THE HISTORY OF CALCULUS*

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Everyone knows that Newton and Leibniz are the founders of Calculus. Some may think it suffices to know just this one fact. But it is worthwhile, indeed, to go into more details and to study the history of the development of Calculus, in particular, up to the time of Newton and Leibniz.

In our courses on Calculus we usually begin with differentiation and then come later to integration. This is entirely justified, since differentiation is simpler and easier than integration. On the other hand, the historical development starts with integration; computing areas, volumes, or lengths of arcs were the first problems occurring in the history of Calculus. Such problems were discussed by ancient Greek mathematicians, especially by Archimedes, whose outstanding and penetrating achievements mark the peak of all ancient mathematics and also the very beginning of the theory of integration. The method applied by Archimedes for his proofs was the so-called method of exhaustion, that is, in the case of plane areas, the method of inscribed and circumscribed polygons with an increasing number of edges. This method was first rigorously applied, in the form of a double *reductio ad absurdum*, by the great Greek mathematician Eudoxus at the beginning of the fourth century B.C. He first proved the facts, previously stated by Democritus, that the volume of a pyramid equals one third of the corresponding prism and the volume of a cone equals one third of the corresponding cylinder. The same method was also used by Euclid and then with the greatest success by Archimedes (third century B.C.). It is well known that Archimedes was the first to determine the area and the length of the circle, that is, to give suitable approximate values of π, and moreover to determine the volume and the area of the surface of the sphere and of cylinders and cones. But he went far beyond this [1]; he found the areas of ellipses, of parabolic segments, and also of sectors of a spiral, the volumes of segments of the solids of revolution of the second degree, the centroids of segments of a parabola, of a cone, of a segment of the sphere, of right segments of a paraboloid of revolution and of a spheroid. These were amazing achievements, indeed. Archimedes proved his results in the classical manner, by the method of exhaustion. Sometimes the type of approximation is just the same as we would use. For instance, in order to obtain the volume of a solid of revolution of the second degree, Archimedes approximates the volume by a sum of cylindrical slabs. But the direct evaluation of the limit of such sums was cumbersome. Hence we may ask: what was the method used by Archimedes for finding his results?

There is an indication of his method in the beginning of his book on the quadrature of the parabola. But a full explanation of his procedure was given by him in a work rediscovered as late as 1906. It is his *Method Concerning Mechanical Theorems, dedicated to Eratosthenes*, known as Archimedes' *Method*

* Based on two addresses given at the Mathematics Club of Purdue University, June 2, 1949, and at the meeting of the Indiana Academy of Science, Crawfordsville, Ind., November 4, 1949.

or ἔφοδος [2]. This manuscript was found in Istanbul as a so-called palimpsest. That is to say, in the 10th century A.D. the manuscript of the *Method* was written on this parchment; later, in the 13th century, since nobody there was interested in it any longer or could even understand it, the *Method* was washed off and a religious text of the orthodox church, a so-called euchologion, was written on the parchment. Fortunately most of the Archimedean text could be restored.

The method of Archimedes may be called a mechanical infinitesimal method. We must remember that Archimedes was also the founder of statics, and of hydrostatics too. Now his method of integration consists in an application of the principle of the lever to elementary parts of the figure. As an example, we shall see how Archimedes determines the area of a segment of a parabola; this indeed is the first example given in his ἔφοδος.

Let $\widehat{AB}$ be the given segment of the parabola, let R be the midpoint of the chord AB, and draw the diameter d (parallel to the axis) through R, intersecting the arc AB in U (cf. Figure 1). We draw the tangent of the parabola at B and the parallel to d through A. So we obtain $\triangle ABC$ and wish to compare the area

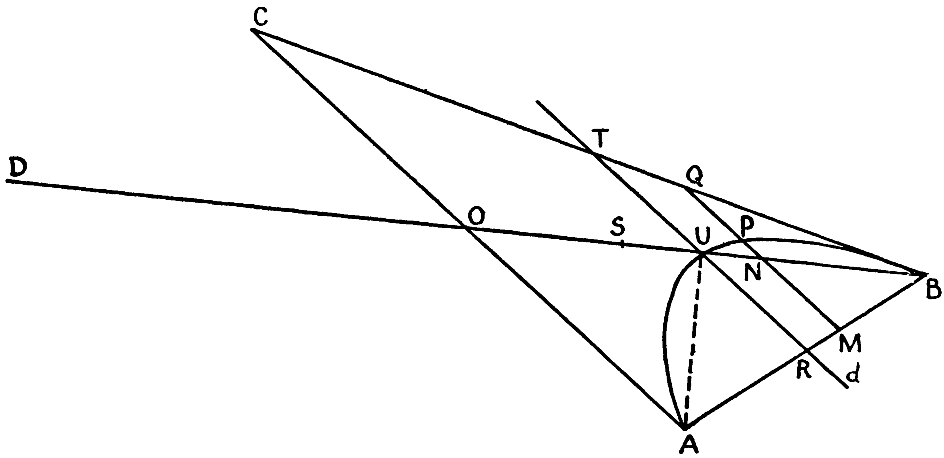

FIG. 1

of the segment $\widehat{AB}$ with the area of this triangle. Let T be the intersection of the tangent BC with the diameter d. From an elementary property of the parabola we obtain: $UT = UR$. Hence the line BU intersects AC in its midpoint O. The centroid S of the $\triangle ABC$ lies on OB with $OS = \frac{1}{3}OB$. Make $OD = OB$. Now through any point P of the arc AUB draw a parallel to the diameter, $MNPQ$. Archimedes proves as a property of the parabola that $MP:MQ = AM:AB$, hence $MP:MQ = ON:OB = ON:OD$. Therefore $MP \times OD = MQ \times ON$. That is, according to the principle of the lever: If DOB is considered to be the bar of a balance with the fulcrum in O (cf. Figure 2), then the line-segment PM suspended at D is in equilibrium with the line-segment MQ which remains, without any change, suspended at N. Since this holds for every line parallel to d, it follows that the total segment $\widehat{AB}$ of the parabola suspended at D is in equilibrium with the $\triangle ABC$ remaining unchanged or, what amounts to the same thing, the segment $\widehat{AB}$ suspended at D is in equilibrium with the $\triangle ABC$ suspended at its centroid S. Therefore, since $OS=\frac{1}{3}OD$, the segment $\widehat{AB}=\frac{1}{3}\triangle ABC$. Moreover, since $RU=\frac{1}{2}RT=\frac{1}{4}AC$, we have $\triangle ABC=4\triangle AUB$, and hence also segment $\widehat{AUB}=\frac{4}{3}\triangle AUB$.

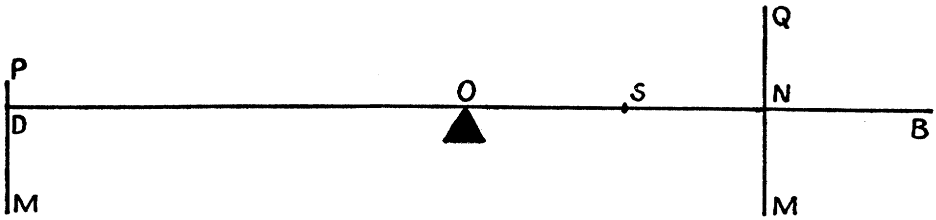

FIG. 2

By this ingenious method of statics, Archimedes found the final result. But he did not regard such reasoning as a proof. Having thus obtained the result, he was then able to give a rigorous formal proof by the method of exhaustion.

One should expect that the wonderful achievements of Archimedes would have become a great stimulus to the further development of Greek mathematics, similar to the great influence of Newton and Leibniz on the mathematical production of succeeding generations. But it is surprising that Archimedes found almost no successors to continue his work. In this connection only one of the subsequent mathematicians is to be mentioned, namely, Dionysodorus who found the volume of the torus. Of course, one has to remember that at the time of Archimedes there lived another outstanding Greek mathematician, Apollonius, about 25 years younger than Archimedes. Apollonius, in a masterly way, completed the Greek theory of conic sections. It is very strange, and I do not understand the reason for it, that soon after Archimedes and Apollonius Greek mathematics declined and that the later development essentially took a different direction. Under the influence of the needs of astronomy, a new branch of mathematics (the roots of which, however, go back also to Archimedes), namely trigonometry, was established and furthermore, much later, the theory of numbers was developed by the work of Diophantus. Original contributions in the direction of Archimedes' work were finally made by one of the latest Greek mathematicians, Pappus (end of the third century A.D.) who stated the important general theorems named after him, in particular, the theorem that the volume of a body of revolution equals the area of the revolving plane figure times the length of the path of the centroid of this area.

When Teutonic tribes, still barbarian at that time, invaded the Roman empire and conquered it, the interest in mathematics almost vanished there; mathematics receded to the Orient, to Byzantium, where at least valuable manuscripts were preserved, to Persia, and afterwards to the Arabian countries, where—on the basis of Greek tradition—mathematics flourished in the period about 800–1200 A.D. One of these mathematicians, the Mesopotamian Ibn Al Haitham (about 1000 A.D.), was able to compute the volume of a solid that is generated by rotation of a segment of a parabola about a line perpendicular to its axis.

Under the influence of the Orient, interest in mathematics was slowly awakened in Europe, in particular in the 12th and 13th centuries. As early as the 16th century great discoveries in algebra were made by Italian mathematicians, namely the solution of the algebraic equations of the third and fourth degree. Simultaneously Archimedes' works were studied and understood again.

Then about the beginning of the 17th century the further development of the ideas of Archimedes starts. This was the same great period in which modern science was first established by Galileo. The Flemish engineer Simon Stevin (as early as 1586) and the Italian mathematician Luca Valerio (1604) were the first ones who, by direct passage to the limit, tended to avoid the double *reductio ad absurdum* of the method of exhaustion. Valerio showed directly that the areas under certain curves can be approximated by sums of circumscribed and inscribed rectangles, whose difference can be made arbitrarily small.

Then, in particular, we have to mention the great German astronomer Johannes Kepler who, in 1615, published a book, *Nova stereometria doliorum vinariorum*, on determining the volumes of wine-casks. Somewhat earlier there had been a year of plenty and there was need of barrels for storing the great supply of wine; moreover, Kepler was puzzled by the rules which dealers applied to estimate the approximate contents of a barrel. So he discussed in a popular manner the volumes of various casks and, in particular, asked which cask has the most economic shape. He found that the Austrian barrel was the most economic one. Kepler used the results and methods of Archimedes, but also discussed quite a few new cases. Because of his popular purpose he replaced the rigorous proofs of Archimedes by an intuitive infinitesimal reasoning, in this way stressing the essential points.

Another mathematician of that time had a great influence on further progress; this was the Italian Bonaventura Cavalieri who published in 1635 an important book on the so-called indivisibles, entitled *Geometria indivisibilibus continuorum nova quadam ratione promota*. Indivisibles mean elements of a given dimension which by their motion generate figures of the next higher dimension. Thus a moving point generates a line, a moving line (parallel to a fixed line) generates a plane figure, a moving plane figure (parallel to a fixed plane) generates a solid. Cavalieri speaks, for instance, about "all lines of a plane figure" ("omnes lineae figurae"). Well known is Cavalieri's principle: Two solids (lying between two parallel planes) have the same volume if they intersect each intermediate parallel plane in two equal areas. Cavalieri's views, influenced by late medieval speculations, have somewhat of the spirit of Archimedes' Method, which, however, was not known at that time.

In connection with Cavalieri we must mention also the Swiss Paul Guldin who, besides criticizing Cavalieri, rediscovered Pappus' theorems on bodies of revolution, the Flemish mathematician Gregorius a St. Vincentio who was the first to observe (1647) that the area between a hyperbola and an asymptote behaves like a logarithm, and also the Italian mathematician and physicist Evangelista Torricelli [3] and the French mathematician Gil Persone de Roberval [4]. The important achievements of these last two men will be discussed presently.

About this time another outstanding event occurred in mathematics, the invention of Analytic Geometry by Descartes (1637) and, simultaneously and independently, by Fermat; this invention, of course, had great influence on the development of Calculus. Both Descartes and, in particular, Fermat also made valuable direct contributions to Calculus.

René Descartes, in his *Géométrie*, gave a method of finding the tangents, or rather the normals, to algebraic curves. He draws a circle with center on the x-axis, which cuts the given curve in two points. If these two points coincide, he obtains the normal. Hence the question is reduced to determining double roots of an algebraic equation. Somewhat later, in a letter, Descartes remarked that, instead of circles, intersecting straight lines could also be used for the same purpose.

Fermat's achievements in Calculus were even more important. In fact, he was the greatest mathematician of the first part of the 17th century, not only in general but particularly in the domain of Calculus. Pierre Fermat was a jurist, a councillor of the parliament at Toulouse in southern France. This position left him enough time for intensive mathematical activity. His outstanding work in the theory of numbers is well known. Now, what was *his* method of finding tangents? His procedure was first applied by him to the particular case of determining maxima and minima [5(a)]. He found this method as early as in 1629, communicated it to Descartes in 1638, and had it published in 1642. In order to find the maximum or minimum of an expression, one replaces the unknown A by $A+E$,* and both expressions obtained in this manner are considered approximately equal. One must cancel on both sides all that is possible to cancel. In this way only terms containing E are left. Now divide by E and then drop all terms still containing E. There remains an equation giving that value of A which yields the desired maximum or minimum. That means, if we write $F(A)$ for the given expression, we have to determine A from the equation

$$\left[\frac{F(A+E)-F(A)}{E}\right]_{E=0} = 0.$$

This is just our usual method. Of course, the condition is only necessary, but not sufficient for the extreme, and the statement of Fermat yields the result only for polynomials F.

* Fermat always used the letter A for the variable and the letter E for its increment.

Fermat [5(a)] gave at the same time a general method for finding the tangent, in the form of determining the subtangent. Let PT (with T lying on the the x-axis) be the tangent line of the given curve $\mathfrak{C}$ at the point P (cf. Figure 3), let P_1 be a point of $\mathfrak{C}$ in the neighborhood of P, let Q and Q_1 be the projections

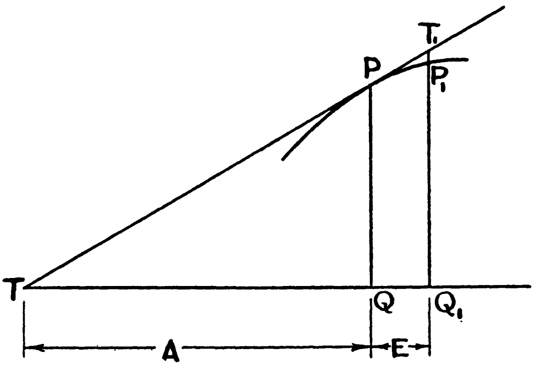

FIG. 3

on the x-axis of P and P_1, respectively, and let T_1 be the point on the tangent line whose projection on the x-axis is Q_1. In order to find the subtangent $A(=TQ)$, whose increment QQ_1 is again designated by E, Fermat uses the similarity of the triangles TQP and TQ_1T_1 and replaces T_1 approximately by P_1. Then he obtains approximately: $A:QP=E:(Q_1P_1-QP)$, that is, in our usual notation if we write the equation of the curve $\mathfrak{C}$ in the form $y=F(x)$,

$$A:F(x) = E:(F(x+E) - F(x));$$

hence

$$A = \frac{F(x)\cdot E}{F(x+E) - F(x)}\cdot$$

Now again one divides the denominator by E and afterwards sets $E=0$.

It should be mentioned here that at this time an entirely different method of constructing tangents of curves, using the parallelogram of velocities, was invented also by Roberval and Torricelli, independently of each other; both of them published it in 1644. On the other hand, somewhat later (before 1659) two Netherlanders Johannes Hudde, for many years mayor of Amsterdam, and René François de Sluse advanced along the road opened up by Descartes and Fermat, giving quite explicit formal rules for finding extremes and subtangents of algebraic curves.

But let us return to Fermat. He had great success also in the theory of integration. He was the first who, by 1636 or earlier, had found and proved the power formula of integration for positive integral exponents n, i.e. a geometrical statement equivalent to the formula which we now write as

$$\int_0^a x^n dx = \frac{a^{n+1}}{n+1} \cdot$$

Roberval also, at the suggestion of Fermat, then found and proved the same theorem. Afterwards Cavalieri discovered it independently,* and he was the first to publish it (1639, 1647); but he proved it explicitly only for the first few cases, including $n=4$, while, as he stated, the general proof which he published was communicated to him by a French mathematician Beaugrand, who quite probably had obtained it from Fermat. At that time Fermat, and then also Torricelli, had already generalized this power formula to rational exponents $n(\neq -1)$. Fermat determined areas under curves which he called "general parabolas" and "general hyperbolas," that is, curves

$$y^m = cx^n, \text{ which, in our notation, leads to } c\int_0^a x^{n/m}dx,$$

and

$$y = \frac{c}{x^m}, \text{ which, in our notation, leads to } c\int_a^{+\infty} \frac{dx}{x^m} \qquad (m > 1).$$

It is remarkable that Fermat in this work does not use subdivisions into equal parts, but subdivisions according to a geometric progression. Other areas were reduced by him to areas under such general parabolas and hyperbolas.

Fermat did interesting work also on the rectification of curves, in a memoir published 1660 [5(b)], where he approximated the arc by segments of tangents, thus using a saw-like figure. At that time various mathematicians obtained rectifications of curves which are now considered classical. In 1645, Torricelli had rectified the logarithmic spiral. The semicubical parabola was rectified independently by the Englishman William Neil (1657), by the Hollander Hendrik van Heuraet, and by Fermat. The rectification of the cycloid was first achieved by the English mathematician and great architect Christopher Wren (1658), and then by Fermat and Roberval, after they had heard of his result. It is noteworthy that Fermat, Neil, van Heuraet, and also Wallis and Huygens, reduced rectifications of curves to the determination of areas of other curves.

Reviewing the achievements of Fermat we see that he was aware of the relation among various problems in differential calculus, and similarly for various problems about definite integrals. But he had not observed the general relation between differentiation and integration.

Another famous French mathematician of that time was Blaise Pascal (a younger friend of Fermat and Roberval), who may be considered a master of integration. Roberval was the first to integrate certain trigonometric functions. Pascal was able to integrate more such trigonometric functions as well as some algebraic functions. As an important means for some of his results Pascal used

* Much later, Blaise Pascal (1654) and Wallis (1656) rediscovered it also.

relations between integrals obtained by interchanging the order of integration in double integrals. Of course, he did this in a geometric form. That is to say, certain volumes were found by means of intersections parallel to one plane and also by intersections parallel to another plane. Since the volume is always the same, Pascal thus obtained a relation between two different integrals, which may be considered to be a kind of integration by parts.

At about the same time important contributions were made by the English mathematician John Wallis, whose book *Arithmetica infinitorum* of 1656 was written in contrast to the geometric work of most of his predecessors. He stressed the notion of the limit. On the other hand, he was very audacious with regard to generalizations and interpolations, but his strong power of intuition kept him on the right track. For instance, he stated the general power formula of integration for any real exponent $n(\neq -1)$.

The notion of limit was carefully considered at that time by the Italian Pietro Mengoli in his book *Geometria speciosa* (1659) [**6**(a)]. In particular, by modifying the procedure of Luca Valerio (see above), he gave in a precise way a representation of the area under certain special curves as limits of sums of rectangles [**6**(b)]. Later the same method was also employed by Newton [**11**, book **1**, lemma 2].

One should now mention two other mathematicians who, like Wallis and Mengoli, were contemporary with Newton and Leibniz, but began their work earlier, so that they too are to be considered, at least partially, as predecessors of Newton and Leibniz. One of these men is the great Dutch physicist and mathematician Christiaan Huygens who, among other important results, introduced the notion of evolutes and involutes. It is remarkable that Huygens used to great extent the classical methods of Archimedes, and only for differentiation employed Fermat's method.

The second of these two mathematicians is the Scotsman James Gregory, who like Torricelli and Pascal died in the prime of life, when only 37 years old, and whose genius has found its full recognition only recently [**7, 8**]. He did excellent work in integration; for instance, in 1668 he published such a difficult result as the following (written in modern notation):

$$\int_0^a \sec x dx = -\log(\sec a - \tan a)$$

and other trigonometric integrals. Moreover, for example, he obtained Newton's interpolation formula, independently of Newton. But perhaps the most important achievements of Gregory belong to the theory of series.

The first great result in the theory of series is due to the German mathematician Nicolaus Mercator (1668), who found the logarithmic series. For this purpose, Mercator used term by term integration of a geometric series. This method was independently discovered, but not published, by Newton. Subsequent contributions were made by William Lord Brouncker, the first president of the Royal Society in London. Then the most outstanding results concerning infinite series were obtained by Newton and Gregory, who worked essentially independent of each other, though Gregory was influenced by the knowledge of some of Newton's statements, but not of his methods. Both discovered the binomial series and also many series for trigonometric and inverse trigonometric functions. In particular, Gregory found the series for arctan x; a special case of it is the series $\pi/4=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$, which was later found independently by Leibniz. Gregory obtained some more complicated series, *e.g.*, for tan x, sec x, log sec x, using processes of differentiation for determining the coefficients of these series, thus anticipating Brook Taylor by more than forty years. This fact has recently been demonstrated by H. W. Turnbull who studied and interpreted notes in Gregory's handwriting (*cf.* [**7**], pp. 168–176, 350–359). It is obvious that many of the contributions to the theory of series were closely connected with the growth of Calculus.

From all that has been discussed so far we have seen that there certainly was an extensive development of the theory of integration and differentiation in the period immediately before Newton and Leibniz, and that many mathematicans of various nations made great contributions. So we shall ask: What then was missing at that time? One very important point still missing was the general fact that differentiation and integration are inverse processes, that is, the so-called fundamental theorem of integral calculus. It is true that a few mathematicians had already come quite close to this knowledge. First in this connection we must mention Torricelli. In manuscripts left at his death (1647) and published as late as 1919 [cf. **3**], he had obtained the distance $s(t)$ of a moving point by means of the quadrature of the velocity $v(t)$, while by the construction of the tangent of the curve $s(t)$ he could recover $v(t)$. It is very doubtful, however, whether he really conceived the significance of this relation. Secondly, as we have already stated, some of the mathematicians of the time knew that the rectification of one curve $\mathfrak{C}_1$ could be obtained by means of the area under another curve $\mathfrak{C}_2$. Fermat in his memoir [**5**(b)], published in 1660, found the relation between the slopes of these two curves. Moreover, James Gregory, in his book *Geometriae pars universalis* (1668), solved the following problem, which is inverse to the rectification of $\mathfrak{C}_1$, just mentioned: Given a certain curve $\mathfrak{C}_2$, find another curve $\mathfrak{C}_1$ whose length equals the area under $\mathfrak{C}_2$. In the solution of this problem Gregory used the quadrature of an auxiliary curve, which indeed corresponds to obtaining the primitive function of a derivative by means of integration [**9**]. In spite of this, one again may doubt whether, at that time, he perceived in general that differentiation and integration are inverse operations. The first who made this important discovery in full generality was Isaac Barrow, the teacher of Newton at Cambridge University. Barrow first was professor of Greek language at Cambridge, then he was professor of mathematics first in London and then again at Cambridge. In 1669 he resigned his chair to his pupil Newton, whose superior genius he had recognized. Barrow then devoted the rest of his life to theology. His principal mathematical work was his *Lectiones Geo-*

metricae, published in 1670, together with the second edition of his *Lectiones Opticae*. These geometrical lectures [**10**] contained important systematic contributions to the theory of differentiation and integration, almost all in purely geometric form. Here, for the first time, the inverse character of differentiation and integration was explicitly stated and proved.

At this remarkable point of the development of the theory we must again ask the same question as above: What more remained to be done? The answer is: What had to be created was just the *Calculus*, a general symbolic and systematic method of analytic operations, to be performed by strictly formal rules, independent of the geometric meaning. Now it is just this Calculus which was established by Newton and Leibniz, independent of each other and using different types of symbolism. Newton's first discoveries were made about ten years before those of Leibniz; on the other hand, Leibniz' publications preceded those of Newton, and—what is more important—Leibniz' symbolism, the same as that used by all mathematicians at present, is superior to that of Newton.

Isaac Newton, influenced by his teacher Barrow and also by the work of Wallis, started his "method of fluxions" in the years 1665–1666, his most creative period, at the age of 23 years. Some of his early manuscripts were known to friends of his and indications of his method were contained in some of his letters. He wrote his *Methodus fluxionum et serierum infinitarum* in 1670–1671; but it was not published until 1736, nine years after his death. In his profound work *Philosophiae naturalis principia mathematica* (1687) Newton avoided his method of fluxions, except for a few indications [**11**, book 2, lemma 2], and presented his great discoveries in the classical geometric form, though simplified by using the notion of limits. The first publication of both the method and the notation of Newton is found as late as 1693 in Wallis' works, where Wallis included two of Newton's letters to him. Newton himself then published an account of his method, entitled *Tractatus de quadratura curvarum*,* as an appendix to his *Optics* in 1704.

Newton, considering motions, took the time t as the independent variable, called the dependent variable x "fluent" and its velocity "fluxion," and wrote $\dot{x}$ for the fluxion, i.e. for the derivative with respect to t. The higher derivatives were then designated by $\ddot{x}$, $\dddot{x}$, etc. For the increment of the independent variable t Newton used the letter o and called $\dot{x}o$ (i.e., the differential of x) the "moment" of x. In the case of the inverse process, that is, if the variable x is given as a fluxion, he first designated the fluent (i.e., the antiderivative of x) by $\square\, x$ or $\boxed{x}$, later by $\acute{x}$, and used then for iterated integration $\acute{\acute{x}}$, $\acute{\acute{\acute{x}}}$, etc. It has to be stressed that Newton was the first to use systematically the results of differentiation in order to obtain antiderivatives, and hence to evaluate integrals.

Our notations, now used generally in Calculus, are due to Gottfried Wilhelm Leibniz. He was a universal spirit, extremely versatile and interested in every kind of knowledge and scholarship, perhaps most famous as a philosopher. He

* Most of it was already written in 1676.

started as a jurist, was soon active in diplomacy, then became librarian and historian at Hanover, and later (1700) founded the Berlin Academy of Sciences. In mathematics, his first publication, as a graduate student, concerned combinations and permutations; then he soon became interested in the theory of differences and in constructing a computing machine. At the suggestion of Huygens (1673), Leibniz thoroughly studied the works of previous mathematicians on integration and differentiation. In particular, he was much influenced by the work of Pascal. The story of Leibniz' own discoveries can be traced in all details, since his manuscripts with his dated sketches were found at the Hanover library (edited and published by C. I. Gerhardt in the middle of the 19th century) [**12, 13, 14, 14a**]. Leibniz' new notation was first introduced by him on October 29, 1675 [**12,** pp. 121–127]. On this day, as on preceding days, Leibniz discussed integrations using Cavalieri's "omnes lineae." Here he abbreviated "omnes" or "omnia" to "omn." and applied quite a few formal operations to this symbol. Then he remarked: "It will be useful to write $\int$ for omn., thus $\int l$ for omn. l, that is, the sum of those l's." Hence $\int$ is derived from the first letter of the word *summa*. Later in the same manuscript, he came to the "contrary calculus" and continued thus: "If $\int l \sqcap ya$" ($\sqcap$ is the equal sign of Leibniz), "then let us set $l \sqcap ya/d$. Certainly as $\int$ will increase the dimensions, so d will diminish them. $\int$, however, designates a sum, d a difference." Hence Leibniz first wrote the differential sign d into the denominator of the variable. But two or three weeks later he writes dx, dy, dx/dy, and the integrals $\int ydy$ or $\int ydx$. So he arrived at the notation, now classical.

In 1684, Leibniz first published his differential calculus in a paper (issued in the newly founded *Acta Eruditorum*) with the title *Nova methodus pro maximis, itemque tangentibus*, $\cdots$. Here he deals with differentials, and it is noteworthy that he introduces dx as an arbitrary finite interval and then defines dy by the proportion $dy:dx=$ordinate:subtangent. Then, in 1686, he published also a paper containing his notation of the integral. (The word "integral" was introduced by Jakob Bernoulli, 1690.)

Later Leibniz was accused by friends and followers of Newton of having plagiarized Newton's ideas. This unfortunate and undignified controversy started first in 1699 and became continuously more and more furious, so that Leibniz' last years were filled with bitterness. But considering Leibniz' manuscripts, nobody can doubt at present that Newton and Leibniz founded their Calculus independently.

The invention of Calculus stimulated an immense and energetic further development. On the English side Taylor and Maclaurin, on the continent the eminent Basel mathematicians, first the brothers Bernoulli, then the prodigious Euler, and later the Frenchman D'Alembert and the Italian Lagrange contributed greatly to this development. Moreover, other closely related parts of mathematics soon originated in connection with the Calculus, e.g., the theory of differential equations, the calculus of variations, differential geometry.

While Newton and Leibniz had rather reasonable (although not always consistent) ideas about the fundamentals of the new Calculus, the extremely rapid further development caused the basic concepts to be neglected or to be treated in a very unsatisfactory manner. In particular, Euler is an example of this tendency. A few mathematicians, among them D'Alembert, stressed the necessity of using the notion of the limit as foundation of the Calculus [**15**]. But it was Cauchy in the beginning of the 19th century who, in such a way, developed the Calculus systematically and consistently. So at last Cauchy and his many successors gave a solid basis to the Calculus.

References

1. T. L. Heath, The Works of Archimedes, Cambridge 1897.

2. The Method of Archimedes, edited by T. L. Heath, Cambridge 1912.

3. E. Bortolotti, I progressi del metodo infinitesimale nell' Opera geometrica di Evangelista Torricelli, Periodico di Matematiche (4) 8 (1928), pp. 19–59.

4. Evelyn Walker, A Study of the Traité des Indivisibles of Gilles Persone de Roberval, Teachers College, Columbia University, New York 1932.

5. Œuvres de Fermat, 4 volumes and supplements, Paris 1891–1922. (a) Methodus ad disquirendam maximam et minimam, Œuvres, vol. 1, pp. 133–136; see also pp. 136–179 (French translation: vol. 3, pp. 121–123 and 124–156); (b) De linearum curvarum cum lineis rectis comparatione, Œuvres, vol. 1, pp. 211–254 (French translation: vol. 3, pp. 181–215).

6. A. Agostini, Peridico di Matematiche (4) 5 (1925). (a) La teoria dei limiti in Pietro Mengoli, pp. 18–30; (b) Il concetto d'integrale definito in Pietro Mengoli, pp. 137–146.

7. James Gregory. Tercentenary Memorial Volume, edited by H. W. Turnbull, London 1939.

8. M. Dehn and E. D. Hellinger, Certain mathematical achievements of James Gregory, Am. Math. Monthly 50 (1943), pp. 149–163.

9. A. Prag, On James Gregory's Geometriae pars universalis, in [7], pp. 487–505.

10. J. M. Child, The Geometrical Lectures of Isaac Barrow, Chicago and London, 1916.

11. Sir Isaac Newton's Mathematical Principles of Natural Philosophy and His System of the World. Translated by A. Motte, revised by F. Cajori, University of California Press, Berkeley, Calif., 1946.

12. C. I. Gerhardt, Die Entdeckung der höheren Analysis, Halle 1855.

13. J. M. Child, The Early Mathematical Manuscripts of Leibniz, Chicago & London 1920.

14. D. Mahnke, Neue Einblicke in die Entdeckungsgeschichte der höheren Analysis, Abhandlungen der Preuss. Akad. der Wissenschaften, Phys.-Math. Klasse, 1925, No. 1.

14a. J. E. Hofmann, Die Entwicklungsgeschichte der Leibnizschen Mathematik während seines Aufenthalts in Paris (1672–1676), Munich 1949.

15. As for the long lasting struggle for a rigorous foundation of the Calculus see C. B. Boyer, The Concepts of the Calculus, New York 1939, 2nd printing 1949.