Coalition Formation Games and Social Ranking Solutions

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ABSTRACT

A social ranking (solution) over a set N is defined as a map assigning to each coalitional relation (i.e. a ranking over subsets of N) another ranking over the single elements in N. Differently, coalition formation situations, and, in particular, hedonic games, mainly focus on partitions of the set N into disjoint coalitions, which are in general referred to as coalition structures. A coalition structure may be stable according to various notions of stability and the objective is to understand under which conditions a coalition structure is stable.

In this paper we merge the framework of coalition formation with the one of social rankings to keep into account the effect of hierarchies within coalitions on the stability of coalition structures. We consider alternative classes of coalition formation games where the preferences of the players over coalitions are induced by a social ranking. More precisely, players compare coalition structures keeping into account both the relative ranking of coalitions to which they belong (according to a coalitional relation) and their position in the social ranking within each coalition. Constructive characterizations of the set of stable coalition structures are provided for alternative classes of hedonic games, together with an impossibility result on the existence of stable coalition structures for (non-hedonic) coalition formation situations.

KEYWORDS

Coalition Formation; Hedonic Games; Social Ranking; Core Stability

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1 INTRODUCTION

In a world where to be on the top of a social hierarchy is rewarded, it could be convenient to belong to a society where it is easier to rise through the ranks. For example, students might prefer to be enrolled in low-ability schools where it is simpler to win a contest, professors might be inclined to stay in little selective academic settings in order to increase their academic self-concept, and even employees seeking for quick promotions might be tempted to make their career in a less attractive organization offering small salaries. In those situations, and in every-day life, humans face the following dilemma: is it better to be a big fish in a small pond, or a small fish in a big pond? In this paper we try to provide a formal model aimed at explaining the tension between the “size” of the fish and the one of the pond in determining how a social group is formed, and how the individual social position affects the stability of a society.

For example, consider a classical (simple) coalitional game, with set of players $N = \{1, 2, 3\}$, where coalitions (subsets of N) may be winning or loosing. A coalition $S \subseteq N$ is winning, if it contains at least two players (i.e., $|S| \geq 2$), and it is losing, otherwise. If players prefer to form winning coalitions than to form losing ones, no matters their composition, then any partition of N containing a winning coalition (i.e., any partition of type $\{\{i, j\}, \{k\}\}$, with $i, j, k \in N$, or partition $\{N\}$) can be considered stable (according to the notion of core stability to be introduced in Section 2), as no player has a strong incentive to leave the element of the partition to which she belongs. On the other hand, if a social ranking over the players exists such that, for instance, player 1 is the leader of any coalition she may form, then players 2 and 3 could have some interest to form a winning coalition excluding player 1 to break free from her leadership, and $\{\{1\}, \{2, 3\}\}$ would end up being the unique stable partition. Obviously, things can be much more complicated: preferences over coalitions could be more general than a dichotomous relation over winning and losing coalitions, and the hierarchy of players induced by a social ranking could depend on the composition of coalitions and on the relative strength of their subsets. For instance, we can assume that players’ preferences use, as a primary criterion to rank coalitions, the strength of coalitions they can form, and then, as a secondary criterion, their social position within each coalition; or the same criteria, but with an opposite priority relation. In this paper, we investigate alternative models of players’ preferences over coalitions, taking into account different combinations of coalitional and social rankings, and we study their impact on the stability of coalition structures.

As a starting point of our analysis, we use a classical framework based on *hedonic games* [3, 5, 9]. In hedonic games, each player has her/his own preferences over coalitions and the goal is finding a coalition structure (i.e. a partition of the set of players) that satisfies certain desiderata and guarantees, according to alternative criteria, the stability of the coalition structure with respect to possible deviations of single individuals or groups. Another important ingredient of our analysis is the notion of *social ranking*, which is a map assigning to each total preorder representing the relative strength of coalitions (namely, a *power relation*) a ranking over the single players representing their *ordinal influence*. Several social rankings have been proposed in the literature (see, for example, [2, 7]). In the paper [7], for instance, the authors axiomatically characterize a social ranking based on the idea that the most influential individuals are those appearing more frequently in the highest positions in the ranking of coalitions. They propose a set of four properties that such a function should satisfy, and they prove that these properties

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are enough to identify a unique social ranking, which is called *lexicographic excellence* (*lex-cel*). The lex-cel has received increasing attention in the recent literature on social rankings. Some generalizations of the lex-cel considering the size of coalitions have been presented in papers [1, 4] following an axiomatic approach. In the paper [2] it has been shown that the lex-cel is resistant to certain forms of manipulability of the power relation by players trying to get higher positions in the social ranking.

The main assumptions made through this paper for players' preferences in hedonic games are the following: (1) the comparison of coalitions is the same for all individuals, who are all aware of a power relation representing the relative strength of coalitions; (2) players share the same social reference to determine a hierarchy within a coalition, which is defined according to a specific social ranking (namely, the lex-cel one); (3) the directionality of players preferences is determined by alternative combinations of two social criteria: being in strong groups and being in high social positions. Based on these hypothesis, we show that the core of hedonic games may be easily characterised using an algorithmic approach. On the other hand, as soon as the framework of hedonic games is abandoned in favour of a more general framework where players' preferences depend on the whole structure of partitions, we show that for non hedonic games with more than seven players, the core is empty.

In the following, we summarize our major contributions over the alternative coalition formation situations considered in this paper:

- in case players' preferences over coalition structures are based on the relative strength of coalitions only (without considering the lex-cel social ranking) we provide an algorithm to find the core of hedonic games using a simple iterative procedure aimed at selecting the "strongest" coalitions through the power relation;
- in case the lex-cel social ranking affects players' preferences as a secondary criterion compared to the power relation, then we show that, in general, the core of the related hedonic games may be empty. However, we prove that the core is not empty and the algorithm introduced for the previous case can be used to find it, if the power relation is *responsive* [8, 11], i.e., the relative ranking of any pair of coalitions differing for just one element (e.g., i and j) should preserve the relative ranking between the two singleton coalitions $\{i\}$ and $\{j\}$.
- in case the lex-cel social ranking is a primary criterion for players' preferences, we show that the core of the related hedonic game is always non-empty and it can be found via a more sophisticated algorithm computing the lex-cel ranking on each restriction of the power relation;
- finally, for players' preferences expressed as a combination of the power ranking and the lex-cel social rankings directly on coalition structures, we provide a general framework based on non-hedonic games boiling down to the model introduced in [10], if the power relation is responsive. Similar to [10], we show that the core of non-hedonic games with at least seven players is empty, if the power relation is responsive, but it can be non-empty, otherwise.

The roadmap of the paper is as follows. We start in the next section with some preliminary notation and notions about hedonic games and the theory of social rankings. We continue in Section 3 with three alternative models for hedonic games where the players' preference are based on the relative strength of coalitions only (Section 3.1), a combination between relative strength of coalitions as a primary criterion, and social rankings as a secondary criterion (Section 3.2) and another combination of those two criteria giving priority to social rankings over the relative strength of coalitions (Section 3.3). Section 4 is devoted to the analysis of a (non-hedonic) model of coalition formation where players' preferences are expressed directly on partitions of the player set, again on the basis of a combination between coalitional strength and social rankings. Section 5 concludes with some perspectives for future research.

2 PRELIMINARIES

Let $N = \{1, \ldots, n\}$ be a finite set of agents/players and $\mathcal{P}(N)$ the set of its non-empty subsets. A *preorder* on N is a reflexive and transitive binary relation on N. A preorder that is total is called *total preorder*, and the family of all total preorders on N is denoted by $\mathcal{T}^N$. An antisymmetric[1] total preorder is called *linear order* or *total order*. A total preorder $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ is said a *power relation*. Given $S, T \in \mathcal{P}(N)$, $S \sqsupseteq T$ will stand for "S is at least as strong as T with respect to the power relation $\sqsupseteq$". We denote by $\sim$ its symmetric part (i.e. $S \sim T$ if $S \sqsupseteq T$ and $T \sqsupseteq S$) and by $\sqsupset$ its asymmetric part (i.e. $S \sqsupseteq T$ and not $T \sqsupseteq S$). So, for each pair of subsets $S, T \in \mathcal{P}(N)$, $S \sqsupset T$ means that S is *strictly stronger* than T, whereas $S \sim T$ means that S and T are *indifferent*.

Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be of the form $S_1 \sqsupseteq S_2 \sqsupseteq S_3 \geq \cdots \sqsupseteq S_{|\mathcal{P}(N)|}$. The *quotient order* of $\sqsupseteq$ is denoted as $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \Sigma_3 \sqsupset \cdots \sqsupset \Sigma_l$ in which the subsets S_j are grouped in the *equivalence classes* Σ_k generated by the symmetric part of $\sqsupseteq$. This means that all the sets in Σ_1 are indifferent to S_1 and are strictly better than the sets in Σ_2 and so on. So, $\Sigma_i = S_i$ for any $i = 1, \ldots, |\mathcal{P}(N)|$ if and only if $\sqsupseteq$ is a linear order.

A *social ranking (solution)* on N is a function $\rho : \mathcal{T}^{\mathcal{P}(N)} \longrightarrow \mathcal{T}(N)$ associating to each power relation $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ a total preorder $\rho(\sqsupseteq)$ (or $\rho^{\sqsupseteq}$) over the elements of N. By this definition, the notion $i\rho^{\sqsupseteq}j$ means that applying the social ranking to the power relation $\sqsupseteq$ gives the result that i is ranked higher than or equal to j. We denote by $I^{\sqsupseteq}$ the symmetric part of $\rho^{\sqsupseteq}$, and by $P^{\sqsupseteq}$ its asymmetric part. A particular social ranking from the literature is the lexicographic excellence one [7]. Given a power relation $\sqsupseteq$ and its associated quotient order $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \Sigma_3 \sqsupset \cdots \sqsupset \Sigma_m$, we denote by $i_k = |\{S \in \Sigma_k : i \in S\}|$ the number of sets in Σ_k containing i for $k = 1, \ldots, l$. Now, let $\theta^{\sqsupseteq}(i)$ be the l-dimensional vector $\theta^{\sqsupseteq}(i) = (i_1, \ldots, i_l)$ associated to $\sqsupseteq$. Consider the lexicographic order $\geq_L$ among vectors $\mathbf{i}$ and $\mathbf{j}$: $\mathbf{i} \geq_L \mathbf{j}$ if either $\mathbf{i} = \mathbf{j}$ or there exists t such that $i_t > j_t$ and $i_r = j_r$ for all $r \in \{1, \ldots, t-1\}$.

Definition 2.1 (Lexicographic-excellence (lex-cel) [7]). Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$. The *lexicographic excellence (lex-cel)* is the binary relation $R_{le}^{\sqsupseteq}$ such that for all $i, j \in N$:

$$i \, R_{le}^{\sqsupseteq} \, j \iff \theta^{\sqsupseteq}(i) \quad \geq_L \quad \theta^{\sqsupseteq}(j).$$

[1] $\forall i, j \in N$, iRj and $jRi \Rightarrow i = j$.

Example 2.2. Let $N = \{1, 2, 3\}$ be the set of agents and let's consider the ranking $\sqsupseteq$ as follows:
$\{1, 2, 3\} \sim \{1, 3\} \sim \{2, 3\} \sqsupset \{1, 2\} \sim \{2\} \sqsupset \{1\} \sqsupset \{3\}$. We have $\theta^{\sqsupseteq}(1) = (2, 1, 1, 0)$, $\theta^{\sqsupseteq}(2) = (2, 2, 0, 0)$ and $\theta^{\sqsupseteq}(3) = (3, 0, 0, 1)$. So, the lex-cel returns the following ranking: $3 \; P^{\sqsupseteq}_{le} \; 2 \; P^{\sqsupseteq}_{le} \; 1$.

A *hedonic game* is defined as the structure $G = (N, (\succeq_i)_{i \in N})$ where $\succeq_i \in \mathcal{T}^{\mathcal{N}_i}$ is a total preorder on $\mathcal{N}_i = \{S \in \mathcal{P}(N) : i \in S\}$, for any $i \in N$. A *coalition structure* $\Pi \subset \mathcal{P}(N)$ is just a partition of N, and we denote by $\Pi(i)$ the element to which an agent $i \in N$ belongs in Π. A coalition $C \subseteq N$ *blocks* a coalition structure Π if $C \succ_i \Pi(i)$ for all $i \in C$. A coalition structure that is not blocked by any coalition is said *stable*. The *core* of a hedonic game G is the set of all coalition structures that are stable, and it is denoted by $C(G)$.

3 HEDONIC GAME MODELS

Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be a power relation with the associated quotient order $\Sigma_1 \sqsupset \Sigma_2 \sqsupset \Sigma_3 \sqsupset \cdots \sqsupset \Sigma_l$. For each element $S \in \mathcal{P}(N)$, the number $\delta^S(\sqsupseteq) \in \{1, 2, ..., l\}$ is defined as the index of the equivalence class to which the coalition S belongs (i.e., $\delta^S(\sqsupseteq)$ is such that $S \in \Sigma_{\delta^S(\sqsupseteq)}$). In this section we analyse three alternative definitions of players' preferences for hedonic games.

3.1 First Case: Preferences Depend Only on the Coalitional Strength

We first consider the case where players are aimed at forming coalition as much strong as possible, with no interest in the social position they cover within coalitions. We therefore define a hedonic game $G^1 = (N, (\succeq_i)_{i \in N})$ where $\succeq_i$, for each player $i \in N$, is the preference relation of player i such that

$$S \succeq_i T \iff \delta^S(\sqsupseteq) \leq \delta^T(\sqsupseteq), \tag{1}$$

for each coalition $S, T \in \mathcal{N}_i$.
In this way, given the preorder on coalitions, each player is only interested in being in the strongest coalitions with respect to the power relation $\sqsupseteq$, completely disregarding any internal balances within the coalitions.
Having defined the hedonic game, we are interested in understanding whether there are stable partitions and, if there are, what they are. For this purpose, we introduce Algorithm 1 that allows us to find all and only stable partitions[2]. Basically Algorithm 1 performs the following steps:

(1) it selects a coalition S from the best equivalence class of the quotient order associated to $\sqsupseteq$;
(2) it removes all coalitions that have non-empty intersection with S and redefines on the remaining coalitions the new quotient order;
(3) it repeats the above steps until there are no more coalitions.

Example 3.1. Let $N = \{1, 2, 3, 4\}$ and consider the power relation: $\{1, 3\} \sim \{2, 3\} \sim \{1, 2, 4\} \sqsupset \{1, 2\} \sqsupset \{2, 4\} \sqsupset \{1\} \sqsupset \{3\} \sqsupset \{2\} \sqsupset \{4\} \sim \{1, 4\} \sim \{3, 4\} \sim \{1, 2, 3\} \sim \{1, 3, 4\} \sim \{2, 3, 4\} \sim \{1, 2, 3, 4\}$. The quotient order[3] of the power relation is shown in Table 1. We denote by $\mathcal{A}(\sqsupseteq)$ the set of all coalition structures that, given a power relation $\sqsupseteq$, are generated by Algorithm 1.

Algorithm 1: finding a stable coalition structure

Input: A quotient order on $\mathcal{P}(N)$ in the form $\Sigma_1 \sqsupset \cdots \sqsupset \Sigma_l$;
Output: A coalition structure $\Pi = \{S_0, \ldots, S_m\} \subseteq \mathcal{P}(N)$;

```
t ← 0;
p ← l;
Π ← ∅;
for k=1 to p do
    Σ_k^(0) ← Σ_k;
end
while Σ_1^(t) ≠ ∅ do
    Π ← Π ∪ {S_t}, with S_t ∈ Σ_1^(t);
    I^(t) ← {C ∈ ⋃_{k=1}^{p} Σ_k^(t) : S_t ∩ C ≠ ∅};
    r ← 0;
    Σ_1^(t+1) ← ∅;
    for k=1 to p do
        if Σ_k^(t) \ I^(t) ≠ ∅;
         then
            r ← r + 1;
            Σ_r^(t+1) ← Σ_k^(t) \ I^(t);
        end
    end
    t ← t + 1;
    p ← r;
end
```

Consider the coalition structure $\Pi = \{\{2, 4\}, \{1\}, \{3\}\}$. It is easy to verify that Π is not stable. Now, using Algorithm 1, we can obtain the following coalition structures:

$$\mathcal{A}(\sqsupseteq) = \{\{\{1, 3\}, \{2, 4\}\},\ \{\{2, 3\}, \{1\}, \{4\}\}\},\ \{\{1, 2, 4\}, \{3\}\}\}.$$

We observe that all the three coalition structures in $\mathcal{A}(\sqsupseteq)$ are stable and form the core of the corresponding hedonic game G^1. In fact, the next theorem shows that, in general, all and only the coalition structures generated by Algorithm 1 are stable, i.e. $\mathcal{A}(\sqsupseteq) = C(\sqsupseteq)$.

Σ_1	Σ_2	Σ_3	Σ_4	Σ_5	Σ_6	Σ_7
13, 23, 124	12	24	1	3	2	4, 14, 34, 123, 134, 234, 1234

Table 1: Quotient order of the power relation of Example 3.1.

THEOREM 3.2. *Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be a power relation and let $G^1 = (N, (\succeq_i)_{i \in N})$ be a hedonic game where the players' preferences are defined according to relation (1). It holds that $C(G^1) = \mathcal{A}(\sqsupseteq)$.*

PROOF. We first prove that $\Pi \in \mathcal{A}(\sqsupseteq) \Longrightarrow \Pi \in C(G^1)$.
Consider a coalition structure $\Pi = \{S_0, S_1, ..., S_m\} \in \mathcal{A}(\sqsupseteq)$.
Suppose that $\Pi \notin C(G^1)$. Then, there must exist a coalition $T \in \Sigma_h$

[2] An algorithm similar to Algorithm 1 has been recently studied in paper [6] to introduce a family of social rankings rooted on the notion of stable coalition structure.
[3] To save space, sets are denoted for short without commas and parentheses, e.g., 123 instead of {1, 2, 3}, and $S \cup i$ instead of $S \cup \{i\}$.

for some $h \in \{1, 2, ..., l\}$ such that $T \succ_i \Pi(i)\ \forall i \in T$.
Since T was not selected by the algorithm, it means that there exists a step t of the algorithm where $S_t \in \Sigma_j$ for some $j \leq h$ with $T \cap S_t \neq \emptyset$. But then $S_t \succeq_i T\ \forall i \in T \cap S_t$, which yields a contradiction.
We now prove the opposite implication $\Pi \in C(G^1) \Longrightarrow \Pi \in \mathcal{A}(\sqsupseteq)$. Consider $\Pi = \{S_0, S_1, ..., S_m\} \in C(G^1)$, adopting as convention:

$$S_0 \sqsupseteq S_1 \sqsupseteq ... \sqsupseteq S_m.$$

We will show that, for every $h \in \{0, 1, ..., m\}$, $S_h \in \Sigma_1^{(h)}$, where $\Sigma_1^{(h)}$ is the first equivalent class of the quotient order $\Sigma_1 \sqsupset \cdots \sqsupset \Sigma_l$ constructed at step h, following the procedure of Algorithm 1, which is equivalent to say that Π can be obtained by Algorithm 1.
One of the following cases holds: (1) either $S_0 \notin \Sigma_1$; (2) or $S_0 \in \Sigma_1$. In the first case there would be a contradiction, since every partition in the core must have an element in Σ_1 (and in particular $S_0 \in \Sigma_1$ since by the convention adopted $S_0 \sqsupseteq S_j\ \forall j = 1, ..., m$). If not, any coalition in Σ_1 would be a blocking coalition. So, the second case holds, and we redefine a new quotient order $\Sigma_1^{(1)} \sqsupset \Sigma_2^{(1)} \sqsupset ... \sqsupset \Sigma_{l_1}^{(1)}$ among the coalitions that have empty intersection with S_0 (similarly to what is done by the algorithm). Since Π is a partition, its elements are disjoint, so S_1 must belong to an equivalent $\Sigma_j^{(1)}$, for some $j = 1, ..., l_1$. So, one of the following cases must hold: (1) either $S_1 \notin \Sigma_1^{(1)}$; (2) or $S_1 \in \Sigma_1^{(1)}$. In the first case we would still have a contradiction, since there would exist a coalition $T \in \Sigma_1^{(1)}$ such that, for every $i \in T$, $T \succ_i \Pi(i)$, since $\Pi(i) \in \Sigma_j^{(1)}$ for $j > 1$.
So, the second case holds, and we redefine a new quotient order $\Sigma_1^{(2)} \sqsupset \Sigma_2^{(2)} \sqsupset ... \sqsupset \Sigma_{l_2}^{(2)}$ among coalitions that have empty intersection with $S_0 \cup S_1$:
Continuing iteratively, if at some step $h \in \{1, ..., m\}$ we have $S_{h-1} \in \Sigma_1^{(h-1)}$, a new quotient order between the coalitions that have empty intersection with $\bigcup_{j=0}^{h-1} S_j$ is defined. So, we have that: (1) either $S_h \notin \Sigma_1^{(h)}$; (2) or $S_h \in \Sigma_1^{(h)}$. Again, in the first case there would be a contradiction, since there would exist $T \in \Sigma_1^{(h)}$ such that $T \succ_i \Pi(i)$ for all $i \in T$, with $\Pi(i) \in \Sigma_j^{(h)}$ and $j > 1$. So, the second case would hold the procedure would continue till step m where $S_m \in \Sigma_1^{(m)}$, and at this point the partition Π is generated according to the algorithm. □

3.2 Second Case: Preferences Depend Lexicographically on Coalitional Strength and on Social Ranking

In this second case we want to study the consequences of inducing a hedonic game based on a power relation, as in the previous case, but this time aimed at breaking some possible ties within coalitions, so that players no longer consider irrelevant their internal social position. To establish an internal ranking for a coalition S, we use a particular notion of social ranking and, for any coalition S, we compute it on the restriction of a power relation to S, which we denote with $\sqsupseteq_S$. So, $\sqsupseteq_S$ is obtained from $\sqsupseteq$ by excluding coalitions that contain players not in S (i.e., $T \sqsupseteq_S U \Leftrightarrow T \sqsupseteq U$ for all $T, U \subseteq S$). In this way given a coalition S and a social ranking ρ, we can define the quotient order of $\rho^{\sqsupseteq_S}$ among the elements of S:

$$\Gamma_1\ P^{\sqsupseteq_S}\ \Gamma_2\ P^{\sqsupseteq_S} \dots P^{\sqsupseteq_S}\ \Gamma_q,$$

where $P^{\sqsupseteq_S}$ is the asymmetric part of the social ranking $\rho^{\sqsupseteq_S}$, that is computed on the restricted power relation $\sqsupseteq_S$. This means that, according to the social ranking ρ computed on the restricted power relation $\sqsupseteq_S$, all the players of S in Γ_1 share the same social position which is strictly higher than players in Σ_2, and so on. In particular, in the following we will consider as a social ranking the lex-cel, i.e. $\rho = R_{le}$. Consider the following values:

- as before, for each coalition $S \in \mathcal{P}(N)$ the value $\delta^S(\sqsupseteq) \in \{1, 2, ..., l\}$ is such that $S \in \Sigma_{\delta^S(\sqsupseteq)}$;
- we introduce the value $\lambda_i(\sqsupseteq_S)$, for each S and each $i \in S$, as the index of the equivalence class to which individual i belongs with respect to the quotient order $\Gamma_1\ P^{\sqsupseteq_S}\ \Gamma_2\ P^{\sqsupseteq_S} \dots P^{\sqsupseteq_S}\ \Gamma_q$ associated to the lex-cel ranking $R_{le}^{\sqsupseteq_S}$ over players in S (i.e., $\lambda_i(\sqsupseteq_S)$ is such that $i \in \Gamma_{\lambda_i(\sqsupseteq_S)}$).

We therefore define a hedonic game $G^2 = (N, (\succeq_i)_{i \in N})$ where $\succeq_i$, for each $i \in N$, is the preference relation of player i such that

$$S \succeq_i T \iff (\delta^S(\sqsupseteq), \lambda_i(\sqsupseteq_S)) \leq_L (\delta^T(\sqsupseteq), \lambda_i(\sqsupseteq_T)) \qquad (2)$$

for each coalition $S, T \in \mathcal{N}_i$, where $\leq_L$ represents the lexicographic relation. Notice that G^2 is now dependent not only on $\sqsupseteq$, but also on the notion of social ranking adopted.

Example 3.3. Consider the power relation of Example 3.1 and let $S = \{1, 2, 3\}$. To define $\lambda_i(\sqsupseteq_S)$ for $i = 1, 2, 3$ we compute the lex-cel on the restriction of $\sqsupseteq$ to S:
$\{1, 3\} \sim_S \{2, 3\} \sqsupset_S \{1, 2\} \sqsupset_S \{1\} \sqsupset_S \{3\} \sqsupset_S \{2\} \sqsupset_S \{1, 2, 3\}$.
It is easy to see that $3P_{le}^{\sqsupseteq_S}1P_{le}^{\sqsupseteq_S}2$. So, $\lambda_1(\sqsupseteq_S) = 2$, $\lambda_2(\sqsupseteq_S) = 3$ and $\lambda_3(\sqsupseteq_S) = 1$. Notice that in general, the lex-cel on $\sqsupseteq$ is different from the lex-cel computed on a restriction of $\sqsupseteq$ to a coalition S. For instance, in this case we have $1P_{le}^{\sqsupseteq}3$ on the power relation $\sqsupseteq$, whereas we have $3P_{le}^{\sqsupseteq_S}1$ on its restriction to S.

Note that with preferences defined according to relation (2), we are still giving more importance to the strength of coalitions (the information provided by the power relation) than to the social ranking within coalitions, since in the lexicographic order the index $\delta^S(\sqsupseteq)$ appears first. Of course, one could decide to use the order on singletons to generate a social ranking ρ_1 such that $i\rho_1^{\sqsupseteq}j \iff \{i\} \sqsupseteq \{j\}$ for all $i, j \in N$. The choice of not using singletons, however, is motivated by the fact that the social ranking ρ_1, considering only the order on coalitions of size one, is somewhat blind to the interaction abilities among players.
However, generating stable coalition structures for hedonic games with preferences defined according to relation (2) is no longer so easy and, in particular, Algorithm 1 cannot guarantee to find a stable coalition structure, as shown in the following example.

Example 3.4. In Table 2, we consider the coalitions that are elements of coalition structures in $\mathcal{A}(\sqsupseteq)$, the set of all coalition structures generated by Algorithm 1, and for each of such coalitions we show the corresponding values of $\delta^S(\sqsupseteq)$ and of $\lambda_i(\sqsupseteq_S)$, $i \in \{1, 2, 3, 4\}$. Coalition $\{1, 3\}$, for instance, belongs to the first equivalence class of the quotient order (i.e., $\delta^{\{1,3\}}(\sqsupseteq) = 1$), and the lex-cel applied to the restriction of $\sqsupseteq$ to $\{1, 3\}$ gives that 1 belongs to the first equivalence class of the quotient order associated to $R_{le}^{\sqsupseteq_{\{1,3\}}}$

S	$\delta^S(\sqsupseteq)$	$\lambda_1(\sqsupseteq_S)$	$\lambda_2(\sqsupseteq_S)$	$\lambda_3(\sqsupseteq_S)$	$\lambda_4(\sqsupseteq_S)$
$\{1,3\}$	1	1	*	2	*
$\{2,4\}$	3	*	1	*	2
$\{2,3\}$	1	*	2	1	*
$\{1\}$	4	1	*	*	*
$\{4\}$	7	*	*	*	1
$\{1,2,4\}$	1	2	1	*	3
$\{3\}$	5	*	*	1	*

Table 2: Values of $\delta^S(\sqsupseteq)$ and of $\lambda_i(\sqsupseteq_S)$, $i \in \{1,2,3,4\}$, for the elements of coalition structures in $\mathcal{A}(\sqsupseteq)$ of Example 3.4.

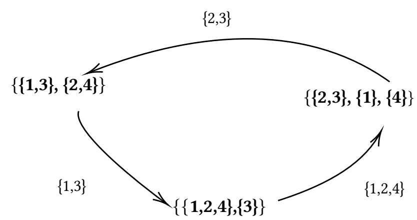

Figure 1: Blocking cycle among coalition structures in $\mathcal{A}(\sqsupseteq)$ (in bold): each coalition structure is blocked by a an element of another coalition structure. Arrows represents the blocking action by an element of the coalition structure from which the arrow starts (the blocking element is indicated close to the corresponding arrow).

(i.e., $\lambda_1(\sqsupseteq_{\{1,3\}}) = 1$), while 3 belongs to the second equivalence class of the quotient order associated to $R_{le}^{\sqsupseteq_{\{1,3\}}}$ (i.e, $\lambda_3(\sqsupseteq_{\{1,3\}}) = 2$). To check that there are no stable coalition structures, it is enough to focus on coalition structures provided by Algorithm 1. Clearly, the coalition structures in $\mathcal{A}(\sqsupseteq)$ are not stable, since each of them is blocked by an element of another one as shown in the diagram in Figure 1. For instance, the coalition structure $\{\{1,3\},\{2,4\}\}$ is blocked by coalition $\{2,3\}$ for player 2 strictly prefers $\{2,3\}$ to $\{2,4\}$ since $\{2,3\} \sqsupset \{2,4\}$ (and, so, $\delta^{\{2,3\}} = 1 < 3 = \delta^{\{2,4\}}$), while $\{1,3\} \sim \{2,3\}$ (and, so, $\delta^{\{2,3\}} = 1 = \delta^{\{1,3\}}$), and the advantage to deviate for player 3 lies in the fact that: $\lambda_3(\sqsupseteq_{\{2,3\}}) = 1 < 2 = \lambda_3(\sqsupseteq_{\{1,3\}})$.

For hedonic games based on preferences defined according to relation (2), even players already in one of the strongest coalitions according to the power relation, can improve their coalition by trying to climb positions within it, and this makes the search for stable coalition structures much more complicated. However, one can consider a subclass of power relations such that it is still possible to generate stable partitions using Algorithm 1. For this reason, we now introduce a well known property for power relations.

Definition 3.5 (Responsiveness [8, 11]). A total preorder $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ is *responsive* if

$$\{i\} \sqsupseteq \{j\} \iff S \cup \{i\} \sqsupseteq S \cup \{j\},$$

for all $i,j \in N$ and all $S \subseteq N \setminus \{i,j\}$.

Notice that the power relation of Example 3.1 is not responsive as, for instance, $\{1\} \sqsupset \{3\}$, but $\{2,3\} \sqsupset \{1,2\}$.
Now, before introducing the next proposition, we study how the lex-cel behaves with respect to responsive power relations.

Lemma 3.6. *Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be a responsive power relation. Then, $R_{le}^{\sqsupseteq} = \rho_1^{\sqsupseteq}$. Precisely, for all $i,j \in N$,*

$$iR_{le}^{\sqsupseteq}j \iff \{i\} \sqsupseteq \{j\}.$$

Proof. We first prove the implication $iR_{le}^{\sqsupseteq}j \implies \{i\} \sqsupseteq \{j\}$. Let $i,j \in N$ be such that $iR_{le}^{\sqsupseteq}j$. Suppose $\{j\} \sqsupset \{i\}$. Then, by responsiveness, $S \cup \{j\} \sqsupset S \cup \{i\}$ for every $S \subseteq N \setminus \{i,j\}$. So, there must exist a particular $T \subseteq N \setminus \{i,j\}$ such that: $T \cup \{j\} \sqsupseteq S \cup \{j\} \sqsupset S \cup \{i\}$ for every $S \subseteq N \setminus \{i,j\}$. But this means that $jP_{le}^{\sqsupseteq}i$ (j is ranked strictly better than i according to the lex-cel), which yields a contradiction.

We now prove the opposite implication $\{i\} \sqsupseteq \{j\} \implies i\,R_{le}^{\sqsupseteq}\,j$. It is easy to check that if $\{i\} \sim \{j\}$ then $i\,I_{le}^{\sqsupseteq}\,j$, for the responsiveness of $\sqsupseteq$ implies $T \cup \{i\} \sim T \cup \{j\}$ for all $T \subseteq N \setminus \{i,j\}$. So, it remains to prove that $\{i\} \sqsupset \{j\} \implies i\,P_{le}^{\sqsupseteq}\,j$.
Let $i,j \in N$ be such that $\{i\} \sqsupset \{j\}$. Suppose that $jR_{le}^{\sqsupseteq}i$. Then, by definition of the lex-cel, there must exist $T \subseteq N \setminus \{i,j\}$ such that $T \cup \{j\} \sqsupseteq S \cup \{i\}$ for every $S \subseteq N \setminus \{i,j\}$.
But this yields a contradiction with the fact that $\sqsupseteq$ is responsive and so it must be $T \cup \{i\} \sqsupset T \cup \{j\}$ for all $T \subseteq N \setminus \{i,j\}$. □

Lemma 3.6 tells us that in responsive power relations, the lex-cel ranking coincides with the one given by the singletons.

Lemma 3.7. *Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be a responsive power relation, $S,T \in \mathcal{P}(N)$ and $a \in S \cap T$ be such that $S_{a_+} = \{z \in S : \{z\} \sqsupset \{a\}\} \neq \emptyset$ and $\widetilde{S}_{a_+} = S_{a_+} \cap T$. The following implication holds*

$$\lambda_a(\sqsupseteq_T) < \lambda_a(\sqsupseteq_S) \implies S_{a_+} \setminus \widetilde{S}_{a_+} \neq \emptyset.$$

Proof. Consider the quotient orders associated to the social ranking $R_{le}^{\sqsupseteq_S}$ over players in S and to the social ranking $R_{le}^{\sqsupseteq_T}$ over players in T, respectively, $\Gamma_1^{(S)} P_{le}^{\sqsupseteq_S} \Gamma_2^{(S)} P_{le}^{\sqsupseteq_S} \ldots P_{le}^{\sqsupseteq_S} \Gamma_{q_S}^{(S)}$ and $\Gamma_1^{(T)} P_{le}^{\sqsupseteq_T} \Gamma_2^{(T)} P_{le}^{\sqsupseteq_T} \ldots P_{le}^{\sqsupseteq_T} \Gamma_{q_T}^{(T)}$. Suppose $a \in \Gamma_h^{(S)}$, for some $h \in \{1,2,\ldots,q_S\}$. By Lemma 3.6, the lex-cel ranking is equivalent to that of the singletons[4]. Then, we have that $h \geq 2$ (because $S_{a_+} \neq \emptyset$) and all the players that are in $\Gamma_j^{(S)}$ for $j < h$ are also elements of S_{a_+}.
Now, suppose $\lambda_a(\sqsupseteq_T) < \lambda_a(\sqsupseteq_S)$ and $S_{a_+} \setminus \widetilde{S}_{a_+} = \emptyset$. Then, it means that $\widetilde{S}_{a_+} = S_{a_+}$, i.e. all players in S, that are ranked strictly higher than a, are also in T. So, it implies that $a \in \Gamma_r^{(T)}$ for $r \geq h$. But that is equivalent to $\lambda_a(\sqsupseteq_T) \geq \lambda_a(\sqsupseteq_S)$, which yields a contradiction. □

Lemma 3.7 says that moving from a coalition S to a coalition T, the internal position of a player a improves only if at least one of the players stronger than a and present in S, is not present in T.

We are now ready to introduce the main result of this section, stating the equivalence between the set of coalition structures produced by Algorithm 1 and the core of a hedonic game where players' preferences are defined according to relation (2) under responsive power relations.

[4]Note also that if we consider a responsive ranking restricted to a coalition S, the obtained ranking is still responsive.

Theorem 3.8. *Let* $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ *be a responsive power relation and let* $G^2 = (N, (\succeq_i)_{i\in N})$ *be a hedonic game where players' preferences are defined according to relation (2). It holds that* $C(G^2) = \mathcal{A}(\sqsupseteq)$.

Proof. We first prove implication $\Pi \in \mathcal{A}(\sqsupseteq) \Longrightarrow \Pi \in C(G^2)$. Consider the coalition structure $\Pi = \{S_0, S_1, ..., S_m\}$ with $S_0 \sqsupseteq S_1 \sqsupseteq, \dots, \sqsupseteq S_m$. Suppose $\Pi \notin C(G^2)$. Then, there exists a blocking coalition $T \in \Sigma_h$ for some $h \in \{1, 2, ..., l\}$ such that $T \succ_i \Pi(i)$ $\forall i \in T$.
Similar to the case of Section 3.1, where preferences were defined according to relation (1) on the basis of the strength of coalitions only, since T is not selected by Algorithm 1, there must exist a step of the algorithm p where $S_p \in \Sigma_j$ was selected for some $j \leq h$ with $T \cap S_p \neq \emptyset$.
In particular, we define $t = \min\{p : T \cap S_p \neq \emptyset\}$, so that we know that $T \cap S_j = \emptyset$ for $j = 1, 2, ..., t-1$.
Notice that $\delta^{S_t}(\sqsupseteq) \leq \delta^T(\sqsupseteq)$. However, since T must block Π it must hold that $\delta^{S_t}(\sqsupseteq) = \delta^T(\sqsupseteq)$ and $\lambda_i(\sqsupseteq_T) < \lambda_i(\sqsupseteq_{S_t})$ $\forall i \in T \cap S_t$.
We now consider the generic element $a \in T \cap S_t$ and write the two sets as follows:

$$\begin{cases} S_t = \{a\} \cup S_{a_+} \cup S_{a_-} \\ T = Q \cup \{a\} \cup \widetilde{S}_{a_+} \cup \widetilde{S}_{a_-} \end{cases}$$

where $S_{a_+} = \{z \in S : \{z\} \sqsupset \{a\}\}$, $S_{a_-} = \{z \in S : \{a\} \sqsupseteq \{z\}, z \neq a\}$, $\widetilde{S}_{a_+} = S_{a_+} \cap T$ and $\widetilde{S}_{a_-} = S_{a_-} \cap T$. The following facts hold:

1. $\lambda_a(\sqsupseteq_T) < \lambda_a(\sqsupseteq_{S_t})$; so, by Lemma 3.7, $S_{a_+} \setminus \widetilde{S}_{a_+} \neq \emptyset$;
2. $T \cap S_j = \emptyset$ for $j = 1, 2, ..., t-1$, since every player in the blocking coalition T cannot be an element of coalitions in a stronger equivalence class than the one to which S_t belongs.

Let $b \in S_{a_+} \setminus \widetilde{S}_{a_+}$. By the responsiveness of $\sqsupseteq$, we have that
$\{b\} \sqsupset \{a\} \Longrightarrow ((T \setminus \{a\}) \cup \{b\}) \sqsupset ((T \setminus \{a\}) \cup \{a\}) = T \sim S_t$.
Moreover, knowing that all elements of T cannot be taken from $S_1, S_2, ..., S_{t-1}$, this leads a contradiction (the algorithm should have selected $((T \setminus \{a\}) \cup \{b\})$ or another coalition stronger than S_t), since S_t was generated by Algorithm 1 and we showed that

$$\begin{cases} ((T \setminus \{a\}) \cup \{b\}) \sqsupset S_t \\ ((T \setminus \{a\}) \cup \{b\}) \cap S_j = \emptyset \ \ \forall j < t. \end{cases}$$

The proof of the opposite implication $\Pi \in C(G^2) \Longrightarrow \Pi \in \mathcal{A}(\sqsupseteq)$, follows immediately from the fact that if $G^1 = (N, (\succeq'_i)_{i\in N})$ is the hedonic game in which the preferences $(\succeq_i)_{i\in N}$ were defined according to relation (1), then it holds: $C(G^2) \subseteq C(G^1) = \mathcal{A}(\sqsupseteq)$. □

3.3 Third Case: Preferences Depend Lexicographically on Social Ranking and on Coalitional Strength

This section is devoted to analyse the opposite case of players' preferences defined according to relation (2). In this case, we assume that players first want to maximize their position within a coalition according to the social ranking and, as a secondary criterion, they prefer to be be in stronger coalitions. To define the corresponding hedonic game $G^3 = (N, (\succeq_i)_{i\in N})$ we use the same ingredients introduced in Section 3.2, but now we define the preference relation $(\succeq_i)_{i\in N}$, for each $i \in N$ as follows

$$S \succeq_i T \iff (\lambda_i(\sqsupseteq_S), \delta^S(\sqsupseteq)) \leq_L (\lambda_i(\sqsupseteq_T), \delta^T(\sqsupseteq)) \qquad (3)$$

for each coalition $S, T \in \mathcal{N}_i$, where $\leq_L$ represents again the lexicographic relation.

Algorithm 2: finding a stable coalition structure

```
Input: A quotient order on P(N) in the form
  Σ_1 ⊐ ··· ⊐ Σ_l;
Output: A coalition structure Π = {S_0, ..., S_m} ⊆ P(N);
t ← 0; p ← l; Π ← ∅;
for k=1 to p do
    Σ_k^(0) ← Σ_k;
end
while Σ_1^(t) ≠ ∅ do
    S_t ← ∅;
    u ← 1;
    while u ≤ p do
        if ∃S ∈ Σ_u^(t) s.t.: λ_i(⊒_S) = 1 ∀i ∈ S;
         then
            S_t ← S;
            u ← p;
        end
        u ← u + 1;
    end
    Π ← Π ∪ {S_t};
    I^(t) ← {C ∈ ⋃_{k=1}^{p} Σ_k^(t) : S_t ∩ C ≠ ∅};
    r ← 0;
    Σ_1^(t+1) ← ∅;
    for k=1 to p do
        if Σ_k^(t) \ I^(t) ≠ ∅;
         then
            r ← r + 1;
            Σ_r^(t+1) ← Σ_k^(t) \ I^(t);
        end
    end
    t ← t + 1;
    p ← r;
end
```

Roughly speaking, Algorithm 2 performs the following steps:

1. starting with the best equivalent class, it looks for a coalition S, in which the players are all equivalent (i.e, $\lambda_i(\sqsupseteq_S) = 1$ for every $i \in S$);
2. it removes all coalitions that have non-empty intersection with S and redefines on the remaining coalitions the new quotient order;
3. it repeats the above steps until there are no more coalitions.

We denote by $\mathcal{A}'(\sqsupseteq)$ the partitions generated by Algorithm 2. One can easily verify that Algorithm 2 always generates at least one partition (i.e: $\mathcal{A}'(\sqsupseteq) \neq \emptyset$). In fact, if we consider the coalition structure $\Pi = \{\{1\}, \{2\}, \dots, \{|N|\}\}$, it holds that $\lambda_i(\sqsupseteq_{\{i\}}) = 1$ for every $i \in N$, which means that if no other coalition structure Π'

exists where all players i gets $\lambda_i(\sqsupseteq_{\Pi'(i)}) = 1$ and where $\Pi'(i) \sqsupset \{1\}$, then $\Pi \in \mathcal{A}'(\sqsupseteq)$.

Example 3.9. Consider again the power relation of Example 3.1. It is immediate to see that $\mathcal{A}'(\sqsupseteq)$ is a singleton containing the unique coalition structure $\{\{1\},\{2\},\{3\},\{4\}\}$.

Theorem 3.10. *Let $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$ be a responsive power relation and let $G^3 = (N, (\succeq_i)_{i\in N})$ be a hedonic game where players' preferences are defined according to relation (3). It holds that $C(G^3) = \mathcal{A}'(\sqsupseteq)$.*

Proof. We prove the implication $\Pi \in \mathcal{A}'(\sqsupseteq) \Longrightarrow \Pi \in C(G^3)$. Consider a coalition structure $\Pi = \{S_0, S_1, ..., S_m\}$, with $S_0 \sqsupseteq S_1 \sqsupseteq \ldots \sqsupseteq S_m$. If Π is not in the core there must exist a blocking coalition T such that $T \succ_i \Pi(i)\ \forall i \in T$.
We define $t = min\{p : S_p \cap T \neq \emptyset\}$. Then, this means that $\lambda_i(\sqsupseteq_T) = 1$ and $T \sqsupset S_t \sqsupseteq \Pi(i)$ for all $i \in T$, which yields a contradiction with the fact that Algorithm 2 has selected S_t instead of T.
We now prove the opposite implication $\Pi \in C(G^3) \Longrightarrow \Pi \in \mathcal{A}'(\sqsupseteq)$. Suppose $\Pi \notin \mathcal{A}'(\sqsupseteq)$. There must exist $t \in \{1, ..., m\}$ with $S_t \in \Sigma_j^{(t)}$ and there must exist $T \in \Sigma_{j'}^{(t)}$ with $j' < j$ such that $\lambda_i(\sqsupseteq_T) = 1$ $\forall i \in T$. However, in this way T would block Π, which yields a contradiction with the fact that Π is in the core. □

As an immediate consequence of Proposition 3.10 and the fact previously remarked that $\mathcal{A}'(\sqsupseteq) \neq \emptyset$ for any power relation $\sqsupseteq$, we have that the core of any hedonic game G^3, with preferences defined according to relation (3), is non-empty (i.e, $C(G^3) \neq \emptyset$).

4 A MODEL WITH PREFERENCES ON COALITION STRUCTURES

In the previous sections, we dealt with purely hedonic games. In this section, we introduce a (non-hedonic) coalition formation situation, in which players' preferences are directly expressed over coalition structures. For this reason, we introduce a more general definition of the notion of blocking coalition. A coalition $C \subseteq N$ blocks a coalition structure $\Pi = \{S_0, S_1, \ldots, S_k\}$ if $\Pi' \succ_i \Pi$ for all $i \in C$, where $\Pi' = \{S_0 \setminus C, S_1 \setminus C, \ldots, S_k \setminus C, C\}$ is obtained by Π adding coalition C and removing the players in C from the elements of Π. Then, a coalition structure that is not blocked by any coalition is said stable and the core $C(\hat{G})$ of a non-hedonic game $\hat{G} = (N, (\succeq_i)_{i\in N})$ is the set of all coalition structures that are stable.

Consider a power relation $\sqsupseteq \in \mathcal{T}^{\mathcal{P}(N)}$. We associate to each partition Π of N a ranking $\succeq_\Pi$ over players in N such that $i \succ_\Pi j$ if

- either $\Pi(i) \sqsupset \Pi(j)$,
- or $\Pi(i) \sim \Pi(j)$ and $i\ P_{le}^{\sqsupseteq_S}\ j$, where $S = \bigcup_{j\in N:\Pi(j)\sim\Pi(i)} \Pi(j)$;

and $i \sim_\Pi j$, otherwise. In words, we say that, within a coalition structure Π, i is strictly "stronger" than j (i.e, $i \succ_\Pi j$) if

- either player i is in a strictly stronger coalition than the one of player j according to $\sqsupseteq$,
- or i and j belong to equivalent coalitions in Π, but i has a strictly better position than j according to the lex-cel computed on the union of the coalitions of Π that are in the same equivalence class as $\Pi(i)$ and $\Pi(j)$;

and i and j are equally strong ($i \sim_\Pi j$) if they belong to equivalent coalitions in Π and the lex-cel ranks them equally on $\sqsupseteq_S$. The quotient order associated to $\succeq_\Pi$ is denoted by $\Gamma_1^\Pi \succeq_\Pi \Gamma_2^\Pi \succeq_\Pi \ldots \succeq_\Pi \Gamma_q^\Pi$. For any $i \in N$, the *position* of i in the quotient order is defined as

$$rk_i^{\sqsupseteq}(\Pi) = \frac{1}{|\Gamma_u^\Pi|} \sum_{k=1+\Xi_{u-1}}^{\Xi_u} k, \tag{4}$$

where $\Xi_u = |\bigcup_{h=1}^{u} \Gamma_h^\Pi| = \sum_{h=1}^{u} |\Gamma_h^\Pi|$. Notice that $rk_i^{\sqsupseteq}(\Pi) \in [1, |N|]$ and $i \succeq_\Pi j \Leftrightarrow rk_i^{\sqsupseteq}(\Pi) \geq rk_j^{\sqsupseteq}(\Pi)$, and the sum of the positions in the ranking is equal to $\sum_{k=1}^{|N|} k = \frac{|N|(|N|+1)}{2}$. We also point out that, although not made explicit, $rk_i^{\sqsupseteq}(\Pi)$ depends on the choice of the social ranking used.

Finally, we define a non-hedonic game $\hat{G} = (N, (\succeq_i)_{i\in N})$ with players' preferences $(\succeq)_{i\in N}$ over coalition structures as follows

$$\Pi \succeq_i \widetilde{\Pi} \iff rk_i^{\sqsupseteq}(\Pi) \leq rk_i^{\sqsupseteq}(\widetilde{\Pi}), \tag{5}$$

for every $i \in N$ and all coalition structures $\Pi, \widetilde{\Pi} \subset \mathcal{P}(N)$. The coalition formation situation represented by a non-hedonic game $\hat{G}$ is similar to the one studied in paper [10]. However, some relevant differences exist between the two models:

- the model studied in [10] is defined on the domain of power relations that are total orders over coalitions, whereas in our case we consider total preorders (i.e., we allow for the indifference between distinct coalitions);
- in [10] the individual positions within a coalition are given exogenously, while in our approach they are determined using a social ranking computed on a power relation;
- we compare each player with all the members in equivalent coalitions and not only in her own coalition, as done in [10].

Example 4.1. Consider $N = \{1,2,3\}$ and the power relation $\sqsupseteq$ defined as: $\{1,2,3\} \sqsupset \{1,2\} \sim \{1,3\} \sqsupset \{2,3\} \sim \{1\} \sqsupset \{2\} \sim \{3\}$. Let $\Pi = \{\{1\},\{2,3\}\}$. Since $\{2,3\} \sim \{1\}$, to determine $\succeq_\Pi$ we must use the lex-cel on the union: $\{2,3\} \cup \{1\} = N$ (i.e., $R_{le}^{\sqsupseteq_N} = R_{le}^{\sqsupseteq}$).

Hence, $1 P_{le}^{\sqsupseteq} 2 I_{le}^{\sqsupseteq} 3$ and $\Gamma_1^\Pi = \{1\}$, $\Gamma_2^\Pi = \{2\}$ and $\Gamma_2^\Pi = \{3\}$. Then,

- $rk_1^{\sqsupseteq}(\Pi) = \frac{1}{|\Gamma_1^\Pi|} \sum_{h=1}^{|\Gamma_1^\Pi|} k = 1$
- $rk_2 \sqsupseteq (\Pi) = rk_3 \sqsupseteq (\Pi) = \frac{1}{|\Gamma_2^\Pi|} \sum_{h=1+|\Gamma_1^\Pi|}^{|\Gamma_1^\Pi|+|\Gamma_2^\Pi|} k = \frac{1}{2}(2+3) = 2.5.$

In this way we can compute, for every partition Π, the associated vector $rk^{\sqsupseteq}(\Pi) = (rk_1^{\sqsupseteq}(\Pi), rk_2^{\sqsupseteq}(\Pi), rk_3^{\sqsupseteq}(\Pi))$:
- $\Pi_0 = \{\{1,2,3\}\} \Longrightarrow rk^{\sqsupseteq}(\Pi_0) = (1, 2.5, 2.5)$;
- $\Pi_1 = \{\{1\},\{2,3\}\} \Longrightarrow rk^{\sqsupseteq}(\Pi_1) = (1, 2.5, 2.5)$
- $\Pi_2 = \{\{2\},\{1,3\}\} \Longrightarrow rk^{\sqsupseteq}(\Pi_2) = (1, 3, 2)$;
- $\Pi_3 = \{\{3\},\{1,2\}\} \Longrightarrow rk^{\sqsupseteq}(\Pi_3) = (1, 2, 3)$;
- $\Pi_4 = \{\{1\},\{2\},\{3\}\} \Longrightarrow rk^{\sqsupseteq}(\Pi_3) = (1, 2.5, 2.5)$.
We can thus deduce the preferences over coalition structures:

(1) $\Pi_0 \sim_1 \Pi_1 \sim_1 \Pi_2 \sim_1 \Pi_3 \sim_1 \Pi_4$;
(2) $\Pi_3 \succ_2 \Pi_0 \sim_2 \Pi_1 \sim_2 \Pi_4 \succ_2 \Pi_2$;
(3) $\Pi_2 \succ_3 \Pi_0 \sim_3 \Pi_1 \sim_3 \Pi_4 \succ_3 \Pi_3$;

So, in this example, every partition is core-stable. In fact, in order to get a better ranking position, players 2 and 3 have to cooperate with player 1, but player 1 has no incentive to deviate, since $rk_1^{\sqsupseteq}(\Pi) = 1$ in every coalition structure Π.

As we already noticed, one of the main differences with respect to paper [10], is that we deduce the individual positions using a social ranking computed on a power relation. So, if we consider a power relation $\sqsupseteq$ that is a linear order (instead of a total preorder) and we use ρ_1 as social ranking (instead of the lex-cel), then we would fall exactly under the assumptions of paper [10]. Consequently, in order to recover the results from model [10] using the lex-cel, we need to consider a responsive total order, for in this case, by Lemma 3.6, the lex-cel coincides with ρ_1.
Next, we introduce a special class of power relations, which, as it will be shown later, do not admit core-stable partitions.

Definition 4.2. A power relation $\sqsupseteq$ on $\mathcal{P}(N)$ is said to be homogeneous if

$$|S| > |T| \Rightarrow S \sqsupset T \quad \text{ for all } S, T \in \mathcal{P}(N).$$

Proposition 4.3. *Let N be such that $|N| \geq 7$. Let $\sqsupseteq$ be a homogeneous and responsive total order on $\mathcal{P}(N)$. Then, the (non-hedonic) game $\hat{G} = (N, (\succeq_i)_{i\in N})$ with preferences over coalition structures defined according to relation* (5) *has an empty core.*

Proof. By Lemma 3.6, the ranking provided by the lex-cel on $\sqsupseteq$ correspons to the total order defined by $\sqsupseteq$ over the singleton coalitions. As a consequence, the players' ranking position $rk_i^{\sqsupseteq}(\Pi)$, for each $i \in N$, coincides precisely with the player i's position in the coalition structure Π according to the definition of player i's position provided on page 6 of the paper [10]. Therefore, the proof follows the same steps of the one of Proposition 3 in the paper [10], and it is omitted in this paper.

□

The assumption of responsiveness for power relations (not assumed in [10]) is crucial for the proof of Proposition 4.3, as shown by the following example with $|N| = 7$ where a total power relation is homogeneous but not responsive and where the core is not empty.

Example 4.4. Let $N = \{1, 2, \dots, 7\}$ and define a homogeneous and total order $\sqsupseteq$ on $\mathcal{P}(N)$ such that $i < j \Rightarrow N \setminus \{j\} \sqsupset N \setminus \{i\}$ for all $i, j \in N$. Notice that $1 \, P_{le}^{\sqsupseteq} \, 2 \, P_{le}^{\sqsupseteq} \dots P_{le}^{\sqsupseteq} \, 7$. Consider the partition $\Pi = \{N\}$. It is easy to check that $rk_i^{\sqsupseteq}(\Pi) = i$ for all $i \in N$. Let $i_S = \min\{j \in S\}$ for all $S \subseteq N$. Moreover, let $\sqsupseteq$ be such that for all $S \subset N$, $|S| < 7-1$, and $j \in S \setminus \{i_S\}$ we have $S \setminus \{i_S\} \sqsupset S \setminus \{j\}$. Notice that $\sqsupseteq$ is not responsive (for all $j \in S \setminus \{i_S\}$, $S \setminus \{i_S\} \sqsupset S \setminus \{j\}$, but $N \setminus \{j\} \sqsupset N \setminus \{i_S\}$). For any coalition $T \subset N$, $T \neq N$, according to the lex-cel applied on the power relation $\sqsupseteq_T$, we have $j \, P_{le}^{\sqsupseteq_T} \, i_T$ for all $j \in T \setminus \{i_T\}$ (player i_T, with the lowest index in T, has the worst position in the lex-cel ranking computed on $\sqsupseteq_T$).

Consider a partition $\Pi_S = \{S, N \setminus S\}$. We distinguish two cases:

- $6 \geq |S| \geq 4$: Notice that $S \cap \{1, 2, 3, 4\} \neq \emptyset$ and $i_S = rk_{i_S}^{\sqsupseteq}(\Pi) \leq 4$. By the homogeneity of $\sqsupseteq$, in the coalition structure Π_S, player i_S is strictly stronger than any player in $N \setminus S$, but any player in S, as we already noticed, is strictly stronger than i_S. So, $rk_{i_S}^{\sqsupseteq}(\Pi_S) \geq 4 \geq rk_{i_S}^{\sqsupseteq}(\Pi)$: S is not blocking for Π.
- $|S| < 4$: By the homogeneity of $\sqsupseteq$, in the coalition structure Π_S, any player in $N \setminus \{i_S\}$ is strictly stronger than i_S. Moreover, $j \, P_{le}^{\sqsupseteq_S} \, i_S$ for all $j \in S \setminus \{i_S\}$. So, $rk_{i_S}^{\sqsupseteq}(\Pi_S) = 7 \geq rk_{i_S}^{\sqsupseteq}(\Pi)$: S is not blocking for Π.

We have shown that Π is in the core of game $\hat{G} = (N, (\succeq_i)_{i\in N})$.

5 CONCLUSIONS

In this paper, we introduced coalition formation models and algorithms aimed at studying the impact of two social attributes, i.e, the strength of coalitions in a society and the position of players within coalitions, on the existence of core-stable coalition structures.

The majority of the results presented in this paper, follows from the basic assumption that the social ranking used to evaluate the position of each player within a coalition is the the lex-cel [7]. Other social rankings have been proposed in the literature (see, for instance, paper [2]), and it would be interesting to analyse and compare the results about the stability of coalition structures adopting alternative notions of social ranking. In a similar fashion, relaxing the hypothesis that a power relation must be a total relation (as a matter of facts, some comparisons among coalitions could be not feasible or not available for practical reasons) could also be useful to better understand the mechanisms governing the construction of a society. Some preliminary results on our general (non-hedonic) framework in Section 4 also suggest new research directions aimed at characterizing new classes of non-responsive power relations with a non-empty core.

In this paper, we focused on the investigation of the core of (non-)hedonic games and its algorithmic characterization. From a practical point of view, however, another reasonable approach could be more oriented to stochastic models and simulations. For instance, considering a population of players having heterogeneous preferences (selected among those proposed in this study). In this direction, an interesting research avenue that we are currently exploring, is the analysis of the recurrence of coalition structures using a Markov-chain, where the states of the chain are the possible partitions of the set of players, and where the probability of transition from one state to another is determined by the presence or absence of blocking coalitions. This approach is particularly relevant for models where the core is empty, or where an efficient procedure for findings stable coalition structures is not available. In fact, it is clear that if no stable coalition structures exist, then, whatever coalition structure is considered, it is blocked by some coalition. So, using the terminology of Markov chains, this is equivalent to say that no absorbing state exists, and therefore it is always possible to end up in a new coalitional structure with positive probability. In this setting, as a possible alternative to the notion of core, one can be interested in looking for coalition structures that are "more recurrent" than others, and in studying the asymptotic behaviour of the chain.

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