Voting for Centrality

Extended Abstract

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ABSTRACT

In network science, centrality indices are used to determine important nodes by virtue of their position in a network. In social choice theory, voting rules are used to aggregate preferences of voters to determine the winners of an election. Exploiting parallels between these two fields, we propose a novel approach to define network centrality indices based on voting rules. Since formal properties of voting rules have been studied in much greater depth, this will not only lead to new applications of social choice theory but also facilitate a deeper understanding of centrality in networks.

KEYWORDS

Network Centrality; Voting

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1 INTRODUCTION

Network science is the study of network models [11] in an ever increasing number of domains including the social sciences, economics, communications, intelligence, engineering, and biology [26, 27, 35]. A particularly important concept is the *centrality of nodes* [9, 22], which, depending on the domain, may concern the identification of influencers in viral marketing, super-spreaders in epidemiology, or criminal actors in covert networks, the placement of beacons in communication networks, finding influential papers in bibliometrics, and many similar tasks. We suggest that *voting*, a powerful tool studied in social choice theory [1, 2] since centuries and currently also in computational social choice [12, 28], may serve to advance the design and analysis of centrality concepts. The key idea is to interpret basic or derived network relations as preferences that nodes may have regarding the other nodes, and to then rank nodes by aggregating the preferences others have about them. This readily leads to equivalent formulations but also to new notions of centrality. Only minor adaptations are necessary to extend elections to rankings and to accommodate ties in preferences.

2 CENTRALITY INDICES

For the purpose of this note, a network is a simple undirected graph $G = (\mathcal{N}, \mathcal{E})$, where $\mathcal{N}$ are the nodes or vertices and $\mathcal{E} \subseteq \binom{\mathcal{N}}{2}$ are the links or edges between them. By n we denote the number of nodes in the network. By $N(i) = \{j \in \mathcal{N} : \{i, j\} \in \mathcal{E}\}$ we denote the neighborhood of i, and by $N[i] = N(i) \cup \{i\}$ its closed neighborhood. A centrality index is a mapping $c : \mathcal{N} \to \mathbb{R}$ such that $c(i) > c(j)$ is interpreted as i being more central than j. Before adding a constraint that justifies this interpretation, we present three examples of common centrality indices.

Degree centrality. The degree of a node $i \in \mathcal{N}$ is the cardinality of its neighborhood, $\deg(i) = |N(i)|$, so that a node is central according to *degree centrality* $c_D(i) = \deg(i)$ if it is linked directly to many others.

Closeness centrality. To take indirect links to other nodes into account, closeness centrality considers shortest-path distances $\delta(s, t)$, i.e., the minimum number of edges in any path between s and t. While typically defined as the inverse of the sum of distances to all other nodes, we here refer to an index called *harmonic closeness* $c_C(i) = \sum_{t \in \mathcal{N} \setminus \{i\}} \frac{1}{\delta(i,t)}$ to simplify the exposition. Note that non-reachable nodes are treated conveniently by defining their distance as infinite and letting $\frac{1}{\infty} = 0$.

Betweenness centrality. Let $\sigma(s, t)$ denote the number of shortest paths from s to t and $\sigma(s, t|i)$ the number of those shortest paths from s to t also passing through a brokering node i. Then *betweenness centrality* [10] is given by $c_B(i) = \sum_{s,t \in \mathcal{N}} \frac{\sigma(s,t|i)}{\sigma(s,t)}$.

Interestingly, these and other centralities can all be unified into sums of weighted relationships, $c(i) = \sum_{j \in \mathcal{N}} \omega(i, j)$, where weights are, for example, binary (adjacent/nonadjacent) for degree, $\omega(i, j) = \frac{1}{\delta(i,j)}$ for (harmonic) closeness, and $\omega(i, j) = \sum_{s \in \mathcal{N}} \frac{\sigma(s,j|i)}{\sigma(s,j)}$ for betweenness. A common intuition is therefore that central nodes are linked more strongly to the others. This can be formalized using the notion of the *vicinal preorder* [21], defined by $i \succcurlyeq j \iff N[i] \supseteq N(j)$ for all $i, j \in \mathcal{N}$. Indeed, rankings obtained from standard centrality indices all respect the vicinal preorder, i.e., $i \succcurlyeq j \implies c(i) \geq c(j)$ for a centrality c [30]. Preservation of the vicinal preorder therefore seems to be a useful minimum requirement. Since all common centrality indices are also invariant under automorphisms, we define centrality more narrowly.

Definition 1 (Centrality). *A node index is a* centrality *if it is invariant under automorphisms and respects the vicinal preorder.*

3 VOTING IN NETWORKS

Social choice theory concerns methods (so-called *voting rules*) for the determination of winners from a set of candidates or alternatives A, based on the preferences of a set of voters V. We propose an interpretation of network centrality based on elections in which nodes are both voters and candidates at the same time, i.e., $A = V = \mathcal{N}$, but are not allowed to vote for themselves.

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While different means of expressing preferences are being considered in the literature, in this short note we focus on an extension of ordinal ranks in which voters' preferences are usually given as linear orders of the candidates. The extension is that we allow for indifference between candidates and thus ties in the ordering.

To obtain preferences from a network structure, assume that voters (nodes) prefer candidates (nodes) that they are closer to. Formally, node v prefers node i over j, denoted as $i \succ_v j$, if $\delta(v,i) < \delta(v,j)$, and is indifferent w.r.t. i and j, or $i \sim_v j$, if $\delta(v,i) = \delta(v,j)$. We refer to these as *distance-based preferences.* The rank position of a node (candidate) i in the preference of node (voter) v can be defined from two perspectives. Top-aligned ranks are defined by $\mathrm{rank}_v(i) = 1 + |\{x \in \mathcal{N} \setminus \{v\} : x \succ_v i\}|$, and bottom-aligned ranks are defined by $\overleftarrow{\mathrm{rank}}_v(i) = 1 + |\{x \in \mathcal{N} \setminus \{v\} : i \succ_v x\}|$. In rankings without ties, $\mathrm{rank}_v(i) = (n-1) - \overleftarrow{\mathrm{rank}}_v(i)$. However, this correspondence does not hold in general.

We consider the following voting rules to illustrate their relation with network centralities.

Plurality. Under the *plurality rule*, all voters assign a score of 1 to the candidates they rank first, and 0 to the others. The associated centrality index becomes $c_{\mathrm{Plu}}(i) = |\{v \in \mathcal{N} : \mathrm{rank}_v(i) = 1\}|$. The resulting centrality ranking is equivalent to the degree centrality ranking, although the exact indices in plurality may be higher in case of isolate nodes, which are indifferent between all nodes and thus contribute 1 to the score of every other node. Plurality can thus be seen as the immediate voting rule correspondence to degree centrality. However, social choice theory is abundant in voting rules with different properties. Two prominent, yet very different rules are those by Borda [7] and Copeland [16], which have been thoroughly studied in computational social choice (see, e.g., [18, 29]). We derive new centrality indices from them.

Borda. A formulation of this rule that also works well in the presence of ties is due to Gärdenfors [24]. It adds 1 to the score of a candidate for every candidate ranked lower, and subtracts 1 for each candidate ranked higher. The same outcome is obtained with combined scores $\overleftarrow{\mathrm{rank}}_v(i) - \mathrm{rank}_v(i)$, and the *Borda score* of node i is defined as

$$c_{\mathrm{Bor}}(i) = \sum_{v \in \mathcal{N}\setminus\{i\}} \overleftarrow{\mathrm{rank}}_v(i) - \mathrm{rank}_v(i) \ .$$

In voting, all candidates with maximum Borda score win. We use the Borda scores simply as a new centrality index.

Theorem 3.1. *Borda induces a centrality ranking.*

Borda scores reverse distances similarly to radiality centrality [32], but since multiple nodes at the same distance reduce the score of nodes farther away by more than one unit, Borda centrality ranks nodes quite differently from existing indices.

Copeland. This rule is a polynomial-time computable Condorcet-consistent method [20]. As all Condorcet-consistent methods, it is based on the pairwise comparison of candidates. We say *i wins against j in a pairwise comparison* (denoted as $i \gg j$) if and only if

$$|\{v \in \mathcal{N} \setminus \{i,j\} : \mathrm{rank}_v(i) < \mathrm{rank}_v(j)\}| >$$
$$|\{v \in \mathcal{N} \setminus \{i,j\} : \mathrm{rank}_v(i) > \mathrm{rank}_v(j)\}| \ .$$

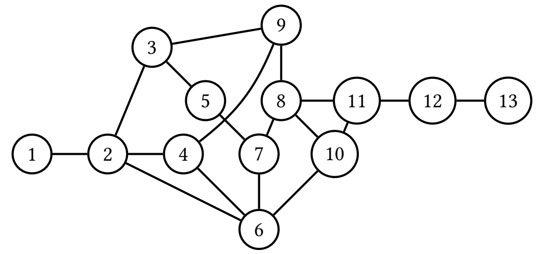

Figure 1: Node 6 has the highest closeness, and 11 the highest betweenness centrality, but 8 is the Condorcet winner.

That is, $i \gg j$ if more nodes are strictly closer to i than to j. A *Condorcet winner* [13] is a candidate that wins against every other candidate in pairwise comparison. However, a Condorcet winner does not always exist (also not in a network). Therefore, the goal behind Condorcet-consistent methods like Copeland is to select the Condorcet winner if it exists, and a node that is as close as possible to being a Condorcet winner otherwise. The *Copeland score* of node i is defined by

$$c_{\mathrm{Cop}}(i) = |\{x \in \mathcal{N} : i \gg x\}| - |\{x \in \mathcal{N} : x \gg i\}| \ .$$

Candidates with maximum Copeland score win an election, but we will simply use these scores as the Copeland centrality index. If a Condorcet winner exists in a network, it has the highest possible Copeland score of $n-1$.

Theorem 3.2. *Copeland induces a centrality ranking.*

To the best of our knowledge, Copeland centrality is the first centrality index ever proposed that ranks the Condorcet winner (if there exists one) first. Figure 1 shows a network where betweenness and closeness select nodes different from the Condorcet winner.

4 OUTLOOK

We expect the rich body of research on voting to provide considerable leverage for conceptual and axiomatic studies of centrality [5, 6, 8, 31]. That voting axioms have immediate counterparts in network science (such as neutrality and invariance under automorphisms, or the Pareto criterion and the vicinal preorder) suggests that further voting axioms may carry over to and make sense in network science. Also the study of manipulating votes [3, 14, 15] and control by adding or deleting candidates or votes [4, 19, 25] offers further directions of research facilitated by our proposal. For instance, how easily can the centrality of a node be manipulated by adding, deleting, or rewiring edges or nodes? This relates to *local hiding* discussed by Waniek et al. [33, 34] and *graph stability* studied by Frei et al. [23]. Further, our approach generalizes beyond distance-based preferences, because many distance and connectivity relations on networks are consistent with the vicinal preorder [30] and therefore suitable for preference rankings.

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