π
A BRIEF HISTORY

by

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The Early History of Pi

Egypt   In 1856 a Scotsman by the name of Henry Rhind purchased a papyrus in Egypt. A copy dating to 1,650 B.C., it consisted of a collection of mathematics problems prepared for a typical mathematics student of the day.

Problems of algebra, linear systems, mensuration, and mathematical amusements were assembled. There was no formal theory, just the enumerated problems.

In one problem we find the statement:

A square of side 8 has the area of a circle of diameter 9.

From this statement, the "student" was expected to compute the area of any circle. Fortunately, the concept of proportion was well understood and thus by scaling any circular area was easily determined.

Effectively, using the fact that the true area of a circle of diameter 9 is

$$\pi \, r^2 = \pi \, (4.5)^2 \approx 63.62$$

and the area of a square of side 8 is 64, we easily see that the *effective* value for π is

$$\pi \approx 3.1605 \quad \text{or} \quad 3\frac{13}{81}$$

Most modern histories give this value to be

$$3\frac{1}{6}$$

From the Bible:

"Also he made a molten sea of ten cubits from brim to brim, round in compass, and five cubits the height thereof; and a line of thirty cubits did compass it round about".

<div align="right">Old Testament, 1 Kings 7:23</div>

Babylon   The Baylonians enjoyed a sophisticated society. Located in the Mesopotamian River Valley, they created large monuments to their gods and kings while engaging in commerce throughout the Mediterranian.

To do this they required some considerable mathematical skills. Excellent records of their whole society, including their mathematics has been preserved in the tens of thousands of clay tablets.

Mathematically ...
For example, they seemed to be well aware of the Pythagorean theorem a thousand years before Pythagoras. They developed a method of square root extraction equivalent to Newton's method. Generally, they used the approximation

$$\pi = 3.$$

However, at Susa (about 200-300 miles from Babylon) a new set of tablets were unearthed in 1936. These tablet were on whole concerned with the ratios of areas and perimeters of regular polygons to their respective side lengths. For example, we find

The ratio of the perimeter of a regular inscribed
hexagon to the circumference of the circle in sex-
agesimal numbers is

$$0; 57, 36 = 0 + \frac{57}{60} + \frac{35}{60^2} \approx .9600$$

The correct value of this ratio is

$$\frac{6}{2\pi}$$

and this gives an *effective* value of π to be

$$3\frac{1}{8}$$

.

Greece   The Greeks, living in an open society free of political and other social restraints, were the first to develop the axiomatic method and **rigor** in their mathematics.

Three problems vexed (and blessed) their mathematical development:

- How to square the circle.

- How to double the cube.

- How to trisect an angle.

What they didn't know is that the first problem depends on the very nature of π.

What they didn't know was that these problems would require more than two millenia to solve.

In Euclid's *Elements*, Book XII, Proposition 2, we find the following result. (It's proof is based on the Eudoxus Method of Exhaustion, the Greek version of taking limits, without formally defining them.)

The ratio of the areas of two circles is as the ratio of the squares upon their diameters.

From this we know that the Greek mathematicians were well aware that there must be a mathematical model equivalent to the one we know as

$$A = \pi \, r^2$$

Archimedes The greatest mathematician of antiquity, Archimedes (287-212 B.C.), proved two fundamental relations about the perimeters and areas of inscribed and circumscribed regular polygons. With respect to a circle of radius r, let

$$b_1 = \text{ an inscribed hexagon with perimeter } p_1 \text{ and area } a_1$$

$$B_1 = \text{ an circumscribed hexagon with perimeter } P_1 \text{ and area } A_1$$

Further, let $b_2 \ldots b_n$ denote the regular inscribed $6 \cdot 2 \ldots 6 \cdot 2^n$ polygons, similarly, $B_2 \ldots B_n$ for the circumscribed polygons.

The following formulae give the relations between the perimeters and areas of these $6 \cdot 2^n$ polygons.

$$P_{n+1} = \frac{2p_n P_n}{p_n + P_n} \qquad p_{n+1} = \sqrt{p_n P_{n+1}}$$

$$a_{n+1} = \sqrt{a_n A_n} \qquad A_{n+1} = \frac{2a_{n+1} A_n}{a_{n+1} + A_n}$$

Using n-gons up to 96 sides he derives

$$3\frac{10}{71} < \pi < 3\frac{1}{7}$$

Following Archemedes, **Apollonius** (250-175 B.C.), in his book, now lost, on rapid calculation techniques, *Quick Delivery*, approximates π as

$$\pi \approx 3.1416$$

No information on how he derived it is known. Apollonius is considered one of the last great mathematicians of antiquity. His book, *On Conics*, did for conics what Euclid did for plane geometry.

The great astronomer and the father of trigonometry **Claudius Ptolemy** (100-178 A.D.) gives
$$\pi \approx \frac{377}{120} = 3.141666666...$$

although his method of derivation is unknown. His book, commonly known as *Almagest*, was studied and considered the source of trigonometry up to the time of Copernicus (fl. 1500)

Rome   The Romans were warriors, engineers, authors, and rulers. They were not mathematicians. Even with all the Greek mathematics available, they computed circular areas using proportion and the statement

A wheel of diameter 4 feet has a perimeter of $12\frac{1}{2}$ feet.

This gives the approximation

$$\pi \approx 3\frac{1}{8}$$

China   Generally, the Chinese, even before 100 B.C., used $\pi \approx 3$. They also used the alternative approximation, $\pi \approx \sqrt{10} \approx 3.16$. By the 5th century more accurate approximations were known. Tsu Ch'ung-Chih gives two sets of approximations
$$\text{inexact} \ \rightarrow \ \frac{22}{7}$$

$$\text{accurate} \rightarrow \frac{355}{113} = 3.14159292035$$

He also shows
$$3.1415926 < \pi < 3.1415927$$

Note that to eleven decimal places $\pi = 3.14159265359$. Results of this accuracy were not achieved in the West until the 15th century.

Hindu   The best approximation to π was given in about 500 A.D. by Āryabhata using the following description

> *Add 4 to 100, multiply by 8, and add 62,000.*
> *The result is approximately the circumference of*
> *a circle of which the diameter is 20,000.*

Of course this gives

$$\frac{(100+4)8 + 62,000}{20,000} = 3\frac{177}{1250} = 3.1416$$

Yet, the Hindu approximation to π was usually $\sqrt{10}$.

Arabic   In 833 A.D., the great arab mathematician **Al-Khwārizmī**, the father of algebra, uses two approximations for π: $3\frac{1}{7}$ and

$$\pi = \frac{62,832}{20,000} = 3.1416$$

In about 1436 **Al-Kashi** gives

$$2\pi \approx 6.2831853071795865,$$

sixteen decimal places, which is prodigiously accurate for the time.

The Modern History of Pi

France The French mathematician **Francois Vieta** (1540-1603), the first mathematician to us symbolic notation for knowns (consonants) and unknowns (vowels) discovered the **first direct formula** for the calculation of π.

$$\frac{\pi}{2} = \cfrac{1}{\sqrt{\frac{1}{2}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}$$

This is easily derived beginning with an inscribed square, and successively doubling the number of sides, getting next the octagon, the 16-gon, ... Using the notation a_n for the area of the inscribed n-gon, apply the formula

$$a_{2n} = a_n \sec \frac{\pi}{n}$$

recursively, beginning with $n = 4$ (the square). Calculating thusly, Vieta obtained ten places accuracy.

Germany **Ludolph van Ceulen** (1540-1610) computed π accurate to 20 places. Later, he computed π to 31 places. So proud was he of this accomplishment that he had the digits engraved on his tombstone. At this time π is sometimes called **Ludolph's** constant. (It is not yet called π.)

> **The known accuracy of π now exceeds any computational need.**

England. A number of mathematical developments come from England. The first approximation to π came from the clergyman, cryptographer and mathematician **John Wallis** (1616-1703). Using a complicated interpolation-like argument based on the quadratures of the functions $(1+x)^n, n = 0, \ldots$, he produced

$$\frac{2}{\pi} = \frac{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdots}{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots}$$

From this **Lord William Brouncker** (1620?-1684), the first president of the Royal Society, discovered the formula

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}$$

James Gregory (1637-1675) derived the well known (to us) formula

$$\int_0^x \frac{dt}{1+t^2} = \arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$$

now called <u>Gregory's formula</u>. Evaluating at $x = 1$, we get

$$\frac{\pi}{4} = 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \cdots$$

This series was used by **Abraham Sharp** (1651-1742) under instructions from Edmund Halley (1656-1743) (of comet fame) to calculate π to 72 places. (Note the convergence is very slow. How many terms are required for 10^{-72} accuracy?)

John Machin (1680-1751), a professor of astronomy at Gresham College in London obtained 100 places of accuracy for the computation of π using Gregory's formula

$$\frac{\pi}{4} = 4 \arctan\frac{1}{5} - \arctan\frac{1}{239}$$

Even **Issac Newton** in 1665-1666 calculated π from the formula

$$\pi = \frac{3\sqrt{3}}{4} + 24(\frac{1}{12} - \frac{1}{5 \cdot 2^5} - \frac{1}{28 \cdot 2^7} + \cdots)$$
$$= \frac{3\sqrt{3}}{4} + 24 \int_0^{\frac{1}{4}} \sqrt{x - x^2}\,dx$$

Said Newton:

"I am ashamed to tell you to how many figures I carried these computations, having not other business at the time."

In 1874, Machin's formula was used by **William Shanks** (1812-1882) to achieve 707 places of accuracy for π. Because the series is alternating and almost geometric, the convergence is very, very fast. Only about 510 terms are required to achieve this accuracy.

Germany.   Perhaps the greatest mathematician that ever lived, **Leonhard Euler** (1707-1783), was born in Basel and studied under James Bernoulli. He eventually found a position at St. Petersburg, originally to serve on the faculty of medicine and physiology.

In 1730, due in part to the death of his colleague and friend, Nicolas Bernoulli, he found himself in the chair of natural philosophy—not medicine. His contributions to mathematics have been well documented and are prodigious.

He also contributed to the derivation of formulas with π.

In 1779 he determined the formula

$$\pi = 20 \arctan\frac{1}{7} - 8 \arctan\frac{3}{79}$$

from which the computation of π can be made using Gregory's formula.

However, Euler derived a different formula for the arc tangent as well. He also derived the formulas

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^6}{945} = \sum_{n=1}^{\infty} \frac{1}{n^6}$$

$$\frac{\pi^8}{9,450} = \sum_{n=1}^{\infty} \frac{1}{n^8}$$

$$\frac{\pi^{10}}{93,955} = \sum_{n=1}^{\infty} \frac{1}{n^{10}}$$

$$\vdots \quad = \quad \vdots$$

as well as formulas for the sum of the remaining reciprocal (even) integers up to to order 26.

Euler coined the modern symbol for pi, π, after John Wallis who first used $\frac{\pi}{\delta}$ for pi. In 1801 he also used the modern symbols for e and $i = \sqrt{-1}$.

Euler proved the most important relation in all of mathematics:

$$e^{i\pi} + 1 = 0$$

The Nature of Pi

Johann Lambert (1728-1777) was born in the Alsace region, the son of a poor tailor. Managing to teach himself elementary mathematics and pursue his mathematical studies while a tutor for a Swiss family, he eventually was able to enjoy the company of the day's greatest mathematicians including Lagrange and Euler. In 1761 he proved that

if x is rational (and non-zero), then neither e^x
nor $\tan x$ can be rational.

He proves this using a continued fraction expansion of e given by Euler in 1737.

Note. At this time there were so many *circle squarers* that in 1775 the Paris Academy passed a resolution that *no more* solutions on the quadrature of the circle should be examined by its officials. This resolution applied as well to angle trisections and the doubling of the cube. The conviction was growing that the solution of squaring the circle was impossible.

It can be shown that by using a straight edge and compass, the only magnitudes that can be constructed from whole magnitudes are numbers involving square roots, square roots of square roots, etc. For example,

$$\sqrt{2}, \quad \sqrt{3 + \sqrt{4}}, \quad \sqrt{5 + \sqrt{6 + \sqrt{7}}}, \quad \text{etc.}$$

Such numbers are also among the solutions of all polynomial equations with rational coefficients. The set of all solutions, real and complex, of polynomial equations with rational coefficients are called the *algebraic numbers*. All other numbers are called *transcendental*.

Transcendental numbers were discovered by **Joseph Liouville** (1809-1882) in 1844 with the number

$$\frac{1}{10} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots$$

In 1873, **Charles Hermite** (1822-1901) proved that e was transcendental by showing that it cannot be the solution of a polynomial equation with rational coefficients.

Finally, in 1882, **C.L.F. Lindemann** (1852-1939) proved that π is transcendental by showing that if $a,\ b,\ c,\ldots$ and $m,\ n,\ r,\ldots$ are algebraic numbers the the equation

$$ae^m + be^n + ce^r + \cdots = 0$$

has no rational solutions. Thus the equation

$$e^{ix} + 1 = 0$$

cannot subsist if x is algebraic. Using Euler's celebrated formula

$$e^{i\pi} + 1 = 0$$

it follows that π is not algebraic. The circle cannot be squared!

Bill No. 246, 1897. State of Indiana

"Be it enacted by the General Assembly of the State of Indiana: It has been found that the circular area is to the quadrant of the circumference, as the area of an equilateral rectangle is to the square on one side."

"In further proof of the value of the author's (E.J.Goodman, M.D.) proposed contribution to education, and offered as a gift to the State of Indiana, is the fact of his solutions of the trisection of the angle, duplication of the cube and quadrature of the circle having been already accepted as contributions to science by the American Mathematical Monthly, the leading exponent of mathematical thought in this country."

Passes three readings, Indiana House, 1897.
(via Committee on Swamp Lands)

Passes first reading, Indiana Senate, 1897.
(via Committee on Temperance)
"The case is perfectly simple. If we pass this bill which establishes a new and correct value of π, the author offer our state without cost the use of this discovery and its free publication in our school textbooks, while everyone else must pay him a royalty."
Professor C.A. Waldo of Purdue intercedes and the bill is tabled.

What we do know!

$\star$ π is irrational.	Lambert (1761)
$\star$ π is transcendental	Lindemann (1882)
$\star$ $\pi + \sqrt{2}\log 2$ is transcendental	Baker (1966)
$\star$ Several billion digits behave regularly.	

Sophisticated Computations
(or what can be done with really fast computers)

Besides the computations already mentioned Machin-like formulae have produced

$\star$ 2037 Digits — Eniac (1949) — (von Neumann, et.al.)
$\star$ 1,000,000 Digits — CDC 7600 (1973) — (Guilloud and Bouyer)
$\star$ 17,000,000 Digits — Symbolics 3670 (1985), (Gosper)
$\star$ 29,000,000 Digits — Cray-2 (1986), (Bailey)
$\star$ 1-4 Billion Digits — (Chudnovskys)
$\star$ 51 Billion Digits — Hitachi's (1997) — (Kanada)

India In 1914 the Indian mathematician **Srinivasa Ramanujan** proved the following representation of π.

$$1/\pi = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)![1103 + 26390n]}{(n!)^4 396^{4n}}.$$

Canada   Even better formulas have been more recently developed. For example, in 1987, **Peter** and **Jonathan Borwein** of the University of Waterloo, formulated a Ramanujan-like formula given below: With

$$a = 212175710912\sqrt{61} + 1657145277365$$

and

$$b = 13773980892672\sqrt{61} + 107578229802750$$

we have

$$\frac{1}{\pi} = 12 \sum_{n=0}^{\infty} \frac{(-1)^n (6n)![a + bn]}{(n!)^4 (3n)![5280(236674 + 30303\sqrt{61})]^{[3n+3/2]}}$$

This formula delivers an impressive 25 digits of accuracy for each value of n computed. Compute to 99 terms and you will obtain π to 2,500 digits!!!

Iterative Formulas are available also. From the Borweins we also have

$$y_0 = \sqrt{2} - 1 \qquad \alpha_0 = 6 - 4\sqrt{2}$$

$$y_{n+1} = \frac{1 - (1 - y_n^4)^{\frac{1}{4}}}{1 + (1 - y_n^4)^{\frac{1}{4}}}$$

$$\alpha_n = (1 + y_{n+1})^4 \alpha_n - 2^{2n+3} y_{n+1}(1 + y_{n+1} + y_{n+1}^2)$$

It follows that

$$\frac{1}{\alpha_n} \to \pi \qquad \text{as } n \to \infty.$$

With n at 15 you are guaranteed over 2 **billion places** of accuracy for π.

The Current Record. 20th September 1999

Japan   The current record is held by Dr. Yasumasa KANADA of the Computer Centre, University of Tokyo Dear folks,

Our latest record was established as the followings;

Declared record: 206,158,430,000 decimal digits

Two independent calculation based on two different algorithms generated $206,158,430,208(= 3*2^36)$ decimal digits of pi and comparison of two generated sequences matched up to 206,158,430,163 decimal digits, e.g., 45 decimal digits difference. Then we are declaring 206,158,430,000 decimal digits as the new world record.

Optimized Main program run:

Job start : 18th September 1999 19:00:52 (JST)

Job end : 20th September 1999 08:21:56 (JST)

Elapsed time : 37:21:04

Main memory : 865 GB (= 6.758 GB * 128)

Algorithm : Gauss-Legendre algorithm

Optimized Verification program run:

Job start : 26th June 1999 01:22:50 (JST)

Job end : 27th June 1999 23:30:40 (JST)

Elapsed time : 46:07:10

Main memory : 817 GB (= 6.383 GB * 128)

Algorithm : Borwein's 4-th order convergent algorithm

Some of interesting digits sequences of pi:

01234567890 : from 53,217,681,704-th of pi

01234567890 : from 148,425,641,592-th of pi

01234567891 : from 26,852,899,245-th of pi

01234567891 : from 41,952,536,161-th of pi

01234567891 : from 99,972,955,571-th of pi

01234567891 : from 102,081,851,717-th of pi

01234567891 : from 171,257,652,369-th of pi

432109876543 : from 149,589,314,822-th of pi

543210987654 : from 197,954,994,289-th of pi

98765432109 : from 123,040,860,473-th of pi

98765432109 : from 133,601,569,485-th of pi

98765432109 : from 150,339,161,883-th of pi

98765432109 : from 183,859,550,237-th of pi

09876543210 : from 42,321,758,803-th of pi

09876543210 : from 57,402,068,394-th of pi

09876543210 : from 83,358,197,954-th of pi

10987654321 : from 89,634,825,550-th of pi

10987654321 : from 137,803,268,208-th of pi

10987654321 : from 152,752,201,245-th of pi
27182818284 : from 45,111,908,393-th of pi

Frequency distribution for pi up to 200,000,000,000 decimal places:
'0' : 20000030841; '3' : 20000069393; '6' : 1999988151; '9' : 19999869180
'1' : 19999914711; '4' : 19999921691; '7' : 19999967594
'2' : 20000136978; '5' : 19999917053; '8' : 20000291044; Chi square = 8.0

Programs are consisted of two sets of routines, e.g. calculation routines and message passing routines. Calculation routines were written by Dr. Daisuke TAKAHASHI, a Research Associate at our Centre and rather speed sensitive message passing routines were written by myself. Calculation routines used were more optimized than these used for the 51.5 billion record establishment.

For establishing this new record, high speed message passing routines were seriously used for both of programs, e.g. main program and verification program. CPU used was HITACHI SR8000 at the Information Technology Center, Computer Centre Division (old Computer Centre,) University of Tokyo. Full of the total CPU, e.g. 128PE's (theoretical peak processing speed for the single PE is eight billion floating operations per second. One trillion floating point operations per second for all PE's), were definitely used as single job and parallel processing for both of programs run.

URL:http://www.lupi.ch/PiSites/Pi-Rekord.html
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Is that it?
I believe so!