilly, 1965, Vol. 2, pp. 501 - 505 and 520). Until second order in $1/c$ it yields a force given by (all quantities calculated and measured at the present time t):

$$\vec{F}_{21}^{Ritz} = \frac{q_1 q_2}{4\pi\varepsilon_o} \frac{1}{r_{12}^2} \left\{ \left[1 + \frac{3-\lambda}{4} \frac{\vec{v}_{12} \cdot \vec{v}_{12}}{c^2} - \frac{3(1-\lambda)}{4} \frac{\dot{r}_{12}^2}{c^2} - \frac{\vec{r}_{12} \cdot \vec{a}_2}{2c^2} \right] \hat{r}_{12} \right.$$

$$-\frac{1+\lambda}{2}\frac{\dot{r}_{12}(\vec{v}_{12})}{c^2}-\frac{r_{12}\vec{a}_2}{2c^2}\Bigg\}\neq -\vec{F}_{12}^{Ritz}\ , \qquad (B10)$$

$$\vec{F}_{12}^{Ritz}=-\frac{q_1q_2}{4\pi\varepsilon_o}\frac{1}{r_{12}^2}\Bigg\{\left[1+\frac{3-\lambda}{4}\frac{\vec{v}_{12}\cdot\vec{v}_{12}}{c^2}-\frac{3(1-\lambda)}{4}\frac{\dot{r}_{12}^2}{c^2}+\frac{\vec{r}_{12}\cdot\vec{a}_{12}}{2c^2}\right]\hat{r}_{12}$$

$$-\frac{1+\lambda}{2}\frac{\dot{r}_{12}(\vec{v}_{12})}{c^2}+\frac{r_{12}\vec{a}_1}{2c^2}\Bigg\}\ . \qquad (B11)$$

In these equations λ is a dimensionless constant which was left unspecified by Ritz. According to O'Rahilly it is probably equal to 3 (O'Rahilly, 1965, Vol. 2, pp. 588, 589 and 616).

This force does not follow the principle of action and reaction. So linear and angular momentum are not conserved. There is also no conservation of energy. There is no Lagrangian energy from which the force can be derived as well.

Following the usual procedure the force between two current elements is found to be

$$d^2\vec{F}_{21}^{Ritz}=-\frac{\mu_o}{4\pi}\frac{I_1I_2}{r_{12}^2}\Bigg\{\left[\frac{3-\lambda}{2}(d\vec{l}_1\cdot d\vec{l}_2)-\frac{3(1-\lambda)}{2}(\hat{r}_{12}\cdot d\vec{l}_1)(\hat{r}_{12}\cdot d\vec{l}_2)\right]\hat{r}_{12}$$

$$-\frac{1+\lambda}{2}\left[(\hat{r}_{12}\cdot d\vec{l}_1)d\vec{l}_2+(\hat{r}_{12}\cdot d\vec{l}_2)d\vec{l}_1\right]\Bigg\}=-d^2\vec{F}_{12}^{Ritz}\ . \qquad (B12)$$

When calculating the force of a closed circuit on a current element of another circuit this yields the same as Ampère's or Grassmann's forces, (4.52), independent of the value of λ.

The main follower of Ritz's ideas was O'Rahilly, who devoted a remarkable book to explore Ritz's electrodynamics. He analysed deeply and critically the fundamentals of electromagnetic theory. Few books of electromagnetism are so critical and full of ideas as this one. His book is highly recommended for its wealth of historical information, his sincerity and courage to express freely his ideas: (O'Rahilly, 1965).

The formulas of this Appendix should be compared with the analogous ones of Weber's electrodynamics, (3.24), (3.25), (3.45) and (4.24); as well as the classical ones: (6.8), (6.9), (6.73), (6.69), (4.28) and (4.29).

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