Capacitated Network Design Games on a Generalized Fair Allocation Model

Extended Abstract

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ABSTRACT

The cost-sharing connection game is a variant of routing games on a network. In this model, given a directed graph with edge costs and capacities, each agent wants to construct a path from a source to a sink with low cost. The cost of each edge is shared by the users based on a cost-sharing function. One of simple cost-sharing functions is defined as the cost divided by the number of users. It models an ideal setting, where no overhead arises when people share things, though it might be quite rare in real life. In this paper, we model more realistic scenarios of cost-sharing connection games by generalizing the cost-sharing function. The arguments do not depend on specific cost-sharing functions and are applicable for a class of all natural cost-sharing scenarios, which include equal divisions with any natural functional overheads. We show that many bounds of the Price of Anarchy and the Price of Stability under sum-cost and max-cost criteria inherit the no-overhead case.

KEYWORDS

Capacitated network design games, Cost-sharing games, Nash equilibrium, Price of anarchy, Price of stability

ACM Reference Format:

Tesshu Hanaka, Toshiyuki Hirose, and Hirotaka Ono. 2022. Capacitated Network Design Games on a Generalized Fair Allocation Model: Extended Abstract. In *Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), Online, May 9–13, 2022*, IFAAMAS, 2 pages.

1 INTRODUCTION

The capacitated symmetric cost-sharing connection game (CSCSG) is a network design model of multiple agents' sharing costs to construct a network infrastructure for connecting a given source-sink pair. In the game, a possible network structure is given, but actual links are not built yet. For example, imagine to build an overlay network structure on a physical network. Each agent wants to construct a path from source s to sink t. To construct a path, each agent builds links by paying the costs associated with them. Two or more agents can commonly use a link if the number of agents is within the capacity associated with the link, and in such a case, the cost of the link is fairly shared by the agents that use it. Thus, the more agents use a common link, the less cost of the link they pay. Under this setting, each agent selfishly chooses a path to construct so that they minimize their costs to pay. The CSCSG can model many real-world situations for sharing the cost of a designed network, such as a virtual overlay, multicast tree, or other sub-network of the Internet [7].

In the previous studies, the link cost is fairly shared, which means that the total cost paid for a link does not vary even if any number of agents use it. However, sharing resources yields more or less extra costs (overheads) in realistic cost-sharing situations; by increasing the number of users, extra commission fees are charged, service degradation occurs, and so on. The existing models are not powerful enough to handle such situations.

In this paper, we model the more realistic scenario of CSCSG by generalizing cost-sharing functions. The arguments on the model do not depend on specific cost-sharing functions, and are applicable for a wide class of cost-sharing functions satisfying certain natural properties. Let p_e and c_e be the cost and capacity associated with link (edge) e, respectively. Suppose that x agents use link e, where $x \leq c_e$. In our model, a cost-sharing function $f_e(x)$ for link e is (1) non-increasing with respect to x, (2) $f_e(x) \geq p_e/x$, and (3) $f_e(1) = p_e$. Condition (1) is a natural property in cost-sharing models, (2) represents the situation that if two or more agents use a link, overheads may arises, and (3) represents that no overhead arises when only an agent uses the edge. We emphasize that this significant generalization does not restrict any nature of fair cost-sharing. We believe that any natural fair cost-sharing function is in this scheme. Note that the cost-sharing function in the previous studies [5–7] is $f_e(x) = p_e/x$, which clearly satisfies (1), (2) and (3).

We investigate the Price of Anarchy (PoA) and the Price of Stability (PoS) of the game. A pure Nash equilibrium (we simply say Nash equilibrium) is a state where no agent can reduce its cost by changing the path that he/she currently chooses. Such a Nash equilibrium does not always exist in a general game, but it does in CSCSG. Thus, a major interest of analyzing games is to measure a goodness of Nash equilibrium. As social goodness measures, we consider two criteria. One is *sum-cost* criterion, where the social cost function is defined as the summation of the costs paid by all the agents, and the other is *max-cost* criterion, where it is defined as the maximum among the costs paid by all agents. Both PoA and PoS are well used measures for evaluating the efficiency of Nash equilibria of games. The PoA is the ratio between the cost of the worst Nash equilibrium and the social optimum, whereas the PoS refers to the ratio between the cost of the best Nash equilibrium and the social optimum.

In this extended abstract, we briefly summarize our contribution and related work. For the detailed proofs, see the full paper at [9].

Proc. of the 21st International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2022), P. Faliszewski, V. Mascardi, C. Pelachaud, M.E. Taylor (eds.), May 9–13, 2022, Online.

Table 1: The summary of PoA and PoS of CSCSG under sum-cost and max-cost. Our results are marked with *.

			parallel-link	series-parallel	DAG	General
Sum-cost	Uncapacitated	PoA	n [1]			
		PoS	1 (trivial)			
	Capacitated	PoA	n (UB [7], LB [1])		**unbounded***	
		PoS	$\log n$ [7][1]			
	Capacitated+General cost	PoA	n^*		**unbounded***	
		PoS	**UB*** : n, **LB*** : $n + 1/n - 1$			
Max-cost	Uncapacitated	PoA	n [1]			
		PoS	1 (trivial)			
	Capacitated	PoA	n [7]		**unbounded***	
		PoS	n (UB [5], LB [7][1])			
	Capacitated+General cost	PoA	n^*		**unbounded***	
		PoS	n^*			

2 OUR CONTRIBUTION

In this paper, we investigate the PoA and the PoS of CSCSG under a generalized cost-sharing scheme as explained above. We address two criteria of social cost: sum-cost and max-cost. As for the sum-cost case, we first show that PoA is unbounded even on directed acyclic graphs (DAGs). On the other hand, on series-parallel graphs (SP graphs), we show that PoA under sum-cost is at most n and it is tight, that is, there is an example whose PoA is n. For PoS, we show that it is at most n and there is an example whose PoS under sum-cost is $n + 1/n - 1$. This gives the difference from the previous study, which shows that PoS is at most $\log n$ and it is tight with ordinary fair cost-sharing functions [7].

Next, we give the results on the max-cost. As with the sum-cost, we show that PoA under max-cost is unbounded on directed acyclic graphs. On SP graphs, we prove that PoA is at most n and it is tight. We also show that PoS is at most n and it is tight. These results imply that the significant generalization does not affect PoA and PoS under max-cost. Table 1 summarizes these results.

We then discuss the capacitated *asymmetric* cost-sharing connection game (CACSG), where agents have different source and sink nodes. We show that the lower bounds of PoA and PoS of CSCSG hold for the asymmetric case, while PoS under sum-cost and max-cost are at most n and n^2, respectively.

Remark that we consider the games on directed graphs, but all the results except for directed acyclic cases can be easily modified to undirected cases. In the sense, our results are generic, which includes the results of [5, 7].

3 RELATED WORK

The cost-sharing connection game (CSG) is firstly introduced by Anshelevich et al. [1]. In the paper, they give the tight bounds of PoA and PoS under sum-cost, which are n and 1, respectively, for *uncapacitated* CSG. They also show that the PoS under sum-cost of *asymmetric* CSG, where agents have different source and sink nodes, can be bounded by $\log n$. Epstein, Feldman and Mansour study the *strong equilibria* of cost-sharing connection games [4].

Feldman and Ron [7] introduce a capacitated variant of CSGs on undirected graphs, and they give the tight bounds of PoA and PoS under both sum-cost and max-cost for several graph classes except the PoS under max-cost for general graphs. Note that their results hold only for symmetric CSG. Erlebach and Radoja fill the gap of the exception for CSG under max-cost [5]. Feldman and Ofir investigate strong equilibria for the capacitated version of CSG [6].

There are vast applications of CSG. A natural application is the decision-making in sharing economy [1–3]. Radko and Laclau mention the relationship between CSG and machine learning [10]. The previous studies for CSCSG do not consider any overhead, but sharing some resource (or tasks) yields some overhead in general. In fact, controlling overheads to share tasks is a major issue in grid/parallel computing fields [8].

[1] In [7], Feldman and Ron gave the lower bounds only for *undirected* parallel-link graphs. They can be easily modified to *directed* parallel link graphs.

ACKNOWLEDGMENTS

This work is partially supported by JSPS KAKENHI Grant Numbers JP17K19960, JP17H01698, JP19K21537, JP20H05967, JP21K17707, JP21H05852, and JP21K19765.

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